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MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
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ERROR ANALYSIS IN L^p, $1 \leqslant p \leqslant \infty$, FOR MIXED FINITE ELEMENT METHODS FOR LINEAR AND QUASI-LINEAR ELLIPTIC PROBLEMS

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Abstract. — *We consider the approximation by mixed finite element method of second order elliptic problems in* $\mathbb{R}^2$. *We show that error estimates in* L^p *follow from stability properties of a weighted* L^2*-projection on the divergence free vectors of the finite element space. Since we work in two dimensions, we show that this projection is related with a Ritz projection and consequently optimal* L^p *estimates for* $1 < p < \infty$ *can be derived easily from the known results for the standard finite element method. Also quasi-optimal* L^∞ *and* L^1 *estimates are obtained. Finally we analyze a quasi-linear problem obtaining similar results than in the linear case.*

Résumé. — *On considère l'approximation par éléments finis mixtes des opérateurs elliptiques de deuxième ordre dans* $\mathbb{R}^2$. *On montre que les estimations d'erreur dans la norme* L^p *peuvent s'obtenir à partir des propriétés de stabilité d'une projection* L^2 *avec des poids dans l'ensemble des vecteurs à divergence nulle appartenant à l'espace des éléments finis.*

Comme on travaille en deux dimensions on montre que cette projection est liée à une projection de Ritz. En conséquence, des estimations d'erreur optimales en norme L^p *pour* $1 < p < \infty$ *peuvent être déduites des résultats connues de la méthode des éléments finis classique. Aussi, estimations d'erreur quasi optimales dans les normes* L^∞ *et* L^1 *ont été obtenues. Finalement, on considère un problème quasi linéaire pour lequel on obtient les mêmes résultats obtenus pour le cas linéaire.*

1. INTRODUCTION

Let Ω be a smooth and simply connected bounded domain in $\mathbb{R}^2$ and consider the Dirichlet problem

$$\begin{cases} -\operatorname{div}\,(a(x)\,\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

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where the coefficient $a(x)$ is assumed to be Lipschitz and bounded by below by a positive constant.

In many applications (see for instance [5], [6]) the variable of interest is

$$\underset{\sim}{q} = -a \nabla u$$

and then it is desirable to use a mixed finite element method which approximates $\underset{\sim}{q}$ and u simultaneously. With this purpose the problem (1.1) is decomposed into a first order system as follows

$$(1.2) \qquad \begin{cases} \underset{\sim}{q} + a \nabla u = 0 & \text{in } \Omega \\ \operatorname{div} \underset{\sim}{q} = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega . \end{cases}$$

If we call $\alpha(x) = a^{-1}(x)$, the weak formulation appropriate for the mixed method is

$$(1.3) \qquad \begin{cases} (\alpha \underset{\sim}{q}, \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, u) = 0, & \forall v \in \underset{\sim}{V} \\ (\operatorname{div} \underset{\sim}{q}, w) = (f, w), & \forall w \in W \end{cases}$$

where (,) denotes the L^2-product,

$$\underset{\sim}{V} = H(\operatorname{div}, \Omega) = \{\underset{\sim}{v} \in [L^2(\Omega)]^2 : \operatorname{div} \underset{\sim}{v} \in L^2(\Omega)\}$$

and $\qquad W = L^2(\Omega)$.

The approximation of (1.3) by finite elements has been studied in several works (see [1], [2], [7], [8], [21]). In [21] Raviart and Thomas introduced finite element spaces $\underset{\sim}{V}_h^k \subset \underset{\sim}{V}$ and $W_h^k \subset W$ satisfying the inf-sup condition of [1]. As a consequence of the abstract theory developed in [1], they obtained optimal order error estimates for the approximation of the Laplace equation using the formulation in (1.3). Their results were generalized by Douglas and Roberts [7], [8], for a more general second order elliptic equation.

A different family of finite element spaces was introduced by Brezzi, Douglas and Marini in [2]. These spaces produce optimal order approximation for the vector variable but reducing the degrees of freedom and consequently simplifying the algebraic problem.

Many works have been devoted to analyze the convergence in L^∞ of mixed finite element methods. For the spaces of [21], Scholz [23], [24], [25], proved optimal convergence but excluding the lowest degree space, one of the most interesting in practice. This case was studied recently by Kwon and Milner [15], [16] who obtained sub-optimal error estimates, and Gastaldi and Nochetto [10], [11] who were able to prove quasi-optimal error estimates. In a more recent work [12] Gastaldi and Nochetto extended their previous results to include the new spaces introduced in [2].

The objective of this paper is to present an error analysis in L^p, $1 \leqslant p \leqslant \infty$, for the mixed method, which relies on the study of a weighted L^2-projection R_h on the divergence free subspace of the finite element spaces. This subspace turns out to be the same for the spaces of [21] than for the spaces of [2] when working with triangular decompositions, and consequently the analysis is the same for both families of spaces.

Since we are considering the two dimensional case, the projection R_h is very related with a Ritz projection and therefore, error estimates for the vector unknown can be derived in a simple way from known results for standard Galerkin methods. Afterwards, error estimates for the scalar variable can be proven by duality as was done for the L^2-case in [7].

When $1 < p < \infty$, the logarithmic factor is removed even for the lowest degree spaces, resembling a similar situation arising in the standard finite element method (see [20]).

We also analyze the following quasi-linear problem

$$\begin{cases} -\operatorname{div}\,(a(x,u)\,\nabla u + \underset{\sim}{b}(x,u)) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

obtaining quasi-optimal uniform estimates and optimal order L^p estimates for $2 \leqslant p \leqslant \infty$ also in this case, when the Raviart-Thomas spaces are used.

The quasi-linear problem (1.4) was analyzed by Milner [17] in L^2 and by Kwon and Milner [16] in L^∞ but obtaining sub-optimal error estimates in the last case.

The paper is organized as follows. In section 2 we introduce the finite element spaces and the projections we are going to work with. Section 3 deals with the error analysis for both vector and scalar variables in the linear case. In section 4 we study the quasi liner problem and finally we make some remarks in section 5.

2. FINITE ELEMENT SPACES AND PROJECTIONS

Let $\{Z_h\}$ be a quasi-uniform family of decompositions of Ω into triangles. We assume that $\Omega = \bigcup_{T \in Z_h} T$ and so, boundary triangles can have a curved side.

Given an integer $k \geqslant 0$, the Raviart-Thomas space of index k [21], associated with Z_h,

$$RT_k = \underset{\sim}{V}_{h,1}^k \times W_{h,1}^k$$

is defined by

$$\underset{\sim}{V}^k_{h,1} = \left\{ \underset{\sim}{v} \in H(\mathrm{div}, \Omega) : \underset{\sim}{v}|_T \in \underset{\sim}{P}_k \oplus \underset{\sim}{x} P_k \right\}$$

and

$$W^k_{h,1} = \left\{ w \in L^2(\Omega) : w|_T \in P_k \right\}$$

where P_k denotes the space of polynomials of degree less than or equal to k and $\underset{\sim}{P}_k = [P_k]^2$.

The Brezzi-Douglas-Marini spaces [2]

$$BDM_k = \underset{\sim}{V}^k_{h,2} \times W^k_{h,2}$$

are defined for $k \geqslant 1$ by,

$$\underset{\sim}{V}^k_{h,2} = \left\{ \underset{\sim}{v} \in H(\mathrm{div}, \Omega) : \underset{\sim}{v}|_T \in \underset{\sim}{P}_k \right\}$$

and

$$W^k_{h,2} = \left\{ w \in L^2(\Omega) : w|_T \in P_{k-1} \right\} .$$

If $\underset{\sim}{V}^k_h$ and W^k_h are either $\underset{\sim}{V}^k_{h,1}$ and $W^k_{h,1}$ or $\underset{\sim}{V}^k_{h,2}$ and $W^k_{h,2}$, a projection operator

$$\Pi^k_h : \underset{\sim}{V} \to \underset{\sim}{V}^k_h$$

such that the following diagram commutes can be constructed (see [2], [9], [21])

$$\begin{tikzcd} \underset{\sim}{V} \arrow[r, "\mathrm{div}"] \arrow[d, "\Pi^k_h"'] & W \arrow[d, "P^k_h"] & \\ \underset{\sim}{V}^k_h \arrow[r] & W^k_h \arrow[r] & 0 \end{tikzcd} \tag{2.1}$$

where P^k_h denotes the L^2-projection. Moreover,

$$\| \underset{\sim}{v} - \Pi^k_h \underset{\sim}{v} \|_{L^p} \leqslant C h^r \| \underset{\sim}{v} \|_{W^{r,p}} , \quad 1 \leqslant p \leqslant \infty , \quad 1 \leqslant r \leqslant k+1 \tag{2.2}$$

(the letter C denotes a constant, not necessarily the same at each occurrence and $W^{k+1,p}$ stands for the usual Sobolev space of functions with derivatives up to the order $k+1$ in L^p).

The operator Π_h^k is the main tool in the L^2-analysis (see [7]) and will play a similar role in the L^p-case.

Let us now introduce the space of piecewise polynomials of degree $k+1$,

$$M_h^{k+1} = \left\{ \phi \in H^1(\Omega) : \phi|_T \in P_{k+1} \quad \text{and} \quad \int_\Omega \phi(x)\, dx = 0 \right\}.$$

Now, we state a lemma that relates the two families of mixed finite elements introduced above.

LEMMA 2.1 : *Let*

$$\mathring{\underset{\sim}{V}}{}_{h,1}^{k} = \{ \underset{\sim}{v} \in \underset{\sim}{V}_{h,1}^{k} : \operatorname{div} \underset{\sim}{v} = 0 \}$$

and

$$\mathring{V}_{h,2}^{k} = \{ \underset{\sim}{v} \in \underset{\sim}{V}_{h,2}^{k} : \operatorname{div} \underset{\sim}{v} = 0 \} \quad \text{then ,}$$

$$\mathring{\underset{\sim}{V}}{}_{h,1}^{k} = \mathring{\underset{\sim}{V}}{}_{h,2}^{k} = \underset{\sim}{\operatorname{curl}}\, M_h^{k+1}$$

where

$$\underset{\sim}{\operatorname{curl}}\, \phi = \left(-\frac{\partial \phi}{\partial x_2}, \frac{\partial \phi}{\partial x_1} \right).$$

Proof: Clearly $\mathring{V}_{h,2}^{k} \subset \mathring{V}_{h,1}^{k}$. If $\underset{\sim}{v} \in V_{h,1}^{k}$ and $\operatorname{div} \underset{\sim}{v} = 0$ it is easy to prove (see for instance [18]) that $\underset{\sim}{v}|_T \in \underset{\sim}{P}_k$ and therefore $\underset{\sim}{v} \in \mathring{\underset{\sim}{V}}{}_{h,2}^{k}$, which proves $\mathring{\underset{\sim}{V}}{}_{h,1}^{k} = \mathring{\underset{\sim}{V}}{}_{h,2}^{k}$.

From now on we drop the subscripts and denote $\mathring{\underset{\sim}{V}}{}_{h}^{k} = \mathring{\underset{\sim}{V}}{}_{h,1}^{k} = \mathring{\underset{\sim}{V}}{}_{h,2}^{k}$. If $\phi \in M_h^{k+1}$ then $\underset{\sim}{v} = \underset{\sim}{\operatorname{curl}}\, \phi$ is divergence free and $\underset{\sim}{v}|_T \in \underset{\sim}{P}_k$.

On the other hand, since Ω is simply connected, given $\underset{\sim}{v} \in H(\operatorname{div}, \Omega)$ such that $\operatorname{div} \underset{\sim}{v} = 0$, there exists $\phi \in H^1(\Omega)$ such that $\underset{\sim}{v} = \underset{\sim}{\operatorname{curl}}\, \phi$. Now, if $\underset{\sim}{v} \in \underset{\sim}{P}_k$, $\phi|_T \in P_{k+1}$ and moreover ϕ can be chosen with mean value equal to zero. Therefore, $\underset{\sim}{\operatorname{curl}}\, M_h^{k+1} = \mathring{\underset{\sim}{V}}{}_{h}^{k}$ as we wanted to prove. □

We end this section by introducing the following weighted L^2-projection which plays an important role in our error analysis,

$$R_h : [L^1(\Omega)]^2 \to \mathring{\underset{\sim}{V}}{}_{h}^{k}.$$

For $\underset{\sim}{v} \in [L^1(\Omega)]^2$, $R_h \underset{\sim}{v} \in \mathring{\underset{\sim}{V}}{}_{h}^{k}$ is defined by

$$(\alpha(\underset{\sim}{v} - R_h \underset{\sim}{v}), \underset{\sim}{r}) = 0 \quad \forall r \in \mathring{\underset{\sim}{V}}{}_{h}^{k}. \tag{2.3}$$

3. ERROR ANALYSIS IN L^p

Let $\underset{\sim}{V}_h^k \times W_h^k$ be any of the finite element spaces introduced in section 2. Then, the mixed finite element approximation to problem (1.1) is given by $(\underset{\sim}{q}_h, u_h) \in \underset{\sim}{V}_h^k \times W_h^k$ satisfying,

$$\begin{cases} (\alpha \underset{\sim}{q}_h, v) - (\operatorname{div} \underset{\sim}{v}, u_h) = 0 & \forall v \in \underset{\sim}{V}_h^k \\ (\operatorname{div} \underset{\sim}{q}_h, w) = (f, w) & \forall w \in W_h^k . \end{cases} \tag{3.1}$$

Subtracting (3.1) from (1.3) we obtain the error equations, namely

$$(\alpha(\underset{\sim}{q} - \underset{\sim}{q}_h), \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, u - u_h) = 0 \quad \forall v \in \underset{\sim}{V}_h^k \tag{3.2}$$

$$(\operatorname{div} (\underset{\sim}{q} - \underset{\sim}{q}_h), w) = 0 \quad \forall w \in W_h^k . \tag{3.3}$$

From (3.3) and the property (2.1) we obtain,

$$(\operatorname{div} (\Pi_h \underset{\sim}{q} - \underset{\sim}{q}_h), w) = 0 \quad \forall w \in W_h^k$$

and since $\operatorname{div} V_h = W_h$ it follows that,

$$\Pi_h \underset{\sim}{q} - \underset{\sim}{q}_h \in \mathring{V}_h^k . \tag{3.4}$$

Now, using (3.2) we get

$$(\alpha[(\underset{\sim}{q} - \Pi_h \underset{\sim}{q}) - (\underset{\sim}{q}_h - \Pi_h \underset{\sim}{q})], \underset{\sim}{v}) = 0 \quad \forall \underset{\sim}{v} \in \mathring{V}_h^k$$

therefore, we have

$$\underset{\sim}{q}_h - \Pi_h \underset{\sim}{q} = R_h(\underset{\sim}{q} - \Pi_h \underset{\sim}{q}) . \tag{3.6}$$

Then, in view of the approximation properties of Π_h (2.2), the convergence analysis is reduced to the study of the stability properties of the projection R_h.

To study this projection we will make use of some known results for a Ritz projection, that we recall now.

Let $\phi \in H^1(\Omega)$, $\int_\Omega \phi \, dx = 0$ and let $\phi_h \in M_h^{k+1}$ such that,

$$(\alpha \underset{\sim}{\operatorname{curl}} (\phi - \phi_h), \underset{\sim}{\operatorname{curl}} \mu) = 0 \quad \forall \mu \in M_h^{k+1} . \tag{3.7}$$

Note that (3.7) can be written as

$$(\alpha \nabla (\phi - \phi_h), \nabla \mu) = 0 \quad \forall \mu \in M_h^{k+1}$$

and therefore, it is known that

$$\|\underset{\sim}{\operatorname{curl}}\, \phi_h\|_{L^p} \leq C \|\underset{\sim}{\operatorname{curl}}\, \phi\|_{L^p}, \quad 2 \leq p \leq \infty . \tag{3.8}$$

In fact (3.8) was proven in a slightly different situation by Nitsche [19] for $k \geq 1$ and by Rannacher and Scott [20] for $k = 0$. We are considering Neumann instead of Dirichlet boundary conditions but their proofs can be carried out for this case with minor modifications. Also, we are working with an operator with variable coefficient but since $\alpha(x)$ is a Lipschitz function, all the regularity properties needed in the proofs are satisfied.

Now we prove the main theorem of this section.

THEOREM 3.1 *Let $2 \leq p < \infty$ and let R_h be the projection defined in* (2.3). *Then, for every $\underset{\sim}{v} \in [L^p(\Omega)]^2$ we have,*

$$\|R_h \underset{\sim}{v}\|_{L^p} \leq C \|\underset{\sim}{v}\|_{L^p} + C_p \|\operatorname{div} \underset{\sim}{v}\|_{W^{-1,p}} \tag{3.9}$$

where C is a constant independent of $\underset{\sim}{v}$, h and p and $C_p = Cp$.

Proof: Given $\underset{\sim}{v} \in [L^p(\Omega)]^2$ we decompose it as

$$\underset{\sim}{v} = \underset{\sim}{\operatorname{curl}}\, \phi + a \nabla \psi \tag{3.10}$$

where ψ is the solution of the problem

$$\begin{cases} \operatorname{div}(a \nabla \psi) = \operatorname{div} \underset{\sim}{v} & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega \end{cases}$$

and ϕ is chosen such that $\int_\Omega \phi \, dx = 0$.

Since $a(x)$ is Lipschitz, it is known (see [4], [13], [22]) that,

$$\|\psi\|_{W^{1,p}} \leq C_p \|\operatorname{div} \underset{\sim}{v}\|_{W^{-1,p}} \tag{3.11}$$

and tracing constants in the proof one can see that $C_p = Cp$ this is the dependence on p for the constant arising in the Calderón-Zygmund theory of singular integral operators (see [3], [26]), which is the main tool for proving the a priori estimate (3.11)).

From (3.10) and (3.11) it follows that,

$$\|\underset{\sim}{\operatorname{curl}}\, \phi\|_{L^p} \leq \|\underset{\sim}{v}\|_{L^p} + C_p \|\operatorname{div} \underset{\sim}{v}\|_{W^{-1,p}} . \tag{3.12}$$

Now, from Lemma 2.1 we have,

$$(\alpha(\underset{\sim}{v} - R_h \underset{\sim}{v}), \underset{\sim}{\mathrm{curl}}\, \mu) = 0 \quad \forall \mu \in M_h^{k+1}. \tag{3.13}$$

Since $R_h \underset{\sim}{v} \in \mathring{V}_h^k$, it can be written as

$$R_h \underset{\sim}{v} = \underset{\sim}{\mathrm{curl}}\, \phi_h ,$$

with $\phi_h \in M_h^{k+1}$.

Then, using (3.10) and (3.13) we get,

$$\begin{aligned}(\alpha(\underset{\sim}{v} - R_h \underset{\sim}{v}), \underset{\sim}{\mathrm{curl}}\, \mu) &= (\alpha(\underset{\sim}{\mathrm{curl}}\, \phi + a\, \nabla\psi - \underset{\sim}{\mathrm{curl}}\, \phi_h), \underset{\sim}{\mathrm{curl}}\, \mu) \\ &= 0 \quad \forall \mu \in M_h^{k+1}\end{aligned} \tag{3.14}$$

but, $(\alpha a\, \nabla\psi, \underset{\sim}{\mathrm{curl}}\, \mu) = (\psi, \mathrm{div}\, \underset{\sim}{\mathrm{curl}}\, \mu) = 0$ and therefore,

$$(\alpha(\underset{\sim}{\mathrm{curl}}\, \phi - \underset{\sim}{\mathrm{curl}}\, \phi_h), \underset{\sim}{\mathrm{curl}}\, \mu) = 0 \quad \forall \mu \in M_h^{k+1}$$

and from (3.8) we obtain,

$$\| R_h \underset{\sim}{v} \|_{L^p} \leq C \, \| \underset{\sim}{\mathrm{curl}}\, \phi \|_{L^p} \tag{3.15}$$

which together with (3.12) proves the theorem.

COROLLARY 3.1 : *For* $2 \leq p < \infty$ *and* $\underset{\sim}{v} \in [L^p(\Omega)]^2$

$$\| R_h \underset{\sim}{v} \|_{L^p} \leq C_p \| \underset{\sim}{v} \|_{L^p} .$$

COROLLARY 3.2 : *Let* $\underset{\sim}{v} \in [L^{\infty}(\Omega)]^2$ *then,*

$$\| R_h \underset{\sim}{v} \|_{L^\infty} \leq C \, \| \underset{\sim}{v} \|_{L^\infty} + C \, |\log h| \, \| \mathrm{div}\, v \|_{W^{-1,\infty}} \tag{3.16}$$

and

$$\| R_h \underset{\sim}{v} \|_{L^\infty} \leq C \, |\log h| \, \| \underset{\sim}{v} \|_{L^\infty} . \tag{3.17}$$

Proof: Using an inverse inequality we have,

$$\| R_h \underset{\sim}{v} \|_{L^\infty} \leq C h^{-2/p} \| R_h \underset{\sim}{v} \|_{L^p} \quad \text{for} \quad 2 \leq p < \infty$$

and from the theorem it follows,

$$\| R_h \underset{\sim}{v} \|_{L^\infty} \leq C h^{-2/p} \| v \|_{L^p} + C_p \, h^{-2/p} \| \mathrm{div}\, \underset{\sim}{v} \|_{W^{-1,p}} \quad 2 \leq p < \infty . \tag{3.18}$$

Now, following the argument in [14] we take $p = |\log h|$ in (3.18) and we obtain (3.16) and consequently (3.17).

As a consequence of Theorem 3.1, we have the following theorem about the error for the vector variable in the mixed method.

THEOREM 3.2 : *Let V_h^k be one of the spaces defined in section* 2. *Let $\underset{\sim}{q}_h \in V_h^k$ be the approximate solution of q defined in* (3.1) *then,*

$$\|\underset{\sim}{q} - \underset{\sim}{q}_h\|_{L^p} \leq C \|\underset{\sim}{q} - \Pi_h \underset{\sim}{q}\|_{L^p} + C_p \|\operatorname{div}(\underset{\sim}{q} - \Pi_h \underset{\sim}{q})\|_{W^{-1,p}}, \quad 2 \leq p < \infty \tag{3.19}$$

and

$$\|\underset{\sim}{q} - \underset{\sim}{q}_h\|_{L^\infty} \leq C \|\underset{\sim}{q} - \Pi_h \underset{\sim}{q}\|_{L^\infty} + C |\log h| \|\operatorname{div}(\underset{\sim}{q} - \Pi_h \underset{\sim}{q})\|_{W^{-1,\infty}} . \tag{3.20}$$

Proof: The results follow immediately from (3.6), (3.9), (3.16) and the triangular inequality.

COROLLARY 3.3 : *Let $\underset{\sim}{q}_h \in \underset{\sim}{V}_h^k$ be as in Theorem* 3.2, *then*

$$\|\underset{\sim}{q} - \underset{\sim}{q}_h\|_{L^p} \leq C \|\underset{\sim}{q} - \Pi_h \underset{\sim}{q}\|_{L^p} + C_p h \|f - P_h f\|_{L^p}, \quad 2 \leq p < \infty \tag{3.21}$$

$$\|\underset{\sim}{q} - \underset{\sim}{q}_h\|_{L^p} \leq C_p \|\underset{\sim}{q} - \Pi_h \underset{\sim}{q}\|_{L^p}, \quad 2 \leq p < \infty \tag{3.22}$$

$$\|\underset{\sim}{q} - \underset{\sim}{q}_h\|_{L^\infty} \leq C \|\underset{\sim}{q} - \Pi_h \underset{\sim}{q}\|_{L^\infty} + Ch |\log h| \|f - P_h f\|_{L^\infty} \tag{3.23}$$

$$\|\underset{\sim}{q} - \underset{\sim}{q}_h\|_{L^\infty} \leq C |\log h| \|\underset{\sim}{q} - \Pi_h \underset{\sim}{q}\|_{L^\infty} . \tag{3.24}$$

Proof: (3.22) and (3.24) follow easily from (3.19) and (3.20). To prove (3.21) and (3.23) observe that,

$$\operatorname{div}(\underset{\sim}{q} - \Pi_h \underset{\sim}{q}) = f - P_h f$$

and by duality we can prove that

$$\|f - P_h f\|_{W^{-1,p}} \leq Ch \|f - P_h f\|_{L^p}, \quad 2 \leq p \leq \infty .$$

COROLLARY 3.4 : *Let $\underset{\sim}{q}_h \in V_h^k$ be as in Theorem* 3.2, *then,* if $\underset{\sim}{q} \in W^{k+1,p}$ *for some p such that $2 \leq p < \infty$,*

$$\|\underset{\sim}{q} - \underset{\sim}{q}_h\|_{L^p} \leq C_p \|q\|_{W^{k+1,p}} h^{k+1} \quad \text{for} \quad 2 \leq p < \infty \tag{3.25}$$

and if $\underset{\sim}{q} \in W^{k+1,\infty}$

$$\|\underset{\sim}{q} - \underset{\sim}{q}_h\|_{L^\infty} \leq C \{ \|q\|_{W^{k+1,\infty}} + |\log h| \|f\|_{W^{k,\infty}} \} h^{k+1} . \tag{3.26}$$

Proof: It follows immediately from the approximation properties of Π_h (2.2).

Remark 3.1 : The estimates (3.21) and (3.22) (or (3.23) and (3.24)) are different in essence. In fact (3.23) has the advantage that the logarithmic factor appears only in the second summand. Since

$$\left\| \underset{\sim}{q} - \Pi_h \underset{\sim}{q} \right\|_{L^\infty} \leqslant C h^{k+1} |\log h| \, \|f\|_{W^{k,\infty}}$$

(as can be shown using an argument of Johnson and Thomee [14] mentioned before) we obtain from (3.23) the following estimate in terms of f,

$$\left\| \underset{\sim}{q} - \underset{\sim}{q}_h \right\|_{L^\infty} \leqslant C h^{k+1} |\log h| \, \|f\|_{W^{k,\infty}} . \tag{3.27}$$

Also, the logarithmic factor can be removed from the estimate (3.25) if we assume that f has some extra regularity (as was pointed out in [10]).

On the other hand, the estimate (3.24) (which is new) is only in terms of $\left\| \underset{\sim}{q} - \Pi_h \underset{\sim}{q} \right\|_{L^\infty}$ and independent of f. Consequently, even when f is not bounded, but $\underset{\sim}{q} \in C^{0,\alpha}$ we obtain an asymptotic estimate in L^∞ of order α.

We now derive some error estimates for the scalar variable u.

THEOREM 3.3 : *Let W_h^k be one of the finite element spaces defined in section* 2 *and assume that the restriction of W_h^k to an element T contains P_1 (so, $k \geqslant 1$ for RT_k and $k \geqslant 2$ for* BDM_k*). Then, if $u_h \in W_h^k$ is the approximate solution defined in* (3.1) *we have,*

$$\|P_h u - u_h\|_{L^p} \leqslant C_p h \left\| \underset{\sim}{q} - \underset{\sim}{q}_h \right\|_{L^p} + C_p h^2 \left\| \operatorname{div} (\underset{\sim}{q} - \underset{\sim}{q}_h) \right\|_{L^p}, \quad 2 \leqslant p < \infty . \tag{3.28}$$

Proof: The proof for the case $p = 2$ given by Douglas and Roberts [7] can be generalized straightforward for our case. In fact, let $\phi \in L^{p^*}$ where $\dfrac{1}{p} + \dfrac{1}{p^*} = 1$ and let $\psi \in W_0^{1,p^*}$ such that

$$\operatorname{div} (a \nabla \psi) = \phi$$

then (see [7]),

$$(P_h u - u_h, \phi) = \\ = (\alpha(\underset{\sim}{q} - \underset{\sim}{q}_h), \Pi_h(a \nabla \psi) - a \nabla \psi) - (\operatorname{div} (\underset{\sim}{q} - \underset{\sim}{q}_h), \psi - P_h \psi)$$

and therefore

$$|(P_h u - u_h, \phi)| \leqslant \\ \leqslant C \left\| \underset{\sim}{q} - \underset{\sim}{q}_h \right\|_{L^p} h \|\psi\|_{W^{2,p^*}} + C \left\| \operatorname{div} (\underset{\sim}{q} - \underset{\sim}{q}_h) \right\|_{L^p} h^2 \|\psi\|_{W^{2,p^*}}$$

and using the a priori estimate

$$\|\psi\|_{W^{2,p^*}} \leq C_p \|\phi\|_{L^{p^*}}$$

we obtain the theorem.

Remark 3.2 : For the spaces RT_0 or BDM_1 we can prove with the same argument that

$$\|P_h u - u_h\|_{L^p} \leq C_p h \left\|\underset{\sim}{q} - \underset{\sim}{q}_h\right\|_{L^p} + C_p h \left\|\operatorname{div}(\underset{\sim}{q} - \underset{\sim}{q}_h)\right\|_{L^p}, \quad 2 \leq p < \infty . \tag{3.29}$$

COROLLARY 3.5 : *Under the hypothesis of Theorem* 3.3 *we have,*

$$\|P_h u - u_h\|_{L^p} \leq C_p^2 h^{k+2} \left\|\underset{\sim}{q}\right\|_{W^{k+1,p}}, \quad 2 \leq p < \infty \tag{3.30}$$

and

$$\|P_h u - u_h\|_{L^\infty} \leq C |\log h|^2 h^{k+2} \left\|\underset{\sim}{q}\right\|_{W^{k+1,\infty}} . \tag{3.31}$$

Proof: Since $\operatorname{div}(\underset{\sim}{q} - \underset{\sim}{q}_h) = f - P_h f$ we have,

$$\left\|\operatorname{div}(\underset{\sim}{q} - \underset{\sim}{q}_h)\right\|_{L^p} \leq C \left\|\underset{\sim}{q}\right\|_{W^{k+1,p}} h^k, \quad 2 \leq p \leq \infty$$

then (3.30) follows from (3.28) and (3.25). Now, since $C_p = Cp$ for $p \geq 2$, (3.31) can be obtained with the argument mentioned in Remark 3.1.

Remark 3.4 : For the spaces RT_0 or BDM_1 we have

$$\|P_h u - u_h\|_{L^p} \leq C_p^2 h^2 \left\{ \left\|\underset{\sim}{q}\right\|_{W^{1,p}} + \|\operatorname{div} q\|_{W^{1,p}} \right\}, \quad 2 \leq p < \infty . \tag{3.32}$$

Remark 3.5 : All the results in this section can be extended to the case $1 < p < 2$. Indeed, the inequality (3.8) is also valid in this case, now with a constant depending on p (see [20]). Therefore the following estimate can be obtained,

$$\|R_h \underset{\sim}{v}\|_{L^p} \leq C_p \|\underset{\sim}{v}\|_{L^p} + C_p^2 \|\operatorname{div} \underset{\sim}{v}\|_{W^{-1,p}}, \quad 1 < p < 2 \tag{3.33}$$

where $C_p = \dfrac{C}{p-1}$.

Moreover, the operator Π_h can be extended to $[W^{1,p}(\Omega)]^2$, $1 \leq p < 2$, straightforward and therefore the convergence results can be derived from (3.33) as we did for $p \geq 2$.

Remark 3.6 : When $p = 1$ we can show by duality that (3.17) implies

$$\|R_h \underset{\sim}{v}\|_{L^1} \leq C\,|\log h|\,\|\underset{\sim}{v}\|_{L^1}$$

and therefore, if $\underset{\sim}{q}_h$ is as in Theorem 3.2 we can prove the quasi-optimal estimate

$$\|\underset{\sim}{q} - \underset{\sim}{q}_h\|_{L^1} \leq C\,|\log h|\; h^{k+1}\|\underset{\sim}{q}\|_{W^{k+1\,1}}\,.$$

4. A QUASI-LINEAR PROBLEM

In this section we consider the following quasi-linear problem,

$$\text{(4.1)} \qquad \begin{cases} -\operatorname{div}\,(a(x,u)\,\nabla u + \underset{\sim}{b}(x,u)) = f & \text{in } \Omega \\ \qquad\qquad\qquad\qquad\qquad u = 0 & \text{on } \partial\Omega \end{cases}$$

We assume that the coefficients $a : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $\underset{\sim}{b} : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}^2$ are twice continuously differentiable functions with bounded derivatives up to the second order and that the problem (4.1) has a unique solution $u \in H^{2+\varepsilon}$ for some positive number ε (in the case $k = 0$ we will assume $u \in H^3$). We also assume that the operator is uniformly elliptic, that is for every $(x, u) \in \bar{\Omega} \times R$, $a(x,u) \geq a_1 > 0$.

In what follows we will write $a(u) = a(x,u)$ and $\underset{\sim}{b}(u) = \underset{\sim}{b}(x,u)$.

Set $\underset{\sim}{q} = -\,(a(u)\,\nabla u + \underset{\sim}{b}(u))$, $\alpha = a^{-1}$ and $\underset{\sim}{\beta} = \alpha\underset{\sim}{b}$. Then the weak formulation appropriate for the mixed method is,

$$\text{(4.2)} \qquad \begin{cases} (\alpha(u)\,\underset{\sim}{q}, \underset{\sim}{v}) - (\operatorname{div}\,\underset{\sim}{v}, u) = -\,(\underset{\sim}{\beta}(u), \underset{\sim}{v})\,, & \forall \underset{\sim}{v} \in \underset{\sim}{V} \\ \qquad\qquad (\operatorname{div}\,\underset{\sim}{q}, w) = (f, w)\,, & \forall w \in W\,. \end{cases}$$

For $k \geq 0$ let $\underset{\sim}{V}_h^k \times W_h^k$ the Raviart-Thomas spaces defined in Section 2. Then, the mixed finite element approximation $(\underset{\sim}{q}_h, u_h) \in \underset{\sim}{V}_h^k \times W_h^k$ is defined by,

$$\text{(4.3)} \qquad \begin{cases} (\alpha(u_h)\,\underset{\sim}{q}_h, \underset{\sim}{v}) - (\operatorname{div}\,\underset{\sim}{v}, u_h) = -\,(\underset{\sim}{\beta}(u_h), \underset{\sim}{v})\,, & \forall \underset{\sim}{v} \in \underset{\sim}{V}_h^k \\ \qquad\qquad (\operatorname{div}\,\underset{\sim}{q}_h, w) = (f, w)\,, & \forall w \in W_h^k\,. \end{cases}$$

The existence of the solution of (4.3) was proved by Milner [17] for h small enough.

Subtracting (4.3) from (4.2) we get the error equations,

$$\begin{cases} (\alpha(u)\, \underset{\sim}{q} - \alpha(u_h)\, \underset{\sim}{q}_h, \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, u - u_h) = (\underset{\sim}{\beta}(u_h) - \underset{\sim}{\beta}(u), \underset{\sim}{v})\,, & \forall \underset{\sim}{v} \in \underset{\sim}{V}_h^k \\ (\operatorname{div} (\underset{\sim}{q} - \underset{\sim}{q}_h), w) = 0\,, & \forall w \in W_h^k \end{cases}$$

or equivalently,

$$(4.4) \quad \begin{cases} (\alpha(u)(\underset{\sim}{q} - \underset{\sim}{q}_h), \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, u - u_h) = \\ = ([\alpha(u_h) - \alpha(u)]\, \underset{\sim}{q}_h, \underset{\sim}{v}) + (\underset{\sim}{\beta}(u_h) - \underset{\sim}{\beta}(u), \underset{\sim}{v}) & \forall \underset{\sim}{v} \in \underset{\sim}{V}_h^k \\ (\operatorname{div} (\underset{\sim}{q} - \underset{\sim}{q}_h), w) = 0 & \forall w \in W_h^k\,. \end{cases}$$

As in the linear case we can see that,

$$\Pi_h \underset{\sim}{q} - \underset{\sim}{q}_h \in \mathring{\underset{\sim}{V}}{}_h^k$$

and using (4.4) we obtain that

$$(\alpha(u)[(\underset{\sim}{q} - \Pi_h \underset{\sim}{q}) - (\underset{\sim}{q}_h - \Pi_h \underset{\sim}{q})], \underset{\sim}{v}) =$$
$$= ([\alpha(u_h) - \alpha(u)]\, \underset{\sim}{q}_h, \underset{\sim}{v}) + (\underset{\sim}{\beta}(u_h) - \underset{\sim}{\beta}(u), \underset{\sim}{v})\,, \quad \forall \underset{\sim}{v} \in \mathring{\underset{\sim}{V}}{}_h^k$$

then, if R_h denotes the L^2-projection on $\mathring{\underset{\sim}{V}}{}_h^k$ with weight $\alpha(u)$ we get,

$$(4.5) \quad \underset{\sim}{q}_h - \Pi_h \underset{\sim}{q} = R_h[a(u)(\alpha(u) - \alpha(u_h))\, \underset{\sim}{q}_h + \\ + a(u)(\underset{\sim}{\beta}(u) - \underset{\sim}{\beta}(u_h) + \underset{\sim}{q} - \Pi_h \underset{\sim}{q})]$$

In [17] it was proved that

$$(4.6) \qquad \|\underset{\sim}{q}_h\|_{L^\infty} \leq C\,(\|u\|_{H^{2+\varepsilon}})\,.$$

In [16] Kwon and Milner proved the following result which generalizes to the quasi-linear case the duality lemma proved by Douglas and Roberts [7].

LEMMA 4.1 : *Let $2 \leq p < \infty$ then, there exists a constant C depending on $\|u\|_{H^{2+\varepsilon}}$ when $k \geq 1$ and on $\|u\|_{H^3}$ when $k = 0$ such that,*

$$(4.7) \quad \|u_h - P_h u\|_{L^p} \leq C C_p \Big\{h \|\underset{\sim}{q} - \underset{\sim}{q}_h\|_{L^p} + h^{2-\delta_{0k}} \|\operatorname{div} (\underset{\sim}{q} - \underset{\sim}{q}_h)\|_{L^p} \\ + h^{k+2} \|u\|_{W^{k+1,p}}\Big\}$$

where $C_p = Cp$.

Using (4 6) and (4 7) we can prove the following theorem, obtaining optimal order error estimates for the approximation of $\underset{\sim}{q}$ in L^p, $2 \leqslant p < \infty$

THEOREM 4 1 *Let $(\underset{\sim}{q}_h, u_h) \in \underset{\sim}{V}_h^k \times W_h^k$ be the approximate solution defined in* (4 3) *and let* $2 \leqslant p < \infty$ *If* $k \geqslant 1$, *assume that* $u \in W^{k+1\,p} \cap H^{2+\varepsilon}$ *and* $\underset{\sim}{q} \in W^{k+1\,p}$ *and if* $k = 0$ *assume that* $u \in H^3$ *(and therefore* $u \in W^{2\,p}$ *by the Sobolev imbedding theorems) and* $\underset{\sim}{q} \in W^{1\,p}$ *Then, for* h *small enough, there exists a constant* C *depending on* $\|u\|_{H^{2+\varepsilon+\delta_{k0}(1-\varepsilon)}}$ $\|u\|_{W^{k+1\,p}}$, *and* $\|q\|_{W^{k+1\,p}}$ *such that*

$$\left\|\underset{\sim}{q} - \underset{\sim}{q}_h\right\|_{L^p} \leqslant C C_p^2 h^{k+1}$$

Proof Since $\alpha(u)$ is Lipschitz we can apply Corollary 3 1 to (4 5) and we obtain

$$\left\|\underset{\sim}{q}_h - \Pi_h \underset{\sim}{q}\right\|_{L^p} \leqslant C C_p \left\{(1 + \left\|\underset{\sim}{q}_h\right\|_{L^\infty}) \|u - u_h\|_{L^p} + \left\|\underset{\sim}{q} - \Pi_h \underset{\sim}{q}\right\|_{L^p}\right\}$$

therefore, using (4 6) we get,

$$\left\|\underset{\sim}{q}_h - \Pi_h \underset{\sim}{q}\right\|_{L^p} \leqslant C C_p \left\{\|u - u_h\|_{L^p} + \left\|\underset{\sim}{q} - \Pi_h \underset{\sim}{q}\right\|_{L^p}\right\}$$

Now, we use Lemma (4 1) to bound the right hand side in the expression above and we have,

$$\left\|\underset{\sim}{q}_h - \Pi_h \underset{\sim}{q}\right\|_{L^p} \leqslant C C_p^2 \left\{h \left\|\underset{\sim}{q} - \underset{\sim}{q}_h\right\|_{L^p} + h^{2-\delta_{0k}}\left\|\operatorname{div}\,(\underset{\sim}{q} - \underset{\sim}{q})\right\|_{L^p} + h^{k+1}\|u\|_{W^{k+1\,p}} + \left\|\underset{\sim}{q} - \Pi_h \underset{\sim}{q}\right\|_{L^p}\right\}$$

and therefore

$$\left\|\underset{\sim}{q} - \underset{\sim}{q}_h\right\|_{L^p} \leqslant C C_p^2 \left\{h \left\|\underset{\sim}{q} - \underset{\sim}{q}_h\right\|_{L^p} + h^{2-\delta_{0k}}\left\|\operatorname{div}\,(\underset{\sim}{q} - \underset{\sim}{q}_h)\right\|_{L^p} + h^{k+1}\|u\|_{W^{k+1\,p}} + h^{k+1}\left\|\underset{\sim}{q}\right\|_{W^{k+1\,p}}\right\}$$

then, for h small enough we obtain,

$$\left\|\underset{\sim}{q} - \underset{\sim}{q}_h\right\|_{L^p} \leqslant C C_p^2 \left\{h^{2-\delta_{0k}}\left\|\operatorname{div}\,(\underset{\sim}{q} - \underset{\sim}{q}_h)\right\|_{L^p} + h^{k+1}\|u\|_{W^{k+1\,p}} + h^{k+1}\left\|\underset{\sim}{q}\right\|_{W^{k+1\,p}}\right\} \tag{4 8}$$

Now, observe that,

$$\left\|\operatorname{div}\,(\underset{\sim}{q} - \underset{\sim}{q}_h)\right\|_{L^p} = \|f - P_h f\|_{L^p} \leqslant Ch^k \|f\|_{W^{k,p}} \leqslant Ch^k \left\|\underset{\sim}{q}\right\|_{W^{k+1,p}}$$

therefore, the theorem follows from (4.8).

THEOREM 4.2 : *Let* $(\underset{\sim}{q}_h, u_h) \in \underset{\sim}{V}_h^k \times W_h^k$ *be the approximate solution defined in* (4.3) *and let* $2 \leqslant p < \infty$. *If* $k \geqslant 1$, *assume that* $u \in W^{k+1,p} \cap H^{2+\varepsilon}$ *and* $\underset{\sim}{q} \in W^{k,p}$, *then for h small enough there exists a constant C depending on* $\|u\|_{H^{2+\varepsilon}}$, $\|u\|_{W^{k+1,p}}$ *and* $\left\|\underset{\sim}{q}\right\|_{W^{k,p}}$ *such that,*

$$\|u - u_h\|_{L^p} \leqslant CC_p^3 h^{k+1} \tag{4.9}$$

and if $k = 0$, $u \in H^3$ *and* $\underset{\sim}{q} \in W^{1,p}$ *there exists a constant C depending on* $\|u\|_{H^3}$ *and* $\left\|\underset{\sim}{q}\right\|_{W^{1,p}}$ *such that for h small enough*

$$\|u - u_h\|_{L^p} \leqslant CC_p^3 h\,. \tag{4.10}$$

Proof: Let $k \geqslant 1$; following the proof of Theorem 4.1 we can see that,

$$\left\|\underset{\sim}{q} - \underset{\sim}{q}_h\right\|_{L^p} \leqslant CC_p^2 \left\{h^2 \left\|\operatorname{div}\,(\underset{\sim}{q} - \underset{\sim}{q}_h)\right\|_{L^p} + h^{k+1}\|u\|_{W^{k+1,p}} + h^k \left\|\underset{\sim}{q}\right\|_{W^{k,p}}\right\}$$

then (4.9) follows by applying Lemma 4.1. When $k = 0$, we apply the result of Theorem 4.1 combined with Lemma 4.1 and we obtain (4.10).

We can prove L^∞ estimates in an analogous way, obtaining the following results.

THEOREM 4.3 : *Under the same hypothesis of Theorem* 4.1 *but with* $p = \infty$ *we have,*

$$\left\|\underset{\sim}{q} - \underset{\sim}{q}_h\right\|_{L^\infty} \leqslant Ch^{k+1} |\log h|^2\,.$$

THEOREM 4.4 : *Under the same hypothesis of Theorem* 4.2 *but with* $p = \infty$ *we have for* $k \geqslant 1$

$$\|u - u_h\|_{L^\infty} \leqslant Ch^{k+1} |\log h|^3$$

and for $k = 0$,

$$\|u - u_h\|_{L^\infty} \leqslant Ch |\log h|^3\,.$$

5. FINAL REMARKS

Remark 5.1 : The results of Sections 3 and 4 can be obtained in the same way for the rectangular elements of Raviart and Thomas [21]. In this case, the space V_h^k is defined locally as $Q_{k+1,k} \times Q_{k,k+1}$ where $Q_{i,j}$ denotes the set of polynomials of degree i in x_1 and j in x_2. Then, it can be easily seen that the associated space M_h will be locally $Q_{k+1,k+1}$.

Remark 5.2 : Although we have considered Dirichlet boundary conditions, the analysis can be carried out for homogeneous Neumann boundary conditions in the same way. Note that in this case, the associated Ritz projection (3.7) will correspond to the problem $-\operatorname{div}(\alpha \nabla u) = f$ with homogeneous Dirichlet boundary conditions.

Moreover, in this case the domain does not need to be simply connected. In fact, it is easy to see that the homogeneous Neumann boundary conditions allow the decomposition (3.10) of the vector field for a general smooth domain.

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