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POINTWISE CONVERGENCE OF SOME BOUNDARY ELEMENT METHODS. Part II (*)

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Abstract. — *This paper is the second part of a work dealing with the approximate solution of strongly elliptic boundary integro-differential equations by the finite element Galerkin method. In Part I, operators of negative and zero order have been considered Here it is shown that as well for operators of positive order, the discrete solutions converge uniformly with almost the same optimal order as is known for their convergence in the mean-square sense As a by-product, these results also yield pointwise convergence estimates for the solutions of ordinary spline collocation boundary element methods for two-dimensional problems As in Part I, the proofs are based on error estimates for discrete Green functions which are derived by using a weighted Sobolev norm technique due to J A Nitsche*

Résumé. — *Cet article est la deuxième partie d'un ouvrage concernant la résolution approchée d'équations intégro-différentielles sur la frontière fortement elliptique par la méthode Galerkin d'éléments finis Dans la première partie on a traité les opérateurs d'ordre négatif ou zéro Ici on démontre que également pour les opérateurs d'ordre positif les solutions discrètes convergent uniformément avec presque le même ordre optimal déja connu pour les convergences dans les sens de moyenne quadratique Ces résultats donnent en outre des estimations de convergence uniformément pour les solutions de méthode « collocation » ordinaire a fonction spline pour les problèmes bi-dimensionnels. Comme dans la première partie les preuves sont basées sur les estimations d'erreur de fonction de Green discrètes obtenues d'après une technique, développée par J A Nitsche, de normes Sobolev à poids*

1. BOUNDARY PSEUDO-DIFFERENTIAL OPERATORS

This paper is the announced extension of [29] to Galerkin methods for operators of positive order. In addition, we also prove pointwise estimates for the ordinary spline collocation of boundary integral equations of arbitrary order.

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This paper is dedicated to Prof Dr. J. A Nitsche on the occasion of his 60th birthday.
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As in Part I, let Γ be a smooth closed regular $(n-1)$-dimensional surface in $\mathbb{R}^n$, $n=2$ or $n=3$. By $L^p(\Gamma)$ and $W^{r,p}(\Gamma)$, $1 \leq p \leq \infty$, $r \in \mathbb{R}$, we denote the Lebesgue and Sobolev spaces on Γ provided with the usual norms ; $(.\,,.)_r$ and $\|.\|_r$ are the inner product and norm of the Hilbert space $H^r(\Gamma) = W^{r,2}(\Gamma)$, respectively. We see $(.\,,.) := (.\,,.)_0$ for the L^2 scalar product and $\|.\| := \|.\|_0$. For convenience, we shall denote by c a generic positive constant which may vary with the context (but will usually be independent of the meshwidth and the solution of the equations in question).

On Γ, we consider a boundary pseudo-differential equation

$$Au = f\,, \tag{1.1}$$

where A is a classical pseudo-differential operator of real order $2\,\alpha$, and is $H^\alpha(\Gamma)$-coercive, i.e., A admits a decomposition

$$A = A_0 + A_1\,, \tag{1.2}$$

where A_0 satisfies the strong coercivity estimate

$$\operatorname{Re}\,(A_0\,v, v) = \operatorname{Re}\,(v, A_0^*\,v) \geq c\,\|v\|_\alpha^2\,, \quad c > 0\,. \tag{1.3}$$

Moreover, A_0 maps $H^{s+\alpha}(\Gamma)$ continuously onto $H^{s-\alpha}(\Gamma)$, and the operator A_1 maps $H^{s+\alpha}(\Gamma)$ continuously into $H^{s-\alpha+1}(\Gamma)$, i.e.,

$$\|A_0\,v\|_{s-\alpha} \leq c\,\|v\|_{s+\alpha}\,, \qquad \|A_1\,v\|_{s-\alpha+1} \leq c\,\|v\|_{s+\alpha}\,,$$

for appropriate $s \in \mathbb{R}$ (see [15]). As is well known, under the above assumptions, the classical Fredholm alternative is valid for equation (1.1). Hence, uniqueness of (1.1) implies unique solvability for any right hand side $f \in H^{s-\alpha}(\Gamma)$, and the unique solution u satisfies the a-priori estimate

$$\|u\|_{s+\alpha} \leq c\,\|f\|_{s-\alpha}\,.$$

Operators of negative and zero order have already been considered in Part I, [29], where also a few examples have been given. Operators of positive order also come up in boundary element Galerkin methods defined via hypersingular integral operators ; see [7], [8], [9], [10], [12], [13], [14], [21], [22], [37]. As an example and model operator let us consider the normal derivative of the double layer potential which is associated to Neumann problems, see [2], [7], [21], [38],

$$\begin{aligned} Au(x) &= \int_\Gamma \left(\frac{\partial}{\partial n_x}\frac{\partial}{\partial n_y}\,\gamma_n(x-y)\right)\,(u(y)-u(x))\,do_y + \\ &\quad + \int_\Gamma K(x,y)\,u(y)\,do_y \\ &= f(x)\,, \quad \text{for} \quad x \in \Gamma\,. \end{aligned} \tag{1.4}$$

Here, γ_n denotes the fundamental solution of the Laplacian in $\mathbb{R}^n$, $n = 2$ or $n = 3$, $\frac{\partial}{\partial n_x}$ and $\frac{\partial}{\partial n_y}$ are the normal derivatives with respect to the exterior unit normal to Γ at x and y, respectively, and $K(x, y)$ is a sufficiently smooth kernel. The integral in (1.4) is defined as a Cauchy principal value integral with respect to the limit of balls $|x - y| \geqslant \varepsilon$, $\varepsilon \to 0$. The operator A defines a classical pseudo-differential operator on Γ of order $2\alpha = 1$. It has the positive definite « principal symbol » (with respect to orthogonal tangential coordinates of Γ at x) ; see [2],

$$a_0(x, \xi) = (n-1)\pi|\xi| , \quad \text{for} \quad \xi \neq 0 ,$$

and is $H^{1/2}(\Gamma)$-coercive satisfying (1.2) and (1.3). For operators of this type, boundary element methods have already been implemented and used in numerical computations for many practical problems.

The $H^{\alpha}(\Gamma)$-coercive operators considered here belong to the slightly more general class of strongly elliptic pseudo-differential operators. It should be noted that, in the case $n = 2$, strong ellipticity is *necessary and sufficient* for optimal order convergence of spline Galerkin and of spline collocation methods ; see Schmidt [31]. Here we prove pointwise convergence estimates for the finite element approximation of problem (1.1), for the case $2\alpha > 0$.

For two-dimensional problems, $n = 2$, the ordinary boundary element method which is nodal collocation using *odd* degree splines has been reduced to equivalent spline Galerkin methods by Arnold and Wendland in [4]. This approach can be used for obtaining also pointwise asymptotic error estimates for these collocation methods involving operators or *arbitrary* order. However, for *even* degree spline collocation which converges with optimal order in Sobolev spaces [5], optimal order pointwise estimates have not been proven yet (except for classical Fredholm integral equations of the second kind).

The essential tool in our pointwise error analysis is the use of weighted Sobolev norms which were introduced by Natterer [19] for pointwise evaluations via L^2-norms. Nitsche in [23] developed this technique to a most powerful instrument for analyzing pointwise errors in finite element methods.

2. THE FINITE ELEMENT GALERKIN METHOD

We continue using the notation introduced in Part I of this work. Let $\pi_h = \{K\}$ be a finite decomposition of the surface Γ into closed subsets K with mutually disjoint interiors $\mathring{K}$; $h \in (0,1/2]$ denotes a discretization parameter corresponding to the maximum diameter of K. We further define

$\Gamma_h = U\{\overset{\circ}{K}, K \in \pi_h\}$ and shall use the corresponding norms $\|\cdot\|_{r,\Gamma_h}$. We consider the family $\{\pi_h\}$ of decompositions depending on h. For properties involving $\{\pi_h\}$, the generic constant c will always be independent of h.

For $\{\pi_h\}$ we assume quasi-regularity in the following sense :

(A.1) *Associated with $\{\pi_h\}$ there exist two positive constants, c_1 and c_2, such that each element $K \in \pi_h$ is contained in the intersection of Γ with some ball $B_1 \subset \mathbb{R}^n$ of radius $c_1 h$, and contains the intersection of Γ with some ball $B_2 \subset \mathbb{R}^n$ of radius $c_2 h$.*

For fixed integers $k \geq 1$ and $m \geq 0$, $m \leq k-1$, let $S_h^{k,m}$ be so-called (k, m)-systems on Γ corresponding to the decompositions π_h (see [6], and [24] for splines, where the notation $S_{k-1}(\pi_h, m-1)$ is used). The first parameter, k, refers to the local approximation order of $S_h^{k,m}$, which usually consists of piecewise polynomial functions (or isoparametric splines) of degree $k-1$; the second parameter, m, indicates the global smoothness of these functions,

$$S_h^{k,m} \subset H^m(\Gamma) \cap H^k(\Gamma_h) .$$

For our purpose, we need to require the following approximation and inverse properties.

(A.2) *There exists an operator p_h: $H^m(\Gamma) \cap H^k(\Gamma_h) \to S_h^{k,m}$, such that for all $v \in H^m(\Gamma) \cap H^k(\Gamma_h)$, there holds the global estimate*

$$\|v - p_h v\|_j \leq ch^{k-j}\|v\|_{k,\Gamma_h}, \quad 0 \leq j \leq m, \tag{2.1}$$

and, in addition, the local estimate

$$\|v - p_h v\|_{H^j(K)} \leq ch^{k-j}\|v\|_{H^k(K_h)}, \quad 0 \leq j \leq k, \tag{2.2}$$

on each $K \in \pi_h$, where K_h may be either K or, if necessary, the union of the open interiors of all neighboring elements of K intersecting a ball $B_k^h \subset \mathbb{R}^n$ of radius ch having its center in K.

(A.3) *For all $\phi_h \in S_h^{k,m}$, there holds, on each $K \in \pi_h$,*

$$\|\phi_h\|_{H^k(K)} \leq c\|\phi_h\|_{H^{k-1}(K)}, \tag{2.3}$$

$$\|\phi_h\|_{H^j(K)} \leq ch^{\ell-j}\|\phi_h\|_{H^\ell(K)}, \tag{2.4}$$

for integers $0 \leq \ell \leq j \leq k-1$.

These are typical properties of (isoparametric) finite element spaces $S_h^{k,m}$ of order k ; for examples from the literature, see Part I, [29] and [34].

Note that (2.3) becomes trivial if ϕ_h is piecewise polynomial of degree $k-1$. Usually, the systems $S_h^{k,m}$ also satisfy the pointwise approximation estimate

$$(2.5) \qquad \inf_{\phi_h \in S_h^{k,m}} \|v - \phi_h\|_{W^{j,\infty}} \leq c h^{k-j} \|v\|_{W^{k,\infty}}, \quad 0 \leq j \leq m,$$

for $v \in W^{k,\infty}(\Gamma)$.

The Galerkin approximations $u_h \in S_h^{k,m}$ to the solution u of problem (1.1) are determined by the finite dimensional analogues of (1.1),

$$(2.6) \qquad (Au_h, \phi_h) = (f, \phi_h) = (Au, \phi_h), \quad \text{for all} \quad \phi_h \in S_h^{k,m}.$$

Due to our assumptions on A and the approximation properties of $S_h^{k,m}$, the problems (2.6) are uniquely solvable for sufficiently small h (see [33]). Furthermore, there holds the convergence estimate

$$(2.7) \qquad \|u - u_h\|_p \leq c h^{q-p} \|u\|_q,$$

for

$$2\alpha - k \leq p \begin{cases} \leq m, & \text{for} \quad n = 3, \\ < m + \frac{1}{2}, & \text{for} \quad n = 2 \end{cases}$$

and $\max\{\alpha, p\} \leq q \leq k$; see [4], [17], [18], [20]. Using the inverse inequality for $S_h^{k,m}$ in the usual manner, the mean square result (2.7) also gives the pointwise error estimate (see also [17])

$$(2.8) \qquad \|u - u_h\|_{W^{r,\infty}} \leq c h^{k-r-n/2} \|u\|_k,$$

for $\max\{0, 2\alpha - k\} \leq r \leq m$. In view of (2.5), this estimate is not of optimal order if $u \in W^{k,\infty}(\Gamma)$. Under the foregoing assumptions (A.1)-(A.3), we shall prove the following stronger result.

THEOREM 1 : *Suppose that $0 < \alpha \leq m \leq k-1$, and that $u \in W^{m,\infty}(\Gamma)$. Then, for the Galerkin solutions $u_h \in S_h^{k,m}$, there holds*

$$(2.9) \qquad \|u - u_h\|_{W^{\ell,\infty}} \leq c \left(\log \frac{1}{h}\right)^{(n-2)/2} h^{-\ell} \inf_{\phi_h \in S^{k\ m}} \sum_{r=0}^{m} h^r \|u - \phi_h\|_{W^{r,\infty}},$$

where

$$(2.10) \qquad \max\{0, 2\alpha - k + (3-n)/2\} \leq \ell \leq m.$$

We emphasize that this Theorem holds for any *real* order $2\alpha > 0$. In view of the approximation estimate (2.5), one obtains from (2.9) the error estimate

$$\|u - u_h\|_{W^{\ell,\infty}} \leq c\left(\log\frac{1}{h}\right)^{(n-2)/2} h^{k-\ell}\|u\|_{W^{k,\infty}}, \tag{2.11}$$

for $\max\{0, 2\alpha - k + (3-n)/2\} \leq \ell \leq m$, provided that $u \in W^{k,\infty}(\Gamma)$. It should also be noted that (2.9) can be refined to a *local* estimate of the form

$$\|u - u_h\|_{W^{\ell,\infty}(K)} \leq c\left(\log\frac{1}{h}\right)^{(n-2)/2} h^{-\ell} \inf_{\phi_h \in S_h^{k,m}} \sum_{r=0}^{m} h^r \times \left\{\|u - \phi_h\|_{W^{r,\infty}(B_K^\rho \cap \Gamma)} + \|u - \phi_h\|_r\right\}, \tag{2.12}$$

where B_K^ρ denotes some ball in $\mathbb{R}^n$ of radius $\rho = 0(1)$, as $h \to 0$, with center in $K \in \pi_h$.

For $n = 2$, the restriction $\alpha \leq m$ can be relaxed to

$$0 < \alpha < m + \frac{1}{2} \leq k - \frac{1}{2}, \tag{2.13}$$

since in this case $S_h^{k,m} \subset H^\sigma(\Gamma)$, for any $\sigma < m + \frac{1}{2}$ (see [4], p. 353).

We did not attempt to avoid the logarithmic factor in our estimate (2.9) in all cases when this were possible ; for a discussion of this question in the case $\alpha = 1$, see [28].

The condition (2.10) is always fulfilled for $\ell \geq 0$ in the case $n = 3$, and for $\ell \geq 1$, in the case $n = 2$, provided that $k \geq 2\alpha$, the latter being even necessary for the optimal order L^2-estimate (see (2.7))

$$\|u - u_h\| \leq ch^k\|u\|_k .$$

In the case $n = 2$, however, the condition (2.10) is not optimal for $\ell = 0$. It excludes, for instance, in the case of a second order operator $(2\alpha = 2)$ the pointwise error estimate $(\ell = 0)$ for continuous piecewise linear finite elements, where $k = 2 = m + 1$. This gap is due to our method of proof which through the use of the commutator inequality (3.10), below, is especially adapted to the case $n = 3$.

However, with L^p-estimates we are able to extend our result for $n = 2$ to the special case $2\alpha - k \leq \ell \leq 2\alpha - k + \frac{1}{2}$. To this end, we now assume that corresponding to $S_h^{k,m}$ there exists an interpolation operator $I_h : H^m(\Gamma) \cap H^k(\Gamma_h) \to S_h^{k,m}$ providing the approximation estimates

$$\|v - I_h v\|_{W^{\rho,p}} \leq c h^{k-\rho} \|v\|_{W^{k,p}}, \tag{2.14}$$

for $0 \leq \rho \leq m$ and $1 \leq p \leq \infty$. Then, we can show the following supplementary result.

THEOREM 2: *Suppose that* $n = 2$, $2\alpha - k \leq \ell < 2\alpha - k + \frac{1}{2}$ *with* $\ell \in \mathbb{N} \cup \{0\}$, (2.13) *and* (2.14). *Then, for the Galerkin solutions* u_h, *there holds*

$$\|u - u_h\|_{W^{\ell,\infty}} \leq c h^{k-\ell} \left(\log \frac{1}{h}\right)^2 \|u\|_{W^{k,\infty}}. \tag{2.15}$$

Again, we did not attempt to avoid the logarithmic factors in (2.15).

Optimal order L^∞-error estimates for the standard finite element method applied to elliptic partial differential equations are known, e.g., from the work of Natterer [19], Nitsche [23], Scott [32], Frehse and Rannacher [11] and Schatz and Wahlbin [30], for second-order operators, and Rannacher [26], [27], for the biharmonic operator. In proving (2.9) for the general case of real $\alpha > 0$, we shall adapt techniques from [11] and [28] in a similar way as in Part I, [29], for the case $\alpha \leq 0$. Again, localization techniques come to work since the strongly coercive part A_0 of A is a pseudo-differential operator of order 2α, and consequently, for any C^∞-multiplier ϕ, the commutator $\phi A_0 - A_0 \phi$ becomes a pseudo-differential operator of order $2\alpha - 1$; see [36], Corollary 4.2, p. 39.

3. PROOFS OF THEOREMS 1 AND 2

We shall use the notation of Part I. In particular, we use the weight-function

$$\sigma(x) = (|x - z|^2 + \kappa^2 h^2)^{1/2}, \quad \kappa \geq 1,$$

and the weighted norms on $H^r(\Gamma_h)$, $r \in \mathbb{N} \cup \{0\}$,

$$\|v\|_{r,\beta} = \left(\sum_{|j| \leq r} \sum_{K \in \pi_h} \int_K \sigma^\beta(x) |D^j v(x)|^2 \, do_x \right)^{1/2}, \quad \beta \in \mathbb{R},$$

where $z \in \Gamma$ is an arbitrary but fixed point; D^j denotes the covariant derivatives of order j on Γ. In the following the generic constant c is always independent of h and of z. If the parameter $\kappa \geq 1$ is chosen sufficiently large, then the local approximation and inverse properties of $S_h^{k,m}$, (A.2) and (A.3), imply the corresponding properties with weighted norms,

$$\|v - p_h v\|_{j,\beta} \leq c_\beta h^{k-j} \|v\|_{k,\beta}, \quad 0 \leq j \leq k, \tag{3.1}$$

for $v \in H^m(\Gamma) \cap H^k(\Gamma_h)$, and

$$\|\phi_h\|_{k,\beta} \leq c_\beta \|\phi_h\|_{k-1,\beta}, \tag{3.2}$$

$$\|\phi_h\|_{r,\beta} \leq c_\beta h^{s-r} \|\phi_h\|_{s,\beta}, \quad 0 \leq s \leq r \leq k-1, \tag{3.3}$$

for $\phi_h \in S_h^{k,m}$; see Nitsche [23].

Next, we fix any of the covariant derivatives D^ℓ on Γ where $\max\{0, 2\alpha - k + (3-n)/2\} \leq |\ell| \leq m$. Then, for any smooth function δ on Γ, let g be the (unique) solution of

$$A_0^* g = (D^\ell)^* \delta \quad \text{on} \quad \Gamma. \tag{3.4}$$

Correspondingly, let $g_h \in S_h^{k,m}$ be the Galerkin approximation to g defined by

$$\begin{aligned}(\phi_h, A_0^* g_h) = (\phi_h, A_0^* g) = (\phi_h, (D^\ell)^* \delta) = \\ = (D^\ell \phi_h, \delta) \quad \text{for all} \quad \phi_h \in S_h^{k,m}.\end{aligned} \tag{3.5}$$

Here, $(D^\ell)^*$ denotes the operator adjoint to D^ℓ. Below, we shall take δ as an approximation to the Dirac functional.

Proof of Theorem 1 : For abbreviation, we set $e = u - u_h$, $\eta = g - g_h$. From the orthogonality properties of e and η, we obtain

$$\begin{aligned}(D^\ell e, \delta) = (e, A_0^* g) = (e, A_0^* \eta) + (e, A_0^* g_h) = \\ = (u - \phi_h, A_0^* \eta) - (e, A_1^* g_h),\end{aligned} \tag{3.6}$$

where $\phi_h \in S_h^{k,m}$ is arbitrary. For the second term on the right hand side we find, using the continuity of A_1 and the error estimate (2.7) for η, that

$$\begin{aligned}|(e, A_1^* g_h)| &\leq |(A_1, e, \eta)| + |(A_1, e, g)| \\ &\leq c\|e\|_\alpha \|\eta\|_{\alpha-1} + c\|e\|_{2\alpha-k} \|g\|_{k-1} \\ &\leq c\left\{h^{k-\alpha}\|e\|_\alpha + \|e\|_{2\alpha-k}\right\} \|g\|_{k-1}.\end{aligned} \tag{3.7}$$

This together with Lemma 3.1, below, gives us

$$|(e, A_1^* g_h)| \leq c h^{k+m-2\alpha} \inf_{\phi_h \in S_h^{k,m}} \|u - \phi_h\|_m \|g\|_{k-1}. \tag{3.8}$$

LEMMA 3.1 : *For any real* β *with* $2\alpha - k \leq \beta \leq \alpha$, *there holds*

$$\|e\|_\beta \leq ch^{m-\beta} \inf_{\phi_h \in S_h^{k,m}} \|u - \phi_h\|_m . \tag{3.9}$$

Proof: The proof is analogous to that of Lemma 3.8 in Part I, [29], and hence omitted.

In order to estimate the first term on the right hand side of (3.6), we provide the following sequence of technical lemmas. For $1 \leq i \leq n$, let $\xi_i = x_i - z_i$.

LEMMA 3.2 : *The commutator* $A_0^* \xi_i - \xi_i A_0^*$ *satisfies*

$$\| [A_0^* \xi_i - \xi_i A_0^*] \phi \|_{r-2\alpha} \leq c \| \phi \|_{r-1} , \quad 1 \leq i \leq n , \tag{3.10}$$

for $\phi \in H^{r-1}(\Gamma)$, $\alpha \leq r \leq k$.

Proof: See Lemma 3.1 of Part I, [29].

LEMMA 3.3 : *There holds the estimate*

$$\| \xi_i \eta \|_{k,0} + \| \xi_i^2 \eta \|_{k,-2} \leq c \| g \|_{k,2} , \quad 1 \leq i \leq n . \tag{3.11}$$

Proof: See Lemma 3.2 of Part I, [29].

LEMMA 3.4 : *For real* β, *with* $0 \leq \beta \leq m$, *and any* $\phi \in H^m(\Gamma)$, *there holds*

$$\| \xi_i \sigma^{-2} \phi \|_\beta \leq ch^{-\beta} \sum_{r=0}^{m} h^r \| \phi \|_{r,-2} , \quad 1 \leq i \leq n . \tag{3.12}$$

Proof: For $\beta = 0$, the properties of the weighted norms and the behaviour of σ^{-2} yield

$$\| \xi_i \sigma^{-2} \phi \|_0 \leq c \| \phi \|_{0,-2} \leq c \sum_{r=0}^{m} h^r \| \phi \|_{r,-2} .$$

On the other hand, one gets

$$| D^r (\xi_i \sigma^{-2}) | \leq c\sigma^{-r-1} ,$$

and hence, with the Leibniz rule,

$$\| \xi_i \sigma^{-2} \phi \|_m \leq ch^{-m} \sum_{r=0}^{m} h^r \| \phi \|_{r,-2} .$$

Now, interpolation between both inequalities gives (3.12).

LEMMA 3.5 : *For functions* $\phi \in H^m(\Gamma)$, *there holds*

$$(3.13) \quad |(\phi, A_0^* \eta)| \leq ch^{-\alpha} \sum_{r=0}^{m} h^r \|\phi\|_{r,-2} \times \\ \times \left\{ \max_{1 \leq \iota \leq n} \|\xi_\iota \eta\|_\alpha + h^{k-\alpha} \|g\|_{k,2} \right\}.$$

Proof: Due to the definition of the weight-function σ there holds

$$\begin{aligned}(\phi, A_0^* \eta) &= \sum_{\iota=1}^{n} (\xi_\iota^2 \sigma^{-2} \phi, A_0^* \eta) + \kappa^2 h^2 (\sigma^{-2} \phi, A_0^* \eta) \\ &= \sum_{\iota=1}^{n} (\xi_\iota \sigma^{-2} \phi, [\xi_\iota A_0^* - A_0^* \xi_\iota] \eta) \\ &\quad + \sum_{\iota=1}^{n} (\xi_\iota \sigma^{-2} \phi, A_0^* [\xi_\iota \eta]) + \kappa^2 h^2 (\sigma^{-2} \phi, A_0^* \eta) \\ &=: \sum_{\iota=1}^{n} a_\iota + \sum_{\iota=1}^{n} b_\iota + c .\end{aligned}$$

The terms a_ι, b_ι, and c will be estimated separately.

Using in an obvious way the properties of weighted norms, the commutator estimate (3.9), and the error estimate (2.7), we conclude that

$$\begin{aligned}|a_\iota| &\leq c \|\xi_\iota \sigma^{-2} \phi\|_m \|[\xi_\iota A_0^* - A_0^* \xi_\iota] \eta\|_{-m} \\ &\leq c \sum_{r=0}^{m} \|\phi\|_{r,2(r-m-1)} \|\eta\|_{2\alpha-m-1} \\ &\leq ch^{k-2\alpha+m+1} \|g\|_k \sum_{r=0}^{m} h^{r-m} \|\phi\|_{r,-2} \\ &\leq ch^{k-2\alpha} \|g\|_{k,2} \sum_{r=0}^{m} h^r \|\phi\|_{r,-2} , \\ |c| &\leq ch^2 \|\sigma^{-2} \phi\|_m \|\eta\|_{2\alpha-m} \\ &\leq ch^{2+k-2\alpha+m} \|g\|_k \sum_{r=0}^{m} \|\phi\|_{r,2(r-m-2)} \\ &\leq ch^{k-2\alpha} \|g\|_{k,2} \sum_{r=0}^{m} h^r \|\phi\|_{r,-2} ,\end{aligned}$$

and with (3.12) and $\beta = \alpha$,

$$|b_\iota| \leq c \|\xi_\iota \sigma^{-2} \phi\|_\alpha \|\xi_\iota \eta\|_\alpha \leq ch^{-\alpha} \|\xi_\iota \eta\|_\alpha \sum_{r=0}^{m} h^r \|\phi\|_{r,-2} .$$

The combination of these three estimates gives (3.13).

LEMMA 3.6 : *There holds the estimate*

$$\max_{1 \le i \le n} \|\xi_i \eta\|_\alpha \le ch^{k-\alpha} \|g\|_{k,2} . \tag{3.14}$$

Proof: In view of the strong coerciveness of A_0^*, we have

$$c_0 \|\xi_i \eta\|_\alpha^2 \le \operatorname{Re} (\xi_i \eta, A_0^*[\xi_i \eta]) , \quad 1 \le i \le n ,$$

and by the orthogonality property of η and Lemma 3.2,

$$\begin{aligned} |(\xi_i \eta, A_0^*[\xi_i \eta])| &= |(\xi_i \eta, [A_0^* \xi_i - \xi_i A_0^*] \eta) + \\ &\quad + (\xi_i^2 \eta - p_h[\xi_i^2 \eta], A_0^* \eta)| \\ &\le c \|\xi_i \eta\|_\alpha \|\eta\|_{\alpha - 1} + |(\xi_i^2 \eta - p_h[\xi_i^2 \eta], A_0^* \eta)| . \end{aligned}$$

Next, we use (3.13) with $\phi = \xi_i^2 \eta - p_h[\xi_i^2 \eta]$ and find that

$$\begin{aligned} |(\xi_i^2 \eta - p_h[\xi_i^2 \eta], A_0^* \eta)| &\le ch^{m-\alpha} \sum_{r=0}^{m} \|\xi_i^2 \eta - p_h[\xi_i^2 \eta]\|_{r, 2(r-m-1)} \times \\ &\quad \times \left\{ \max_{1 \le i \le n} \|\xi_i \eta\|_\alpha + h^{k-\alpha} \|g\|_{k,2} \right\} . \end{aligned}$$

Using (3.1) and (3.11), there holds, for $0 \le r \le m$,

$$\begin{aligned} \|\xi_i^2 \eta - p_h[\xi_i^2 \eta]\|_{r, 2(r-m-1)} &\le ch^{k-r} \|\xi_i^2 \eta\|_{k, 2(r-m-1)} \\ &\le ch^{k-m} \|\xi_i^2 \eta\|_{k,-2} \\ &\le ch^{k-m} \|g\|_{k,2} . \end{aligned}$$

Furthermore, in view of (2.7),

$$\|\eta\|_{\alpha-1} \le ch^{k-\alpha+1} \|g\|_k \le ch^{k-\alpha} \|g\|_{k,2} .$$

Combining the foregoing estimates and (3.13), we eventually obtain

$$\|\xi_i \eta\|_\alpha^2 \le ch^{k-\alpha} \|g\|_{k,2} \left\{ \max_{1 \le j \le n} \|\xi_j \eta\|_\alpha + h^{k-\alpha} \|g\|_{k,2} \right\} ,$$

which completes the proof of (3.14).

From the estimates (3.13) and (3.14) of Lemma 3.6, respectively, it follows that

$$|(\phi, A_0^* \eta)| \le ch^{k-2\alpha} \|g\|_{k,2} \sum_{r=0}^{m} h^r \|\phi\|_{r,-2} ,$$

for $\phi \in H^m(\Gamma)$. Consequently, taking $\phi = u - \phi_h$ and using the definition of weighted norms, we obtain the estimate

$$(3.15) \quad |(u - \phi_h, A_0^* \eta)| \leq c h^{k-2\alpha} \|g\|_{k,2}\, h^{(n-3)/2} \left(\log \frac{1}{h}\right)^{(n-2)/2} \times \sum_{r=0}^{m} h^r \|u - \phi_h\|_{W^{r,\infty}},$$

for the firts term on the right hand side of (3.6).

Combining (3.15) with (3.7) and (3.6), there follows the preliminary result

$$(3.16) \quad |(D^\ell e, \delta)| \leq c h^{k-2\alpha} h^{(n-3)/2} \left(\log \frac{1}{h}\right)^{(n-2)/2} \times \sum_{r=0}^{m} h^r \|u - \phi_h\|_{W^{r,\infty}} \{\|g\|_{k,2} + \|g\|_{k-1}\},$$

with any arbitrary $\phi_h \in S_h^{k,m}$.

Now we are prepared to prove the pointwise error estimate (2.9). To this end, we take the function δ as a *regularized* Dirac function at $z \in K \in \pi_h$.

LEMMA 3.7 : *There exists a function $\delta \in C_0^\infty(K)$, such that*

$$(3.17) \qquad D^\ell \phi_h(z) = (D^\ell \phi_h, \delta) \quad \text{for all} \quad \phi_h \in S_h^{k,m},$$

where D^ℓ is the prefixed covariant derivative of order ℓ with $\max\{0, 2\alpha - k + (3-n)/2\} \leq |\ell| \leq m$. *Further, there hold the estimates*

$$(3.18) \qquad \|\delta\|_{L^1} \leq c,$$

$$(3.19) \qquad h^r \|\delta\|_{r-2\alpha+|\ell|} \leq c h^{2\alpha + 1/2 - n/2 - |\ell|},$$

for $2\alpha - |\ell| \leq r \leq k$, *and*

$$(3.20) \qquad h^k \|\xi_i \delta\|_{k-2\alpha+|\ell|} \leq c h^{2\alpha + 3/2 - n/2 - |\ell|}, \quad 1 \leq i \leq n.$$

Proof: The function $\delta \in C_0^\infty(K)$ satisfying (3.17) and (3.18) can be constructed following the arguments used in the proof of Lemma 3.7 in Part I, [29], for the special case $|\ell| = 0$. The estimates (3.19) and (3.20) then follow from this construction observing that $r - 2\alpha + |\ell| \geq 0$.

LEMMA 3.8 : *There hold the a-priori estimates*

$$(3.21) \qquad \|g\|_{k-1} \leq c \|\delta\|_{k-1-2\alpha+|\ell|},$$

$$(3.22) \quad \|g\|_{k,2} \leq c \left\{ h \|\delta\|_{k-2\alpha+|\ell|} + \sum_{i=1}^{n} \|\xi_i \delta\|_{k-2\alpha+|\ell|} + \|\delta\|_{k-1-2\alpha+|\ell|} \right\}.$$

Proof: See the proof of Lemma 3.6 in Part I [29].

Now, let δ be taken as the function given by Lemma 3.7. Then, in view of Lemma 3.8, there holds the estimate

$$\|g\|_{k,2} + \|g\|_{k-1} \leqslant ch^{2\alpha - k - |\ell| + (3-n)/2}. \tag{3.23}$$

(Note that $2\alpha - k + (3-n)/2 - |\ell| \leqslant 0$ due to the restriction (2.10) on $|\ell|$.)

Inserting the estimate (3.23) into (3.16), we obtain

$$|D^\ell e, \delta| \leqslant ch^{-|\ell|}\left(\log \frac{1}{h}\right)^{(n-2)/2} \sum_{r=0}^{m} h^r \|u - \phi_h\|_{W^{r,\infty}}, \tag{3.24}$$

with an arbitrary $\phi_h \in S_h^{k,m}$.

Now, using the property (3.17) of δ, we have

$$\begin{aligned} |D^\ell e(z)| &= |D^\ell(u - \phi_h)(z) + D^\ell(\phi_h - u_h)(z)| \\ &\leqslant \|D^\ell(u - \phi_h)\|_{L^\infty} + |(D^\ell e, \delta)| + \|D^\ell(u - \phi_h)\|_{L^\infty}\|\delta\|_{L^1} \end{aligned}$$

with an arbitrary $\phi_h \in S_h^{k,m}$. Consequently,

$$|D^\ell e(z)| \leqslant c\|D^\ell(u - \phi_h)\|_{L^\infty} + |(D^\ell e, \delta)|. \tag{3.25}$$

This estimate together with (3.24) eventually proves (2.9).

Proof of Theorem 2: From (3.5) we find with (2.5) that

$$|D^\ell e(z)| \leqslant |(D^\ell e, \delta)| + ch^{k-\ell}\|u\|_{W^{k,\infty}}.$$

Then, as in (3.6), we get with (3.4)

$$\begin{aligned} (D^\ell e, \delta) &= (e, A_0^*(g - I_h g)) + (e, A_0^* I_h g) \\ &= (e, A_0^*(g - I_h g)) - (e, A_1^* I_h g). \end{aligned}$$

The last term can again be estimated as in (3.7) giving (3.8). For the remaining term we use the Bessel operator $\Lambda = (1 - d^2/dt^2)^{1/2}$ and Hölder's inequality as follows,

$$\begin{aligned} |(e, A_0^*(g - I_h g))| &= |(\Lambda^{l+1} e, \Lambda^{-\ell-1} A_0^*(g - I_h g))| \\ &\leqslant \|\Lambda^{\ell+1} e\|_{L^q}\|\Lambda^{-\ell-1} A_0^*(g - I_h g)\|_{L^p}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Since $\Lambda^{-\ell-1} A_0^*$ is a classical pseudo-differential operator of order $2\alpha - \ell - 1$, we can use its continuity properties in L^p, according to the Marcinkiewicz inequality for $p > 1$ (see [35], p. 30 ff.),

$$\|\Lambda^{-\ell-1} A_0^*(g - I_h g)\|_{L^p} \leq \frac{c}{p-1} \|g - I_h g\|_{W^{2\alpha-\ell-1,p}}.$$

Observe that $2\alpha - \ell - 1 \geq 0$, since $\ell < 2\alpha - k + \frac{1}{2}$ and $k \geq 2$. Now, with (2.14) we get

$$(3.26) \quad |(e, A_0^*(g - I_h g))| \leq \frac{c}{p-1} \|e\|_{W^{\ell+1,q}} \|g\|_{W^{k,p}} h^{k+\ell+1-2\alpha}.$$

Again, for $g = A_0^{*-1}(D^\ell)^* \delta$ we use the Marcinkiewicz inequality, since $A_0^{*-1} D^\ell$ is also a classical pseudo-differential operator being the composition of the inverse of a strongly elliptic coercive classical pseudo-differential operator and a differential operator (see [35], Theorem 1.3, p. 61). Thus,

$$(3.27) \qquad \|g\|_{W^{k,p}} \leq \frac{c}{1-p} \|\delta\|_{W^{k+\ell-2\alpha,p}}.$$

From Lemma 3.7 of Part I [29], one easily sees that

$$(3.28) \qquad \|\delta\|_{W^{k+\ell-2\alpha,p}} \leq c h^{2\alpha-k-\ell+1/p},$$

where c is also independent of p. Inserting (3.28) into (3.26) and (3.27) and using Theorem 1 with $\ell + 1$ instead of ℓ we eventually obtain

$$(3.29) \qquad |D^\ell e(z)| \leq h^{k-\ell} \left\{ c_1 + \frac{c_2 h^{-1}}{(p-1)^2} \right\} \|u\|_{W^{k,\infty}}.$$

Now, set $p - 1 = |\ln h|^{-1}$ in (3.29) and observe the elementary inequality

$$h^{-1/(1+|\ln h|)} \leq c,$$

for all $0 < h \leq \frac{1}{2}$, to obtain the desired inequality (2.15).

4. THE NODAL COLLOCATION METHOD WITH ODD DEGREE SPLINES

The naive spline collocation with odd degree splines for boundary element methods in two dimensions, $n = 2$, has been analyzed in [4] as a modified Galerkin method. Therefore, our results on the pointwise error estimates can be applied to that modification. In the following, α may be arbitrary real.

Let now Γ in $\mathbb{R}^2$ be a finite system of mutually disjoint Jordan curves Γ_j, where as in [4], every Γ_j can be considered to be the smooth image of a corresponding homeomorphism of the unit circle which is parametrized by $y = (\cos 2\pi t, \sin 2\pi t)^T$. Then, the original equation (1.1) is equivalent to a system of equations of the form

$$Au = f , \tag{4.1}$$

for a 1-periodic vector valued function u and given 1-periodic vector valued function f. Let $\Delta = \{t_j\}_{j=-\infty}^{+\infty}$ be an increasing sequence of mesh points, $t_j \in \mathbb{R}$, satisfying $t_{j+M} = t_j + 1$, for fixed $M \in \mathbb{N}$ and all $j \in \mathbb{Z}$. Let us choose $k = m+1$ and let $S_h^{k,m} = S_m(\Delta)$ be the family of *odd* degree m-splines subordinate to the partitions Δ.

Then, $k/2 = (m+1)/2$ is an integer. Let $h = \max(t_j - t_{j-1})$. For regular families of meshes Δ, all the properties (2.1)-(2.5), and (2.14) are available. The naive collocation method for (4.1) reads

Find $u_\Delta \in S_m(\Delta)$ such that the collocation equations :

$$Au_\Delta(t_j) = f(t_j)\,, \quad j = 1, \ldots, M \tag{4.2}$$

are satisfied. For Au_Δ being continuous, we now require as in [4] that

$$2\alpha < m = k - 1\,. \tag{4.3}$$

Furthermore, let Λ be a strongly elliptic system of pseudo-differential operators of degree 2α, i.e., corresponding to A there exists a regular smooth matrix-valued function θ on Γ such that

$$\theta A = A_0 + A_1\,, \tag{4.4}$$

where A_0 satisfies the strong coercivity condition

$$\operatorname{Re}\,(A_0 v, v)_{H^{k/2}(\Gamma)} \geq c\,\|v\|^2_{H^{\alpha+k/2}(\Gamma)}\,, \quad c > 0\,, \tag{4.5}$$

and A_1 is continuous from $H^{s+\alpha}(\Gamma)$ into $H^{s-\alpha+1}(\Gamma)$, for all $s \in \mathbb{R}$.

Note that this concept of strong ellipticity is weaker than the coercivity assumption (1.2), (1.3) for the Galerkin method (see also [4], Section 2.3).

In [4] it has been shown that, under the above assumptions (4.3)-(4.5), the collocation equations (4.2) are uniquely solvable for sufficiently small $h > 0$, and that there holds the convergence estimate

$$\|u - u_\Delta\|_p \leq c h^{q-p} \|u\|_q\,, \tag{4.6}$$

for $2\alpha \leq p < m + \dfrac{1}{2}$ and $\max\{\alpha, p\} \leq q \leq k$.

Here we can also show pointwise error estimates.

THEOREM 3 : *Let A be a strongly elliptic pseudo-differential operator of order $2\alpha < m$, let* (4.1) *be uniquely solvable and let m be odd. Then the collocation method* (4.2) *converges pointwise as*

$$\|u - u_\Delta\|_{W^{\ell,\infty}} \leqslant c h^{k-\ell} \|u\|_{W^{k,\infty}}, \tag{4.7}$$

provided that

$$2\alpha + \frac{1}{2} \leqslant \ell \leqslant m \quad \text{and} \quad 0 \leqslant \ell . \tag{4.8}$$

For $2\alpha \leqslant \ell < 2\alpha + \frac{1}{2}$ and $\ell \geqslant 0$, we have

$$\|u - u_\Delta\|_{W^{\ell,\infty}} \leqslant c h^{k-\ell} \left(\log \frac{1}{h}\right)^2 \|u\|_{W^{k,\infty}} . \tag{4.9}$$

The estimates (4.7) and (4.9) are new for our class of equations. Let us discuss these estimates for three special cases.

4.1. Symm's integral equation of the first kind

These equations have logarithmic principal part ; see [29], Appendix A.1, and [4], Equation (2.3.12). There, $\alpha = -\frac{1}{2}$, A is strongly elliptic with $\theta = 1$, and we see from (4.8) that (4.7) holds for $0 \leqslant \ell \leqslant m$. E.g., for piecewise linears, where $m = 1$, we find pointwise convergence of quadratic order which is optimal and higher than in [1] and [3]. For piecewise cubics (4.7) gives pointwise convergence of order four. Both orders have been observed in numerical computations [16].

4.2. Cauchy singular integral equations

Here, $\alpha = 0$ with odd $m \geqslant 1$. For $1 \leqslant \ell \leqslant m$, (4.7) provides optimal order pointwise convergence for the ℓ-th order derivatives and (4.9) shows almost optimal order pointwise convergence for the function values. These estimates for point collocation improve those in [25].

4.3. Hypersingular boundary integral equations

In acoustics [21], elasticity [22], electromagnetics [8] and flow problems [14] one finds hypersingular strongly elliptic boundary integral equations of order $2\alpha = 1$. Here (4.7) provides the pointwise quadratic order convergence of the second derivatives and almost cubic order pointwise convergence of the first derivatives and the function values with piecewise cubic spline collocation.

5. PROOF OF THEOREM 3

The proof of Theorem 3 rests on the equivalence of the collocation method (4.2) with the modified Galerkin equations

$$((I - J + J_\Delta)\,\theta A\,(u_\Delta - u),\ \varphi_h)_{H^{k/2}(\Gamma)} = 0\ , \tag{5.1}$$

for all test splines $\varphi_h \in S_m(\Delta)$ and the desired collocation solution $u_\Delta \in S_m(\Delta)$. This was also the basic idea in [4]. Here, I denotes the identity, and J and J_Δ are defined by

$$Jv = \int_0^1 v(t)\,dt\ ,$$
$$J_\Delta\, v = \sum_{j=1}^{M} v(t_j)\,(t_{j+1} - t_{j-1})/2\ .$$

Integration by parts in (5.1) yields the equivalent Galerkin scheme

$$(Bu_\Delta,\ \varphi_h)_{L^2} = (B[u + B^{-1}(J - J_\Delta)\,\theta A\,(u_\Delta - u)],\ \varphi_h)_{L^2} = \\ = (Bw,\ \varphi_h)_{L^2}\ , \tag{5.2}$$

for all $\varphi_h \in S_m(\Delta)$, where

$$B = \sum_{r=0}^{k/2} \left(i\,\frac{d}{dt}\right)^{2r} \theta A$$

is a strongly elliptic coercive pseudo-differential operator of order $2\,\beta = 2\,\alpha + k$ providing the decomposition (1.2). In addition, B is invertible on the Sobolev spaces of 1-periodic functions, since A is invertible.

For (5.2), we now apply our result from Part I [29], for $\beta \leq 0$, or the estimate (2.9) of Theorem 1, for $\beta > 0$, respectively, to obtain

$$\|u - u_\Delta\|_{W^{\ell,\infty}} \leq c \inf_{\varphi_h \in S_m(\Delta)} \sum_{r=0}^{m} h^{r-\ell}\,\|w - \varphi_h\|_{W^{r,\infty}}\ , \tag{5.3}$$

where, according to (2.13), $\alpha + \frac{k}{2} = \beta < m + \frac{1}{2}$, and $\max\left\{0,\ 2\,\alpha + \frac{1}{2}\right\} = \max\left\{0,\ 2\,\beta - k + \frac{1}{2}\right\} \leq \ell \leq m$. These two conditions are equivalent to the conditions in (4.8).

For the right hand side in (5.3) we now use the approximation properties (2.5) and the definition of w to obtain

$$\|u - u_\Delta\|_{W^{\ell,\infty}} \leq ch^{k-\ell}\left\{ \|u\|_{W^{k,\infty}} + \|B^{-1}(J - J_\Delta)\,\theta A(u_\Delta - u)\|_{W^{k,\infty}}\right\} .$$

Since $(J - J_\Delta)\,\theta A(u_\Delta - u) \in \mathbb{R}$, the right hand side can further be estimated as

$$\begin{aligned}\|u - u_\Delta\|_{W^{\ell,\infty}} &\leq ch^{k-\ell}\|u\|_{W^{k,\infty}} + ch^{k-\ell}|(J - J_\Delta)\,\theta A(u_\Delta - u)| \\ &\leq ch^{k-\ell}\|u\|_{W^{k,\infty}} + ch^{k-\ell}\|u_\Delta - u\|_{2\alpha} .\end{aligned}$$

For $\|u_\Delta - u\|_{2\alpha}$ we now use (4.6), with $p = 2\alpha$ and $q = k$, to obtain

$$\|u - u_\Delta\|_{W^{\ell,\infty}} \leq c\{h^{k-\ell} + ch^{2k-2\alpha-\ell}\}\,\|u\|_{W^{k,\infty}} . \tag{5.4}$$

Since $2\alpha < m < k$ we have $2k - 2\alpha - \ell > k - \ell$, and (5.4) gives (4.7).

In the case $\ell \geq 0$ and $2\alpha \leq \ell < 2\alpha + \frac{1}{2}$, we use Theorem 2, with α replaced by $\alpha + \frac{k}{2}$. We find from (2.15) as in the previous case that

$$\begin{aligned}\|u - u_\Delta\|_{W^{\ell,\infty}} &\leq ch^{k-\ell}\left(\log\frac{1}{h}\right)^2 \|w\|_{W^{k,\infty}} \\ &\leq ch^{k-\ell}\left(\log\frac{1}{h}\right)^2 \left\{\|u\|_{W^{k,\infty}} + \|u_\Delta - u\|_{2\alpha}\right\} \\ &\leq ch^{k-\ell}\left(\log\frac{1}{h}\right)^2 \{1 + ch^{k-2\alpha}\}\,\|u\|_{W^{k,\infty}} .\end{aligned}$$

This gives (4.9), since $k > 2$. The proof is completed.

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