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INCREMENTAL METHODS IN NONLINEAR, THREE-DIMENSIONAL, INCOMPRESSIBLE ELASTICITY (*)

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Communiqué par P G. CIARLET

Abstract — *In this paper, we apply the incremental methods to approximate the equilibrium equations of nonlinearly elastic incompressible bodies, subject to dead or live loads, with pure displacement or traction boundary conditions We establish the convergence of the methods*

Résume — *Dans cet article, nous appliquons des méthodes incrémentales pour approcher les équations d'équilibre de corps élastiques non linéaires incompressibles, soumis à des forces mortes ou a des forces vives, avec des conditions aux limites de déplacement pur ou de traction pure Nous établissons la convergence de ces méthodes*

INTRODUCTION

We first specify the notation we shall use, concerning notably vector fields, matrices, function spaces, derivatives and norms. In what follows, Latin indices take their values in the set $\{1, 2, 3\}$ and the repeated index convention is used.

p : pressure field,
$\underset{\sim}{u} = (u_i)$: displacement field,
$\underset{\sim}{\phi} = (\phi_i)$: deformation field,
$\underset{\sim}{\nu} = (\nu_i)$: unit outer normal vector to the boundary of a domain,
$\underset{\sim}{u} \cdot \underset{\sim}{v} = u_i v_i$: Euclidean inner product,
$|\underset{\sim}{u}| = (\underset{\sim}{u} \cdot \underset{\sim}{u})^{1/2}$: Euclidean vector norm,
$\underset{\sim}{A} = (A_{ij})$: matrix with element A_{ij} (i : row index, j: column index),

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$\underset{\sim}{A} : \underset{\sim}{B} = A_{ij} B_{ij}$	: matrix inner product,
$\Vert \underset{\sim}{A} \Vert = (\underset{\sim}{A} : \underset{\sim}{A})^{1/2}$	: matrix norm associated with the matrix inner product,
$\underset{\sim}{\mathrm{adj}}\, \underset{\sim}{A}$	: adjugate of the matrix $\underset{\sim}{A}$ = transpose of the cofactor matrix,
$\mathbb{M}^3$	: set of all matrices of order 3,
skew	: set of skew symmetric matrices of order 3,
sym	: set of symmetric matrices of order 3,
$S^3_>$	: set of symmetric positive definite matrices,
O^3	: set of orthogonal matrices,
O^3_+	: set of orthogonal matrices, with determinant equals to one,
$\hat{\underset{\sim}{T}}$	: first Piola-Kirchhoff stress,
$\hat{\underset{\sim}{\sigma}}$	: second Piola-Kirchhoff stress,
$\underset{\sim}{b}$	: volume force operator
$\underset{\sim}{\tau}$	: surface force operator,
$\underset{\sim}{\ell} = (\underset{\sim}{b}, \underset{\sim}{\tau})$	: loading operator,
$\underset{\sim}{\ell} = (\underset{\sim}{b}, \underset{\sim}{\tau})$	: a dead load,
$\underset{\sim}{W}^{m,q} = (W^{m,q}(\Omega))^3$	: for some integer $m \geqq 0$ and some real number $q \geqq 1$,

$$\underset{\sim}{W}^{m-1/q,q} = (W^{m-1/q,q}(\Gamma))^3,$$
$$\underset{\sim}{W}_0^{1,q} = (W_0^{1,q}(\Omega))^3,$$
$$\underset{\sim}{H}^m = \underset{\sim}{W}^{m,2},$$
$$\underset{\sim}{H}^m_{\mathrm{sym}} = \{\underset{\sim}{\phi} \in \underset{\sim}{H}^m,\ \underset{\sim}{\phi}(0) = 0,\ \underset{\sim}{\nabla}\underset{\sim}{\phi}(0) = \underset{\sim}{\nabla}\underset{\sim}{\phi}(0)^{\mathrm{T}}\},$$
$$\underset{\sim}{V}^{m,q} = \underset{\sim}{W}^{m,q} \cap \underset{\sim}{W}_0^{1,q},$$
$$W^{m,q,0} = \left\{p \in W^{m,q},\ \int_\Omega p = 0\right\},$$
$$\underset{\sim}{M}^{m,q} = (W^{m,1}(\Omega))^9,$$
$$\underset{\sim}{C}_{\mathrm{sym}} = \{\underset{\sim}{\phi} \in \underset{\sim}{W}^{m+2,q},\ \underset{\sim}{\phi}(0) = 0,\ \underset{\sim}{\nabla}\underset{\sim}{\phi}(0) = \underset{\sim}{\nabla}\underset{\sim}{\phi}(0)^{\mathrm{T}}\},$$
$$\underset{\sim}{L} = \left\{(\underset{\sim}{b}, \underset{\sim}{\tau}) \in \underset{\sim}{W}^{m,q} \times \underset{\sim}{W}^{m+1-1/q,q},\ \int_\Omega \underset{\sim}{b} + \int_\Gamma \underset{\sim}{\tau} = 0\right\},$$

$\partial_i = \dfrac{\partial}{\partial x_i}$	: usual partial derivatives,
$\partial_{\underset{\sim}{A}} = \dfrac{\partial}{\partial \underset{\sim}{A}}$	: partial derivative with regard to the matrix variable $\underset{\sim}{A}$,
$\underset{\sim}{\nabla}\underset{\sim}{\phi} = (\partial_i \phi_j) \in \mathbb{M}^3$	: gradient of a mapping $\underset{\sim}{\phi} : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$,
$\underset{\sim}{\mathrm{div}}\, \underset{\sim}{T} = (\partial_j T_{ij}) \in \mathbb{R}^3$	: divergence of a tensor field $\underset{\sim}{T} : \Omega \subset \mathbb{R}^3 \to \mathbb{M}^3$,
$\underset{\sim}{\theta}', \underset{\sim}{\theta}''$	: first and second Fréchet derivatives of an operator $\underset{\sim}{\theta} : X \to Y$, $\theta'(u) \in L(X, Y)$, $\theta''(u) \in L_2(X, Y)$, X and Y being two normed vector spaces.

We denote by C_1, C_2, etc., constants which are independent of the various functions found in a given inequality.

Let Ω be a bounded open connected subset of $\mathbb{R}^3$ with a sufficiently smooth boundary Γ. We assume that $\bar{\Omega}$ is the *reference configuration* occupied by an *elastic incompressible body* in the absence of any applied force. The constitutive law of the material is given by

$$\hat{\underset{\sim}{T}} = \underset{\sim}{a}(\underset{\sim}{\nabla}\underset{\sim}{\phi}) - p\, \underset{\sim}{\mathrm{adj}}\, \underset{\sim}{\nabla}\underset{\sim}{\phi}^{\mathrm{T}}\,, \tag{0.1}$$

for a given C^{∞} matrix-valued function $\underset{\sim}{a} : (x, \underset{\sim}{A}) \in \bar{\Omega} \times \mathbb{M}^3 \to \mathbb{M}^3$, $\hat{\underset{\sim}{T}}$ being the first Piola-Kirchhoff tensor. For details about the constitutive law, the interested reader should refer to [14].

The equilibrium equations for such a body subject to body force $\underset{\sim}{b}$ are of the form

$$\underset{\sim}{\mathrm{div}}\, \hat{\underset{\sim}{T}} + \underset{\sim}{b} = 0 \quad \text{in } \Omega\,, \tag{0.2}$$

$$\det \underset{\sim}{\nabla}\underset{\sim}{\phi} - 1 = 0 \quad \text{in } \Omega\,, \tag{0.3}$$

the deformation $\underset{\sim}{\phi}$ and the pressure p being the unknowns. The equations (0.2) and (0.3) together with the boundary condition

$$\underset{\sim}{\phi} = \underset{\sim}{\mathrm{id}} \quad \text{on } \Gamma \tag{0.4}$$

constitute the pure displacement problem while the boundary condition

$$\{\underset{\sim}{a}(\underset{\sim}{\nabla}\underset{\sim}{\phi}) - p \,\mathrm{adj}\, \underset{\sim}{\nabla}\underset{\sim}{\phi}^{\mathrm{T}}\}\underset{\sim}{\nu} = \underset{\sim}{\tau} \quad \text{on } \Gamma \tag{0.5}$$

together with the equations (0.2)-(0.3) constitute the pure traction problem.

In what follows the real number $q > 3$. Taking into account equation (0.3) leads to the definition of the Sobolev submanifold

$$\underset{\sim}{\Sigma}_m^q = \{\underset{\sim}{\phi} \in \underset{\sim}{W}^{m+2,q}, \det \underset{\sim}{\nabla}\underset{\sim}{\phi} = 1 \text{ in } \Omega\}$$

and two nonlinear elasticity operators $\underset{\sim}{\theta}_1$ and $\underset{\sim}{\theta}_2$ corresponding respectively to the pure displacement and pure traction problems. When we consider the displacement $\underset{\sim}{u}$ and the pressure p as the unknowns, the first operator $\underset{\sim}{\theta}_1$ maps $\underset{\sim}{\Sigma}_m^q \cap \underset{\sim}{W}_0^{1,q} \times W^{m+1,q}$ into $\underset{\sim}{W}^{m,q}$ for each integer $m \geqq 0$ and real number $q > 3$. With the same conditions on (m, q) the second operator $\underset{\sim}{\theta}_2$ maps $\underset{\sim}{\Sigma}_m^q \times W^{m+1,q}$ into a submanifold of the set of loads $\underset{\sim}{L}$. The pure displacement and traction problems respectively reduce to solving the equations

$$\underset{\sim}{\theta}_1(\underset{\sim}{u}, p) = \underset{\sim}{b} \tag{0.6}$$

and

$$\underset{\sim}{\theta}_2(\underset{\sim}{\phi}, p) = (\underset{\sim}{b}, \underset{\sim}{\tau})\,. \tag{0.7}$$

The incremental methods consist in approximating the above nonlinear equations by a sequence of linear problems. Unfortunately these operators are not defined between affine manifolds and moreover they do not always possess the inverse function theorem property, a crucial condition on which depends the applicability and the convergence of the method as proved by Bernadou-Ciarlet-Hu [5]. We therefore have to modify the incremental methods according to the problem we want to solve.

In Section 1 we study the pure displacement problem. The corresponding nonlinear operator $\underset{\sim}{\theta}_1$ defines a local diffeomorphism between the spaces $\underset{\sim}{\Sigma}^q_m \cap \underset{\sim}{W}^{1,q}_0 \times W^{m+1,q,0}$ and $\underset{\sim}{W}^{m,q}$ but since the space $\underset{\sim}{\Sigma}^q_m$ is not an affine manifold, we define a second operator $\hat{\underset{\sim}{\theta}}_1$ between $\underset{\sim}{V}^{m+2,q} \times W^{m+1,q,0}$ and $\underset{\sim}{W}^{m,q} \times W^{m+1,q,0}$ which are both linear spaces. This operator satisfies the crucial inverse function theorem property. We then describe the relation between both operators $\underset{\sim}{\theta}_1$ and $\hat{\underset{\sim}{\theta}}_1$ and we apply the incremental methods to the second operator, in order to approximate the local solution for the pure displacement problem in the case of a dead load.

In Section 2 we consider the pure traction problem. The associated operator $\underset{\sim}{\theta}_2$ is defined on a non affine manifold. As in Section 1 we define a second operator $\hat{\underset{\sim}{\theta}}_2$ between the spaces $\underset{\sim}{W}^{m+2,q} \times W^{m+1,q}$ and $\underset{\sim}{L} \times W^{m+1,q}$. Using results due to Chillingworth-Marsden-Wan [6] and Wan, Y. H. [21], we prove that this operator defines a local diffeomorphism between an open neighborhood of $(\underset{\sim}{\mathrm{id}}, 0)$ in the linear space $\underset{\sim}{C}_{\mathrm{sym}} \times W^{m+1,q}$ and a non affine Sobolev submanifold N of $\underset{\sim}{L} \times W^{m+1,q}$. From the decomposition of the load space as $\underset{\sim}{L} = \underset{\sim}{L}_e \oplus$ skew, we show that the submanifold N is the graph of a function G defined from $\underset{\sim}{L}_e \times W^{m+1,q}$ into Skew. Let $\underset{\sim}{\ell}$ be a dead load in $\underset{\sim}{L}_e$ with no axis of equilibrium ; then locally there exists a unique rotation Q such that the element $(Q\underset{\sim}{\ell}, 0) \in N$. Consequently we obtain the local unique solution corresponding to the load $\underset{\sim}{\ell}$. The solution corresponding to the dead load $Q\underset{\sim}{\ell}$ is obtained from the solution of an appropriate differential equation between the above considered spaces, with $Q\underset{\sim}{\ell}$ as a parameter in the vector field. The approximation comprises two steps : we first approach the parameter $Q\underset{\sim}{\ell}$, then we apply Euler's method to the differential equation in which the approximate value of $Q\underset{\sim}{\ell}$ is a parameter.

Finally we consider a class of live loads. Under some additional assumptions on the loading operator $\underset{\sim}{\ell}$ we first prove the existence and local uniqueness of a solution *via* the fixed point theorem, using in a crucial way the polar decomposition of invertible matrices. The approximation of the solution consists in approaching a finite sequence of solutions for traction problems of the dead load type.

1. APPROXIMATION OF A PURE DISPLACEMENT BOUNDARY-VALUE PROBLEM FOR INCOMPRESSIBLE MATERIALS

1.1. Existence

Let Ω be an open bounded and connected subset of $\mathbb{R}^3$ with a sufficiently smooth boundary Γ. We assume that $\bar{\Omega}$ is the *reference configuration* of an *incompressible body* in the absence of any applied forces. The equilibrium equations of the body, when it is subjected to a body force $\underset{\sim}{b}$, are :

$$-\underset{\sim}{\operatorname{div}}\, \hat{\underset{\sim}{T}} = \underset{\sim}{b} \quad \text{in } \Omega\,, \tag{1.1-1}$$
$$\det \underset{\sim}{\nabla} \underset{\sim}{\phi} = 1 \quad \text{in } \Omega\,, \tag{1.1-2}$$
$$\underset{\sim}{\phi} = \underset{\sim}{\mathrm{id}} \quad \text{on } \Gamma\,. \tag{1.1-3}$$

Given a constitutive law for the first Piola-Kirchhoff stress $\hat{\underset{\sim}{T}}$ as defined in [14]

$$\hat{\underset{\sim}{T}} = \underset{\sim}{a}(\underset{\sim}{\nabla}\underset{\sim}{\phi}) - p\, \underset{\sim}{\operatorname{adj}}\, \underset{\sim}{\nabla}\underset{\sim}{\phi}^{\mathrm{T}}\,, \tag{1.1-4}$$

where p is the pressure and the constitutive law is a matrix-valued C^{∞} mapping $\underset{\sim}{a} : \bar{\Omega} \times M^3 \to M^3$, the problem then consists in solving the following boundary-value problem for the displacement field $\underset{\sim}{u} = (u_i)$ and the pressure p :

$$-\underset{\sim}{\operatorname{div}}\, \{\underset{\sim}{a}(I + \underset{\sim}{\nabla}\underset{\sim}{u}) - p\, \underset{\sim}{\operatorname{adj}}\, (I + \underset{\sim}{\nabla}\underset{\sim}{u})^{\mathrm{T}}\} = \underset{\sim}{b} \quad \text{in } \Omega\,, \tag{1.1-5}$$
$$\det\, (I + \underset{\sim}{\nabla}\underset{\sim}{u}) = 1 \quad \text{in } \Omega\,, \tag{1.1-6}$$
$$\underset{\sim}{u} = 0 \quad \text{on } \Gamma\,. \tag{1.1-7}$$

The dependence on x in the constitutive law has been omitted in the formulas. The second Piola-Kirchhoff stress

$$\hat{\underset{\sim}{\sigma}} = (I + \underset{\sim}{\nabla}\underset{\sim}{u})^{-1}\, \hat{\underset{\sim}{T}} \tag{1.1-8}$$

could be equivalently used in these equations as in [5].

In what follows the different Sobolev spaces are defined for integers m and real numbers q satisfying the condition

$$m \geqq 0 \quad \text{and} \quad q > 3\,. \tag{1.1-9}$$

Considering the equations (1.1-6)-(1.1-7) and the condition (1.1-9) we define the set of *admissible displacements*

$$\underset{\sim}{\Sigma}{}_m^q = \{\underset{\sim}{u} \in \underset{\sim}{V}^{m+2,q},\ \det\, (I + \underset{\sim}{\nabla}\underset{\sim}{u}) = 1 \text{ in } \Omega\}\,. \tag{1.1-10}$$

This set has a C^{∞}-submanifold structure because of the condition (1.1-9) as shown in [14]. The pure displacement problem reduces to solving the equations

$$(\underset{\sim}{u}, p) \in \underset{\sim}{\Sigma}_m^q \times W^{m+1,q}, \tag{1.1-11}$$

$$-\underset{\sim}{\operatorname{div}} \left\{\underset{\sim}{a}(I + \underset{\sim}{\nabla} \underset{\sim}{u}) - p\, \underset{\sim}{\operatorname{adj}}\, (I + \underset{\sim}{\nabla} \underset{\sim}{u})^{\mathrm{T}}\right\} = \underset{\sim}{b} \quad \text{in } \Omega . \tag{1.1-12}$$

In order to analyse the properties of the left hand side of the equation (1.1-12) which will be useful in the sequel, we gather here all the properties of the constitutive law $\underset{\sim}{a}$ under our hypotheses :

- *the matrix-valued function* $\underset{\sim}{a} : \bar{\Omega} \times \mathbb{M}^3 \to \mathbb{M}^3$ *is of class* C^{∞}, (1.1-13)
- *the reference configuration is a natural state, i.e.* $\underset{\sim}{a}(I) = 0$, (1.1-14)
- *the principle of material frame-indifference is satisfied, i.e.*

$$\underset{\sim}{a}(QF) = Q\underset{\sim}{a}(F)\,, \quad \textit{for all } Q \textit{ in } O_+^3 \tag{1.1-15}$$

- *letting* $\underset{\sim}{C} = \partial_{\underset{\sim}{A}} \underset{\sim}{a}(I)$, *there exists a real number* $\beta > 0$ *such that*

$$C_{ijk\ell}\, e_{ij}\, e_{k\ell} \geqq \beta e_{ij}\, e_{ij} \tag{1.1-16}$$

where $\underset{\sim}{e} = (e_{ij})$ *is a symmetric matrix.*

From the above hypotheses we deduce that the equations (1.1-11) and (1.1-12) are equivalent to solving

$$\underset{\sim}{\theta}_1(\underset{\sim}{u}, p) = \underset{\sim}{b} \tag{1.1-17}$$

where the nonlinear operator $\underset{\sim}{\theta}_1$ is defined by

$$\underset{\sim}{\theta}_1 : (\underset{\sim}{u}, p) \in \underset{\sim}{\Sigma}_m^q \times W^{m+1,q} \to$$
$$\to -\underset{\sim}{\operatorname{div}} \left\{\underset{\sim}{a}(I + \underset{\sim}{\nabla} \underset{\sim}{u}) - p\, \underset{\sim}{\operatorname{adj}}\, (I + \nabla \underset{\sim}{u})^{\mathrm{T}}\right\} \in \underset{\sim}{W}^{m,q} . \tag{1.1-18}$$

This operator is C^{∞} and locally bounded as a composition of the mappings $\underset{\sim}{u} \mapsto \underset{\sim}{\operatorname{adj}}\, (I + \underset{\sim}{\nabla} \underset{\sim}{u})$, $(\underset{\sim}{u}, p) \mapsto p\, \underset{\sim}{\operatorname{adj}}\, (I + \underset{\sim}{\nabla} \underset{\sim}{u})^{\mathrm{T}}$ and the Nemytsky operator $\underset{\sim}{\phi} \mapsto \underset{\sim}{a}(\underset{\sim}{\nabla} \underset{\sim}{\phi})$ which are all C^{∞} and locally bounded ([13], [20]). It is clear from (1.1-14) that $\underset{\sim}{\theta}_1(\underset{\sim}{0}, 0) = \underset{\sim}{0}$.

In order to apply the inverse function theorem we must compute $\underset{\sim}{\theta}_1'(\underset{\sim}{0}, 0)$ and verify that it is an isomorphism between $T_{\underset{\sim}{0}}\, \underset{\sim}{\Sigma}_m^q \times W^{m+1,q,0}$ and $\underset{\sim}{W}^{m,q}$, where

$$T_{\underset{\sim}{0}}\, \underset{\sim}{\Sigma}_m^q = \left\{\underset{\sim}{u} \in \underset{\sim}{V}^{m+2,q},\ \operatorname{div} \underset{\sim}{u} = 0\right\} \tag{1.1-19}$$

is the tangent space to $\underset{\sim}{\Sigma}_m^q$ at the origin $\underset{\sim}{0}$ and

$$W^{m+1,q,0} = \left\{ p \in W^{m+1,q},\ \int_\Omega p = 0 \right\}. \tag{1.1-20}$$

The equation

$$\underset{\sim}{\theta}_1'(\underset{\sim}{0}, 0)(\underset{\sim}{u}, p) = \underset{\sim}{b} \tag{1.1-21}$$

is equivalent to the linear pure displacement boundary-value problem

$$-\underset{\sim}{\operatorname{div}} \{C \in (\underset{\sim}{u}) - pI\} = \underset{\sim}{b} \quad \text{in } \Omega\,, \tag{1.1-22}$$

$$\operatorname{div} \underset{\sim}{u} = g \quad \text{in } \Omega\,, \tag{1.1-23}$$

$$\underset{\sim}{u} = \underset{\sim}{h} \quad \text{on } \Gamma\,, \tag{1.1-24}$$

with

$$g = 0 \quad \text{and} \quad \underset{\sim}{h} = \underset{\sim}{0}\,. \tag{1.1-25}$$

The tensor $\in(\underset{\sim}{u})$ is the linearized strain tensor defined by

$$\in(\underset{\sim}{u}) = (\in_{ij}(\underset{\sim}{u}))\,, \quad \text{with} \quad 2 \in_{ij}(\underset{\sim}{u}) = \partial_i u_j + \partial_j u_i\,. \tag{1.1-26}$$

Clearly the operator defined by (1.1-21) is continuous. Under the hypotheses (1.1-13)-(1.1-16) and the additional hypothesis :

• *The complementarity condition of Agmon-Douglis-Nirenberg* ([3], [4]) *holds for the equations* (1.1-22)-(1.1-24), (1.1-27)

it follows ([14]) that the operator $\underset{\sim}{\theta}_1'(\underset{\sim}{0}, 0)$ is an isomorphism between the spaces $T_0\,\underset{\sim}{\Sigma}_m^q \times W^{m+1,q,0}$ and $\underset{\sim}{W}^{m,q}$. We keep these various hypotheses throughout ; we can now easily deduce existence results :

THEOREM 1 : *For each pair of numbers* (m, q) *satisfying condition* (1.1-9), *there exist a neighborhood* B *of* $\underset{\sim}{0}$ *in* $\underset{\sim}{W}^{m,q}$ *and a neighborhood* $U \times P$ *of* $(\underset{\sim}{0}, 0)$ *in* $\underset{\sim}{\Sigma}_m^q \times W^{m+1,q,0}$ *such that, for each load* $\underset{\sim}{b}$ *in* B, *the pure displacement boundary-value problem* (1.1-5)-(1.1-7) *has exactly one solution in* $U \times P$. □

Such a solution therefore satisfies the equation (1.1-17) ; however the incremental method cannot be directly described. Indeed in the description of the method [5], we compute the approximate solution $(\underset{\sim}{u}^{n+1}, p^{n+1})$ at step $n+1$ from $(\underset{\sim}{u}^n, p^n)$ by a formula of the form

$$\underset{\sim}{u}^{n+1} = \underset{\sim}{u}^n + \delta \underset{\sim}{u}^n \quad \text{and} \quad p^{n+1} = p^n + \delta p^n \tag{1.1-28}$$

where $\delta \underset{\sim}{u}^n$ and δp^n are « small » increments of the displacement and the pressure respectively. *Since* $\underset{\sim}{\Sigma}_m^q$ *is not a linear space* $\underset{\sim}{u}^{n+1} = \underset{\sim}{u}^n + \delta \underset{\sim}{u}^n$ *does not* *belong to* $\underset{\sim}{\Sigma}_m^q$ *in general.* We overcome this difficulty by defining a second operator $\hat{\underset{\sim}{\theta}}_1$ between linear spaces such that

$$\hat{\underset{\sim}{\theta}}_1(\underset{\sim}{u}, p) = (\underset{\sim}{b}, 0) \tag{1.1-29}$$

be equivalent to (1.1-17) for the load $\underset{\sim}{b}$. A natural definition for $\hat{\underset{\sim}{\theta}}_1$ is

$$\hat{\underset{\sim}{\theta}}_1 : (\underset{\sim}{u}, p) \in \underset{\sim}{V}^{m+2, q} \times W^{m+1, q, 0} \to$$
$$\to (\underset{\sim}{\theta}_1(\underset{\sim}{u}, p), \det(I + \underset{\sim}{\nabla}\underset{\sim}{u}) - 1) \in \underset{\sim}{W}^{m, q} \times W^{m+1, q, 0}. \tag{1.1-30}$$

We must therefore verify that

$$\det(I + \underset{\sim}{\nabla}\underset{\sim}{u}) - 1 \in W^{m+1, q, 0}. \tag{1.1-31}$$

We may write

$$\det(I + \underset{\sim}{\nabla}\underset{\sim}{u}) - 1 = \operatorname{div} \underset{\sim}{u} + \{\partial_{i+2} u_{i+2}\, \partial_{i+1} u_{i+1} -$$
$$- \partial_{i+1} u_{i+2}\, \partial_{i+2} u_{i+1}\} + \frac{1}{3}\, \partial_j u_i (\underset{\sim}{\operatorname{adj}}\, \underset{\sim}{\nabla}\underset{\sim}{u}^{\mathrm{T}})\, ij\,. \tag{1.1-32}$$

We find that the integral of the above expression vanishes using Green's formula, the Piola identity and the boundary condition. Clearly this operator is also C^{∞} and locally bounded with all its derivatives. It is easy to conclude that

$$\hat{\underset{\sim}{\theta}}_1'(\underset{\sim}{0}, 0)(\underset{\sim}{u}, p) = (\underset{\sim}{b}, g) \tag{1.1-33}$$

defines an isomorphism between $\underset{\sim}{V}^{m+2, q} \times W^{m+1, q, 0}$ and $\underset{\sim}{W}^{m, q} \times W^{m+1, q, 0}$. Indeed the equation (1.1-33) is equivalent to the equations (1.1-22)-(1.1-24) for the particular case in which $\underset{\sim}{h} = \underset{\sim}{0}$. The operator $\hat{\underset{\sim}{\theta}}_1'(\underset{\sim}{0}, 0)$ is also continuous and one-to-one. Its image is the subset

$$\left\{(\underset{\sim}{b}, g, \underset{\sim}{h}) \in \underset{\sim}{W}^{m, q} \times W^{m+1, q, 0} \times \underset{\sim}{W}^{m+2-1/q, q}\,,\ \int_{\Omega} g - \int_{\Gamma} \underset{\sim}{h} \cdot \underset{\sim}{\nu} = 0\right\}, \tag{1.1-34}$$

which contains $\underset{\sim}{W}^{m, q} \times W^{m+1, q, 0} \times \{\underset{\sim}{0}\}$, ([14]). We easily conclude using the Banach theorem.

We can now deduce a theorem similar to theorem 1.

THEOREM 2 : *For each pair of numbers* (m, q) *satisfying* (1.1-9), *there exist a neighborhood* $B \times G$ *of* $(\underset{\sim}{0}, 0)$ *in* $\underset{\sim}{W}^{m, q} \times W^{m+1, q, 0}$ *and a neighborhood* $U \times P$ *of* $(\underset{\sim}{0}, 0)$ *in* $\underset{\sim}{V}^{m+2, q} \times W^{m+1, q, 0}$ *such that* $\hat{\underset{\sim}{\theta}}_1$ *defines a diffeomorphism between the two neighborhoods.* □

Clearly the operator $\hat{\underset{\sim}{\theta}}_1$ satisfies equation (1.1-29) in these neighborhoods. For a given dead load $\underset{\sim}{b}$ in B, we shall use the incremental method to approximate the solution of the equation

$$\hat{\underset{\sim}{\theta}}_1(\underset{\sim}{u}, p) = (\underset{\sim}{b}, 0) \tag{1.1-35}$$

which is exactly the solution for the pure displacement boundary-value problem (1.1-5)-(1.1-7).

1.2. Description and convergence of the method.

We begin by specifying some subsequent notations. We denote by

$$\|\underset{\sim}{u}\|_{m,q} = \left\{\int_\Omega \sum_{|\alpha| \leqq m} |\partial_\alpha \underset{\sim}{u}|^q\right\}^{1/q}$$

the norm of the Sobolev spaces $\underset{\sim}{W}^{m,q}$, by

$$\|(\underset{\sim}{u}, p)\| = \left\{\|\underset{\sim}{u}\|^q_{m+2,q} + \|p\|^q_{m+1,q}\right\}^{1/q}$$

the norm of the product space $\underset{\sim}{W}^{m+2,q} \times W^{m+1,q}$ and its subspaces. We shall denote by the same symbol $\|.\|$ the norms of the spaces $L(X, Y)$, $L(Y, X)$ and $L_2(X, Y)$ for the normed spaces X and Y. We shall not consider the case of a gradient vector field $\underset{\sim}{b} \in C^1(\Omega, \mathbb{R}^3)$ for which a trivial solution is given by the pair $(\underset{\sim}{0}, p)$,

$$p = \int_0^1 \underset{\sim}{b}(tx) \, . \, x \, dt \, . \tag{1.2-1}$$

We can now describe the approximation of the boundary-value problem (1.1-5)-(1.1-7) by an incremental method. Given a dead load $\underset{\sim}{b}$ in the neighborhood B, as defined in theorem 2, we approximate the solution $(\underset{\sim}{u}, p)$, $(\hat{\underset{\sim}{\theta}}_1(\underset{\sim}{u}, p) = (\underset{\sim}{b}, 0))$ as follows : let there be given any partition

$$0 = \lambda^0 < \lambda^1 < \cdots < \lambda^N = 1 \tag{1.2-2}$$

of the interval $[0, 1]$. We let

$$\Delta\underset{\sim}{b}^n = (\lambda^{n+1} - \lambda^n)\, \underset{\sim}{b} \,, \quad 0 \leqq n \leqq N-1 \,, \tag{1.2-3}$$

$$\underset{\sim}{u}^0 = \underset{\sim}{0} \,, \quad p^0 = 0 \,, \tag{1.2-4}$$

then assuming $\underset{\sim}{u}^n$ and p^n are known, we solve the linear problem

$$\hat{\underset{\sim}{\theta}}{}'_1(\underset{\sim}{u}^n, p^n)(\delta\underset{\sim}{u}^n, \delta p^n) = (\Delta\underset{\sim}{b}^n, 0) \,, \tag{1.2-5}$$

$$(\delta\underset{\sim}{u}^n, \delta p^n) \in \underset{\sim}{V}^{m+2,q} \times W^{m+1,q,0} \, . \tag{1.2-6}$$

We define the $(n+1)$-st approximate displacement and pressure

$$\underset{\sim}{u}^{n+1} = \underset{\sim}{u}^{n} + \delta \underset{\sim}{u}^{n} \quad \text{and} \quad p^{n+1} = p^{n} + \delta p^{n} . \tag{1.2-7}$$

We end this algorithm by computing the N-th approximate Piola-Kirchhoff second stress tensor from the N-th approximate displacement and pressure by

$$\hat{\underset{\sim}{\sigma}}^{N} = \hat{\underset{\sim}{\sigma}}(\underset{\sim}{u}^{N}, p^{N}) = (I + \underset{\sim}{\nabla} \underset{\sim}{u}^{N})^{-1} \times \times \left\{ \underset{\sim}{a}(I + \underset{\sim}{\nabla}\underset{\sim}{u}^{N}) - p^{N} \underset{\sim}{\operatorname{adj}} (I + \underset{\sim}{\nabla}\underset{\sim}{u}^{N})^{\mathsf{T}} \right\} . \tag{1.2-8}$$

Before proving the convergence of this method, we recall some relevant results proved so far :

We showed that the operator $\hat{\underset{\sim}{\theta}}_1$ is C^{∞} and has locally bounded derivatives of all orders. We deduced (theorem 2) from the hypotheses (1.1-13)-(1.1-16) and (1.1-27) that $\hat{\underset{\sim}{\theta}}_1$ defines a local diffeomorphism between a neighborhood $U \times P$ of $(\underset{\sim}{0}, 0)$ in $\underset{\sim}{V}^{m+2,q} \times W^{m+1,q,0}$ and a neighborhood $B \times G$ of $(\underset{\sim}{0}, 0)$ in $\underset{\sim}{W}^{m,q} \times W^{m+1,q,0}$. For each element $(\underset{\sim}{u}, p)$ in $U \times P$, $\hat{\underset{\sim}{\theta}}_1'(\underset{\sim}{u}, p)$ is an isomorphism between the above spaces which are both Banach spaces. We now describe these neighborhoods in the following theorem.

THEOREM 3 (Bernadou-Ciarlet-Hu 1983 theorem 2) : *For each pair of numbers* (m, q) *satisfying* (1.1-9), *there exists a real number* $\rho_0 > 0$ *depending on* m *and* q *such that if*

$$0 \leqq \rho \leqq \rho_0 , \tag{1.2-9}$$

then

$$\hat{\underset{\sim}{\theta}}_1'(\underset{\sim}{u}, p) \textit{ is an isomorphism for each } (\underset{\sim}{u}, p) \in U_\rho \times P_\rho , \tag{1.2-10}$$

where

$$U_\rho \times P_\rho = \left\{ (\underset{\sim}{u}, p) \in \underset{\sim}{V}^{m+2,q} \times W^{m+1,q,0},\ \| (\underset{\sim}{u}, p) \| \leqq \rho \right\} , \tag{1.2-11}$$

$$\gamma_\rho = \sup_{(\underset{\sim}{u}, p) \in U_\rho \times P_\rho} \left\| \left\{ \hat{\underset{\sim}{\theta}}_1'(\underset{\sim}{u}, p) \right\}^{-1} \right\| < +\infty , \tag{1.2-12}$$

$$L_\rho = \sup_{\substack{(\underset{\sim}{u}_1, p_1),\ (\underset{\sim}{u}_2, p_2) \in U_\rho \times P_\rho \\ (\underset{\sim}{u}_1, p_1) \neq (\underset{\sim}{u}_1, p_2)}} \frac{\left\| \left\{ \hat{\underset{\sim}{\theta}}_1'(\underset{\sim}{u}_1, p_1) \right\}^{-1} - \left\{ \hat{\underset{\sim}{\theta}}_1'(\underset{\sim}{u}_2, p_2) \right\}^{-1} \right\|}{\| (\underset{\sim}{u}_1 - \underset{\sim}{u}_2, p_1 - p_2) \|} < +\infty . \qquad \square \tag{1.2-13}$$

We deduce from this theorem and a classical result in differential equations (e.g. Crouzeix-Mignot [9]) :

THEOREM 4 (Bernadou-Ciarlet-Hu 1983 theorem 3) : *For each real number* ρ *which satisfies the inequality* (1.2-9) *and for each load* $(\underset{\sim}{b}, g)$ *which satisfies*

$$\| (\underset{\sim}{b}, g) \| \leqq \rho \gamma_\rho^{-1} , \tag{1.2-14}$$

the differential equation defined in the closed ball $U_\rho \times P_\rho$ *by*

$$\frac{d}{d\lambda} (\underset{\sim}{\tilde{u}}(\lambda), \tilde{p}(\lambda)) = \left\{ \underset{\sim}{\hat{\theta}}_1' (\underset{\sim}{\tilde{u}}(\lambda), \tilde{p}(\lambda)) \right\}^{-1} (\underset{\sim}{b}, g), \quad 0 \leqq \lambda \leqq 1 , \tag{1.2-15}$$

$$\underset{\sim}{\tilde{u}}(0) = \underset{\sim}{0} \quad and \quad \tilde{p}(0) = 0 \tag{1.2-16}$$

has a unique solution which satisfies

$$\underset{\sim}{\hat{\theta}}_1 (\underset{\sim}{\tilde{u}}(\lambda), \tilde{p}(\lambda)) = \lambda (\underset{\sim}{b}, g) , \quad 0 \leqq \lambda \leqq 1 . \qquad \square \tag{1.2-17}$$

Consequently, one has

$$\| \underset{\sim}{u}^N - \underset{\sim}{u} \|_{m+2, q} \leqq C_1 \max_{0 \leqq n \leqq N-1} \{ \lambda^{n+1} - \lambda^n \} , \tag{1.2-18}$$

$$\| p^N - p \|_{m+1, q} \leqq C_1 \max_{0 \leqq n \leqq N-1} \{ \lambda^{n+1} - \lambda^n \} , \tag{1.2-19}$$

and, as a by-product,

$$\| \underset{\sim}{\hat{\sigma}}^N - \underset{\sim}{\hat{\sigma}} \|_{m+1, q} \leqq C_2 \max_{0 \leqq n \leqq N-1} \{ \lambda^{n+1} - \lambda^n \} , \tag{1.2-20}$$

where C_1 and C_2 are constants which depend only on ρ, L_ρ and $\| (\underset{\sim}{b}, g) \|$.

2. APPROXIMATION OF A PURE TRACTION BOUNDARY-VALUE PROBLEM FOR INCOMPRESSIBLE MATERIALS

2.1. Preliminary

To begin with, let us recall some definitions that we shall use in the sequel. The mapping,

$$k : (\underset{\sim}{b}, \tau) \in \underset{\sim}{L} \to \int_\Omega \underset{\sim}{b} \otimes x + \int_\Gamma \underset{\sim}{\tau} \otimes x \in \mathbb{M}^3 \tag{2.1-1}$$

is called the *astatic load.* A load $\underset{\sim}{\ell} = (\underset{\sim}{b}, \underset{\sim}{\tau})$ is *equilibrated* relative to the natural state if and only if

$$k(\underset{\sim}{\ell}) \in \text{sym} . \tag{2.1-2}$$

We define two supplementary subsets of the load space $\underset{\sim}{L}$,

$$\underset{\sim}{L}_e = \{\underset{\sim}{\ell} \in \underset{\sim}{L}, k(\underset{\sim}{\ell}) \in \text{sym}\} , \tag{2.1-3}$$
$$\text{skew} = \{\underset{\sim}{\ell} \in \underset{\sim}{L}, k(\underset{\sim}{\ell}) \in \text{skew}\} . \tag{2.1-4}$$

For a given load $\underset{\sim}{\ell}$ in $\underset{\sim}{L}$, skew $(\underset{\sim}{\ell})$ will denote the component of $\underset{\sim}{\ell}$ in the subspace skew. A load $\underset{\sim}{\ell}$ in $\underset{\sim}{L}_e$ is *without axis of equilibrium* if the following equivalent conditions are satisfied :

$$\det\,(k(\underset{\sim}{\ell}) - \operatorname{tr} k(\underset{\sim}{\ell})\, I) \neq 0 \tag{2.1-5}$$

or if the mapping

$$w \in \text{skew} \to k\,(\underset{\sim}{\ell})\, w + wk\,(\underset{\sim}{\ell}) \in \text{skew} \tag{2.1-6}$$

is an isomorphism. We define two canonical projections

$$\Lambda_e : \underset{\sim}{\ell} = (\underset{\sim}{\ell}_e, \text{skew}\ (\underset{\sim}{\ell})) \in \underset{\sim}{L} \to \underset{\sim}{\ell}_e \in \underset{\sim}{L}_e , \tag{2.1-7}$$
$$\Lambda : \underset{\sim}{\ell} = (\underset{\sim}{\ell}_e, \text{skew}\ (\underset{\sim}{\ell})) \in \underset{\sim}{L} \to \text{skew}\ (\underset{\sim}{\ell}) \in \text{skew} . \tag{2.1-8}$$

A *loading operator* is a continuous mapping

$$\underset{\sim}{\ell} : \phi \in \underset{\sim}{W}^{m+2,q} \to \underset{\sim}{\ell}(\phi) = (\underset{\sim}{b}(\phi), \underset{\sim}{\tau}(\phi)) \in \underset{\sim}{L} . \tag{2.1-9}$$

A loading operator is said to be a *dead load* if it is constant $(\underset{\sim}{\ell}(\phi) = (\underset{\sim}{b}, \underset{\sim}{\tau}))$, otherwise it is called a *live load.* One should refer to Chillingworth-Marsden-Wan [6] for details on the above definition and assumptions.

We can now describe the pure traction boundary-value problem. We consider the constitutive law of an incompressible material as defined in Section 1 by the relation (1.1-4) and we keep the hypotheses (1.1-13)-(1.1-16). The problem consists in finding a deformation ϕ and a pressure p that satisfy

$$(\phi, p) \in \underset{\sim}{W}^{m+2,q} \times W^{m+1,q} , \tag{2.1-10}$$
$$-\,\underset{\sim}{\operatorname{div}}\, \{\underset{\sim}{a}(\underset{\sim}{\nabla}\Phi) - p\, \underset{\sim}{\operatorname{adj}}\, \underset{\sim}{\nabla}\Phi^{\mathrm{T}}\} = \underset{\sim}{b} \text{ in } \Omega , \tag{2.1-11}$$
$$\{\underset{\sim}{a}(\underset{\sim}{\nabla}\phi) - p\, \underset{\sim}{\operatorname{adj}}\, \underset{\sim}{\nabla}\phi^{\mathrm{T}}\}\, \underset{\sim}{\nu} = \underset{\sim}{\tau} \text{ on } \Gamma , \tag{2.1-12}$$
$$\det \underset{\sim}{\nabla}\phi - 1 = 0 \text{ in } \Omega . \tag{2.1-13}$$

Green's formula implies that the load $(\underset{\sim}{b}, \underset{\sim}{\tau})$ must be of total force zero.

Similarly as in Section 1 these equations can be put in the form of an operator

$$\underset{\sim}{\theta}_2 = (\underset{\sim}{\phi}, p) \in W^{m+2,q} \times W^{m+1,q} \to (-\underset{\sim}{\operatorname{div}} \{\underset{\sim}{a}(\underset{\sim}{\nabla}\underset{\sim}{\phi}) - p \,\underset{\sim}{\operatorname{adj}}\, \underset{\sim}{\nabla}\underset{\sim}{\phi}^{\mathsf{T}}\},$$
$$\{\underset{\sim}{a}(\underset{\sim}{\nabla}\underset{\sim}{\phi}) - p\,\underset{\sim}{\operatorname{adj}}\,\underset{\sim}{\nabla}\underset{\sim}{\phi}^{\mathsf{T}}\}\, \underset{\sim}{\nu}, \det \underset{\sim}{\nabla}\underset{\sim}{\phi} - 1) \in \underset{\sim}{L} \times W^{m+1,q}, \quad (2.1\text{-}14)$$

which is also C^{∞} and has locally bounded derivatives of all orders. Clearly one has

$$\underset{\sim}{\theta}_2(\underset{\sim}{\mathrm{id}}, 0) = (\underset{\sim}{0}, 0). \quad (2.1\text{-}15)$$

In order to eliminate the indetermination due to rigid body motions we shall restrict this operator to the set of $\underset{\sim}{C}_{\mathrm{sym}} \times W^{m+1,q}$. In this way we define

$$\hat{\underset{\sim}{\theta}}_2 : (\underset{\sim}{\varphi}, p) \in \underset{\sim}{C}_{\mathrm{sym}} \times W^{m+1,q} \to \underset{\sim}{\theta}_2(\underset{\sim}{\varphi}, p) \in \underset{\sim}{L} \times W^{m+1,q}. \quad (2.1\text{-}16)$$

The equation

$$\hat{\underset{\sim}{\theta}}{}_2'(\underset{\sim}{\mathrm{id}}, 0)(\underset{\sim}{v}, p) = (\underset{\sim}{b}, \underset{\sim}{\tau}, g) \quad (2.1\text{-}17)$$

is equivalent to

$$-\underset{\sim}{\operatorname{div}} \{\underset{\sim}{C}\, \epsilon\, (\underset{\sim}{v}) - pI\} = \underset{\sim}{b} \text{ in } \Omega, \quad (2.1\text{-}18)$$
$$\{\underset{\sim}{C}\, \epsilon\, (\underset{\sim}{v}) - pI\}\, \underset{\sim}{\nu} = \underset{\sim}{\tau} \text{ on } \Gamma, \quad (2.1\text{-}19)$$
$$\operatorname{div} \underset{\sim}{v} = g \text{ in } \Omega. \quad (2.1\text{-}20)$$

In addition to the hypotheses (1.1-13)-(1.1-16) we assume

• *the complementarity condition of Agmon-Douglis-Nirenberg* ([3], [4]) holds for the equations (2.1-18)-(2.1-20). □ (2.1-21)

THEOREM 5 : *Under the hypotheses* (1.1-13)-1.1-16) *and* (2.1-21), *the linear operator* $\hat{\underset{\sim}{\theta}}{}_2'(\underset{\sim}{\mathrm{id}}, 0)$ *defines an isomorphism between the spaces* $\underset{\sim}{C}_{\mathrm{sym}} \times W^{m+1,q}$ *and* $\underset{\sim}{L}_e \times W^{m+1,q}$.

Proof: Clearly $\hat{\underset{\sim}{\theta}}{}_2'(\underset{\sim}{\mathrm{id}}, 0)$ is a continuous one-to-one mapping between the spaces $\underset{\sim}{C}_{\mathrm{sym}} \times W^{m+1,q}$ and $\underset{\sim}{L}_e \times W^{m+1,q}$. Let $\underset{\sim}{v}_0 \in \underset{\sim}{W}^{m+1,q}$ denote the right inverse of g (see [14]) for the divergence operator. One has

$$\operatorname{div} \underset{\sim}{v}_0 = g. \quad (2.1\text{-}22)$$

We let

$$\underset{\sim}{v}_1 = \underset{\sim}{v}_0 - \underset{\sim}{v}_0(0). \quad (2.1\text{-}23)$$

We still have

$$\operatorname{div} \underset{\sim}{v}_1 = g \ , \quad \text{but} \quad \underset{\sim}{v}_1(0) = 0 \ . \tag{2.1-24}$$

The linear mapping

$$\underset{\sim}{w} : x \in \Omega \to (\text{skew } \underset{\sim}{A}) \, x \in \mathbb{R}^3 \ , \tag{2.1-25}$$

where the matrix

$$\underset{\sim}{A} = \underset{\sim}{\nabla} \underset{\sim}{v}_1(0) \ , \tag{2.1-26}$$

satisfies the condition

$$\operatorname{div} \underset{\sim}{w} = 0 \quad \text{and} \quad \underset{\sim}{w}(0) = 0 \ . \tag{2.1-27}$$

It follows that the vector field

$$\underset{\sim}{v}_2 = \underset{\sim}{v}_1 - \underset{\sim}{w} \tag{2.1-28}$$

is the right inverse of g in the space $\underset{\sim}{C}_{\text{sym}}$. The change of variable in the equations (2.1-18)-(2.1-20) leads to an equivalent system

$$-\underset{\sim}{\operatorname{div}} \{ \underset{\sim}{C} \in (\underset{\sim}{u}) - qI \} = \underset{\sim}{b}' \quad \text{in } \Omega \ , \tag{2.1-30}$$

$$\{ \underset{\sim}{C} \in (\underset{\sim}{u}) - qI \} \underset{\sim}{v} = \underset{\sim}{\tau}' \quad \text{on } \Gamma \ , \tag{2.1-31}$$

$$\underset{\sim}{\operatorname{div}} \, \underset{\sim}{u} = 0 \quad \text{in } \Omega \ , \tag{2.1-32}$$

where

$$\underset{\sim}{b}' = \underset{\sim}{b} + \underset{\sim}{\operatorname{div}} \{ \underset{\sim}{C} \in (\underset{\sim}{v}) \} \ , \tag{2.1-33}$$

$$\underset{\sim}{\tau}' = \underset{\sim}{\tau} - \underset{\sim}{C} \in (\underset{\sim}{v}) \, \underset{\sim}{v} \ . \tag{2.1-34}$$

Evidently the load $(\underset{\sim}{b}', \underset{\sim}{\tau}')$ belongs to $\underset{\sim}{L}_e$. For $m > 3/2 - 1$ it is known ([21]) that this system possesses a unique solution in $\underset{\sim}{H}^{m+2}_{\text{sym}} \times H^{m+1}$. The hypothesis (2.1-21) guarantees that the regularity results hold for $m \geqq 0$ and $q > 3$; *cf.* Geymonat [10]. □

THEOREM 6 : *There exists a neighborhood* $U \times P$ *of* $(\underset{\sim}{\text{id}}, 0)$ *in* $\underset{\sim}{C}_{\text{sym}} \times W^{m+1,q}$ *whose image under the operator* $\hat{\underset{\sim}{\theta}}_2$ *is a* C^∞ *submanifold* N *in* $\underset{\sim}{L} \times W^{m+1,q}$.

Proof: First, we redefine the operator $\hat{\underset{\sim}{\theta}}_2$, using the projections Λ_e and Λ of the load space $\underset{\sim}{L}$ onto its supplementary subspaces $\underset{\sim}{L}_e$ and skew, by letting

$$\hat{\underset{\sim}{\theta}}_2 : (\underset{\sim}{\varphi}, p) \in \underset{\sim}{C}_{\text{sym}} \times W^{m+1,q} \to$$

$$\to (\Lambda_e \, \hat{\underset{\sim}{\theta}}_2(\underset{\sim}{\varphi}, p), \Lambda \hat{\underset{\sim}{\theta}}_2(\underset{\sim}{\varphi}, p), \det \, \underset{\sim}{\nabla} \underset{\sim}{\varphi} - 1) \in \underset{\sim}{L}_e \times \text{skew} \times W^{m+1,q} \ . \tag{2.1-35}$$

For convenience we let

$$\hat{\underset{\sim}{\theta}}_e(\underset{\sim}{\varphi}, p) = (\Lambda_e \hat{\underset{\sim}{\theta}}_2(\underset{\sim}{\varphi}, p), \det \underset{\sim}{\nabla} \underset{\sim}{\varphi} - 1) \tag{2.1-36}$$

and rewrite

$$\hat{\underset{\sim}{\theta}}_2(\underset{\sim}{\varphi}, p) = (\hat{\underset{\sim}{\theta}}_e(\underset{\sim}{\varphi}, p), \Lambda \hat{\underset{\sim}{\theta}}_2(\underset{\sim}{\varphi}, p)) . \tag{2.1-37}$$

Clearly $\hat{\underset{\sim}{\theta}}'_e(\underset{\sim}{\mathrm{id}}, 0)$ is an isomorphism between $\underset{\sim}{C}_{\mathrm{sym}} \times W^{m+1,q}$ and $\underset{\sim}{L}_e \times W^{m+1,q}$ and $\hat{\underset{\sim}{\theta}}_e$ defines a local diffeomorphism. For a given load $\underset{\sim}{\ell}$, we shall write

$$\underset{\sim}{\ell}_e = \Lambda_e \underset{\sim}{\ell} . \tag{2.1-38}$$

The mapping

$$\chi : (\underset{\sim}{\varphi}, p) \in \underset{\sim}{C}_{\mathrm{sym}} \times W^{m+1,q} \to$$
$$\to \left\{\hat{\underset{\sim}{\theta}}'_e(\underset{\sim}{\mathrm{id}}, 0)\right\}^{-1} \hat{\underset{\sim}{\theta}}_e(\underset{\sim}{\varphi}, p) \in \underset{\sim}{C}_{\mathrm{sym}} \times W^{m+1,q} \tag{2.1-39}$$

is of class C^∞ and also defines a local diffeomorphism in a neighborhood of $(\underset{\sim}{\mathrm{id}}, 0)$ in $\underset{\sim}{C}_{\mathrm{sym}} \times W^{m+1,q}$ that we denote $U \times P$. For all $(\underset{\sim}{\varphi}, p)$ in $U \times P$, we have

$$\hat{\underset{\sim}{\theta}}_e(\underset{\sim}{\varphi}, p) = \hat{\underset{\sim}{\theta}}'_e(\underset{\sim}{\mathrm{id}}, 0) \, \chi(\underset{\sim}{\varphi}, p) . \tag{2.1-40}$$

We may then write

$$\hat{\underset{\sim}{\theta}}_2(\underset{\sim}{\varphi}, p) = (\underset{\sim}{\ell}_e, \Lambda \hat{\underset{\sim}{\theta}}_2 \{\hat{\underset{\sim}{\theta}}_e\}^{-1}(\underset{\sim}{\ell}_e, g), g) , \tag{2.1-41}$$

where

$$(\underset{\sim}{\ell}_e, g) = \hat{\underset{\sim}{\theta}}'_e(\underset{\sim}{\mathrm{id}}, 0) \, \chi(\underset{\sim}{\varphi}, p) . \tag{2.1-42}$$

We deduce immediately that the image N of the neighborhood $U \times P$ under $\hat{\underset{\sim}{\theta}}_2$ is the graph of the function

$$G : (\underset{\sim}{\ell}_e, g) \in \underset{\sim}{L}_e \times W^{m+1,q} \to \Lambda \hat{\underset{\sim}{\theta}}_2 \{\hat{\underset{\sim}{\theta}}_e\}^{-1}(\underset{\sim}{\ell}_e, g) \in \text{skew} , \tag{2.1-43}$$

which is also C^∞ and satisfies

$$G(\underset{\sim}{0}, 0) = (\underset{\sim}{0}, 0) \quad \text{and} \quad G'(\underset{\sim}{0}, 0)(\underset{\sim}{\ell}_e, g) = 0$$
$$\text{for all} \quad (\underset{\sim}{\ell}_e, g) \in \underset{\sim}{L}_e \times W^{m+1,q} . \tag{2.1-44}$$

□ We have thus shown that N is a submanifold of class C^∞.

In the sequel, we choose the neighborhood $U \times P$ to be the closed ball defined by

$$U \times P = \bar{B}(\underset{\sim}{\mathrm{id}}, \rho_{0/2}) \times \bar{B}(0, \rho_{0/2}) . \tag{2.1-45}$$

Let a load $(\underset{\sim}{\ell}, 0)$ be in N, then we infer from theorem 5 that a solution for the pure traction boundary-value problem (2.1-10)-(2.1-13) is

$$(\underset{\sim}{\varphi}, p) = \left\{\hat{\underset{\sim}{\theta}}_e\right\}^{-1}(\underset{\sim}{\ell}_e, 0) . \tag{2.1-46}$$

Arguing as in Section 1, we find that there exists a constant C_0 such that for each load $(\underset{\sim}{\ell}, g) \in N$

$$\| (\underset{\sim}{\ell}_e, g) \| \leqq C_0^{-1} \rho_0 , \tag{2.1-47}$$

where

$$C_0 = \sup_{(\underset{\sim}{\varphi}, p) \in U \times P} \left\| \left\{ \hat{\underset{\sim}{\theta}}_e'(\underset{\sim}{\varphi}, p) \right\}^{-1} \right\| . \tag{2.1-48}$$

Thus the projection of N onto $\underset{\sim}{L}_e \times W^{m+1, q}$ is contained in the ball $\bar{B}(\underset{\sim}{0}, C_0^{-1} \rho_0)$.

2.2. Approximation of the pure traction boundary-value problem with dead load

We first consider the case of a dead load that belongs to the submanifold N (the results obtained in this case will be used later). Let there be given a load $(\underset{\sim}{\ell}, 0)$ in N ; then the solution of the problem is such that

$$\hat{\underset{\sim}{\theta}}_e(\underset{\sim}{\varphi}, p) = (\underset{\sim}{\ell}_e, 0) . \tag{2.2-1}$$

We can now proceed as in Section 1. We consider a regular partition of the interval [0, 1],

$$0 = \lambda^0 < \lambda^1 < \cdots < \lambda^N = 1 \tag{2.2-2}$$

such that

$$\Delta\lambda^n = \lambda^{n+1} - \lambda^n = \frac{1}{N}, \quad 0 \leqq n \leqq N - 1 , \tag{2.2-3}$$

and we let N approach infinity. The incremental method is defined as in Section 1 : we let

$$\underset{\sim}{\varphi}^0 = \underset{\sim}{\mathrm{id}} \quad \text{and} \quad p^0 = 0 , \tag{2.2-4}$$

and, assuming $\underset{\sim}{\varphi}^n$ and p^n are known, we solve the linear problem

$$\hat{\underset{\sim}{\theta}}_e'^n(\delta\underset{\sim}{\varphi}^n, \delta p^n) = (\Delta \underset{\sim}{\ell}_e^n, 0)\,, \tag{2.2-5}$$

$$(\delta\underset{\sim}{\varphi}^n, \delta p^n) \in \underset{\sim}{C}_{\text{sym}} \times W^{m+1,q}\,, \tag{2.2-6}$$

where

$$\hat{\underset{\sim}{\theta}}_e'^n = \hat{\underset{\sim}{\theta}}_e'(\underset{\sim}{\varphi}^n, p^n)\,. \tag{2.2-7}$$

Then we define the $(n+1)$-st approximate solution

$$\underset{\sim}{\varphi}^{n+1} = \underset{\sim}{\varphi}^n + \delta\underset{\sim}{\varphi}^n\,, \quad p^{n+1} = p^n + \delta p^n\,, \tag{2.2-8}$$

and in particular we obtain the N-th approximate solution $(\underset{\sim}{\varphi}^N, p^N)$.

The convergence of this method also follows from classical result about ordinary differential equation. It suffices to note that the solution of the equation

$$\hat{\underset{\sim}{\theta}}_e(\underset{\sim}{\varphi}, p) = (\underset{\sim}{\ell}_e, 0) \tag{2.2-9}$$

is the value at the point 1 of the solution of the differential equation

$$\frac{d}{dt}(\underset{\sim}{\psi}(t), q(t)) = \left\{\hat{\underset{\sim}{\theta}}_e'(\underset{\sim}{\psi}(t), q(t))\right\}^{-1}(\underset{\sim}{\ell}_e, 0)\,, \tag{2.2-10}$$

$$\underset{\sim}{\psi}(0) = \underset{\sim}{\mathrm{id}} \quad \text{and} \quad q(0) = 0\,. \tag{2.2-11}$$

The vector field defined in the ball $\bar{B}(\mathrm{id}, \rho_{0/2}) \times \bar{B}(0, \rho_{0/2})$ by

$$\underset{\sim}{v}_1(\underset{\sim}{\varphi}, p, \underset{\sim}{\ell}) = \left\{\hat{\underset{\sim}{\theta}}_e'(\underset{\sim}{\varphi}, p)\right\}^{-1}(\underset{\sim}{\ell}_e, 0) \tag{2.2-12}$$

for $\underset{\sim}{\ell}$ in N, with

$$\underset{\sim}{\ell}_e = \Lambda_e\,\underset{\sim}{\ell} \in \bar{B}(\underset{\sim}{0}, C_0^{-1}\rho_0)\,, \tag{2.2-13}$$

is of class C^∞ with respect to all its arguments. Moreover it is locally bounded because of the local boundedness of the different operators it is composed of. The equilibrated part of the load, $\underset{\sim}{\ell}_e$ (or $\underset{\sim}{\ell}$, evidently) appears as a parameter in the vector field (2.2-12) and the solution of the equations (2.2-12)-(2.2-13) depends continuously on this parameter. More precisely, there exists a constant C depending on the vector field $\underset{\sim}{v}_1$ such that for given parameters $\underset{\sim}{\ell}_{1e}$ and $\underset{\sim}{\ell}_{2e}$, if $(\underset{\sim}{\varphi}_1, p_1)$ and $(\underset{\sim}{\varphi}_2, p_2)$ are the solutions of the equations (2.2-10) and (2.2-11), we have

$$\left\|(\underset{\sim}{\varphi}_1 - \underset{\sim}{\varphi}_2, p_1 - p_2)\right\| \leqq C(\underset{\sim}{v}_1)\,\|\underset{\sim}{\ell}_{1e} - \underset{\sim}{\ell}_{2e}\|\,. \tag{2.2-14}$$

We shall now use these results to treat the case of an equilibrated load $\underset{\sim}{\ell}$, ($\underset{\sim}{\ell} \in \underset{\sim}{L}_e$). We begin by recalling essential results due to Chillingworth-Marsden-Wan [6, Sections 4, 5], which will be necessary for constructing the incremental method. We adapt them to the case of incompressible materials in the following form :

THEOREM 7 : *Let $\underset{\sim}{\ell}_0$ be a load in $\underset{\sim}{L}_e$ without axis of equilibrium. Then there exist a real number $\eta > 0$, a neighborhood O_0 of $\underset{\sim}{\ell}_0$ in $\underset{\sim}{L}_e$ and a neighborhood V_I of I in O^3, such that :*

- *each load $\underset{\sim}{\ell} \in O_0$ is without axis of equilibrium,* (2.2-15)
- *for each pair $(\underset{\sim}{\ell}, \lambda) \in O_0 \times [0, \eta]$, there exists a unique $Q \in V_I$ such that*

$$\lambda(Q\underset{\sim}{\ell}, 0) \in N , \tag{2.2-16}$$

- *Q is differentiable with respect to $\underset{\sim}{\ell}$ and Q is the unique solution of the implicit equation* (2.2-17)

$$Q \in O^3 , \tag{2.2-18}$$

$$H(\lambda, \underset{\sim}{\ell}, Q) = 0 \tag{2.2-19}$$

where H is the mapping defined by

$$H : (\lambda, \underset{\sim}{\ell}, Q) \in \mathbb{R} \times \underset{\sim}{L}_e \times O^3 \to \Lambda Q\underset{\sim}{\ell} - \frac{1}{\lambda} G(\lambda\Lambda_e\, Q\underset{\sim}{\ell}, 0) \in \text{skew} . \tag{2.2-20}$$

Proof: It follows from the decomposition of $\underset{\sim}{L}$ onto $\underset{\sim}{L}_e \oplus$ skew that

$$\Lambda Q\underset{\sim}{\ell} = \text{skew } (Q\underset{\sim}{\ell}) . \tag{2.2-21}$$

The results then follow from [6, theorem 5.1]. □

From the above theorem, we conclude that if a load $\underset{\sim}{\ell}$ in the neighborhood O_0 is sufficiently small, we can define in O^3 a curve $Q(t\underset{\sim}{\ell})$ that is defined for $t = 1$. This is equivalent to saying that we can choose the pair $(O_0, [0, 1])$ such that the condition (2.2-16) is satisfied for each element. We can therefore define a mapping

$$Q : \underset{\sim}{\ell} \in O_0 \to Q(\underset{\sim}{\ell}) \in O^3 \tag{2.2-22}$$

such that

$$Q(\underset{\sim}{\ell})\, \underset{\sim}{\ell} \in N . \tag{2.2-23}$$

It is clear that the neighborhood O_0 is contained in the closed ball

$\bar{B}(\underset{\sim}{0}, C_0^{-1}\rho_0)$ (2.1-48). For each load $\underset{\sim}{\ell}$ in O_0 there exists a unique solution $(\underset{\sim}{\varphi}, p) \in \bar{B}(\underset{\sim}{\mathrm{id}}, \rho_0/2) \times \bar{B}(0, \rho_{0/2})$ of the equation

$$\hat{\underset{\sim}{\theta}}_2(\underset{\sim}{\varphi}, p) = (Q(\underset{\sim}{\ell})\, \underset{\sim}{\ell}, 0)\,. \tag{2.2-24}$$

This solution depends continuously on the load $\underset{\sim}{\ell}$. Letting

$$\underset{\sim}{\phi} = Q(\underset{\sim}{\ell})^{\mathrm{T}}\, \underset{\sim}{\varphi}\,, \tag{2.2-25}$$

we deduce from the principe of material frame-indifference and from the relation

$$\det \underset{\sim}{\nabla} \underset{\sim}{\phi} = \det \underset{\sim}{\nabla} \underset{\sim}{\varphi}\,, \tag{2.2-26}$$

that

$$\underset{\sim}{\theta}_2(\underset{\sim}{\phi}, p) = (\underset{\sim}{\ell}, 0)\,. \tag{2.2-27}$$

We have thus shown that the pair $(\underset{\sim}{\phi}, p)$ is a solution of the pure traction boundary-value problem. We obtain the same results for the general case of a load $(\underset{\sim}{\ell}, g)$, with $g \in W^{m+1,q}$ not necessary zero.

Let $\underset{\sim}{\ell}$ be in $\underset{\sim}{L}_e$. The orbit of $\underset{\sim}{\ell}$, viz.,

$$\theta_{\underset{\sim}{\ell}} = \{Q\underset{\sim}{\ell}, Q \in O^3\}\,, \tag{2.2-28}$$

which we identify with the set

$$\{(Q\underset{\sim}{\ell}, 0), Q \in O^3\}\,, \tag{2.2-29}$$

intersects the manifold N for at least four rotations [6], but if we restrict ourself to the neighborhood V_I, the set of such rotations reduces to the single rotation that we have denoted $Q(\underset{\sim}{\ell})$. In the construction of an incremental algorithm, we must successively approximate $Q(\underset{\sim}{\ell})$, then the solution $(\underset{\sim}{\varphi}, p)$ and finally $\underset{\sim}{\phi}$. We approximate $Q(\underset{\sim}{\ell})$ by an incremental method. For this purpose we here recall the following well known results (Lang, S [12]).

The exponential mapping

$$\exp : w \in \text{skew} \to \exp(w) \in O^3 \tag{2.2-30}$$

defines a diffeomorphism in a neighborhood V_0 of the matrix 0. It is no loss of generality to assume that the neighborhood V_I defined in (2.2-16) satisfies the condition

$$V_I = \exp(V_0)\,. \tag{2.2-31}$$

Then the implicit equations (2.2-18)-(2.2-20) are equivalent to the equations $w \in$ skew,

$$\bar{H}(\lambda, w, \underset{\sim}{\ell}) = 0 \,, \quad 0 \leqq \lambda \leqq 1 \,, \tag{2.2-33}$$

where the function $\bar{H}$ is defined by

$$\bar{H}(\lambda, w, \ell) = H(\lambda, \exp w, \underset{\sim}{\ell}) \,. \tag{2.2-34}$$

Clearly we have

$$\frac{\partial \bar{H}}{\partial w}(0, 0, \underset{\sim}{\ell})\, w = \frac{\partial H}{\partial Q}(0, I, \underset{\sim}{\ell})\, w \,. \tag{2.2-35}$$

We deduce from Theorem 5.1 of [6] that the linear mapping (2.2-35) is an isomorphism in the space skew for each load $\underset{\sim}{\ell}$ in O_0.

We can choose the neighborhood V_0 (equivalently V_I) and O_0 in such a way that :

$$\frac{\partial \bar{H}}{\partial w}(\lambda, w, \underset{\sim}{\ell}) \textit{ is an isomorphism for } (\lambda, w, \underset{\sim}{\ell}) \in [0, 1] \times V_0 \times O_0 \,. \tag{2.2-36}$$

Then for a fixed load $\underset{\sim}{\ell}$ in O_0, the solution of the equation (2.2-32)-(2.2-34) is a curve which is the solution of the differential equation

$$\frac{dw}{d\lambda} = -\left\{\frac{\partial \bar{H}}{\partial w}\right\}^{-1} \frac{\partial \bar{H}}{\partial \lambda}(\lambda, w(\lambda, \underset{\sim}{\ell}), \underset{\sim}{\ell}) \,, \quad 0 \leqq \lambda \leqq 1 \,, \tag{2.2-37}$$

together with the initial condition

$$w(0) = 0 \,. \tag{2.2-38}$$

The vector field $\underset{\sim}{v}_2$ of this equation is differentiable with respect to its arguments and its derivatives are also locally bounded. The load $\underset{\sim}{\ell}$ then appears to be a parameter in this vector field. consequently the mapping

$$Q : \underset{\sim}{\ell} \in O_0 \to \exp w(1, \underset{\sim}{\ell}) \in O^3 \tag{2.2-39}$$

is Lipschitzian. Therefore there exists a constant C depending on the vector field $\underset{\sim}{v}_2$ such that if $\underset{\sim}{\ell}_1$ and $\underset{\sim}{\ell}_2$ belong to O_0, we have

$$\|\exp w(1, \underset{\sim}{\ell}_1) - \exp w(1, \underset{\sim}{\ell}_2)\| \leqq C(\underset{\sim}{v}_2)\|\underset{\sim}{\ell}_1 - \underset{\sim}{\ell}_2\| \,. \tag{2.2-40}$$

The full expression of the vector field $\underset{\sim}{v}_2$ is quite involved. We shall instead consider an approximation obtained from a truncated Taylor expansion of the function $\bar{H}$. More precisely we write (see [12])

$$\bar{H}(\lambda, w(1, \underset{\sim}{\ell}), \underset{\sim}{\ell}) = \Lambda \exp w\underset{\sim}{\ell} - \frac{\lambda}{2} G''(\underset{\sim}{0}, 0)(\Lambda_e \exp w\underset{\sim}{\ell}, 0)^2 + \lambda^2 T(w, \underset{\sim}{\ell}), \tag{2.2-41}$$

since

$$G(\underset{\sim}{0}, 0) = 0 \quad \text{and} \quad G'(\underset{\sim}{0}, 0) = 0. \tag{2.2-42}$$

We use the fact that G and all its derivatives are locally bounded to conclude that there exists a constant C such that

$$\| T(w, \underset{\sim}{\ell}) \| \leqq C \| \underset{\sim}{\ell} \|^3. \tag{2.2-43}$$

We let

$$\bar{H}_2(\lambda, w, \underset{\sim}{\ell}) = \Lambda \exp w\underset{\sim}{\ell} - \frac{\lambda}{2} G''(\underset{\sim}{0}, 0)(\Lambda_e \exp w\underset{\sim}{\ell}, 0)^2. \tag{2.2-44}$$

We still have

$$\frac{\partial \bar{H}_2}{\partial w}(0, 0, \underset{\sim}{\ell}) = \frac{\partial \bar{H}}{\partial w}(0, 0, \underset{\sim}{\ell}). \tag{2.2-45}$$

So, we can again consider the neighborhoods V_0 and O_0 such that $\bar{H}$ and $\bar{H}_2$ satisfy the condition (2.2-36). We therefore approximate the curve $w(\lambda, \underset{\sim}{\ell})$ by a curve $w_2(\lambda, \underset{\sim}{\ell})$ which is the solution of the equations

$$\frac{dw_2}{d\lambda} = -\left\{ \frac{\partial \bar{H}_2}{\partial w} \right\}^{-1} \frac{\partial \bar{H}_2}{\partial \lambda}(\lambda, w_2, \underset{\sim}{\ell}), \quad 0 \leqq \lambda \leqq 1 \tag{2.2-46}$$

and

$$w_2(0) = 0. \tag{2.2-47}$$

Indeed, the operator $\bar{H}_2$ (unlike $\bar{H}$) can be computed, since its expression is deduced from linearized incompressible elasticity. Moreover we have the following estimate :

LEMME 1 : *Let w and w_2 be the exact solutions of the equations* (2.2-37)-(2.2-38), *and* (2.2-46)-(2.2-47), *respectively. Then there exists a constant C independent of $\underset{\sim}{\ell}$ such that :*

$$\operatorname*{Sup}_{\lambda \in [0,1]} \| w(\lambda, \underset{\sim}{\ell}) - w_2(\lambda, \underset{\sim}{\ell}) \| \leqq C \| \underset{\sim}{\ell} \|^3. \tag{2.2-48}$$

Proof: We write

$$\bar{H}(\lambda, w, \underset{\sim}{\ell}) = \bar{H}_2(\lambda, w, \underset{\sim}{\ell}) + \lambda^2 T(w, \underset{\sim}{\ell}). \qquad (2.2\text{-}49)$$

Using equations (2.2-37)-(2.2-38) and (2.2-46)-(2.2-47), the mean value theorem and the condition (2.2-36), we obtain

$$\| w_2(\lambda, \underset{\sim}{\ell}) - w(\lambda, \underset{\sim}{\ell}) \| \leqq \lambda^2 \left\| \left\{ \frac{\partial \bar{H}_2}{\partial w}(\lambda, \bar{w}, \underset{\sim}{\ell}) \right\}^{-1} T(w, \ell) \right\|, \qquad (2.2\text{-}50)$$

where $\bar{w}$ belongs to the interval $[w_1, w]$. The conclusion follows from the local boundedness of the derivatives of $\bar{H}_2$. □

Notice the effect on the estimates of the remaining term T in the Taylor's expansion of $\bar{H}$.

We now establish some estimates we shall use in the sequel.

LEMME 2 : *Let* $(\underset{\sim}{\varphi}, p)$ *be the solution of the equation*

$$\hat{\underset{\sim}{\theta}}_2(\underset{\sim}{\varphi}, p) = (\exp w(1, \underset{\sim}{\ell})\, \underset{\sim}{\ell}, 0) \qquad (2.2\text{-}51)$$

and let $(\bar{\underset{\sim}{\varphi}}, \bar{p})$ *denote the exact value of the solution of the differential equations* (2.2-10)-(2.2-11) *at the point* 1, *with* $\bar{\underset{\sim}{\ell}}_e$ *as the parameter, where*

$$\bar{\underset{\sim}{\ell}}_e = \Lambda_e \exp w_2(1, \underset{\sim}{\ell})\, \underset{\sim}{\ell}. \qquad (2.2\text{-}52)$$

Then there exists a constant C independent of $\underset{\sim}{\ell}$ *such that*

$$\left\| (\underset{\sim}{\varphi} - \bar{\underset{\sim}{\varphi}}, p - \bar{p}) \right\| \leqq C \| \underset{\sim}{\ell} \|^4. \qquad (2.2\text{-}53)$$

Proof: Recall that $(\underset{\sim}{\varphi}, p)$ and $(\bar{\underset{\sim}{\varphi}}, \bar{p})$ are equal to $(\underset{\sim}{\psi}(1), q(1))$ for the parameters $\underset{\sim}{\ell}_e$ and $\bar{\underset{\sim}{\ell}}_e$ respectively, where $(\underset{\sim}{\psi}(t), q(t))$ is the solution of the differential equations (2.2-10)-(2.2-11). The result then follows from the estimates (2.2-14) and (2.2-48). We have

$$\left\| (\underset{\sim}{\varphi} - \bar{\underset{\sim}{\varphi}}, p - \bar{p}) \right\| \leqq C(\underset{\sim}{v}_1) \left\| \underset{\sim}{\ell}_e - \bar{\underset{\sim}{\ell}}_e \right\|, \qquad (2.2\text{-}54)$$

which yields

$$\left\| (\underset{\sim}{\varphi} - \bar{\underset{\sim}{\varphi}}, p - \bar{p}) \right\| \leqq C(\underset{\sim}{v}_1) \| \underset{\sim}{\ell} \| \, \| w(1, \underset{\sim}{\ell}) - w_2(1, \underset{\sim}{\ell}) \|. \qquad (2.2\text{-}55)$$

Then we conclude from lemma 2. □

We can now describe the method. We first start by approximating

w_2 by Euler's method. Let w_2^M denote the M-th approximation of $w_2(1, \underset{\sim}{\ell})$. We approximate the solution of the equations (2.2-10)-(2.2-11), in which the parameter is computed from w_2^M. We then define the method as follows. Let M and N be two positive integers. We define two regular partitions of the interval [0, 1],

$$0 = \mu^0 < \mu^1 < \cdots < \mu^M = 1 , \qquad (2.2\text{-}56)$$

$$0 = \lambda^0 < \lambda^1 < \cdots < \lambda^N = 1 . \qquad (2.2\text{-}57)$$

We let

$$w_2^0 = 0 , \qquad (2.2\text{-}58)$$

then assuming that w_2^m is known, we solve the linear problem

$$\frac{\partial \bar{H}_2}{\partial w} (\mu^m, w_2^m, \underset{\sim}{\ell})\, \delta w^m = - \Delta\mu^m \frac{\partial \bar{H}_2}{\partial \mu} (\mu^m, w_2^m, \underset{\sim}{\ell}) , \qquad (2.2\text{-}59)$$

$$\delta w^m \in \text{skew} , \qquad (2.2\text{-}60)$$

and we define the $(m+1)$-st approximation

$$w^{m+1} = w^m + \delta w^m . \qquad (2.2\text{-}61)$$

We compute the approximate parameter from the M-th approximation of $w_2(1, \underset{\sim}{\ell})$

$$\underset{\sim e}{\bar{\ell}^M} = \Lambda_e \exp w_2^M \underset{\sim}{\ell} . \qquad (2.2\text{-}62)$$

Applying the incremental method (2.2-4)-(2.2-8), we compute the N-th approximation $(\underset{\sim}{\bar{\varphi}}^N, \bar{p}^N)$. Finally we let

$$\underset{\sim}{\bar{\Phi}}^N = (\exp w_2^M)^{\top} \underset{\sim}{\bar{\varphi}}^N . \qquad (2.2\text{-}63)$$

We next establish the convergence of the method.

THEOREM 8 : *Let $\underset{\sim}{\ell}$ be a dead load in O_0. Let $(\underset{\sim}{\varphi}, p)$ be the solution of*

$$\underset{\sim}{\theta}_2(\underset{\sim}{\phi}, p) = (\underset{\sim}{\ell}, 0) \qquad (2.2\text{-}64)$$

and let $(\underset{\sim}{\bar{\Phi}}^N, \bar{p}^N)$ be the approximate solution computed by the incremental method. Then there exists a constant C such that

$$\left\| (\underset{\sim}{\phi} - \underset{\sim}{\bar{\Phi}}^N, p - \bar{p}^N) \right\| \leqq C \left(\| \underset{\sim}{\ell} \| \left(\frac{1}{N} + \frac{1}{M} \right) + \| T \| \right) . \qquad (2.2\text{-}65)$$

Proof: From the proof of convergence of Euler's method we know that there exists a constant C such that

$$\| w_2^M - w_2 \| \leqq \frac{C}{M} \| \underset{\sim}{\ell} \| . \tag{2.2-66}$$

We may write

$$\underset{\sim}{\phi} - \bar{\underset{\sim}{\phi}}^N = (\exp w(1)^{\mathrm{T}} - \exp w_2(1)^{\mathrm{T}} + \exp w_2(1)^{\mathrm{T}}) - \\ - (\exp (w_2^M)^{\mathrm{T}}) \underset{\sim}{\varphi} + (exw_2^M)^{\mathrm{T}} (\underset{\sim}{\varphi} - \bar{\underset{\sim}{\varphi}}^N) . \tag{2.2-67}$$

Now we can also write

$$\underset{\sim}{\varphi} - \bar{\underset{\sim}{\varphi}}^N = \underset{\sim}{\varphi} - \bar{\underset{\sim}{\varphi}} + \bar{\underset{\sim}{\varphi}} - \bar{\underset{\sim}{\varphi}}^N . \tag{2.2-68}$$

Using the estimate (2.2-53) and the convergence result established in Section 1, we have

$$\left\| (\exp w_2^M)^{\mathrm{T}} (\underset{\sim}{\varphi} - \bar{\underset{\sim}{\varphi}}^N) \right\| \leqq C \left(\frac{\| \underset{\sim}{\ell} \|}{N} + \| T \| \right) . \tag{2.2-69}$$

From (2.2-48) and (2.2-67) we deduce the estimate

$$\left\| (\exp w(1)^{\mathrm{T}} - (\exp w_2^M)^{\mathrm{T}}) \underset{\sim}{\varphi} \right\| \leqq C \left(\frac{\| \underset{\sim}{\ell} \|}{M} + \| T \| \right) . \tag{2.2-70}$$

We combine both inequalities to obtain the desired result. A similar estimate holds for $\| p - \bar{p}^N \|$. □

To sum sup, our approach is based on an essential result on compressible materials due to Chillingworth-Marsden-Wan [6] : given a sufficient « small » dead load $\underset{\sim}{\ell}_0$ in $\underset{\sim}{L}_e$ without axis of equilibrium, there exists a neighborhood O_0 of $\underset{\sim}{\ell}_0$ in $\underset{\sim}{L}_e$, in which each load is without axis of equilibrium and has an orbit which intersects N. We have adapted this result to incompressible materials and finally we have constructed an incremental method for each load $\underset{\sim}{\ell}$ in O_0. We shall also use this result to treat live loads in the next paragraph.

2.3. Approximation of the pure traction boundary-value problem with live loads

In this paragraph we keep all the hypotheses made on the nonlinear operator $\underset{\sim}{\theta}_2$. Let us consider a loading operator $\underset{\sim}{\ell}$ as define by (2.1-7). The problem consists in solving the equation

$$\underset{\sim}{\theta}_2(\underset{\sim}{\phi}, p) = (\underset{\sim}{\ell}(\underset{\sim}{\phi}), 0) \tag{2.3-1}$$

where

$$\underline{\ell}(\phi) = (\underline{b}(\phi), \underline{\tau}(\phi)) \in \underset{\sim}{L} . \tag{2.3-2}$$

In order to guarantee the existence of a solution for this problem, we first assume that

$$\underset{\sim}{h}_0 = \underline{\ell}(\underset{\sim}{\mathrm{id}}) \in \underset{\sim}{L}_e \tag{2.3-3}$$

satisfies the condition

$$\det k(\underset{\sim}{h}_0) > 0 . \tag{2.3-4}$$

Then we establish a lemma which will be usefull in the sequel.

LEMMA 3 : *Let $\underset{\sim}{h}$ be a dead load in $\underset{\sim}{L}_e$ which satisfies the inequality* (2.3-4). *Then $R^{\top} \underset{\sim}{h}$ is without axis of equilibrium, where R is the polar decomposition rotation of $k(\underset{\sim}{h})$.*

Proof: Clearly $R^{\top} k(\underset{\sim}{h})$ belongs to the set of symmetric, positive definite matrices $S^3_>$. It suffices to consider a diagonalizing orthonormal matrix P of $R^{\top} k(\underset{\sim}{h})$. We have

$$PR^{\top} k(\underset{\sim}{h}) P^{\top} = D(\alpha_1, \alpha_2, \alpha_3) , \quad \alpha_1 , \quad \alpha_2 , \quad \alpha_1 > 0 . \tag{2.3-5}$$

Consequently we have

$$\det (\operatorname{tr} R^{\top} k(\underset{\sim}{h}) I - R^{\top} k(\underset{\sim}{h})) = \det ((\alpha_1 + \alpha_2 + \alpha_3) I - D) \neq 0 \tag{2.3-6}$$

which is an equivalent definition of a load without axis of equilibrium. □

We easily conclude that there exists a neighborhood U of $\underset{\sim}{h}_0$ in $\underset{\sim}{L}$ in which each load satisfies the inequality (2.3-4). Consequently for each load $\underset{\sim}{h}$ in U, we can define the polar decomposition $R(\underset{\sim}{h})$ of the matrix $k(\underset{\sim}{h})$. This leads to the definition of the mapping

$$\zeta : \underset{\sim}{h} \in U \to R(\underset{\sim}{h})^{\top} \underset{\sim}{h} \in \underset{\sim}{L}_e \tag{2.3-7}$$

since the orthogonal matrix $R(\underset{\sim}{h})$ satisfies the equation

$$R(\underset{\sim}{h})[k(\underset{\sim}{h})^{\top} k(\underset{\sim}{h})]^{1/2} = k(\underset{\sim}{h}) . \tag{2.3-8}$$

From lemma 3 we infer that the loads $\zeta(\underset{\sim}{h})$ is without axis of equilibrium for each load $\underset{\sim}{h}$ in the neighborhood U. If we choose the load $\underset{\sim}{h}_0$ to be in the closed ball $\bar{B}(\underset{\sim}{0}, C_0^{-1} \rho_{0/2})$ (2.2-13), then we can choose a closed neighborhood U_e of $\underset{\sim}{h}_0$ contained in $\bar{B}(\underset{\sim}{0}, C_0^{-1} \rho_0)$ such that

$$U \subset \zeta^{-1}(U_e) . \tag{2.3-9}$$

We deduce from Theorem 7 that for each load $\underset{\sim}{h}$ in U there exists a rotation $Q(\underset{\sim}{h})$ such that

$$Q(\underset{\sim}{h})\,R(\underset{\sim}{h})^{\mathrm{T}}\,\underset{\sim}{h} \in N\,. \tag{2.3-10}$$

Consequently there exists a pair $(\underset{\sim}{\varphi}, p)$ such that

$$\hat{\underset{\sim}{\theta}}_2(\underset{\sim}{\varphi}, p) = (Q(\underset{\sim}{h})\,R(\underset{\sim}{h})^{\mathrm{T}}\,\underset{\sim}{h}, 0)\,. \tag{2.3-11}$$

Let

$$\underset{\sim}{\phi} = R(\underset{\sim}{h})\,Q(\underset{\sim}{h})^{\mathrm{T}}\,\underset{\sim}{\varphi}\,, \tag{2.3-12}$$

then, because of the material frame indifference, we have

$$\underset{\sim}{\theta}_2(\underset{\sim}{\phi}, p) = (\underset{\sim}{h}, 0)\,. \tag{2.3-13}$$

We have thus shown that there exists a local unique solution for the pure traction problem for each dead load in the neighborhood U. In the previous paragraph we established such an existence result and the possibility of approximating the solution only for equilibrated loads. We have established a stronger result, which is essential because in the case of the live load problem we shall show that the solution is nothing but that corresponding to a particular dead load in U. This will be possible if at least locally the image of the live loading operator $\underset{\sim}{\ell}$ is contained in the neighborhood U. We therefore need an additional assumption on the loading operator $\underset{\sim}{\ell}$.

We begin by considering the set of deformations

$$\left\{\underset{\sim}{\phi},\ \underset{\sim}{\phi} = R(\underset{\sim}{h})\,Q(\underset{\sim}{h})^{\mathrm{T}}\,\underset{\sim}{\varphi},\ \underset{\sim}{h} \in U,\ \underset{\sim}{\varphi} \in \bar{B}(\underset{\sim}{\mathrm{id}}, \rho_0/2)\right\}, \tag{2.3-14}$$

where $(\underset{\sim}{\varphi}, p)$ is the solution for the load $Q(h)\ R(\underset{\sim}{h})\,\underset{\sim}{h}^{\mathrm{T}}$ as shown in paragraph (2.1). Using relation (2.1-45) and the fact that the mappings Q and R are all locally bounded, we clearly see that the set of deformations (2.3-14) is bounded. We consider a closed ball $\bar{B}(\underset{\sim}{\mathrm{id}}, \delta)$ which contains this set and we assume that the loading operator restricted to this ball is Lipschitzian with a Lipschitz constant d such that

$$d\delta < r_0\,, \quad r_0 = C_0^{-1}\,\rho_0/2\,. \tag{2.3-15}$$

Consequently we have the following inclusions :

$$\underset{\sim}{\ell}(\bar{B}(\underset{\sim}{\mathrm{id}}, \delta)) \subset U \subset \zeta^{-1}(U_e)\,. \tag{2.3-16}$$

Using this inclusion, we define a mapping

$$\chi : \underset{\sim}{h} \in U \to \underset{\sim}{\ell}(\underset{\sim}{\phi}) \in U\,, \tag{2.3-17}$$

where

$$\underset{\sim}{\phi} = R(\underset{\sim}{h})\, Q(\underset{\sim}{h})^{\mathrm{T}}\, \underset{\sim}{\varphi} \quad \text{and} \quad \underset{\sim}{\theta}_2(\underset{\sim}{\phi}, p) = (\underset{\sim}{h}, 0)\,. \tag{2.3-18}$$

We easily show that this mapping is also Lipschitzian.
More precisely, for given loads $\underset{\sim}{h}_1$ and $\underset{\sim}{h}_2$ in U, we have

$$\|\chi(\underset{\sim}{h}_1) - \chi(\underset{\sim}{h}_2)\| \leqq d\gamma\, \|\underset{\sim}{h}_1 - \underset{\sim}{h}_2\|\,, \tag{2.3-19}$$

with

$$\gamma = C\,(C\,(R) + C\,(Q)\,C\,(\zeta) + C\,(\zeta)\,C\,(\underset{\sim}{v}_1))\,. \tag{2.3-20}$$

We observe that the constants $C(R)$ and $C(\zeta)$ can be computed independently of the nature of the material, while the constants $C(Q)$ and $C(\underset{\sim}{v}_1)$ depend only on the constitutive law. Hence the constant γ can be computed if the constitutive law is known. Let

$$\beta = \max\,(\gamma, \delta/r_0)\,, \tag{2.3-21}$$

and assume finally that the constant d is such that

$$d\beta < 1\,. \tag{2.3-22}$$

We can now establish an existence result.

THEOREM 9 : *Under the additional condition* (2.3-22) *on the loading operator* $\underset{\sim}{\ell}$, *the live load pure traction boundary-value problem possesses a local unique solution.*

Proof: Clearly the condition (2.3-22) implies that the mapping χ admits a unique fixed point. If $\underset{\sim}{h}$ denotes the fixed point then we have

$$\underset{\sim}{\theta}_2(\underset{\sim}{\phi}, p) = (\underset{\sim}{h}, 0) = (\underset{\sim}{\ell}(\underset{\sim}{\phi}), 0)\,. \qquad \square \tag{2.3-23}$$

We have thus shown that the solution of the live load problem is nothing but that of the dead load problem in which the fixed point $\underset{\sim}{h}$ is the load. We also note that the polar decomposition is used in a crucial way in the construction of the mapping χ. This was possible because we assumed condition (2.3-4).

It is now easy to describe an approximation scheme of the solution based on the dead load case. Let us consider the sequence of dead loads recursively defined in U as follows :

$$\underset{\sim}{h}_0,\ \underset{\sim}{h}_{i+1} = \chi(\underset{\sim}{h}_j)\,. \tag{2.3-24}$$

This sequence converges to the fixed point $\underset{\sim}{h}$ and consequently the corresponding sequence of solutions

$$(\phi_j, p_j) \quad \text{with} \quad \underset{\sim}{\theta}_2(\phi_j, p_j) = (\underset{\sim}{h}_j, 0)\,, \tag{2.3-25}$$

converges to the exact solution of (2.3-23). The incremental method will consist in approximating the sequence (2.3-25) of traction problems with dead load, and in using the methods described in paragraph 2.2. We shall consider a finite number of terms $\underset{\sim}{h}_j$, $j = 0, 1, \dots, J$ and estimate the distance between the exact solution of the problem and the approximate solution corresponding to $\underset{\sim}{h}_J$.

The method is described as follows. We let

$$\underset{\sim}{h}^0 = \underline{\ell}(\underset{\sim}{\mathrm{id}})\,, \tag{2.3-26}$$

then assuming $\underset{\sim}{h}^j$ is known, we set

$$\underset{\sim}{f}^j = R(\underset{\sim}{h}^j)^{\mathrm{T}}\, \underset{\sim}{h}^j\,. \tag{2.3-27}$$

The load $\underset{\sim}{f}^j$ is in $\underset{\sim}{L}_e$. We therefore use results of paragraph 2.2 to approximate the corresponding solution, i.e. we apply the method (2.2-57)-(2.2-64) to approximate the problem

$$\underset{\sim}{\theta}_2(\underset{\sim}{\psi}_j, p_j) = (\underset{\sim}{f}^j, 0)\,. \tag{2.3-28}$$

We let

$$\bar{\underset{\sim}{\phi}}^j = R(\underset{\sim}{h}^j)\, \bar{\underset{\sim}{\psi}}^N\,, \tag{2.3-29}$$

where $(\bar{\underset{\sim}{\psi}}^N, \bar{p}^N)$ is the approximate solution of (2.3-28). We compute the $(j+1)$-st approximate load

$$\underset{\sim}{h}^{j+1} = \underline{\ell}(\bar{\underset{\sim}{\phi}}^N)\,. \tag{2.3-30}$$

In this fashion we obtain the J-th approximate deformation and pressure $(\bar{\underset{\sim}{\phi}}^J, \bar{p}^J)$. Upper indices are used for denoting approximate terms.

We now establish convergence results. We let

$$[T] = \sup_{\underset{\sim}{\ell} \in U} \|T\| \tag{2.3-31}$$

denote the bound of the remainder in the Taylor's expansion of the function $\bar{H}$, when restricted to the closed ball U.

THEOREM 10 : *Let $(\underset{\sim}{\phi}, p)$ be the local exact solution obtained in theorem 9, and let $(\bar{\underset{\sim}{\phi}}^J, \bar{p}^J)$ be the approximate solution. Then there exists a constant C such that*

$$\left\|(\underset{\sim}{\phi} - \bar{\underset{\sim}{\phi}}^J, p - \bar{p}^J)\right\| \leqq C(J+1)\left\{\frac{1}{N} + \frac{1}{M} + [T]\right\} + \frac{C(d\gamma)^J}{1 - d\gamma}. \quad (2.3\text{-}32)$$

Proof: Consider again the sequence $(\underset{\sim}{h}_j)\ j \geqq 0$ defined in (2.3-24). Let $\underset{\sim}{h}$ be the fixed point of the contracting mapping χ. Clearly we have

$$\|(\underset{\sim}{h}_J - \underset{\sim}{h})\| \leqq \frac{(d\gamma)^J}{1 - d\gamma}\|\underset{\sim}{h}_1 - \underset{\sim}{h}_0\| \leqq C(d\gamma)^J. \quad (2.3\text{-}33)$$

Let $(\underset{\sim}{\phi}, p)$ and $(\underset{\sim}{\phi}_J, p_J)$ be such that

$$\underset{\sim}{\theta}_2(\underset{\sim}{\phi}, p) = (\underset{\sim}{h}, 0) \quad \text{and} \quad \underset{\sim}{\theta}_2(\underset{\sim}{\phi}_J, p_J) = (\underset{\sim}{h}_J, 0). \quad (2.3\text{-}34)$$

By construction we have

$$\underset{\sim}{\phi}_J = R(\underset{\sim}{h}_J)\, Q(\underset{\sim}{\ell}_J)^{\mathrm{T}}\, \underset{\sim}{\varphi}_J \quad \text{and} \quad \underset{\sim}{\phi} = R(\underset{\sim}{h})\, Q(\underset{\sim}{\ell})^{\mathrm{T}}\, \underset{\sim}{\varphi}, \quad (2.3\text{-}35)$$

where

$$\underset{\sim}{\ell}_J = R(\underset{\sim}{h}_J)^{\mathrm{T}}\, \underset{\sim}{h}_J \quad \text{and} \quad \underset{\sim}{\ell} = R(\underset{\sim}{h})^{\mathrm{T}}\, (\underset{\sim}{h}_J). \quad (2.3\text{-}36)$$

Hence we deduce from (2.3-19) and inequality (2.2-14) that

$$\|(\underset{\sim}{\phi}_J - \underset{\sim}{\phi}, p_J - p)\| \leqq \gamma\|\underset{\sim}{h}_J - \underset{\sim}{h}\| \leqq C(d\gamma)^J. \quad (2.3\text{-}37)$$

We want to estimate $\left\|(\bar{\underset{\sim}{\phi}}^J - \underset{\sim}{\phi}, \bar{p}^J - p)\right\|$. We may write

$$\left\|(\bar{\underset{\sim}{\phi}}^J - \underset{\sim}{\phi}, p - \bar{p}^J)\right\| \leqq \left\|(\bar{\underset{\sim}{\phi}}^J - \underset{\sim}{\phi}_J, \bar{p}^J - p_J)\right\| + \|(\underset{\sim}{\phi}_J - \underset{\sim}{\phi}, p_J - p)\|. \quad (2.3\text{-}38)$$

It therefore suffices to estimate the first term in the right hand side. We show by induction that for each integer j we have

$$\left\|(\bar{\underset{\sim}{\phi}}^j - \underset{\sim}{\phi}_j, \bar{p}^j - p_j)\right\| \leqq C(j+1)\left\{\frac{1}{N} + \frac{1}{M} + [T]\right\}. \quad (2.3\text{-}39)$$

For $j = 0$, the result holds (*cf.* § 2.2). Suppose that it is true for some $j \geqq 0$. Consider $(\underset{\sim}{\psi}_{j+1}, q_{j+1})$ and $(\underset{\sim}{\phi}_{j+1}, p_{j+1})$ satisfying

$$\underset{\sim}{\theta}_2(\underset{\sim}{\psi}_{j+1}, q_{j+1}) = (\underset{\sim}{h}^{j+1}, 0). \quad (2.3\text{-}40)$$

Arguing as in (2.3-19) and (2.3-37) we also obtain

$$\left\|(\underset{\sim}{\psi}_{j+1} - \underset{\sim}{\phi}_{j+1}, q_{j+1} - p_{j+1})\right\| \leqq \gamma \left\|\underset{\sim}{h}^{j+1} - \underset{\sim}{h}_{j+1}\right\| . \tag{2.3-41}$$

But

$$\underset{\sim}{h}^{j+1} = \underline{\ell}(\bar{\underset{\sim}{\phi}}^{j}) \quad \text{and} \quad \underset{\sim}{h}_{j+1} = \underline{\ell}(\underset{\sim}{\phi}_{j}) , \tag{2.3-42}$$

hence we have

$$\left\|\underset{\sim}{h}^{j+1} - \underset{\sim}{h}_{j+1}\right\| \leqq d \left\|\bar{\underset{\sim}{\phi}}^{j} - \underset{\sim}{\phi}_{j}\right\| , \tag{2.3-43}$$

and by induction the following inequality holds

$$\left\|(\underset{\sim}{\psi}_{j+1} - \underset{\sim}{\phi}_{j+1}, q_{j+1} - p_{j+1})\right\| \leqq \left\|\bar{\underset{\sim}{\phi}}^{j} - \underset{\sim}{\phi}_{j}\right\| \leqq C(j+1)\left(\frac{1}{N} + \frac{1}{M} + [T]\right) \tag{2.3-44}$$

since $d\gamma < 1$. Finally we write

$$\begin{aligned} \left\|(\underset{\sim}{\psi}_{j+1} - \underset{\sim}{\phi}_{j+1}, q_{j+1} - p_{j+1})\right\| \leqq & \left\|(\bar{\underset{\sim}{\phi}}^{j+1} - \underset{\sim}{\psi}_{j+1}, \bar{p}^{j+1} \quad q_{j+1})\right\| \\ & + \left\|(\underset{\sim}{\psi}_{j+1} - \underset{\sim}{\phi}_{j+1}, q_{j+1} - p_{j+1})\right\| . \end{aligned} \tag{2.3-45}$$

We then apply the estimate established in the incremental method for dead load to the first term in the right hand side. This yields

$$\left\|(\underset{\sim}{\phi}^{j+1} - \underset{\sim}{\phi}_{j+1}, \bar{p}^{j+1} - p_{j+1})\right\| \leqq C(j+2)\left(\frac{1}{N} + \frac{1}{M} + [T]\right) . \tag{2.3-46}$$

Hence we conclude that the result also holds at the $(j+1)$-st step and the proof is complete. □

We infer from the above theorem that for a given $\varepsilon > 0$, we may choose J such that

$$C(d\gamma)^{J} < \varepsilon/3 , \tag{2.3-47}$$

then we choose M and N such that

$$C(J+1)\left\{\frac{1}{N} + \frac{1}{M}\right\} < \varepsilon/3 \tag{2.3-48}$$

and we obtain

$$\left\|(\bar{\underset{\sim}{\phi}}^{J} - \underset{\sim}{\phi}, \bar{p}^{J} - p)\right\| \leqq \frac{2\varepsilon}{3} + C(J+1)[T] , \tag{2.3-49}$$

hence the smaller is $[T]$, the better is the approximation.

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