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**ON THE APPLICATION OF MIXED FINITE ELEMENT METHODS  
 TO THE WAVE EQUATIONS (\*)**

by Tunc GEVECI (1)

Abstract — *The convergence of certain semidiscrete approximation schemes based on the « velocity-stress » formulation of the wave equation and spaces such as those introduced by Raviart and Thomas is discussed. The discussion also applies to similar schemes for the equations of elasticity.*

Resumé. — *La convergence de certains schémas d'approximation semi-discrète basés sur la formulation « vitesse-contrainte » de l'équation d'onde et d'espace tel que ceux introduits par Raviart et Thomas est discuté. La discussion s'applique également pour les schémas similaires aux équations d'élasticité.*

**1. THE « VELOCITY-STRESS » FORMULATION OF THE WAVE EQUATION AND A SEMIDISCRETE VERSION**

Let us consider the following initial-boundary value problem for the wave equation :

$$\begin{aligned}
 (1.1) \quad & D_t^2 u(t, x) - \Delta u(t, x) = f(t, x), \quad t > 0, \quad x \in \Omega \subset \mathbb{R}^2, \\
 & u(t, x) = 0, \quad t > 0, \quad x \in \Gamma, \\
 & u(0, x) = u_0(x), \quad D_t u(0, x) = v_0(x), \quad x \in \Omega,
 \end{aligned}$$

where  $\Omega$  is a bounded domain with boundary  $\Gamma$ , and  $f, u_0, v_0$  are given functions. Introducing the « stress »  $\sigma = \nabla u$ , (1.1) may be reformulated as

$$\begin{aligned}
 (1.2) \quad & D_t^2 u(t, x) - \operatorname{div} \sigma(t, x) = f(t, x), \quad t > 0, \quad x \in \Omega, \\
 & \sigma(t, x) = \nabla u(t, x), \quad t > 0, \quad x \in \Omega, \\
 & u(t, x) = 0, \quad t > 0, \quad x \in \Gamma, \\
 & u(0, x) = u_0(x), \quad D_t u(0, x) = v_0(x), \quad x \in \Omega.
 \end{aligned}$$

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We use the notation of Johnson and Thomée [12]:

$$V = L_2(\Omega), \quad H = \{\chi \in L_2(\Omega)^2 : \operatorname{div} \chi \in L_2(\Omega)\}.$$

Using Green's formula

$$\int_{\Omega} u \operatorname{div} \chi \, dx = \int_{\Gamma} u \chi \cdot n \, ds - \int_{\Omega} \nabla u \cdot \chi \, dx$$

where  $n$  is the unit exterior normal to  $\Gamma$ , a Galerkin version of (1.2) is to seek  $u(t) \in V$ ,  $\sigma(t) \in H$ ,  $t > 0$ , satisfying

$$(1.3) \quad \begin{aligned} (D_t^2 u(t), w) - (\operatorname{div} \sigma(t), w) &= (f(t), w), \quad w \in V, \\ (\sigma(t) \chi) + (u(t), \operatorname{div} \chi) &= 0, \quad \chi \in H, \\ u(0) = u_0, \quad D_t u(0) &= v_0, \end{aligned}$$

where the parentheses denote the appropriate inner products ( $L_2$ -inner product in  $V$ ,  $L_2(\Omega)^2$ -inner product in  $H$ ). If  $V_h \subset V$  and  $H_h \subset H$  are finite dimensional subspaces, such as the spaces introduced by Raviart and Thomas [13], and by Brezzi, Douglas, Jr. and Marini [6], a semidiscrete version of (1.3) seeks  $u_h(t) \in V_h$ ,  $\sigma_h(t) \in H_h$ ,  $t > 0$ , satisfying

$$(1.3_h) \quad \begin{aligned} (D_t^2 u_h(t), w_h) - (\operatorname{div} \sigma_h(t), w_h) &= (f(t), w_h), \quad w_h \in V_h, \\ (\sigma_h(t), \chi_h) + (u_h(t), \operatorname{div} \chi_h) &= 0, \quad \chi_h \in H_h, \\ u_h(0) = u_{0,h}, \quad D_t u_h(0) &= v_{0,h}, \end{aligned}$$

where  $u_{0,h}$ ,  $v_{0,h} \in V_h$  are approximations to  $u_0$  and  $v_0$ , respectively.

Johnson and Thomée [12] have discussed the parabolic counterpart of (1.3<sub>h</sub>). The analysis of convergence of (1.3<sub>h</sub>) can be carried out along similar lines, parallel to Baker and Bramble [4], for example. (1.3<sub>h</sub>) is treated essentially as a non-conforming « displacement » model for the wave equation (1.1). The purpose of this note is to discuss the convergence of the « velocity-stress » models based on pairs of spaces  $(V_h, H_h)$  such as those in [6], [12], [13]. Thus, defining  $v = D_t u$ ,  $v_h = D_t u_h$ , (1.3) and (1.3<sub>h</sub>) are transformed, respectively, to

$$(1.4) \quad \begin{aligned} (D_t v(t), w) - (\operatorname{div} \sigma(t), w) &= (f(t), w), \quad w \in V, \\ (D_t \sigma(t), \chi) + (v(t), \operatorname{div} \chi) &= 0, \quad \chi \in H, \\ v(0) = v_0, \quad \sigma(0) = \sigma_0 = \nabla u_0, \end{aligned}$$

where  $v(t) \in V$ ,  $\sigma(t) \in H$ ,  $t \geq 0$ , and

$$(D_t v_h(t), w_h) - (\operatorname{div} \sigma_h(t), w_h) = (f(t), w_h), \quad w_h \in V_h,$$

$$(1.4_h) \quad (D_t \sigma_h(t), \chi_h) + (v_h(t), \operatorname{div} \chi_h) = 0, \quad \chi_h \in H_h, \\ v_h(0) = v_{0,h}, \quad \sigma_h(0) = \sigma_{0,h},$$

where  $v_h(t) \in V_h, \sigma_h(t) \in H_h, t \geq 0$ .

We now list the basic features of the space  $V_h, H_h$  which lead to a straightforward analysis of the convergence of  $v_h$  to  $v$  and  $\sigma_h$  to  $\sigma$  :

(H.1) There exists a linear operator  $\Pi_h : H \rightarrow H_h$  such that

$$(1.5) \quad (\operatorname{div} \Pi_h \chi, w_h) = (\operatorname{div} \chi, w_h) \quad \forall w_h \in V_h, \quad \chi \in H,$$

$$(1.6) \quad \|\Pi_h \chi - \chi\| \leq Ch^s \|\chi\|_s \quad \text{for } 1 \leq s \leq r, \quad r \geq 2$$

( $\|\cdot\|$  is the  $L_2(\Omega)^2$ -norm, and  $\|\cdot\|_s$  is the  $H^s(\Omega)^2$ -norm).

(H.2) There exists a linear operator  $P_h : V \rightarrow V_h$  such that

$$(1.7) \quad (P_h v, \operatorname{div} \chi_h) = (v, \operatorname{div} \chi_h) \quad \forall \chi_h \in H_h, \quad v \in V,$$

$$(1.8) \quad \|P_h v - v\| \leq Ch^s \|v\|_s, \quad 1 \leq s \leq r, \quad r \geq 2$$

( $\|\cdot\|$  is the  $L_2(\Omega)$ -norm, and  $\|\cdot\|_s$  is the  $H^s(\Omega)$ -norm, and, as usual,  $C$  denotes a generic constant which depends only on the data and on the particular discretization scheme).

If  $\Omega$  is a polygonal domain and  $V_h, H_h$  are the Raviart-Thomas spaces [12], [13], or if these spaces are the pairs introduced in the paper by Brezzi, Douglas, Jr., and Marini [6],  $\operatorname{div} \chi_h \in V_h$ , and  $P_h$  can be taken to be the  $L_2$ -projection. For an example of a pair  $(V_h, H_h)$  satisfying the above hypotheses (with  $r = 2$ ), where  $P_h$  is not the  $L_2$ -projection, we refer the reader to the paper by Johnson and Thomée [12]. We would also like to point out that (H.1) and (H.2) are valid for the mixed method that has been introduced by Arnold, Douglas, Jr., and Gupta [3] to approximate solution of plane elasticity problems. Our analysis is readily adapted to the corresponding (genuine) velocity-stress formulation of the time-dependent problem.

We can now state and prove our convergence result :

**THEOREM :** *If  $u$  is the solution of (1.1),  $v = D_t u, \sigma = \nabla u$ , and if the pair  $\{v_h, \sigma_h\}$  is the solution of (1.4<sub>h</sub>), under the hypotheses (H.1) and (H.2) we have, for  $1 \leq s \leq r, r \geq 2$ ,*

$$(1.9) \quad \|v_h(t) - v(t)\| + \|\sigma_h(t) - \sigma(t)\| \leq C (\|v_0 - v_{0,h}\| + \|\sigma_0 - \sigma_{0,h}\|) + \\ + Ch^s \left( \|v_0\|_s + \|\sigma_0\|_s + \int_0^t (\|D_\tau v(\tau)\|_s + \|D_\tau \sigma(\tau)\|_s) d\tau \right).$$

*Proof:* Let us denote by  $X$  the space  $V \times H$ , the elements of which will be designated as  $\xi = \{v, \sigma\}$  or  $\zeta = \{w, \chi\}$  and set

$$\begin{aligned} ((\xi, \zeta)) &= (v, w) + (\sigma, \chi), \\ \|\xi\| &= \sqrt{((\xi, \xi))}. \end{aligned}$$

Let  $X_h = V_h \times H_h$  be equipped with  $((\cdot, \cdot))$  and the induced norm  $\|\cdot\|$ . Elements of  $X_h$  will be designated as  $\xi_h = \{v_h, \sigma_h\}$  or  $\zeta_h = \{w_h, \chi_h\}$ . We define the bilinear form  $a(\cdot, \cdot)$  on  $X$  by

$$(1.10) \quad a(\xi, \zeta) = -(\operatorname{div} \sigma, w) + (v, \operatorname{div} \chi)$$

for  $\xi = \{v, \sigma\}$ ,  $\zeta = \{w, \chi\}$ .

We can now express (1.4) as

$$(1.11) \quad \begin{aligned} ((D_t \xi(t), \zeta)) + a(\xi(t), \zeta) &= (f(t), w), \quad \zeta \in X \\ (\xi(t) = \{v(t), \sigma(t)\}, \zeta = \{w, \chi\}), \end{aligned}$$

and we can express (1.4<sub>h</sub>) as

$$(1.11_h) \quad \begin{aligned} ((D_t \xi_h(t), \zeta_h)) + a(\xi_h(t), \zeta_h) &= (f(t), w_h), \quad \zeta_h \in X_h \\ (\xi_h(t) = \{v_h(t), \sigma_h(t)\}, \zeta_h = \{w_h, \chi_h\}). \end{aligned}$$

Let us define  $\underline{P}_h \xi = \{P_h v, \Pi_h \sigma\}$  for  $\xi = \{v, \sigma\} \in X$ , and observe that

$$(1.12) \quad a(\underline{P}_h \xi, \zeta_h) = a(\xi, \zeta_h), \quad \zeta_h \in X_h$$

by (H.1) ((1.5)) and (H.2) ((1.7)).

Therefore we obtain from (1.11)

$$(1.13) \quad \begin{aligned} ((D_t \underline{P}_h \xi(t), \zeta_h)) + a(\underline{P}_h \xi(t), \zeta_h) \\ = (f(t), w_h) + ((\underline{P}_h D_t \xi(t) - D_t \xi(t), \zeta_h)), \quad \zeta_h \in X_h. \end{aligned}$$

Setting  $\varepsilon_h(t) = \underline{P}_h \xi(t) - \xi_h(t)$ , (1.11<sub>h</sub>) and (1.13) yield

$$(1.14) \quad \begin{aligned} ((D_t \varepsilon_h(t), \zeta_h)) + a(\varepsilon_h(t), \zeta_h) = \\ = ((\underline{P}_h D_t \xi(t) - D_t \xi(t), \zeta_h)), \quad \zeta_h \in X_h. \end{aligned}$$

Let us define  $\Lambda_h : X_h \rightarrow X_h$  by

$$(1.15) \quad ((\Lambda_h \xi_h, \zeta_h)) = a(\xi_h, \zeta_h), \quad \xi_h, \zeta_h \in X_h.$$

Since

$$(1.16) \quad a(\xi_h, \zeta_h) = -a(\zeta_h, \xi_h), \quad \xi_h, \zeta_h \in X_h,$$

as is readily seen (cf. (1.10)),  $\Lambda_h$  is skew-adjoint,

$$(1.17) \quad ((\Lambda_h \xi_h, \zeta_h)) = -((\xi_h, \Lambda_h \zeta_h)), \quad \xi_h, \zeta_h \in X_h,$$

and  $-\Lambda_h$  generates the unitary group  $e^{-t\Lambda_h}$ . In particular,

$$(1.18) \quad \|e^{-t\Lambda_h} \xi_h(0)\| = \|\xi_h(0)\|, \quad t \in \mathbb{R}.$$

Let us denote by  $P_h^0: X \rightarrow X_h$  the projection with respect  $((\cdot, \cdot))$ .

We can now express (1.14) as

$$(1.19) \quad D_t \varepsilon_h(t) + \Lambda_h \varepsilon_h(t) = P_h^0(P_h D_t \xi(t) - D_t \xi(t))$$

so that

$$(1.20) \quad \varepsilon_h(t) = e^{-t\Lambda_h} \varepsilon_h(0) + \int_0^t e^{-(t-\tau)\Lambda_h} P_h^0(P_h D_\tau \xi(\tau) - D_\tau \xi(\tau)) d\tau.$$

(1.18) and (1.20) yield the estimate

$$(1.21) \quad \|\varepsilon_h(t)\| \leq \|\varepsilon_h(0)\| + \int_0^t \|P_h D_\tau \xi(\tau) - D_\tau \xi(\tau)\| d\tau$$

( $P_h^0$  is the  $((\cdot, \cdot))$ -projection).

(1.21) is readily translated to

$$\begin{aligned} & \|P_h v(t) - v_h(t)\| + \|\Pi_h \sigma(t) - \sigma_h(t)\| \\ & \leq C (\|P_h v_0 - v_{0,h}\| + \|\Pi_h \sigma_0 - \sigma_{0,h}\| \\ & \quad + \int_0^t (\|P_h D_\tau v(\tau) - D_\tau v(\tau)\| + \|\Pi_h D_\tau \sigma(\tau) - D_\tau \sigma(\tau)\|) d\tau) \\ & \leq C (\|v_0 - v_{0,h}\| + \|\sigma_0 - \sigma_{0,h}\| + \|P_h v_0 - v_0\| + \|\Pi_h \sigma_0 - \sigma_0\| \\ & \quad + \int_0^t (\|P_h D_\tau v(\tau) - D_\tau v(\tau)\| + \|\Pi_h D_\tau \sigma(\tau) - D_\tau \sigma(\tau)\|) d\tau), \end{aligned}$$

and this, together with (1.6) and (1.8), yields

$$(1.22) \quad \begin{aligned} & \|P_h v(t) - v_h(t)\| + \|\Pi_h \sigma(t) - \sigma_h(t)\| \\ & \leq C (\|v_0 - v_{0,h}\| + \|\sigma_0 - \sigma_{0,h}\| + h^s (\|v_0\|_s + \|\sigma_0\|_s)) \\ & \quad + Ch^s \int_0^t (\|D_\tau v(\tau)\|_s + \|D_\tau \sigma(\tau)\|_s) d\tau. \end{aligned}$$

Since

$$\begin{aligned} & \|v(t) - v_h(t)\| + \|\sigma(t) - \sigma_h(t)\| \\ & \leq \|v(t) - P_h v(t)\| + \|P_h v(t) - v_h(t)\| \\ & \quad + \|\sigma(t) - \Pi_h \sigma(t)\| + \|\Pi_h \sigma(t) - \sigma_h(t)\| \\ & \leq Ch^s(\|v(t)\|_s + \|\sigma(t)\|_s) + \|P_h v(t) - v_h(t)\| + \|\Pi_h \sigma(t) - \sigma_h(t)\|, \end{aligned}$$

by (1.6) and (1.8), and

$$\begin{aligned} \|v(t)\|_s & \leq \|v_0\|_s + \int_0^t \|D_\tau v(\tau)\|_s d\tau, \\ \|\sigma(t)\|_s & \leq \|\sigma_0\|_s + \int_0^t \|D_\tau \sigma(\tau)\|_s d\tau, \end{aligned}$$

(1.22) leads to (1.9), the assertion of the theorem.

## 2. SOME OBSERVATIONS IN REGARD TO THE TIME-DIFFERENCING OF THE SEMIDISCRETE MODEL

(1.4<sub>h</sub>) leads to a system of ordinary differential equations in the form

$$(2.1) \quad \begin{aligned} M_0 D_t W - D\Sigma &= F, \\ M_1 D_t \Sigma + D^T W &= 0, \end{aligned}$$

where  $W$  corresponds to  $v_h$ ,  $\Sigma$  corresponds to  $\sigma_h$ ,  $M_0, M_1$  are symmetric, positive-definite matrices, and  $D^T$  denotes the transpose of  $D$ . The application of implicit Euler time-differencing

$$(2.2) \quad \begin{aligned} M_0 \frac{W^{n+1} - W^n}{k} - D\Sigma^{n+1} &= F^{n+1}, \\ M_1 \frac{\Sigma^{n+1} - \Sigma^n}{k} + D^T W^{n+1} &= 0, \end{aligned}$$

( $k$  denotes the time step), necessitates the solution of

$$(2.3) \quad \begin{aligned} M_0 W^{n+1} - kD\Sigma^{n+1} &= kF^{n+1} + M_0 W^n, \\ M_1 \Sigma^{n+1} + kD^T W^{n+1} &= M_1 \Sigma^n. \end{aligned}$$

$M_0$  is in block-diagonal form if  $V_h$  consists of functions with no continuity requirement across inter-element boundaries, as is the case in [6], [12], [13], and the elimination of  $W^{n+1}$  in (2.3) is efficiently implementable. This leads to a system in the form

$$(2.4) \quad (M_1 + k^2 D^T M_0^{-1} D) \Sigma^{n+1} = G,$$

where  $M_1$  is symmetric, positive definite and  $D^T M_0^{-1} D$  is symmetric, positive-semidefinite, for the determination of  $\Sigma^{n+1}$ .

On the other hand, (1.3<sub>h</sub>) leads to

$$(2.5) \quad \begin{aligned} M_0 D_t^2 U - D \Sigma &= F, \\ M_1 \Sigma + D^T U &= 0, \end{aligned}$$

where  $U$  corresponds to  $u_h$ . (2.5) can be expressed as

$$(2.6) \quad M_0 D_t^2 U + D M_1^{-1} D^T U = F,$$

where  $M_0$ ,  $D M_1^{-1} D^T$  are symmetric, positive-definite [12]. If (2.6) is expressed as a system in  $\{U, W\}$ ,

$$(2.7) \quad \begin{aligned} D_t U - W &= 0 \\ M_0 D_t W + D M_1^{-1} D^T U &= F, \end{aligned}$$

and implicit Euler time-differencing is applied to (2.7),

$$(2.8) \quad \begin{aligned} \frac{U^{n+1} - U^n}{k} - W^{n+1} &= 0 \\ M_0 \frac{W^{n+1} - W^n}{k} + D M_1^{-1} D^T U^{n+1} &= F^{n+1}, \end{aligned}$$

elimination of  $W^{n+1}$  leads to a system in the form

$$(2.9) \quad (M_0 + k^2 D M_1^{-1} D^T) U^{n+1} = \tilde{G}.$$

The matrix in (2.9) is symmetric, positive-definite, so that (2.9) is solvable. But  $M_1$  is not block-diagonal, unlike  $M_0$ , so that deriving the reduced system (2.9), which includes inverting  $M_1$ , is more expensive than forming the reduced system (2.4). The time-independent counterpart of (2.5),

$$(2.10) \quad \begin{aligned} -D \Sigma &= F, \\ M_1 \Sigma + D^T U &= 0, \end{aligned}$$

led Arnold and Brezzi [2] to relax the requirement that  $\text{div } \sigma_h \in L_2(\Omega)$  in order to have a block-diagonal matrix instead of  $M_1$  and be able to eliminate  $\Sigma$  efficiently. This approach has to introduce a multiplier corresponding to the relaxation of the requirement  $\text{div } \sigma_h \in L_2(\Omega)$ .

The above considerations suggest that the « velocity-stress » formulation (1.4<sub>h</sub>) may be preferable to (1.3<sub>h</sub>) if the approximation of the « stress »  $\sigma$  is of primary concern.

The application of diagonally implicit Runge-Kutta methods (see, for example, Crouzeix [8], Crouzeix and Raviart [9], Alexander [1], Burrage



[7], Dougalis and Serbin [10]) to (2.1) leads to systems similar to (2.4) so that our discussion is relevant to higher-order time differencing as well. We will not prove error estimates for such full-descrete approximation schemes based on  $(1.4)_h$ . Such estimates should be obtainable by employing techniques that have been utilized in [5] or [11], for example.

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