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C. FOIAS
O. MANLEY
R. TEMAM

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MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
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MODELLING OF THE INTERACTION OF SMALL AND LARGE EDDIES IN TWO DIMENSIONAL TURBULENT FLOWS (*)

by C. FOIAS [1], O. MANLEY [2] and R. TEMAM [3]

Résumé. — *Notre objet dans cet article est de présenter quelques résultats concernant la modélisation de l'interaction des petites et grandes structures d'écoulements bidimensionnels turbulents. Nous montrons que l'amplitude des petits tourbillons décroît exponentiellement vers une valeur petite et nous en déduisons une loi d'interaction simplifiée des petits et grands tourbillons. Outre leur intérêt concernant la compréhension de la physique de la turbulence, ces résultats conduisent à des schémas numériques nouveaux qui seront étudiés dans un travail séparé.*

Abstract. — *Our aim in this article is to present some results concerning the modeling of the interaction of small and large eddies in two dimensional turbulent flows. We show that the amplitude of small structures decays exponentially to a small value and we infer from this a simplified interaction law of small and large eddies. Beside their intrinsic interest for the understanding of the physics of turbulence, these results lead to new numerical schemes which will be studied in a separate work.*

CONTENTS

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[1] Department of Mathematics, Indiana University, Bloomington, IN 47405.
[2] Department of Energy, Washington, DC 20545.
[3] Laboratoire d'Analyse Numérique, Université Paris-Sud, Bât. 425, 91405 Orsay, France.

INTRODUCTION

The conventional theory of turbulence in space dimension three asserts the existence of a length l_d which is small in comparison with the macroscopical length l_0 determined by the geometry, and which is such that the eddies of size less than l_d are damped by the effect of viscosity and become rapidly small in amplitude ; the length l_d is called the Kolmogorov dissipation length [9]. In space dimension two the situation is similar but l_d is replaced by the larger length l_χ introduced by Kraichnan [10]. It is one of our aims in this article to derive directly from the Navier-Stokes equations and without any phenomenological consideration, a mathematically rigorous proof of this property : the exponential decay of the small eddies toward a small limiting value. Note however that our estimate of the eddy sizes below which viscous damping is effective is much smaller than l_χ or even l_d ; this is due in part to the high level of generality allowed here which includes singular flows such as those generated by flows in nonsmooth cavities, e.g. the flow in a rectangular cavity. A physical discussion of the necessary cut-off length is presented hereafter.

Our approach is as follows : the Navier-Stokes equations of two dimensional viscous incompressible flows are written as

$$\frac{\partial u}{\partial t} - \nu \,\Delta u + (u \,.\, \nabla)\, u - \nabla \varpi = f \quad \text{in} \quad \Omega \times \mathbb{R}_+ \tag{0.1}$$

$$\nabla \,.\, u = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+ \tag{0.2}$$

where $u = u(x, t) = \{u_1, u_2\}$ is the velocity vector, $\varpi = \varpi(x, t)$ is the pressure, f represents volume forces, $\nu > 0$ is the kinematic viscosity. As usual (0.1), (0.2) are supplemented by boundary conditions which could be for instance

$$u = 0 \quad \text{on} \quad \partial\Omega \tag{0.3a}$$

or

$$u \,.\, \nu = 0, \ \nu \times \text{curl} u = 0 \quad \text{on} \quad \partial\Omega\,, \tag{0.3b}$$

ν the unit outward normal on $\partial\Omega$, or

$$\Omega = (0, L_1) \times (0, L_2) \tag{0.3c}$$

and u, ϖ are periodic of period L_i in the direction x_i, $i = 1, 2$.

Here our emphasis will be on the space periodic case (0.3c) but the other boundary conditions will be considered as well. In all cases (0.1)-(0.3) reduces to an abstract evolution equation for u in an appropriate Hilbert space H:

$$\frac{du}{dt} + \nu A u + B(u) = f . \tag{0.4}$$

The operator A linear, self-adjoint unbounded positive in H with domain $D(A) \subset H$, is the Stokes operator. Since A^{-1} is compact self-adjoint, A possesses a complete family of eigenvectors w_j which is orthonormal in H

$$\begin{gathered} A w_j = \lambda_j w_j , \quad j = 1, 2, \ldots \\ 0 < \lambda_1 \leqslant \lambda_2, \ldots, \quad \lambda_j \to \infty \quad \text{as} \quad j \to \infty . \end{gathered} \tag{0.5}$$

Of course in the space periodic case (0.3c) the w_j's are directly related to the appropriate sine and cosine functions of the Fourier series expansion (see [13]). The operator B is a quadratic operator ; $B(u) = B(u, u)$, where $B(\cdot, \cdot)$ is a bilinear compact operator from $D(A)$ into H.

For fixed m we denote by $P = P_m$ the projector in H onto the space spanned by $w_1, \ldots, w_m$, and we write $Q = Q_m = I - P_m$. We set

$$u = p + q , \quad p = Pu , \quad q = Qu ,$$

and we show that, after a transient period, and for various norms, p is comparable to u and q is small in comparison with p and u (see Sec. 1).

We then project equation (0.4) on PH and QH ; this yields a coupled system of equations for p and q :

$$\frac{dp}{dt} + \nu A p + PB(p + q) = Pf \tag{0.6}$$

$$\frac{dq}{dt} + \nu A q + QB(p + q) = Qf . \tag{0.7}$$

Since q is small in comparison with p one can speculate that $B(q, q) = B(q)$ is small in comparison with $B(p, q)$ and $B(q, p)$ and that in turn these quantities are small in comparison with $B(p, p) = B(p)$. Also the relaxation time for the linear part of (0.7) of the order of $(\nu\lambda_{m+1})^{-1}$ is much smaller than that of (0.6) which is of order $(\nu\lambda_1)^{-1}$. This suggests that an acceptable approximation to (0.7) is given by

$$\nu A q + QB(p) = Qf . \tag{0.8}$$

This leads us to introduce in H the finite dimensional manifold $\mathcal{M}_0$ with equation

$$(0.9)\qquad \begin{cases} q = \Phi_0(p) = (\nu A)^{-1}\,(Qf - QB(p)) \\ p = Pu\,, \quad q = Qu\,. \end{cases}$$

It is one of our aims to justify this approximation : for large times, i.e., after a sufficiently long transient period, the ratio of q to u is of the order of $\dfrac{\lambda_1}{\lambda_{m+1}} \sim (m+1)^{-1}$ for large m, while the distance of q to $\mathcal{M}_0$ (compared to a quantity of the order of u), is of the order of $\left(\dfrac{\lambda_1}{\lambda_{m+1}}\right)^{3/2}$ for large m. The proof of this result appears in Section 2. Hence, for large time, an orbit $u(t) = p(t) + q(t)$ corresponding to any solution of (0.4) becomes closer to $\mathcal{M}_0$ than to the linear space $q = 0$. In a subsequent work we intend to construct a whole family of explicitly defined manifolds $\mathcal{M}_j$ providing better and better approximations to the orbits as j increases (*cf.* [3]). The manifold $\mathcal{M}_0$ (as well as the future manifolds $\mathcal{M}_j$) plays the role of *approximate inertial manifolds* to the two dimensional Navier-Stokes equations, and constitute a substitute for exact manifolds in situations where we cannot prove their existence.

In Section 3 we recall and improve significantly a result in [8] : this leads us to introduce a Lipschitz manifold Σ of finite dimension similar to $\mathcal{M}_0$; it has the property that eventually all the orbits of (0.4) are not further from it than $\exp(-c\lambda_{m+1}/\lambda_1)$. Hence Σ provides a much better approximation than $\mathcal{M}_0$ but, unfortunately for now, the proof of existence is nonconstructive and hence does not provide an explicit expression like (0.9). Nevertheless it offers an interesting complementary aspect. Let us mention also that another type of approximate manifold containing all the stationary solutions has been exhibited by E. Titi [15].

This article ends with an Appendix providing a technical but totally new method of estimating certain norms of the solutions of an evolution equation such as (0.4) : taking advantage of the analyticity of the solutions with respect to time, we estimate the domain of analyticity in the complex time plan and using Cauchy's formula, we readily deduce estimates on the derivatives $d^k u/dt^k$ from the estimates on u in the domain of analyticity ; these estimates on the time derivatives of u are much sharper than those obtained by real variable methods. The results presented here were announced in [2].

1. FAST DECAY OF SMALL EDDIES

In Sections 1.1 and 1.2 we briefly recall the functional setting of the Navier-Stokes equations and some useful estimates. Then in Section 1.3 we derive the estimates on the magnitude of the small eddies.

1.1. Preliminaries

As we recalled in the Introduction, the Navier-Stokes equations (0.1), (0.2) associated to one of the boundary conditions (0.3) are equivalent to an evolution equation

$$\frac{du}{dt} + \nu A u + B(u) = f \tag{1.1}$$

in an appropriate Hilbert space H. Here $f \in H$, $\nu > 0$, A is a linear self-adjoint positive operator with domain $D(A) \subset H$, and whose inverse A^{-1} is compact ; we have $B(u) = B(u, u)$ where $B(\cdot\,,\cdot\,)$ is a bilinear compact operator from $D(A)$ (endowed with the norm $|A\,\cdot\,|$) into H ; H is a Hilbert subspace of $L^2(\Omega)^2$. Its norm and scalar product are denoted $|\cdot\,|$, $(\cdot\,,\cdot\,)$ as those of $L^2(\Omega)^2$ or $L^2(\Omega)$; for the details see [12], [13].

We recall that for u_0 given in H the initial value problem (1.1), (1.2) :

$$u(0) = u_0\,, \tag{1.2}$$

possesses a unique solution u defined for all $t > 0$ and such that

$$u \in C(\mathbb{R}_+\,;H) \cap L^2(0, T\,; V)\,, \quad \forall T > 0\,; \tag{1.3}$$

here $V = D(A^{1/2})$ and the norm $|A^{1/2}\cdot\,| = \|\cdot\,\|$ on V is equivalent to the L^2 norm of grad u. If $u_0 \in V$ then

$$u \in C(\mathbb{R}_+\,; V) \cap L^2(0, T\,; D(A))\,, \quad \forall T > 0\,. \tag{1.4}$$

In both cases ($u_0 \in H$ or V), $u(\cdot\,)$ is analytic in t with values in $D(A)$; the domain of analyticity of u in the complex plane $\mathbb{C}_t$ comprises a band around $\mathbb{R}_+$ and is described in more detail in the Appendix.

It is useful here to reproduce some a priori estimates satisfied by the solutions u of (1.1), (1.2). But first we recall some inequalities (continuity properties) concerning B (see [8]) : for every u, v, $w \in D(A)$:

$$|B(u, v)| \leq c_1 \begin{cases} |u|^{1/2}\,\|u\|^{1/2}\,\|v\|^{1/2}\,|Av| \\ |u|^{1/2}\,|Au|^{1/2}\,\|v\| \end{cases} \tag{1.5}$$

$$(1.6) \qquad |(B(u,v),w)| \leqslant c_2 |u|^{1/2} \|u\|^{1/2} \|v\| \; |w|^{1/2} \|w\|^{1/2} ,$$

where c_1, c_2 like the quantities c_i, c_i', which will appear subsequently are dimensionless constants ([1]). Also we recall from [1], [4] the inequality

$$(1.7) \qquad |\phi|_{L^\infty(\Omega)^2} \leqslant c_3 \|\phi\| \left(1 + \log \frac{|A\phi|^2}{\lambda \|\phi\|^2}\right)^{1/2} , \quad \forall \phi \in D(A) ,$$

from which we deduce that

$$|B(u,v)| \leqslant |(u \,.\, \nabla)\, v| \leqslant \begin{cases} |u|_{L^\infty(\Omega)} |\nabla v| \\ |u| \; |\nabla v|_{L^\infty(\Omega)} \end{cases}$$

and using (1.7)

$$(1.8) \qquad |B(u,v)| \leqslant c_4 \begin{cases} \|u\| \; \|v\| \left(1 + \log \dfrac{|Au|^2}{\lambda_1 \|u\|^2}\right)^{1/2} \\[2ex] |u| \; |Av| \left(1 + \log \dfrac{|A^{3/2} v|^2}{\lambda_1 |Av|^2}\right)^{1/2} . \end{cases}$$

1.2. Behavior of small eddies

As mentioned in the Introduction we fix an integer $m \in N$ and denote by $P = P_m$ the projector in H onto the space spanned by the first m eigenvectors of A, $w_1, \ldots, w_m$; we set also $Q = Q_m = I - P_m$, and for the sake of simplicity

$$(1.9) \qquad \lambda = \lambda_m \, , \quad \Lambda = \lambda_{m+1} \, .$$

We write $p = Pu$, $q = Qu$; p represents a superposition of « large eddies » of size larger than $\lambda_m^{-1/2}$, and q represents « small eddies » of size smaller than $\lambda_{m+1}^{-1/2}$. By projecting (1.1) on PH and QH we find since $PA = AP$ and $QA = AQ$:

$$(1.10) \qquad \frac{dp}{dt} + \nu Ap + PB(p+q) = Pf$$

$$(1.11) \qquad \frac{dq}{dt} + \nu Aq + QB(p+q) = Qf \, .$$

We take the scalar product of (1.10) with q in H :

$$(1.12) \qquad \frac{1}{2} \frac{d}{dt} |q|^2 + \nu \|q\|^2 = (Qf, q) - (B(p+q), q) \, .$$

([1]) These constants can be absolute constants or they may depend on the shape of Ω : by this we mean that they are invariant by translation or homothety of Ω.

Thanks to the orthogonality property

$$(1.13)\qquad (B(\phi, \psi), \psi) = 0\,, \quad \forall \phi, \psi \in V\,,$$

the right hand side of (1.11) reduces to

$$(Qf, q) - (B(p, p), q) - (B(q, p), q)\,.$$

Using (1.6) and Schwarz inequality we majorize it by

$$|Qf|\;|q| + c_4 \|p\|^2\,|q| \left(1 + \log \frac{|Ap|^2}{\lambda_1 \|p\|^2}\right)^{1/2} + c_2 |q|\,\|q\|\,\|p\| \leqslant$$

$$\leqslant (\text{since } \|p\| \leqslant \|u\|)$$

$$\leqslant |Qf|\,|q| + c_4 \|p\|^2\,|q| \left(1 + \log \frac{|Ap|^2}{\lambda_1 \|p\|^2}\right)^{1/2} + c_2\,\Lambda^{-1/2}\,\|q\|^2\,\|u\|\,.$$

We denote now a bound of $|u|$ (resp. $\|u\|$, $|Au|$), on the interval of time $I = (t_0, \infty)$ under consideration, by M_0 (resp. M_1, M_2)

$$(1.14)\quad M_0 = \operatorname*{Sup}_{s \in I} |u(s)|\,, M_1 = \operatorname*{Sup}_{s \in I} \|u(s)\|\,, M_2 = \operatorname*{Sup}_{s \in I} |Au(s)|\;;$$

we observe that

$$|Ap|^2 \leqslant \lambda_m \|p\|^2 = \lambda \|p\|^2$$

and set

$$(1.15)\qquad L = \left(1 + \log \frac{\lambda_{m+1}}{\lambda_1}\right).$$

We obtain

$$(1.16)\quad \frac{d}{dt}|q|^2 + (2\,\nu - c_2\,\Lambda^{-1/2}\,M_1)\|q\|^2 \leqslant |Qf|\,|q| + c_4\,M_1^2\,L^{1/2}\,|q|\,.$$

Hence, assuming that $c_2\,\Lambda^{-1/2}\,M_1 \leqslant \nu$, i.e.,

$$(1.17)\qquad \lambda_{m+1} = \Lambda \geqslant \left(\frac{2\,c_2\,M_1}{\nu}\right)^2$$

(1.16) yields

$$\frac{d}{dt}|q|^2 + \frac{3\,\nu}{2}\|q\|^2 \leqslant \Lambda^{-1/2}(|Qf| + 4\,M_1^2\,L^{1/2})\|q\|$$

$$\leqslant \frac{\nu}{2}\|q\|^2 + \frac{1}{\nu\Lambda}(|Qf|^2 + c_4^2\,M_1^4\,L)$$

$$(1.18)\qquad \frac{d}{dt}|q|^2 + \nu\|q\|^2 \leqslant \frac{1}{\nu\Lambda}(|Qf|^2 + c_4^2\,M_1^4\,L)$$

$$\frac{d}{dt}|q|^2 + \nu\Lambda|q|^2 \leqslant \frac{1}{\nu\Lambda}(|Qf|^2 + c_4^2 M_1^4 L). \tag{1.19}$$

We infer easily from (1.19) that for $t \geqslant t_1$, $t_1, t \in I$:

$$|q(t)|^2 \leqslant |q(t_1)|^2 \exp(-\nu\Lambda(t - t_1)) + \frac{1}{\nu^2\Lambda^2}(|Qf|^2 + c_4^2 M_1^4 L). \tag{1.20}$$

Before interpreting this inequality, we derive a similar inequality for the (H^1) V norm. Taking the scalar product of (1.11) with Aq in H we find

$$\frac{1}{2}\frac{d}{dt}\|q\|^2 + \nu|Aq|^2 = (Qf, Aq) - (B(p+q), Aq).$$

We expand and use Schwarz inequality together with (1.6)-(1.8) to majorize the right hand side of this equation by

$$\begin{aligned}|Qf|\,|Aq| &+ c_2\|p\|\,L^{1/2}\,|Aq|\,(\|p\| + \|q\|) + \\ &+ c_4|q|^{1/2}\,|Aq|^{3/2}\,(\|p\| + \|q\|)\\ &\leqslant \text{(with Young's inequality)}\\ &\leqslant \frac{\nu}{2}|Aq|^2 + \frac{1}{\nu}|Qf|^2 + \frac{c_1' M_1^4 L}{\nu} + \frac{c_2'}{\nu^3} M_0^2 M_1^4.\end{aligned}$$

Thus,

$$\frac{d}{dt}\|q\|^2 + \nu|Aq|^2 \leqslant c_3'\left(\frac{1}{\nu}|Qf|^2 + \frac{M_1^4 L}{\nu} + \frac{M_0^2 M_1^4}{\nu^3}\right) \tag{1.21}$$

$$\frac{d}{dt}\|q\|^2 + \nu\Lambda\|q\|^2 \leqslant c_3'\left(\frac{1}{\nu}|Qf|^2 + \frac{M_1^4 L}{\nu} + \frac{M_0^2 M_1^4}{\nu^3}\right) \tag{1.22}$$

and we conclude that

$$\begin{aligned}\|q(t)\|^2 \leqslant{}& \|q(t_1)\|^2 \exp(-\nu\Lambda(t - t_1))\\ &+ \frac{c_3'}{\nu\Lambda}\left(\frac{1}{\nu}|Qf|^2 + \frac{M_1^4}{\nu}L + \frac{M_0^2 M_1^4}{\nu^3}\right).\end{aligned} \tag{1.23}$$

In (1.20) and (1.23) we can bound $|q(t_1)|^2$ and $\|q(t_1)\|^2$ by M_0^2 and M_1^2 respectively. Then after a time depending only on M_0 (or M_1), ν and $\Lambda = \lambda_{m+1}$, the term involving t becomes negligible and we obtain

$$\left\{\begin{aligned}|q(t)|^2 &\leqslant \frac{2}{\nu^2\Lambda^2}(|Qf|^2 + c_4^2 M_1^4 L),\\ \|q(t)\|^2 &\leqslant \frac{2c_3'}{\nu^2\Lambda}\left(|Qf|^2 + M_1^4 L + \frac{M_0^2 M_1^4}{\nu^2}\right)\end{aligned}\right. \tag{1.24}$$

for t large. Alternatively, denoting by κ, κ_i, κ_i', some quantities which depend only on the data ν, f, Ω, and M_0, M_1, M_2, we rewrite (1.24) as

$$(1.25)\qquad |q(t)|^2 \leq \kappa L \delta^2,\ \|q(t)\|^2 \leq \kappa L \delta \quad \text{for } t \text{ large}\,,$$
$$\delta = \frac{\lambda_1}{\Lambda} = \frac{\lambda_1}{\lambda_{m+1}}\,,\quad L = 1 + \log \frac{\lambda_{m+1}}{\lambda_1}\,.$$

Using also the results in the Appendix we conclude the following

THEOREM 1.1 : *We assume that m is sufficiently large so that* (1.17) *holds. Then for any orbit of* (1.1), *after a time t_* which depends only on the data ν, f, Ω and the initial value $u(0) = u_0$, the small eddies component of u, $q = Q_m u$, is small in the following sense*

$$\begin{aligned} |q(t)| &\leq \kappa_0 L^{1/2} \delta\,, & \|q(t)\| &\leq \kappa_1 L^{1/2} \delta^{1/2} \\ |q'(t)| &\leq \kappa_0' L^{1/2} \delta\,, & |Aq(t)| &\leq \kappa_2 L^{1/2}\,, \quad t \geq t_*\,. \end{aligned}$$

The first two inequalities in (1.26) follow from (1.25) ; the third one follows from (1.25) and the analog of (A.15) for q [1]. The fourth inequality is obtained by writing

$$\nu A q = Qf - q' - QB(p+q)$$
$$|Aq| \leq \frac{1}{\nu}\,|Qf| + \frac{1}{\nu}\,|q'| + \frac{1}{\nu}\,|QB(p+q)|$$

and utilizing (1.5), (1.6), (1.8).

In Section 1.3 hereafter we intend to provide a more explicit form of the constants κ in the case of space periodic flows.

1.3. The space periodic case

We first review the well-known a priori estimates for the solutions of (1.1). This will yield more explicit expressions for M_0, M_1, M_2.

We take the scalar product of (1.1) with u in H ; using the orthogonality property (1.13) we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\,|u|^2 + \nu \|u\|^2 = (f, u) &\leq |f|\,|u| \\ &\leq \lambda_1^{-1/2} |f|\,\|u\| \\ &\leq \frac{\nu}{2}\,\|u\|^2 + \frac{1}{2\,\nu\lambda_1}\,|f|^2 \end{aligned}$$

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[1] Note that q is analytic in the same region of the complex plan as u. We write $q' = dq/dt$.

(1.26) $$\frac{d}{dt}|u|^2 + \nu\|u\|^2 \leq \frac{1}{\nu\lambda_1}|f|^2$$

(1.27) $$\frac{d}{dt}|u|^2 + \nu\lambda_1|u|^2 \leq \frac{1}{\nu\lambda_1}|f|^2;$$

(1.27) yields

(1.28) $$|u(t)|^2 \leq |u(0)|^2 \exp(-\nu\lambda_1 t) + \frac{|f|^2}{\nu^2\lambda_1^2}(1 - \exp(-\nu\lambda_1 t)), \quad \forall t > 0.$$

If we assume that $|u(0)| \leq R_0$, then after a time $t_0 = t_0(R_0)$ depending only on R_0 and the data ν, f, λ_1, we have

(1.29) $$|u(t)|^2 \leq \frac{2|f|^2}{\nu^2\lambda_1^2}, \quad \forall t \geq t_0(R_0).$$

We can introduce as in [4] the nondimensional Grashof number ([1])

(1.30) $$G = \frac{|f|}{\nu^2\lambda_1}$$

and rewrite (1.29) in the form

(1.31) $$|u(t)|^2 \leq \frac{2|f|G}{\lambda_1}, \quad \forall t \geq t_0(R_0);$$

(1.31) expresses the fact that the ball of H centered at 0 of radius $(2|f|G/\lambda_1)^{1/2}$ is *absorbing* in H (*cf.* [14]).

We now restrict ourselves to the case of the space periodic boundary condition (0.3*c*). In this case we have [13] the identity

(1.32) $$(B(\phi, \phi), A\phi) = 0, \quad \forall \phi \in D(A);$$

hence on taking the scalar product of (1.1) with Au in H we find

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu|Au|^2 = (f, Au) \leq \frac{\nu}{2}|Au|^2 + \frac{1}{2\nu}|f|^2$$

([1]) Some authors prefer to introduce a nondimensional number proportional to ν^{-1}:

$$\mathrm{Re} = G^{1/2} = \frac{|f|^{1/2}}{\nu\lambda_1^{1/2}}$$

and call it the Reynolds number of the flow. However, there is no evidence that $|f|^{1/2}$ (which has the dimension of a velocity) is a characteristic velocity of the flow under consideration.

$$\frac{d}{dt}\|u\|^2 + \nu|Au|^2 \leqslant \frac{1}{\nu}|f|^2$$

$$\frac{d}{dt}\|u\|^2 + \nu\lambda_1\|u\|^2 \leqslant \frac{1}{\nu}|f|^2 . \tag{1.33}$$

Thus,

$$\|u(t)\|^2 \leqslant \|u(t_1)\|^2 \exp(-\nu\lambda_1(t-t_1)) + \frac{1}{\nu^2\lambda_1}|f|^2(1-\exp(-\nu\lambda_1(t-t_1))), \quad \forall t \geqslant t_1 \geqslant 0 . \tag{1.34}$$

If $u_0 \in H$, $|u_0| \leqslant R_0$, then at any time $t_1 \geqslant 0$, $u(t_1) \in V$, with a bound on $\|u(t_1)\|$ depending only on t_1, R_0 and the data (f, ν, Ω). Thus, after a time $t_2 = t_2(R_0)$ depending only on R_0, f, ν, Ω, the terms involving t become negligible and there remains

$$\|u(t)\|^2 \leqslant \frac{2}{\nu^2\lambda_1}|f|^2 , \quad \forall t \geqslant t_2 . \tag{1.35}$$

Since we are not interested in transient flows but rather in permanent regimes, our emphasis will be on large time behaviors. Thus we can restrict ourselves to $I = (t_2, \infty)$ and take

$$M_0 = \left(\frac{2|f|}{\lambda_1}G\right)^{1/2}, \quad M_1 = (2|f|G)^{1/2} . \tag{1.36}$$

The estimate of $|u'(t)|$ for $t \geqslant t_2$ follows promptly from (1.34), (1.35) and is established in the Appendix by utilization of Cauchy's formula :

$$|u'(t)| \leqslant M_0' , \quad \forall t \geqslant t_2$$

$$M_0' = c\,|f|\,G^2 \log G . \tag{1.37}$$

Now we can give a more explicit form of (1.17) :

$$\frac{\lambda_{m+1}}{\lambda_1} \geqslant \frac{4c_2^2}{\nu^2\lambda_1}M_1^2$$

$$\frac{\lambda_{m+1}}{\lambda_1} \geqslant 8c_2^2G^2 . \tag{1.38}$$

Since $\lambda_m \sim m$ as $m \to \infty$, (1.38) means that we need to retain for p, at least G^2 modes which is higher than what is predicted by Kolmogorov (cG) and Kraichnan ($cG^{2/3}$) theories ; the inequality (1.38) below shows that for such a value of m, $m \sim cG^2$, $|q|$ is small, of the order of $c\left(\frac{|f|}{\lambda_1}\right)^{1/2} G^{-1/2}$. Then we rewrite (1.24) in the form

$$|q(t)|^2 \leqslant \frac{\lambda_1^2}{\Lambda^2} L \cdot \frac{2}{\nu^2 \lambda_1^2} \left(|Qf|^2 + c_4^2 4 \, |f|^2 G^2\right),$$

$$\leqslant cL\delta^2 \frac{|f|}{\lambda_1} (G + G^3), \quad t \geqslant t_2,$$

$$\|q(t)\|^2 \leqslant \frac{\lambda_1}{\Lambda} L \cdot \frac{2 c_3'}{\nu^2 \lambda_1} \left(|Qf|^2 + 4 \, |f|^2 G^2 + \frac{8}{\nu^2 \lambda_1} |f|^3 G^3 \right)$$

$$\leqslant \delta L c \, |f| (G + G^3 + G^5).$$

We then take in (1.26)

$$\kappa_0 = c \left(\frac{|f|}{\lambda_1} \right)^{1/2} (1 + G^{3/2}), \quad \kappa_1 = c \, |f|^{1/2} (1 + G^{5/2}), \tag{1.38}$$

c an absolute constant and as explained before, the time t_* in Theorem 1.1 depends only on R_0 ($|u(0)| \leqslant R_0$) and the data ν, f, Ω.

Remark 1.1 : In the case of the boundary conditions (0.3a, b), (1.32) fails ; one can derive a time-uniform bound for the norm of u in V by using the uniform Gronwall Lemma (see [6], [14]), but M_1 and then m, κ_0, κ_1, are unrealistically high functions of G, exponentials of G. It is an open problem whether M_1 can be expressed as a polynomial function of G in this case.

2. THE APPROXIMATE MANIFOLD

In this section we show that the orbits of (1.1) converge, as $t \to \infty$, to the vicinity of a very simply defined manifold $\mathcal{M}_0$. In Section 2.1 we derive the equation of the manifold and in Section 2.2 we estimate the distance of the orbits to this manifold.

2.1. Equations of the manifold

As indicated in the Introduction, the results of Section 1 show that q is small so that $B(p, q)$ and $B(q, p)$ are small by comparison with $B(p, p)$ and $B(q, q)$ is small in comparison with $B(p, q)$ and $B(q, p)$. Therefore, one can expect to approximate reasonably (1.11) by replacing $QB(p + q)$ by $QB(p)$ (1). Also the relaxation time in (1.11) for the linear part of the equation is of the order of $(\nu\Lambda)^{-1} = (\nu\lambda_{m+1})^{-1}$ and is therefore

(1) Performing the same approximations in (1.10) i.e., replacing $PB(p + q)$ by $PB(p)$ leads to totally different difficulties which will not be contemplated in this article.

much smaller than the relaxation time in (1.10) for the linear part of this equation, $(\nu\lambda_1)^{-1}$. Hence it is reasonable to consider that the evolution in (1.11) is quasi-static and this leads us to replace (1.11) by the approximate equation

$$\nu A q + QB(p) = Qf \,. \tag{2.1}$$

For p given the resolution of (1.12) is straightforward ; we denote by $q = q_m$ its solution

$$q_m = \Phi_0(p) = (\nu A)^{-1}\,[Qf - QB(p)]\,. \tag{2.2}$$

The graph of the function $\Phi_0 : PH \to QH$ defines in H a smooth (analytic) manifold $\mathcal{M}_0$ of dimension m. Our task is now to show that all the solutions of (1.1) (or (1.10), (1.11)) are attracted by a thin neighborhood of $\mathcal{M}_0$. This will be proved in Section 2.1 ; for the moment we conclude Section 2.1 by establishing some a priori estimates on q_m similar to those on q : we recall that $u = p + q$ is a solution of (1.1) (or (1.10), (1.11)) whereas q_m is defined in terms of p by (2.2).

We infer from (2.2), (1.8) that

$$\begin{aligned} |\nu A q_m| &\leqslant |Qf| + |QB(p)| \\ &\leqslant |Qf| + c_4\|p\|^2\left(1 + \log \frac{|Ap|^2}{\lambda_1\|p\|^2}\right)^{1/2} \\ &\leqslant |Qf| + c_4 M_1^2 L^{1/2} \end{aligned}$$

$$|Aq_m| \leqslant \frac{1}{\nu}\,|Qf| + \frac{c_4}{\nu}\,M_1^2 L^{1/2}\,. \tag{2.3}$$

Hence

$$\begin{aligned} |q_m| &\leqslant \kappa_{0m}\,\delta L^{1/2} \\ \|q_m\| &\leqslant \kappa_{1m}\,\delta^{1/2} L^{1/2} \end{aligned} \tag{2.4}$$

$\kappa_{0m} = \kappa_{1m} = \dfrac{1}{\nu\lambda_1}\,(|Qf| + c_4 M_1^2)$. These bounds are precisely of the same order as the bounds (1.25) on q.

2.2. Estimates on the distance of the orbits to $\mathcal{M}_0$

While the orbit $u(t) = p(t) + q(t)$ lies anywhere in H, the associated orbit $u_m(t) = p(t) + q_m(t)$ lies on $\mathcal{M}_0$. Thus, at each time t,

$$\begin{aligned} \text{dist}\,(u(t), \mathcal{M}_0) &\leqslant \text{norm}\,(u_m(t) - u(t)) \\ &= \text{norm}\,(q_m(t) - q(t)) \end{aligned}$$

and evaluating the distance in H or V of $u(t)$ to $\mathcal{M}_0$ amounts to evaluate the norm in H or V of $\chi_m = q_m - q$. Substracting (1.11) from (2.1) (where $q = q_m$) we find

$$\nu A\chi_m = QB(p, q) + QB(q, p) + QB(q) + q' . \tag{2.5}$$

Hence, as we did for q_m, we write

$$|A\chi_m| \leqslant \frac{1}{\nu} \{|B(p, q)| + |B(q, p)| + |B(q)| + |q'|\} .$$

By utilization of (1.5), (1.8), (1.27) this yields, for t large :

$$\begin{aligned} |A\chi_m| &\leqslant \frac{c_4}{\nu} \|p\| L^{1/2} \|q\| + \frac{c_1}{\nu} |q|^{1/2} \|q\|^{1/2} \|p\|^{1/2} |Ap| \\ &\quad + \frac{c_1}{\nu} |q|^{1/2} \|q\| |Aq|^{1/2} + \kappa_0' L^{1/2} \delta \\ &\leqslant \frac{c_4 \kappa_1}{\nu} M_1 L\delta^{1/2} + \frac{c_1}{\nu} \|q\| \|p\| \\ &\quad + \frac{c_1}{\nu} (\kappa_0 \kappa_2)^{1/2} \kappa_1 L\delta + \kappa_0' L^{1/2} \delta \\ &\leqslant \kappa L\delta^{1/2} + \kappa L^{1/2} \delta^{1/2} + \kappa L\delta + \kappa L^{1/2} \delta \\ &\leqslant \kappa L\delta^{1/2} . \end{aligned} \tag{2.6}$$

Since $\chi_m \in QH$ and

$$|A^{-1/2}|_{\mathscr{L}(QH)} \leqslant \Lambda^{-1/2}, \ |A^{-1}|_{\mathscr{L}(QH)} \leqslant \Lambda^{-1} ,$$

we can write

$$\|\chi_m\| \leqslant \kappa L\delta , \quad |\chi_m| \leqslant \kappa L\delta^{3/2} \tag{2.7}$$

and with the methods of the Appendix

$$|\chi_m'| \leqslant \kappa L\delta^{3/2} . \tag{2.8}$$

All the bounds of the norms of χ_m are smaller than those on the corresponding norms of q_m and q by a factor $(L\delta)^{1/2}$. Hence for t large, an orbit $u(t)$ comes closer to $\mathcal{M}_0$ than to the flat space $q = 0$, by this factor $(L\delta)^{1/2}$.

We have proved the :

THEOREM 2.1 : *For t sufficiently large, $t \geqslant t_*$, any orbit of* (1.1) *remains at a distance in H of $P_m H$ of the order of $\kappa L^{1/2}\delta$ and at a distance in H of $\mathcal{M}_0$ of the order of $\kappa L\delta^{3/2}$. In the norm of V, the corresponding distances are* *of order* $\kappa\delta^{1/2} L^{1/2}$ *and* $\kappa L\delta$; *the constants* κ *depend on the data* ν, λ_1, $|f|$, *and* t_* *depends on these quantities and on* R_0, *when* $|u(0)| \leq R_0$.

3. A NONCONSTRUCTIVE RESULT

Our aim in this last section is to exhibit a manifold Σ which is Lipschitz, has finite dimension and captures the solutions of (1.1) in a much narrower neighborhood than $\mathcal{M}_0$ does. However, the existence of Σ is proved in a nonconstructive way, in contrast with the very simple and explicit equation (2.2) available for $\mathcal{M}_0$. Sections 3.1 and 3.2 provide preliminary results and Section 3.3 contains the main one.

3.1. Quotient of norms

We consider two solutions u, v of (1.1) and set $w = u - v$:

$$\frac{du}{dt} + \nu A u + B(u) = f\,, \quad u(0) = u_0\,, \tag{3.1}$$

$$\frac{dv}{dt} + \nu A v + B(v) = f\,, \quad v(0) = v_0\,, \tag{3.2}$$

$$\frac{dw}{dt} + \nu A w + B(u, w) + B(w, v) = 0\,. \tag{3.3}$$

Let σ denote the quotient of norms $\|w\|^2/|w|^2$; then

$$\begin{aligned}\frac{d\sigma}{dt} &= \frac{2((w', w))}{|w|^2} - \frac{2\,\|w\|^2}{|w|^4}(w', w) = \frac{2}{|w|^2}(w', Aw - \sigma w) \\ &= -\frac{2}{|w|^2}(\nu A w + B(u, w) + B(w, v), Aw - \sigma w)\,.\end{aligned}$$

Since $(Aw, Aw - \sigma w) = |Aw - \sigma w|^2$, we conclude, using (1.5), that

$$\begin{aligned}&\frac{d\sigma}{dt} + \frac{2\,\nu}{|w|^2}|Aw - \sigma w|^2 = \\ &\quad = -\frac{2}{|w|^2}(B(u, w) + B(w, v), Aw - \sigma w) \\ &\quad \leq \frac{2}{|w|^2}|Aw - \sigma w|(|B(u, w)| + |B(w, v)|) \\ &\quad \leq \frac{2\,c_1}{|w|^2}|Aw - \sigma w|^2 (|u|^{1/2}|Au|^{1/2}\|w\| + |w|^{1/2}|Aw|^{1/2}\|v\|) \\ &\quad \leq \frac{\nu}{|w|^2}|Aw - \sigma w|^2 + \frac{2\,c_1^2}{\nu}(|u|\,|Au| + \|v\|\,|Av|\,\lambda_1^{-1/2})\,\sigma\,.\end{aligned}$$

Hence

$$\frac{d\sigma}{dt} + \frac{\nu}{|w|^2} |Aw - \sigma w|^2 \leqslant \rho\sigma \tag{3.4}$$

where

$$\rho = \rho_u + \rho_v , \quad \rho_u = \frac{2 c_1^2}{\nu \lambda_1^{1/2}} \|u\| \, |Au| . \tag{3.5}$$

By integration of the differential inequality $\sigma' \leqslant \rho\sigma$, we find that for $t_1 < t < \tau < t_1 + T$

$$\frac{\|w(\tau)\|^2}{|w(\tau)|^2} \leqslant \frac{\|w(t)\|^2}{|w(t)|^2} \exp\left(\int_t^\tau \rho(s)\, ds \right) . \tag{3.6}$$

Now we estimate the integral of ρ in terms of the data ; as in (1.16) we assume that on the interval of time under consideration

$$\|u(t)\| \leqslant M_1 , \quad \|v(t)\| \leqslant M_1 . \tag{3.7}$$

With an appropriate value of M_1 (3.7) will be valid on some finite interval of time $[0, T]$, or on some interval of time (t_0, ∞), once the orbits have entered the absorbing set.

We have

$$\begin{aligned} \int_t^\tau \rho_u \, ds &\leqslant \frac{2 c_1}{\nu \lambda_1^{1/2}} \int_t^\tau \|u\| \, |Au| \, ds \\ &\leqslant \frac{2 c_1^2}{\nu \lambda_1^{1/2}} M_1 (\tau - t)^{1/2} \left(\int_t^\tau |Au|^2 \, ds \right)^{1/2} . \end{aligned}$$

An estimate on Au is obtained by taking the scalar product of (3.1) with Au in H :

$$\begin{aligned} \frac{d}{dt} \|u\|^2 + 2\nu\, |Au|^2 &= -2(B(u), Au) - 2(f, Au) \\ &\leqslant 2\, |B(u)|\, |Au| + 2\, |f|\, |Au| \\ &\leqslant \text{(with (1.5))} \\ &\leqslant 2 c_1\, |u|^{1/2}\, \|u\|\, |Au|^{1/2} + 2\, |f|\, |Au| \\ &\leqslant \nu\, |Au|^2 + \frac{c_1'}{\nu^3} |u|^2\, \|u\|^4 + \frac{2}{\nu} |f|^2 \end{aligned}$$

$$\frac{d}{dt} \|u\|^2 + \nu |Au|^2 \leqslant \frac{1}{\nu} |f|^2 + \frac{c_1'}{\nu^3 \lambda_1} M_1^6 . \tag{3.8}$$

Thus

$$\int_{t_1}^{t_1+\tau} |Au|^2 \, ds \leqslant \frac{\|u(t_1)\|^2}{\nu} + \frac{\tau}{\nu^2}\left(|f|^2 + \frac{c_1'}{\nu^2 \lambda_1} M_1^6\right)$$

$$\int_{t_1}^{t_1+\tau} |Au|^2 \, ds \leqslant \frac{M_1^2}{\nu} + \frac{\tau |f|^2}{\nu^2} + \frac{c_1' \tau}{\nu^4 \lambda_1} M_1^6 \tag{3.9}$$

and

$$\int_t^\tau \rho_u \, ds \leqslant \frac{1}{2}(\tau - t)^{1/2} \kappa_3$$

$$\kappa_3 = \frac{c_2'}{\nu \lambda_1^{1/2}} M_1 \left(\frac{M_1^2}{\nu} + \frac{T|f|^2}{\nu^2} + \frac{T M_1^6}{\nu^4 \lambda_1}\right)^{1/2}. \tag{3.10}$$

Since the estimates on v and ρ_v are the same, we have

$$\int_t^\tau \rho \, ds \leqslant (\tau - t)^{1/2} \kappa_3 . \tag{3.11}$$

3.2. The squeezing property

The squeezing property is an important property of the solutions of the Navier-Stokes equations which has been introduced in [7]. A stronger form of it, called the strong squeezing property or the cone property was proven in [5] for some other, more strongly dissipative equations. For the two dimensional Navier-Stokes equations, we derive here a form of the squeezing property sharper than in [7].

We take the scalar product of (3.3) with w in H and thanks to (1.13), (1.16) we find

$$\begin{aligned}
\frac{d}{dt}|w|^2 + 2\nu\|w\|^2 &= -2\,b(w, v, w) \\
&\leqslant 2\,c_2 |w|\,\|w\|\,\|v\| \\
&\leqslant \nu\|w\|^2 + \frac{c_2^2}{\nu}|w|^2\,\|v\|^2 \\
&\leqslant \nu\|w\|^2 + \frac{c_2^2}{\nu} M_1^2 |w|^2
\end{aligned}$$

$$\frac{d}{dt}|w|^2 + \left(\nu\frac{\|w\|^2}{|w|^2} - \frac{c_2^2}{\nu} M_1^2\right)|w|^2 \leqslant 0 . \tag{3.12}$$

We consider t_0, t, $0 < t < t_0 \leqslant T$ and write, using (3.6), (3.11)

$$\gamma_0 = \frac{\|w(t_0)\|^2}{|w(t_0)|^2} \leqslant \exp(\kappa_3(t_0 - t)^{1/2}) \frac{\|w(t)\|^2}{|w(t)|^2}. \tag{3.13}$$

Thus,

$$\frac{d}{dt}|w|^2 + \left(\nu\gamma_0 \exp(-\kappa_3 t_0^{1/2}) - \frac{c_2^2}{\nu} M_1^2\right)|w|^2 \leqslant 0 \tag{3.14}$$

and by integration

$$|w(t_0)|^2 \leqslant |w(0)|^2 \times \exp\left(-\nu\gamma_0 t_0 \exp(-\kappa_3 t_0^{1/2}) + \frac{c_2^2}{\nu} M_1^2 t_0\right). \tag{3.15}$$

Now if $|Q_m w(t_0)| > |P_m w(t_0)|$, we write

$$\gamma_0 = \frac{\|P_m w(t_0)\|^2 + \|Q_m w(t_0)\|^2}{|P_m w(t_0)|^2 + |Q_m w(t_0)|^2}$$
$$\geqslant \frac{\|Q_m w(t_0)\|^2}{2|Q_m w(t_0)|^2} \geqslant \frac{\lambda_{m+1}}{2}$$

and

$$\begin{gathered} |w(t_0)|^2 \leqslant |w(0)|^2 \exp(-\nu\lambda_{m+1}\kappa_5 t_0 + \kappa_4 t_0) \\ \kappa_4 = \frac{c_2^2}{\nu} M_1^2, \quad \kappa_5 = \frac{1}{2}\exp(-\kappa_3 t_0^{1/2}). \end{gathered} \tag{3.16}$$

Of course the interval $(0, t_0)$ can be replaced by any interval $(t_1, t_1 + t_0)$ on which the bound (3.7) is valid.

In conclusion (this is the *squeezing property*), whenever (3.7) is valid on some interval $(t_1, t_1 + t_0)$, then $w = u - v$ satisfies one of the following conditions :

$$|Q_m w(t_0 + t_1)| \leqslant |P_m w(t_0 + t_1)| \tag{3.17a}$$

or

$$|w(t_0 + t_1)|^2 \leqslant |w(t_1)|^2 \exp(-\nu\lambda_{m+1}\kappa_5 t_0 + \kappa_4). \tag{3.17b}$$

Since κ_4, κ_5 are independent of m, the exponential term in (3.17*b*) can be made arbitrarily small by choosing m sufficiently large ; we will take advantage of this remark in Section 3.3.

A slightly more explicit form of κ_4, κ_5 can be derived by using the Grashof number $G = |f|/\nu^2 \lambda_1$ and the Reynolds type number $R_n = M_1/\nu\lambda_1^{1/2}$. We find $(\tau = t_0)$:

$$\begin{aligned}(3.18)\quad \kappa_3 &= c_2' R_n(\nu\lambda_1)^{1/2} (R_n^2 + t_0 \nu\lambda_1 G^2 + t_0 \nu\lambda_1 R_n^6)^{1/2}\\ \kappa_4 &= c_2^2 R_n^2(\nu\lambda_1)\\ \kappa_5 &= \frac{1}{2}\exp((-c_2' R_n(\nu\lambda_1 t_0))^{1/2} (R_n^2 + t_0 \nu\lambda_1 G^2 + t_0 \nu\lambda_1 R_n^6)^{1/2}).\end{aligned}$$

In the space periodic case we have seen that, for large times, we can take $M_1 = (2|f|\ G)^{1/2}$. Then $R_n = \sqrt{2}\,G$ and the above quantities become

$$\begin{aligned}(3.19)\quad \kappa_3 &= c_3'(\nu\lambda_1)^{1/2} (G^4 + t_0 \nu\lambda_1 G^4 + t_0 \nu\lambda_1 G^8)^{1/2}\\ \kappa_4 &= 2\,c_2^2(\nu\lambda_1)\,G^2\\ \kappa_5 &= \frac{1}{2}\exp(-c_3'(\nu\lambda_1 t_0)^{1/2} (G^4 + t_0 \nu\lambda_1 G^4 + t_0 \nu\lambda_1 G^8)^{1/2}).\end{aligned}$$

3.3. The approximate manifold

We denote by $S(t)$, $t > 0$ the operator in H: $u_0 \to u(t)$, where $u(\cdot)$ is the unique solution of (1.1) satisfying $u(0) = u_0$. The operators $S(t)$, $t \geqslant 0$, form a semigroup in H.

The squeezing property tells us that if $u(\cdot)$, $v(\cdot)$ are two solutions of (1.1) lying in the ball $\{\phi \in v,\ \|\phi\| \leqslant M_1\}$, for $0 \leqslant t \leqslant T$, then at each time $t \in [0, T]$ and for every $m \in \mathbb{N}$, we have either

$$|Q_m(S(t)\,u_0 - S(t)\,v_0)| \leqslant |P_m(S(t)\,u_0 - S(t)\,v_0)|$$

or

$$|S(t)\,u_0 - S(t)\,v_0| \leqslant |u_0 - v_0| \exp\frac{1}{2}(-\nu\lambda_{m+1}\kappa_5 t_0 + \kappa_4 t_0)$$

κ_4, κ_5 as above.

Now we choose $t_0 \in [0, T]$, $m \in \mathbb{N}$, and consider a subset $\Sigma = \Sigma(m)$ of

$$S(t_0)\{u_0 \in V,\ \|u_0\| \leqslant M_1\}$$

which is maximal under the property

$$(3.20)\qquad |Q_m(u - v)| \leqslant |P_m(u - v)|\,.$$

By this we mean that if $u \in \Sigma(m)$ then

$$\{v \in V,\ v \text{ satisfies (3.20)}\} \subset \Sigma(m)\,.$$

Showing the existence of such a maximal set is easy.

We then apply the squeezing property : whenever $\|u(s)\| \leq M_1$, we see that $S(t_0)\,u(s) = u(t_0+s)$ either belongs to $\Sigma(m)$ i.e.,

$$|Q_m(S(t_0)\,u(s) - S(t_0)\,\phi)| \leq |P_m(S(t_0)\,u(s) - S(t_0)\,\phi)| \,,$$

for some $\phi \in V$ such that $\|\phi\| \leq M_1$ and $S(t_0)\,\phi \in \Sigma(m)$ or, if not, then for every such ϕ

$$\begin{aligned} |S(t_0)\,u(s) - S(t_0)\,\phi|^2 &\leq |u(s) - \phi|^2 \exp(-\nu\lambda_{m+1}\,\kappa_5\,t_0 + \kappa_4\,t_0) \\ &\leq \frac{4\,M_1^2}{\lambda_1} \exp(-\nu\lambda_{m+1}\,\kappa_5\,t_0 + \kappa_4\,t_0)\,. \end{aligned}$$

In all cases the distance of $S(t_0)\,u(s)$ to $\Sigma(m)$ is bounded by

$$\frac{2\,M_1}{\lambda_1^{1/2}} \exp\left(\frac{t_0}{2}\,(\kappa_4 - \nu\lambda_{m+1}\,\kappa_5)\right)\,.$$

We can choose $t_0 = (\nu\lambda_1)^{-1}$ and the bound becomes

$$\frac{2\,M_1}{\lambda_1^{1/2}} \exp\left(-\frac{\kappa_5}{4}\frac{\lambda_{m+1}}{\lambda_1}\right)$$

provided that

$$\frac{\lambda_{m+1}}{\lambda_1} \geq \frac{2\,\kappa_4}{\kappa_5\,\nu\lambda_1}\,. \tag{3.21}$$

By translation in time $(t \to t - t_*)$, we conclude that once the orbit u has entered the absorbing set $\{\|\phi\| \leq M_1\}$, which happens for $t \geq t_* = t_*(R_0)$ (for $|u(0)| \leq R_0$), the distance of $S(t)\,u_0$ to $\Sigma(m)$ is bounded by a given quantity E,

$$\operatorname{dist}_H\,(S(t)\,u_0,\,\Sigma(m)) \leq E \tag{3.22}$$

provided $t \geq t_* + (\nu\lambda_1)^{-1}$, and

$$\exp\left(-\frac{\kappa_5}{4}\frac{\lambda_{m+1}}{\lambda_1}\right) \leq E$$

i.e.,

$$\frac{\lambda_{m+1}}{\lambda_1} \geq -\frac{4}{\kappa_5}\log E\,. \tag{3.23}$$

By definition the set $\Sigma(m)$ enjoys the property that

$$|Q_m(u - v)| \leq |P_m(u - v)|\,, \quad \forall u, v \in \Sigma(m)\,.$$

Hence, $\Sigma(m)$ is the graph of a Lipschitz function

$$\psi : P_m \Sigma(m) \to Q_m H$$

$$|\Psi(P_m u) - \Psi(P_m v)| \leq |P_m u - P_m v| , \quad \forall P_m u, P_m v \in P_m \Sigma(m) .$$

By the Kirszbaum extension Theorem [16] Ψ can be extended as a Lipschitz function (with the same constant) from $P_m H$ into $Q_m H$, that we still denote by Ψ. Now Ψ is defined from $P_m H$ into $Q_m H$, and its graph is a Lipschitz manifold above all of $P_m H$.

In conclusion we have proved the following theorem

THEOREM 3.1 : *If m is sufficiently large so that* (3.21) *is satisfied* ([1]) *then there exists a Lipschitz manifold $\Sigma(m)$ of dimension m, which enjoys the following property : for a solution $u(\cdot)$ of* (1.1), *for t sufficiently large* ($t \geq t_{**}(R_0, \nu, f, \Omega)$, *for* $|u_0| \leq R_0$), *the distance in H of $u(t)$ to $\Sigma(m)$ is majorized by*

$$\frac{2 M_1}{\lambda_1^{1/2}} \exp\left(- \frac{\kappa_5}{4} \frac{\lambda_{m+1}}{\lambda_1} \right) .$$

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APPENDIX

ESTIMATES IN THE COMPLEX TIME PLANE

It was proved in [7] (see also [13]) that the solutions to the Navier-Stokes equations are analytic in time ; we want to show how one can then use Cauchy's formula to get a priori estimates on the time derivatives of the solutions. The main point in the proof is to determine the width of the band

([1]) κ_4, κ_5 as above with $t_0 = (\nu\lambda_1)^{-1}$, and M_1 the radius of an absorbing set in V for (1.1).

of analyticity of the solution around the real axis $\mathbb{R}_+$; this will follow as in [7, 13] from a priori estimates on the solution in the complex plan.

The complex time is denoted $\zeta = s\, e^{i\theta}$; $\mathbb{H}$, $\mathbb{V}$, $\mathbb{D}(A)$ are the complexified spaces of H, V, $D(A)$; A, B are respectively extended as linear and bilinear operators from $\mathbb{D}(A)$ into $\mathbb{H}$:

$$A(u_1 + iu_2) = Au_1 + Au_2 , \tag{A.1}$$

$$B(u_1 + iu_2, v_1 + iv_2) = B(u_1, v_1) - B(u_2, v_2) + i\,[B(u_2, v_1) + B(u_1, v_2)] \tag{A.2}$$

$\forall u = u_1 + iu_2, v = v_1 + iv_2 \in \mathbb{D}(A)$. The Navier-Stokes equation (1.1) becomes $(u = u(\zeta))$:

$$\frac{du}{d\zeta} + \nu Au + B(u) = f \tag{A.3}$$

$$u(0) = u_0 . \tag{A.4}$$

Assuming that $u_0 \in V$ (or $\mathbb{V}$), then $u\big|_{\mathbb{R}_+} \in L^\infty(\mathbb{R}_+ ; \mathbb{V})$ as in (1.14), we denote by M_0, M_1, the supremum of $|u(t)|$ and $\|u(t)\|$, $t \in \mathbb{R}_+$. We take the scalar product in $\mathbb{H}$ of (A.3) with Au ; we multiply the resulting equation by $e^{i\theta}$ and takes its real part. This yields

$$\begin{aligned} \frac{1}{2}\frac{d}{ds}\, \|u(s\, e^{i\theta})\|^2 + \nu \cos\theta\, |Au(s\, e^{i\theta})|^2 &= \\ &= -\operatorname{Re} e^{i\theta}(B(u), Au) - \operatorname{Re} e^{i\theta}(f, Au) \\ &\leqslant |(B(u), Au)| + |f|\,|Au| . \end{aligned} \tag{A.5}$$

We expand by bilinearity (using (A.2)) and bound the resulting expressions with the help of (1.8) :

$$|(B(u), Au)| \leqslant c\,\|u\|^2 \left(1 + \log \frac{|Au|^2}{\lambda_1 \|u\|^2}\right)^{1/2} |Au| .$$

Also

$$|f|\,|Au| \leqslant \frac{\nu\cos\theta}{2}\, |Au|^2 + \frac{|f|^2}{2\,\nu\cos\theta} .$$

Hence (with $u = u(s\, e^{i\theta})$) :

$$\begin{aligned} \frac{d}{ds}\, \|u\|^2 + \nu\cos\theta\, |Au|^2 &\leqslant \\ &\leqslant \frac{|f|^2}{\nu\cos\theta} + c_5 \|u\|^2\, |Au| \left(1 + \log \frac{|Au|^2}{\lambda_1 \|u\|^2}\right)^{1/2} \end{aligned} \tag{A.6}$$

We write $z = \dfrac{|Au|}{\lambda_1^{1/2}\|u\|} \geq 1$ and consider the function

$$z \to \phi(z) = -\frac{\lambda_1 \nu \cos\theta}{2} z^2 + c_5 \|u\| \lambda^{1/2} z (1 + \log z^2)^{1/2}.$$

By elementary computations ([1])

$$\text{(A.7)} \qquad \phi(z) \leq \frac{c_5^2 \|u\|^2}{2 \nu \cos\theta} \log\left(\frac{4 c_5^2 \|u\|^2}{\lambda_1 \nu^2 \cos^2\theta} \right), \quad \text{for} \quad z \geq 1,$$

and (A.6) yields

$$\text{(A.8)} \quad \frac{d}{ds} \|u\|^2 + \frac{\nu \cos\theta}{2} |Au|^2 \leq$$

$$\leq \frac{|f|^2}{\nu \cos\theta} + \frac{c_5^2}{2 \nu \cos\theta} \|u\|^4 \left(\log \frac{4 c_5^2 \|u\|^2}{\lambda_1 \nu^2 \cos^2\theta} \right).$$

Setting $y(s) = \dfrac{(1 + 4 c_5^2)}{\lambda_1 \nu^2 \cos^2\theta} (|f| + \|u(s e^{i\theta})\|^2)$ we infer from (A.8) that

$$\frac{dy}{ds} \leq c_1' \lambda_1 \nu \cos\theta y^2 \log y,$$

where c_1' is an appropriate nondimensional constant. As long as $y(s) \leq 2 y_0 = 2 y(0)$, we have

$$y' \leq c_1' \lambda_1 \nu \cos\theta y^2 \log(2 y_0)$$

$$y(s) \leq \frac{y_0}{1 - c_1' \lambda_1 \nu \cos\theta \log(2 y_0) s}$$

([1]) Looking for the maximum of $-\alpha^2 z^2 + \beta^2(1 + \log z^2)$, we find

$$\beta^2(1 + \log z^2) \leq \alpha^2 z^2 + \beta^2 \log \frac{\beta^2}{\alpha^2}$$

$$z\beta(1 + \log z^2)^{1/2} \leq \alpha z^2 + \beta z \left(\log \frac{\beta^2}{\alpha^2} \right)^{1/2}$$

$$\leq 2 \alpha z^2 + \frac{1}{4} \frac{\beta^2}{\alpha} \left(\log \frac{\beta^2}{\alpha^2} \right).$$

We then choose $\alpha = \dfrac{\lambda_1 \nu \cos\theta}{4}$, $\beta = c_5 \|u\| \lambda_1^{1/2}$.

and this is indeed $\leqslant 2\, y_0$ as long as $s \leqslant T_*$:

$$T_* = \frac{3}{2\, c_1'\, \lambda_1\, \nu \cos\theta\, y_0 \log(2\, y_0)}\,.$$

For $\|u_0\| \leqslant M_1$, we replace T_* by

(A.9) $T_*(M_1) =$

$$= \frac{3}{2\, c_1'\, \lambda_1\, \nu \cos\theta \left(\dfrac{G}{\cos^2\theta} + \dfrac{M_1^2}{\lambda_1\, \nu^2 \cos^2\theta}\right) \log 2 \left(\dfrac{G}{\cos^2\theta} + \dfrac{M_1^2}{\lambda_1\, \nu^2 \cos^2\theta}\right)}\,.$$

Thus

$$\|u(s\, e^{i\theta})\|^2 \leqslant 2(|f| + \|u_0\|^2) \leqslant 2(|f| + M_1^2) \tag{A.10}$$

for

$$0 \leqslant s \leqslant \frac{3\cos\theta}{2\, c_1'\, \lambda_1\, \nu \left(G + \dfrac{M_1^2}{\lambda_1\, \nu^2}\right) + \log 2 \left(\dfrac{G}{\cos^2\theta} + \dfrac{M_1^2}{\lambda_1\, \nu^2 \cos^2\theta}\right)}$$

and in particular for

$$0 \leqslant s \leqslant \frac{3\cos\theta}{2\, c_1'\, \lambda_1\, \nu \left(G + \dfrac{M_1^2}{\lambda_1\, \nu^2}\right) + \log 4 \left(G + \dfrac{M_1^2}{\lambda_1\, \nu^2}\right)} \tag{A.11}$$

when $\cos^2\theta \geqslant \dfrac{1}{2}$.

Following the method developed in [7] we conclude that the solution u of (A.3) (or (1.1)) is analytic in the region

$$\Delta(u_0) = \left\{ s\, e^{i\theta},\ s \leqslant \alpha \cos\theta,\ \cos\theta \geqslant \frac{\sqrt{2}}{2} \right\} \tag{A.12}$$

$$\alpha = \frac{3}{2\, c_1'\, \lambda_1\, \nu \left(G + \dfrac{M_1^2}{\lambda_1\, \nu^2}\right) + \log 4 \left(G + \dfrac{M_1^2}{\lambda_1\, \nu^2}\right)}$$

which comprises the regions

$$|\operatorname{Im}\zeta| \leqslant \operatorname{Re}\zeta\,, \quad 0 < \operatorname{Re}\zeta \leqslant \frac{\alpha}{2}$$

and

$$|\operatorname{Im}\zeta| \leqslant \frac{\alpha}{2}\,, \quad \operatorname{Re}\zeta \geqslant \frac{\alpha}{2}\,. \tag{A.13}$$

At any point $t \in \mathbb{R}_+$, $t \geqslant \alpha$, we can apply Cauchy's formula to the circle Γ centered at t of radius $\alpha/4$:

$$\frac{d^k u(t)}{dt^k} = \frac{k!}{2\pi i} \int_\Gamma \frac{u(\zeta)}{(t-\zeta)^{k+1}} d\zeta . \tag{A.14}$$

Thus,

$$\sup_{t \geqslant \alpha} \left| \frac{d^k u(t)}{dt^k} \right| \leqslant \frac{4^k}{\alpha^k} k! \, M_0 \tag{A.15}$$

$$\sup_{t \geqslant \alpha} \left\| \frac{d^k u}{dt^k}(t) \right\| \leqslant \frac{4^k}{\alpha^k} k! \, M_1 . \tag{A.16}$$

Explicit values of M_0 and M_1 were derived in (1.36) for the two dimensional space periodic case : $M_1 = (2|f|\,G)^{1/2}$ $(t \geqslant t_2)$. This yields (assuming $G \geqslant 1$):

$$\alpha = \frac{2}{2 c_1' \lambda_1 \nu (G + 2G^2) \log 4 (G + 2G^2)}$$

$$\alpha \geqslant \frac{c_2'}{\lambda_1 \nu G^2 \log G} \tag{A.17}$$

and we deduce from (A.15), (A.16) that for t sufficiently large ([1])

$$\begin{aligned} \left| \frac{d^k u(t)}{dt^k} \right| &\leqslant c \frac{|f|^{1/2}}{\lambda_1^{1/2}} (|f|\,\lambda_1)^{k/2} (G^2 \log G)^k \\ \left\| \frac{d^k u(t)}{dt^k} \right\| &\leqslant c |f|^{1/2} (|f|\,\lambda_1)^{k/2} (G^2 \log G)^k . \end{aligned} \tag{A.18}$$

In particular ($k = 1$):

$$\begin{aligned} \left| \frac{du(t)}{dt} \right| &\leqslant c |f| \, G^2 \log G \\ \left\| \frac{du(t)}{dt} \right\| &\leqslant c |f| \, \lambda_1^{1/2} G^2 \log G , \quad t \geqslant T_* . \end{aligned} \tag{A.19}$$

This produces an interesting bound on $|Au(t)|$ for t large :

$$\nu A u = f - B(u) - u'$$

$$|Au| \leqslant \frac{1}{\nu} |f| + \frac{c_1}{\nu} |u|^{1/2} \|u\| \, |Au|^{1/2} + \frac{1}{\nu} |u'|$$

$$|Au| \leqslant \frac{2}{\nu} |f| + \frac{c_1^2}{\nu^2} |u| \, \|u\|^2 + \frac{2}{\nu} |u'|$$

([1]) This means as in Theorem 1.1 and elsewhere $t \geqslant T_*(R_0, \nu, \lambda_1, |f|)$, for $|u_0| \leqslant R_0$.

$$\leqslant c(|f|\lambda_1)^{1/2}(G^{1/2}+G+G^{5/2}\log G)$$

$$|Au(t)|\leqslant c(|f|\lambda_1)^{1/2}G^{5/2}\log G, \quad \text{for} \quad t\geqslant T_* . \tag{A.20}$$

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