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SUPERCONVERGENCE OF THE GRADIENT OF GALERKIN APPROXIMATIONS FOR ELLIPTIC PROBLEMS (*)

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Dedicated to Prof. Seiiti Huzino on the occasion of his 60th birthday.

Abstract. — *We consider superconvergence for an averaged gradient in a Garlerkin method, for the Dirichlet problem on a square, based on tensor products of continuous piecewise polynomial spaces. We prove that, when we use odd degree polynomials, the approximation by averaging yields superconvergence of order* h^{r+1} *in* L^{∞}. *The theoretical results are illustrated by numerical examples.*

Résumé. — *Nous considérons la superconvergence pour un gradient moyen de la méthode de Galerkin, pour les problèmes de Dirichlet sur un carré, basée sur les produits tensoriels des espaces de fonctions continues, polynômes par morceaux. On montre que, lorsque nous employons les polynômes de degré impair, l'approximation par la technique moyenne donne la superconvergence d'ordre* h^{r+1} *en norme* L^{∞}. *Les résultats théoriques sont accompagnés d'exemples numériques.*

1. INTRODUCTION

It is known that, when we construct a Galerkin approximation for the boundary value problem using piecewise polynomials, various superconvergence phenomena are observed at certain specific points in the domain ([2] ~ [14], [18]). Particularly, superconvergence properties for the derivative of the approximate solution are considered in [3, 4, 8, 9, 10, 13, 14, 18]. In these studies, Krizek & Neittaanmäki [8], using the result in [15], presented a theoretical result for the technique of averaging gradients at the mesh points which improves the accuracy of the derivative of the Galerkin finite element solution using linear triangular elements. That is, they proved that

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the new approximation obtained by the averaging technique is a superconvergent approximation to the exact gradient in the L^2-norm sense. Also Levine [10] proposed another averaging method which admits superconvergence estimates in the mean-square sense of the gradient at the midpoint of element edges for linear finite element methods.

In this paper, we consider a similar problem described in [8] for a Galerkin method to the elliptic equations on the unit square with Dirichlet boundary condition based on tensor products of continuous piecewise polynomial spaces. We attempt to improve upon the estimates derived in [8] and generalize the results to the case of higher order elements. The main result of the paper is that the superconvergence phenomenon of the gradient occurs, rather surprisingly, only in case of using odd degree polynomials.

In the following section, we present the elliptic boundary value problem and some notation to be used in later sections, and then define the Galerkin approximation. In § 3, first we show that, in the one dimensional case with the use of odd degree piecewise polynomials, the average values of the left and right limits of the approximate derivatives at the internal mesh points are superconvergent. On the other hand we prove that the global convergence rate of the gradient of the difference between the one dimensional projection of the exact solution and the Galerkin approximation is one order higher than the optimal rate. Next, we describe, in § 4, an a posteriori method to obtain the global superconvergence approximation utilizing the results in the previous section and the superconvergence estimates at Gauss points. Finally, in § 5, we illustrate some numerical examples which confirm the superconvergence properties derived in § 3 and § 4. We also present a counterexample which shows that the averaging technique does not yield superconvergence in the case of even degree polynomials.

2. THE ELLIPTIC PROBLEM AND THE GALERKIN METHOD

Consider the following elliptic boundary value problem on a rectangular domain $\Omega = (0,1) \times (0,1)$ in R^2.

$$\begin{cases} Lu \equiv -\Delta u + bu = f & \text{in } \Omega, \\ \qquad\qquad\quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We assume that $b(x,y)$ is in $L^\infty(\Omega)$ and nonnegative. Then for each $f \in L^p(\Omega)$, $1 < p < \infty$, (2.1) has a unique solution $u \in W_p^2(\Omega)$. Here $W_p^m(\Omega)$ denotes the usual L^p-Sobolev space of order m on Ω.

Now, in order to define the Galerkin approximation to (2.1), we introduce the approximation spaces. Let $r \geqq 1$ be a fixed integer and $I = (0,1)$. For each set $E \subset I$, $P_r(E)$ denotes the set of polynomials of degree at most r on E. Let $\delta_x : 0 = x_0 < x_1 < \cdots < x_N = 1$ and δ_y : $0 = y_0 < y_1 < \cdots < y_M = 1$ be quasiuniform partitions of I. For simplicity we set $\delta_x = \delta_y$. Let $I_i = (x_{i-1}, x_i)$, $h_i = x_i - x_{i-1}$ and $h = \max_{1 \leqq i \leqq N} h_i$. Also set $\mathscr{R} = \{I_i \times I_j ; 1 \leqq i, j \leqq N\}$. Further let

$$\mathscr{M}_0^r(\delta_x) = \left\{v \in C(I) ; v|_{I_i} \in P_r(I_i), \quad 1 \leqq i \leqq N, \quad v(0) = v(1) = 0\right\}$$

and

$$P_r^0(I_i) = \{v \in P_r(I_i) ; v(x_{i-1}) = v(x_i) = 0\} .$$

We now define the partition of Ω by $\delta = \delta_x \otimes \delta_y$ and let

$$\mathscr{M}(\delta) = \mathscr{M}_0^r(\delta_x) \otimes \mathscr{M}_0^r(\delta_y) .$$

Here δ, δ_x and δ_y will be usually suppressed. Then we define the Galerkin approximation $u \in \mathscr{M}$ to (2.1) by

$$B(U, v) = (f, v) , \qquad v \in \mathscr{M} , \tag{2.2}$$

where

$$B(U, v) \equiv (\nabla U, \nabla v) + (bU, v)$$

and $(.,.)$ is the L^2-inner product on Ω. From now on, for any domain A in R^1 or R^2, $(.,.)_A$ will denote the L^2-inner product on A. Also denote the usual L^p-Sobolev space of order m on A by $W_p^m(A)$ or $W_{p,0}^m(A)$ for $1 \leqq p \leqq \infty$ and an integer $m \geqq 0$. Particularly for $p = 2$, by convention, $W_2^m(A)$ or $W_{2,0}^m(A)$ are written as $H^m(A)$ or $H_0^m(A)$, respectively. Further, we adopt the usual Sobolev norm as the norm in $W_p^m(A)$ and, when $A = \Omega$, $\|\cdot\|_{W_p^m(\Omega)}$ is simply denoted by $\|\cdot\|_{W_p^m}$.

Now we give the definition of the Gauss points. First, r Gauss points on I : $0 < \tau_1 < \cdots < \tau_r < 1$ are the roots of the following Jacobi polynomial.

$$J_r(x) \equiv \frac{1}{c} \cdot \frac{d^r}{dx^r} [x^r(1 - x)^r] , \tag{2.3}$$

where c is a constant chosen so the coefficient of x^r in (2.3) is 1. Next, for each $1 \leqq i \leqq N$, the Gauss points x_{ik} on I_i are defined as the affine transformation of τ_k to I_i :

$$x_{ik} = x_{i-1} + \tau_k h_i , \qquad 1 \leqq k \leqq r . \tag{2.4}$$

Then for each subrectangle $\rho = I_i \times I_j \in \mathscr{R}$, the Gauss points on ρ are the set of all points of the form $(x_{ik}, y_{j\ell})$, $1 \leqq k, \ell \leqq r$. Hereafter, we use the symbol C to denote a generic positive constant independent of h and not necessarily the same at any two places.

3. SUPERCONVERGENCE AT INTERNAL MESH POINTS

For any $g \in H_0^1(I)$, we define a projection $Pg \in \mathcal{M}_0^r$ by

$$(g' - (Pg)', v')_I = 0 , \quad v \in \mathcal{M}_0^r . \tag{3.1}$$

Then, notice that the following property holds [6]

$$(Pg)(x_i) = g(x_i) , \quad 0 \leqq i \leqq N . \tag{3.2}$$

For each i, $1 \leqq i \leqq N - 1$, set $I_i^* \equiv (x_{i-1}, x_{i+1}) = I_i \cup I_{i+1} \cup \{x_i\}$. We now define for a function ψ which is smooth on I_i^* except at x_i,

$$\tilde{D}_i \psi = \frac{1}{1 + \alpha_i^r} \{\psi'(x_i -) + \alpha_i^r \psi'(x_i +)\} ,$$

where $\alpha_i = h_i / h_{i+1}$.

The following estimates play an essential role in the superconvergence results in this paper.

THEOREM 1 : *If r is odd and $g \in W_\infty^{r+2}(I_i^*)$, $1 \leqq i \leqq N - 1$, then*

$$\left| g'(x_i) - \tilde{D}_i(Pg) \right| \leqq C \bar{h}_i^{r+1} \| g \|_{W_\infty^{r+2}(I_i^*)} ,$$

where $\bar{h}_i = \max (h_i, h_{i+1})$.

Proof: First, define a linear functional ℓ_i on $W_\infty^{r+2}(I_i^*)$ by

$$\ell_i(\phi) \equiv \phi'(x_i) - \tilde{D}_i(Q_r \phi) , \tag{3.3}$$

where $Q_r \phi \in P_r(I_i) \cap P_r(I_{i+1})$ is determined by

$$\begin{cases} (\phi' - (Q_r \phi)', v')_{I_j} = 0 , & v \in P_r^0(I_j) , \quad j = i , \quad i + 1 , \\ (Q_r \phi)(x_j) = \phi(x_j) , & j = i - 1 , \quad i, \quad i + 1 . \end{cases} \tag{3.4}$$

Notice that, from (3.2), $P\phi = Q_r \phi$ on I_i^* for any $\phi \in H_0^1(I)$.

We now fix $\phi \in P_{r+1}(I_i^*)$. Then, clearly $\phi = Q_{r+1} \phi$.
For any $w \in P_{r-1}(I_j)$, $j = i, i + 1$, let

$$v = \int_{x_{j-1}}^{x} w \, dx - \frac{x - x_{j-1}}{h_j} \int_{I_j} w \, dx .$$

Noting that $v \in P_r^0(I_j)$ and $v' = w - \text{const.}$, we have by (3.4)

$$(\phi' - (Q_r \phi)', w)_{I_j} = 0 .$$

This implies that $(\phi' - (Q_r \phi)')(x) = K_j J_r(h_j^{-1}(x - x_{j-1}))$ for $x \in I_j$, where J_r is Jacobi polynomial defined by (2.3) and K_j is some constant. Therefore, we have for each Gauss point on I_i^*

$$\begin{cases} (\phi' - (Q_r \phi)')(x_{ik}) = 0 , & 1 \leqq k \leqq r , \\ (\phi' - (Q_r \phi)')(x_{i+1,k}) = 0 , & 1 \leqq k \leqq r , \end{cases} \tag{3.5}$$

where x_{ik} and $x_{i+1,k}$ are defined by (2.4). Hence, we have the representations

$$\begin{cases} \phi'(x) = a_i(x - x_{i1}) \times \cdots \times (x - x_{ir}) + (Q_r \phi)'(x) , \quad x \in I_i , \\ \phi'(x) = a_{i+1}(x - x_{i+1,1}) \times \cdots \times (x - x_{i+1,r}) \\ \qquad + (Q_r \phi)'(x) , \quad x \in I_{i+1} , \end{cases} \tag{3.6}$$

where a_i and a_{i+1} are constants. However, $\phi'(x)$ must be a single polynomial throughout I_i^*. Thus, we have $a_i = a_{i+1}$ and, by (2.4)

$$\begin{aligned} \phi'(x_i -) &= a_i(x_i - x_{i1}) \times \cdots \times (x_i - x_{ir}) + (Q_r \phi)'(x_i -) \\ &= a_i h_i^r(1 - \tau_1) \times \cdots \times (1 - \tau_r) + (Q_r \phi)'(x_i -) . \end{aligned} \tag{3.7}$$

Furthermore,

$$\begin{aligned} \phi'(x_i +) &= a_i(x_i - x_{i+1,1}) \times \cdots \times (x_i - x_{i+1,r}) + (Q_r \phi)'(x_i +) \\ &= (-1)^r a_i h_{i+1}^r \tau_1 \times \cdots \times \tau_r + (Q_r \phi)'(x_i +) . \end{aligned} \tag{3.8}$$

Notice that, from the property of the roots of Jacobi polynomial (2.3), $\tau_1 \times \cdots \times \tau_r = (1 - \tau_1) \times \cdots \times (1 - \tau_r)$. Therefore, if r is odd and $a_i \neq 0$, then by (3.7) and (3.8) we obtain

$$\begin{aligned} \phi'(x_i) &= \frac{1}{h_i^r + h_{i+1}^r} \{h_{i+1}^r (Q_r \phi)'(x_i -) + h_i^r (Q_r \phi)'(x_i +)\} \\ &= \frac{1}{1 + \alpha_i^r} \{(Q_r \phi)'(x_i -) + \alpha_i^r (Q_r \phi)'(x_i +)\} \\ &= \tilde{D}_i(Q_r \phi) . \end{aligned}$$

If $a_i = 0$ then clearly $Q_r \phi = \phi$ which yields $\phi'(x_i) = \tilde{D}_i(Q_r \phi)$. Consequently, for any $\phi \in P_{r+1}(I_i^*)$, we have $\ell_i(\phi) = 0$. Thus, applying the Peano kernel theorem or the Bramble-Hilbert Lemma, we can easily obtain the desired estimates.

Now, for a function $\phi \in H_0^1(\Omega)$ we define a projection $P_x \phi \in \mathscr{M}_0^r(\delta_x)$ by $P_x u = P\phi(.,y)$ for each fixed $y \in I$. $P_y u \in \mathscr{M}_0^r(\delta_y)$ is similarly defined. Then, clearly, we have $P_y P_x \phi = P_x P_y \phi \in \mathscr{M}(\delta)$. The following lemma is obtainable using the results in [16] and [17]. However, we would like to present a complete proof, in order that the argument be self-contained and since the estimates can be derived by a considerably simpler technique in the present case.

LEMMA 1 : *Let u and U be solutions to* (2.1) *and* (2.2), *respectively. If $u \in W_p^{r+2}(\Omega)$, $2 \leqq p < \infty$, then, for sufficiently small h,*

$$\|u - U\|_{L^p} \leqq Ch^{r+1}\|u\|_{W_p^{r+2}}.$$

Proof: By the triangle inequality, we have

$$\|u - U\|_{L^p} \leqq \|u - P_y P_x u\|_{L^p} + \|P_y P_x u - U\|_{L^p},$$

and the first term of the right hand side can be estimated from well-known results in the one dimensional case ; that is,

$$\|u - P_y P_x u\|_{L^p} \leqq Ch^{r+1}\|u\|_{W_p^{r+1}}. \tag{3.9}$$

We now estimates the second term. Set $\eta = P_y P_x u - U$. For each q with $1 < q \leqq 2$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, and for any $\psi \in L^q(\Omega)$, consider the solution $\phi \in W_q^2(\Omega)$ of the following auxiliary problem :

$$\begin{cases} L\phi = \psi & \text{in} \quad \Omega\,, \\ \phi = 0 & \text{on} \quad \partial\Omega\,. \end{cases} \tag{3.10}$$

Then for any $\hat{\phi} \in \mathscr{M}$, we have by (2.2)

$$\begin{aligned}(\eta, \psi) &= (\eta, L\phi) \\ &= B(\eta, \phi) \\ &= B(\eta, \phi - \hat{\phi}) + B(\eta, \hat{\phi}) \\ &= B(\eta, \phi - \hat{\phi}) + B(P_y P_x u - u, \hat{\phi})\,. \end{aligned} \tag{3.11}$$

Let $\hat{\phi}$ be the solution of

$$B(v, \phi - \hat{\phi}) = 0\,, \qquad v \in \mathscr{M}\,. \tag{3.12}$$

Then, by (3.11) and (3.1)

$$\begin{aligned}(\eta, \psi) &= (P_y u_x - u_x, \hat{\phi}_x) + (P_x u_y - u_y, \hat{\phi}_y) + (P_y P_x u - u, b\hat{\phi}) \\ &\leqq C(\|P_y u_x - u_x\|_{L^p} + \|P_x u_y - u_y\|_{L^p} \\ &\quad + \|P_y P_x u - u\|_{L^p})\|\hat{\phi}\|_{W_q^1}. \end{aligned} \tag{3.13}$$

Further, by virtue of well-known estimates for the solution of (3.12) and elliptic regularity related to the equation (3.10), we obtain, for sufficiently small h,

$$\begin{aligned}\|\hat{\phi}\|_{W_q^1} \leqq \|\hat{\phi}\|_{H^1} &\leqq C\|\phi\|_{H^1}\\ &\leqq C\|\phi\|_{W_q^2}\\ &\leqq C\|\psi\|_{L^q},\end{aligned}$$

where we have used the Sobolev estimate $\|\phi\|_{H^1} \leqq C\|\phi\|_{W_q^2}$. Combining this with (3.13) and (3.9), we have

$$\|\eta\|_{L^p} \leqq Ch^{r+1}\|u\|_{W_p^{r+2}},$$

which completes the proof.

Next, we show that the gradient of the difference between the composite projection $P_y P_x u$ and the approximate solution U has the rate of convergence with one order higher than the optimal rate. Although this fact is easily derived for $r \geqq 3$ from the estimates in [7] and the quasi-uniformity of the partition, however, the arguments in [7] are not applicable to the present case, i.e. for $r \geqq 1$.

LEMMA 2 : *Let u and U be solutions to* (2.1) *and* (2.2), *respectively. If* $u \in W_p^{r+3}(\Omega)$, $2 < p \leqq \infty$, *then for sufficiently small h*

$$\|\nabla(P_y P_x u - U)\|_{L^\infty} \leqq Ch^{r+1}\|u\|_{W_p^{r+3}}.$$

Proof: For fixed $1 \leqq i, j \leqq N$, set $\rho = I_i \times I_j$. Let $\eta = P_y P_x u - U$ and define $\hat{\eta} \in L^\infty(\Omega)$ by

$$\hat{\eta} = \begin{cases}\eta & \text{on } \rho,\\ 0 & \text{otherwise}.\end{cases}$$

We now choose $\hat{\phi} \in \mathcal{M}$ satisfying

$$(\nabla(\hat{\eta} - \hat{\phi}), \nabla v) = 0, \qquad v \in \mathcal{M}. \tag{3.14}$$

Here, $\nabla\hat{\eta}$ is not the gradient in the sense of distributions but we interpret it as follows :

$$\nabla\hat{\eta} = \begin{cases}\nabla\eta & \text{on } \rho,\\ 0 & \text{otherwise}.\end{cases}$$

From (3.14), (2.2) and (3.1) we have

$$\begin{aligned}(\nabla\eta, \nabla\dot{\eta})_\rho &= (\nabla\eta, \nabla(\hat{\eta} - \hat{\phi})) + (\nabla\eta, \nabla\hat{\phi}) \\ &= (\nabla\eta, \nabla\hat{\phi}) \\ &= (\nabla(P_y P_x u - u), \nabla\hat{\phi}) - ((u - U), b\hat{\phi}) \\ &= (u_{xx} - P_y u_{xx} + u_{yy} - P_x u_{yy}, \hat{\phi}) - (u - U, b\hat{\phi}) \\ &\leqq C(\|u_{xx} - P_y u_{xx}\|_{L^p} + \|u_{yy} - P_x u_{yy}\|_{L^p} + \|u - U\|_{L^p})\|\hat{\phi}\|_{L^q}, \end{aligned} \tag{3.15}$$

where q is a positive number such that $\frac{1}{p} + \frac{1}{q} = 1$. We now estimate $\|\hat{\phi}\|_{L^q}$. For any $\psi \in L^p(\Omega)$, let $\phi \in H_0^1(\Omega) \cap W_p^2(\Omega)$ be a solution of the following problem.

$$\begin{cases} -\Delta\phi = \psi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.16}$$

Further, choose $v \in \mathcal{M}$ satisfying

$$(\nabla(\phi - v), \nabla w) = 0, \qquad w \in \mathcal{M}. \tag{3.17}$$

Then, by (3.14) we have

$$\begin{aligned}(\hat{\phi}, \psi) &= (\hat{\phi}, -\Delta\phi) \\ &= (\nabla\hat{\phi}, \nabla(\phi - v)) + (\nabla\hat{\phi}, \nabla v) \\ &= (\nabla\eta, \nabla v)_\rho \\ &\leqq \|\nabla\eta\|_{L^1(\rho)} \|\nabla\eta\|_{L^\infty(\rho)}. \end{aligned} \tag{3.18}$$

Using the W_∞^1-stability of the finite element solution which is implied by the results in [16, 17] and an inverse property, we have

$$\|\nabla v\|_{L^\infty(\rho)} \leqq \|v\|_{W_\infty^1} \leqq C\|\phi\|_{W_\infty^1}.$$

Furthermore, by the Sobolev's lemma and elliptic regularity for (3.16),

$$\|\phi\|_{W_\infty^1} \leqq C\|\phi\|_{W_p^2} \leqq C\|\psi\|_{L^p}.$$

Thus, from (3.18), we obtain

$$\|\hat{\phi}\|_{L^q} \leqq C\|\nabla\eta\|_{L^1(\rho)}, \quad 1 < q \leqq 2. \tag{3.19}$$

On the other hand, notice that

$$\begin{aligned}\|\nabla\eta\|_{L^\infty(\rho)}\|\nabla\eta\|_{L^1(\rho)} &\leqq Ch^{-1}\|\nabla\eta\|_{L^2(\rho)}h\|\nabla\eta\|_{L^2(\rho)}\\ &= C(\nabla\eta,\nabla\eta)_\rho\,.\end{aligned}$$

Combining the above with (3.15) and (3.19), and applying Lemma 1 concludes the proof of the lemma.

Now, for $1 \leqq i,\ j \leqq N-1$ and a continuous function ζ on $I_i^* \times I_j^*$ which is smooth on each $\rho \subset I_i^* \times I_j^*$, where ρ is in $\mathcal{R}$, we define

$$\tilde{D}_{ix}\,\zeta(x,y) = \frac{1}{1+\alpha_i^r}\left\{\frac{\partial\zeta}{\partial x}(x-,y) + \alpha_i^r\frac{\partial\zeta}{\partial x}(x+,y)\right\}$$

$$\tilde{D}_{jy}\,\zeta(x,y) = \frac{1}{1+\alpha_j^r}\left\{\frac{\partial\zeta}{\partial y}(x,y-) + \alpha_j^r\frac{\partial\zeta}{\partial y}(x,y+)\right\}.$$

Furthermore, let

$$G(\zeta\,;i,j) = (\tilde{D}_{ix}\,\zeta(x_i,y_j),\ \tilde{D}_{jy}\,\zeta(x_i,y_j))\,.$$

Then, as stated in the following theorem, $G(U\,;i,j)$ admits a superconvergent approximation to ∇u at internal mesh points (x_i, y_j). From now on, we set $|\gamma| \equiv \max(|\gamma_1|, |\gamma_2|)$ for any $\gamma = (\gamma_1, \gamma_2) \in R^2$.

THEOREM 2 : *Let $r \geqq 1$ be an odd integer. And let u and U be solutions to* (2.1) *and* (2.2) *respectively. If $u \in W_p^{r+3}(\Omega)$, $2 < p \leqq \infty$, then for sufficiently small h,*

$$\max_{1 \leqq i,j \leqq N-1} |\nabla u(x_i,y_j) - G(U\,;i,j)| \leqq Ch^{r+1}\|u\|_{W_p^{r+3}}\,.$$

Proof: Observe that

$$\begin{aligned}|\nabla u(x_i\,y_j) - G(U\,;i,j)| \leqq\ & |\nabla u(x_i,y_j)\\ &- (\tilde{D}_{ix}\,P_x\,u(x_i,y_j),\ \tilde{D}_{jy}\,P_y\,u(x_i,y_j))|\\ &+ |(\tilde{D}_{ix}\,P_x\,u(x_i,y_j),\ \tilde{D}_{jy}\,P_y\,u(x_i,y_j))\\ &- G(P_y\,P_x\,u\,;i,j)| + |G(P_y\,P_x\,u\,;i,j)\\ &- G(U\,;i,j)|\,. \end{aligned} \tag{3.20}$$

Now, each term of the right hand side of (3.20) can be estimated as follows. By Theorem 1,

$$[\text{the first term}] \leqq Ch^{r+1}\|u\|_{W_\infty^{r+2}(I_i^*\times I_j^*)}\,.$$

Taking into account (3.2), it is easily seen that the second term vanishes. Furthermore, from Lemma 2 we have

$$[\text{the third term}] \leqq \|\nabla(P_y P_x u - U)\|_{L^\infty} \leqq Ch^{r+1}\|u\|_{W_p^{r+3}}.$$

Thus, noting that $\|u\|_{W_\infty^{r+2}} \leqq C\|u\|_{W_p^{r+3}}$ by the Sobolev's lemma, we obtain the desired estimate.

The estimate in Theorem 2 is one order better than the global optimal estimates for $\nabla(u - U)$.

4. SUPERCONVERGENCE FOR ARBITRARY POINTS

In this section, it is shown that a posteriori local procedures, utilizing the results in previous section, can be carried out so as to provide $O(h^{r+1})$ approximations to ∇u at arbitrary points in the domain. Also, simple quadratures, using these local approximations, are exhibited which yield $O(h^{r+2})$ convergence to u itself.

Now let P be the projection defined by (3.1). Then, the following superconvergence estimates at Gauss points are obtained from the property (3.5).

$$|g'(x_{ik}) - (Pg)'(x_{ik})| \leqq Ch_i^{r+1}\|g\|_{W_\infty^{r+2}(I_i)} \tag{4.1}$$

for $1 \leqq i \leqq N$ and $1 \leqq k \leqq r$, where x_{ik} is defined by (2.4). We extend this result for two dimensional case to get the following estimate.

LEMMA 3 : *Let u and U be solutions to* (2.1) *and* (2.2), *respectively. If $u \in W_\infty^{r+3}(\Omega)$ and h is sufficiently small, then*

$$|\nabla(u - U)(x_{ik}, y_{j\ell})| \leqq Ch^{r+1}\|u\|_{W_\infty^{r+3}}$$

for $1 \leqq i, j \leqq N$ and $1 \leqq k, \ell \leqq r$, where $(x_{ik}, y_{j\ell})$ is a Gauss point on $I_i \times I_j$ defined in § 2.

Proof: First, observe that

$$|\nabla(u - U)(x_{ik}, y_{j\ell})| \leqq |\nabla(u - P_y P_x u)(x_{ik}, y_{j\ell})| + \|\nabla(P_y P_x u - U)\|_{L^\infty}. \tag{4.2}$$

Next, by noting that

$$\nabla(u - P_y P_x u) = \left((u_x - P_y u_x) + \frac{\partial}{\partial x}(P_y u - P_x P_y u),\right.$$
$$\left.(u_y - P_x u_y) + \frac{\partial}{\partial y}(P_x u - P_y P_x u)\right)$$

we have

$$|\nabla(u - P_y P_x u)(x_{ik}, y_{j\ell})| \leqq \|u_x - P_y u_x\|_{L^\infty} + \|u_y - P_x u_y\|_{L^\infty} + \left|\frac{\partial}{\partial x}(P_y u - P_x P_y u)(x_{ik}, y_{j\ell})\right| + \left|\frac{\partial}{\partial y}(P_x u - P_y P_x u)(x_{ik}, y_{j\ell})\right| . \tag{4.3}$$

Therefore, the desired estimates follow by (4.2), (4.3), Lemma 2, (4.1) and the error estimates for the solution to (3.1).

Now, by the use of the identity

$$u - U = (u - P_x u) + (P_x u - P_y P_x u) + (P_y P_x u - U), \tag{4.4}$$

we have, for each $y \in I$,

$$\left|\frac{\partial}{\partial x}(u - U)(x_{ik}, y)\right| \leqq Ch^{r+1}\|u\|_{W_\infty^{r+3}} .$$

Hence, Lemma 3 can be extended in the following way. That is, under the same assumptions in Lemma 3,

$$\left|\frac{\partial}{\partial x}(u - U)(x_{ik}, y)\right| + \left|\frac{\partial}{\partial y}(u - U)(x, y_{j\ell})\right| \leqq Ch^{r+1}\|u\|_{W_\infty^{r+3}} \tag{4.5}$$

for $1 \leqq i, j \leqq N$, $1 \leqq k, \ell \leqq r$ and any $(x, y) \in \Omega$. Furthermore, from Theorem 1 and Lemma 2 we have

$$\begin{aligned}\left|\left(\frac{\partial u}{\partial x} - \tilde{D}_{ix} U\right)(x_i, y)\right| &\leqq C\left(\left|\left(\frac{\partial u}{\partial x} - \tilde{D}_{ix} P_x u\right)(x_i, y)\right|\right. \\ &\quad + \left\|\frac{\partial}{\partial x}(P_x u) - P_y \frac{\partial}{\partial x}(P_x u)\right\|_{L^\infty} \\ &\quad \left. + \|\nabla(P_y P_x u - U)\|_{L^\infty}\right) \\ &\leqq Ch^{r+1}\|u\|_{W_\infty^{r+3}} .\end{aligned}$$

Since similar estimates are also derived with respect to y, we can extend Theorem 2 in previous section and obtain that

$$\left|\left(\frac{\partial u}{\partial x} - \tilde{D}_{ix} U\right)(x_i, y)\right| + \left|\left(\frac{\partial u}{\partial y} - \tilde{D}_{jy} U\right)(x, y_j)\right| \leqq Ch^{r+1}\|u\|_{W_\infty^{r+3}} \tag{4.6}$$

for $1 \leqq i, j \leqq N$, $1 \leqq k, \ell \leqq r$ and $(x, y) \in \Omega$.

Now, using these results we construct a superconvergent approximation to the gradient of u on each subrectangle $\rho \in \mathscr{R}$. For fixed $1 \leqq i$,

$j \leqq N$, we define $G(U) \equiv (U_x^*, U_y^*)$, where $U_x^*, U_y^* \in P_r(I_i) \otimes P_r(I_j)$, as the solution of following linear equations

$$\begin{cases} G(U)(x_i, y_j) = G(U; i, j), \\ G(U)(x_{ik}, y_{j\ell}) = \nabla U(x_{ik}, y_{j\ell}), \quad 1 \leqq k, \ell \leqq r, \\ G(U)(x_{ik}, y_j) = \left(\dfrac{\partial U}{\partial x}(x_{ik}, y_j), \tilde{D}_{jy} U(x_{ik}, y_j) \right), \quad 1 \leqq k \leqq r, \\ G(U)(x_i, y_{j\ell}) = \left(\tilde{D}_{ix} U(x_i, y_{j\ell}), \dfrac{\partial U}{\partial y}(x_i, y_{j\ell}) \right), \quad 1 \leqq \ell \leqq r. \end{cases} \tag{4.7}$$

Here, when $i = N$ we replace x_i by x_{N-1} in (4.7) and y_i by y_{N-1}.

Thus we can determine $G(U)$ for all $\rho \in \mathscr{R}$. Therefore, $G(U)$ is considered as a function on Ω having, in general, discontinuity on each mesh line. The following theorem implies that $G(U)$ is an $O(h^{r+1})$ superconvergent approximation to ∇u in the L^∞-norm sense.

THEOREM 3: *Let $r \geqq 1$ be an odd integer. And let u, U and $G(U)$ be solutions to* (2.1), (2.2) *and* (4.7), *respectively. If $u \in W_\infty^{r+3}(\Omega)$, then for sufficiently small h*

$$\|\nabla u - G(U)\|_{L^\infty(\rho)} \leqq C h^{r+1} \|u\|_{W_\infty^{r+3}}, \quad \rho \in \mathscr{R}.$$

Proof: In order to express clearly the degree of polynomials used, we denote P by P^r. Thus, P_x is denoted by P_x^r and P_y by P_y^r. Noting that $\dfrac{\partial}{\partial x} P_x^{r+1} P_y^r u \in P_r(I_i) \otimes P_r(I_j)$ we have

$$\left\| U_x^* - \frac{\partial}{\partial x} P_x^{r+1} P_y^r u \right\|_{L^\infty(\rho)} \leqq C \max_{(x,y) \in S_\rho} \left| \left(U_x^* - \frac{\partial}{\partial x} P_x^{r+1} P_y^r u \right)(x, y) \right|, \tag{4.8}$$

where S_ρ denotes the set of all interpolation points which determines $G(U)$ on $\rho \in \mathscr{R}$ in (4.7). For each $(x, y) \in S_\rho$, by (4.7), Theorem 2, (4.5) and (4.6) it follows that

$$\left| \left(U_x^* - \frac{\partial u}{\partial x} \right)(x, y) \right| \leqq C h^{r+1} \|u\|_{W_\infty^{r+3}}. \tag{4.9}$$

Also it is easily seen, from the error estimates for the projections, that

$$\begin{aligned} \left| \left(\frac{\partial u}{\partial x} - \frac{\partial}{\partial x} P_x^{r+1} P_y^r u \right)(x, y) \right| &\leqq \left\| \frac{\partial u}{\partial x} - P_y^r \left(\frac{\partial u}{\partial x} \right) \right\|_{L^\infty(\rho)} \\ &\quad + \left\| \frac{\partial}{\partial x} (P_y^r u - P_x^{r+1} P_y^r u) \right\|_{L^\infty(\rho)} \\ &\leqq C h^{r+1} \|u\|_{W_\infty^{r+3}(\rho)}. \end{aligned} \tag{4.10}$$

Thus, (4.8), (4.9) and (4.10) and the triangle inequality yield

$$\left\| U_x^* - \frac{\partial}{\partial x} P_x^{r+1} P_y u \right\|_{L^\infty(\rho)} \leqq C h^{r+1} \|u\|_{W_\infty^{r+3}} .$$

Combining this with the following triangle inequality we conclude the desired estimates for x-derivative

$$\begin{aligned}\left\| U_x^* - \frac{\partial u}{\partial x} \right\|_{L^\infty(\rho)} \leqq & \left\| U^* - \frac{\partial}{\partial x} P_x^{r+1} P_y^r u \right\|_{L^\infty(\rho)} \\ & + \left\| \frac{\partial}{\partial x} (P_x^{r+1} P_y^r u - P_y^r u) \right\|_{L^\infty(\rho)} + \left\| P_y^r \left(\frac{\partial u}{\partial x} \right) - \frac{\partial u}{\partial x} \right\|_{L^\infty(\rho)} .\end{aligned}$$

Since we can estimate similarly for $U_y^* - \dfrac{\partial u}{\partial y}$, this completes the proof.

Further, in particular for $r \geqq 3$, using $G(U)$ defined in (4.7) we can provide a superconvergent approximation $\tilde{U}$ for the solution u of (2.1) itself which is determined locally by

$$\tilde{U}(x, y) = \int_{x_{i-1}}^{x} U_x^*(\xi, y)\, d\xi + \int_{y_{j-1}}^{y} U_y^*(x_{i-1}, \eta)\, d\eta + U(x_{i-1}, y_{j-1}) \tag{4.11}$$

for each $1 \leqq i, j \leqq N$ and $(x, y) \in I_i \times I_j$.

THEOREM 4: *Assume the hypotheses of Theorem 3 and let $\tilde{U}$ be the function defined by* (4.11) *on each* $\rho \in \mathcal{R}$. *If* $r \geqq 3$ *and* h *is sufficiently small, then*

$$\| u - \tilde{U} \|_{L^\infty(\rho)} \leqq C h^{r+2} \|u\|_{W_\infty^{r+3}}, \quad \rho \in \mathcal{R} .$$

Proof: For any $(x, y) \in \rho = I_i \times I_j$, by (4.11) we have

$$\begin{aligned} u(x, y) - \tilde{U}(x, y) = & \int_{x_{i-1}}^{x} \left(\frac{\partial u}{\partial x} - U_x^* \right)(\xi, y)\, d\xi + \int_{y_{j-1}}^{y} \left(\frac{\partial u}{\partial y} - U_y^* \right) \\ & \times (x_{i-1, x})\, d\eta + (u - U)(x_{i-1}, y_{j-1}) \\ \leqq & \; h \| \nabla u - G(U) \|_{L^\infty(\rho)} + | (u - U)(x_{i-1}, y_{j-1}) | .\end{aligned}$$

On the other hand, by estimates in [7] for $r \geqq 3$

$$| (u - U)(x_{i-1}, y_{j-1}) | \leqq C h^{r+2} \|u\|_{H^{r+3}} .$$

Therefore, the proof of the theorem immediately follows from Theorem 3.

Remark : It might be expected that the superconvergence phenomena described in the paper are also valid for somewhat more general elliptic equations on the rectangular domain. However, observing the proof of Lemma 2, the techniques used here are peculiar to the equations of the form (2.1).

5. NUMERICAL EXAMPLES

In order to illustrate the results described in § 3 and § 4, we present some numerical examples.

Example :

$$\begin{cases} -\Delta u = \sin \pi x \,.\, \sin \pi y \,, & (x, y) \in \Omega \,, \\ u = 0 \,, & (x, y) \in \partial\Omega \,, \end{cases} \tag{5.1}$$

where $\Omega = (0, 1) \times (0, 1)$.

The exact solution to (5.1) is

$$u(x, y) = \frac{1}{2\pi^2} \sin \pi x \,.\, \sin \pi y \,.$$

We solved (5.1) numerically using the scheme (2.2) and calculated the various errors for several r and N. We show these results in Tables 1 to 4. The meanings of each symbol are as follows :

$$\text{LEFT} = \max_{1 \leq i, j \leq N-1} \left\{ \left| \frac{\partial}{\partial x} (u - U)(x_i -, y_j) \right| , \left| \frac{\partial}{\partial y} (u - U)(x_i, y_j -) \right| \right\} ,$$

$$\text{RIGHT} = \max_{1 \leq i, j \leq N-1} \left\{ \left| \frac{\partial}{\partial x} (u - U)(x_i +, y_j) \right| , \left| \frac{\partial}{\partial y} (u - U)(x_i, y_j +) \right| \right\} ,$$

$$\text{MEAN} = \max_{1 \leq i, j \leq N-1} \left\{ \left| \nabla u(x_i, y_j) - G(U ; i, j) \right| \right\} ,$$

$$\text{GAUSS} = \max_{\substack{1 \leq i, j \leq N \\ 1 \leq k, \ell \leq r}} \left\{ \left| \nabla (u - U)(x_{ik}, y_{j\ell}) \right| \right\} ,$$

$$\text{OTHER} = \max_{1 \leq i, j \leq N} \left\{ \left| \nabla (u - U)(x_i - h/4, y_j - h/4) \right| \right\} ,$$

$$\text{NEWGRAD} = \max_{1 \leq i, j \leq N} \left\{ \left| (\nabla u - G(U))(x_i - h/4, y_j - h/4) \right| \right\} .$$

Here, we used the partition of I into N equal parts.

TABLE 1

Improvement of errors by averaging (for $r = 1$)

N	LEFT (= RIGHT)	MEAN
8	$0.312E$-1	$0.190E$-2
12	$0.208E$-1	$0.880E$-3
16	$0.156E$-1	$0.501E$-3

TABLE 2

Non-improvement of errors by averaging (for $r = 2$)

N	LEFT (= RIGHT)	MEAN
4	$0.723E$-2	$0.549E$-2
6	$0.346E$-2	$0.308E$-2
8	$0.198E$-2	$0.187E$-2

TABLE 3

Errors at Gauss points (= *GAUSS*)

N	$r = 1$ (*)	$r = 2$ (**)
4	$0.713E$-2	$0.562E$-3
6	$0.342E$-2	$0.176E$-3
8	$0.198E$-2	$0.755E$-4

TABLE 4

Improvement of errors by procedure (4.7) (*for* $r = 1$)

N	OTHER	NEWGRAD
8	$0.153E$-1	$0.332E$-2
12	$0.103E$-1	$0.154E$-2
16	$0.777E$-2	$0.877E$-3

Tables 1 and 4 illustrate the superconvergence asserted in Theorems 2 and 3, respectively. Table 2 suggests that we cannot expect the improvement of the errors by the averaging procedures in case of using even degree

(*) $\tau_1 = 0.5$, (**) $\tau_1 = 0.5 - 1/\sqrt{12}$, $\tau_2 = 0.5 + 1/\sqrt{12}$.

polynomials. Further, Table 3 illustrates the superconvergence of derivatives at Gauss points proved in Lemma 3.

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