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MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
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# ENERGY ERROR ESTIMATES FOR A LINEAR SCHEME TO APPROXIMATE NONLINEAR PARABOLIC PROBLEMS (*)

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Abstract. — *This paper is concerned with a time-discrete algorithm which arises in the theory of nonlinear semigroups of contractions. It allows one to approximate a (strongly) nonlinear parabolic* P.D.E. *by solving at each time step a linear elliptic* P.D.E. *and making then an algebraic correction on account of the nonlinearity. This framework is so general as to include multidimensional Stefan problems and porous medium equations. Several energy error estimates are derived for the physical unknowns and for both degenerate and non-degenerate equations ; most of these orders are optimal. A variational technique is used.*

Résumé. — *On considère un schéma de discrétisation en la variable de temps qui vient de la théorie des semi-groupes non linéaires de contractions et qui conduit à l'approximation de problèmes (fortement) non linéaires paraboliques de* E.D.P. ; *à chaque pas de temps on doit résoudre une* E.D.P. *elliptique linéaire et calculer après une fonction non linéaire. Parmi les problèmes ici étudiés rentrent le problème de Stefan et l'équation de la diffusion du gaz dans les milieux poreux. On obtient plusieurs estimations de l'erreur en normes de l'énergie, la plupart desquelles sont optimales pour les équations soit dégénérées soit non dégénérées.*

## 1. INTRODUCTION

The aim of this paper is to analyze the accuracy of a linear semi-discrete scheme suggested by the theory of nonlinear semigroups of contractions.

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The scheme is used to approximate the (strongly) nonlinear parabolic problem

$$\frac{\partial u}{\partial t} + A\beta(u) = f(\beta(u)), \quad 0 < t < T, \quad u(0) = u_0, \tag{1.1}$$

where $A$ stands for a linear elliptic differential operator in the space variable $x \in \Omega \subset R^d$ $(d \geq 1)$ and $\beta$ for a non-decreasing Lipschitz continuous function defined on $R$. This framework is so general as to include the Stefan problem and the porous medium equation as a model of singular parabolic problems as well as a wide class of mildly nonlinear parabolic equations.

The idea of studying an abstract P.D.E. like (1.1) by means of the theory of nonlinear semigroups of contractions in Banach spaces was first proposed by Brézis [4]. This theory not only leads to existence, uniqueness and global regularity results but also suggests some time-discrete algorithms to approximate the P.D.E.. The first one comes directly from the definition itself of semigroup and it is known as Crandall-Liggett formula [7, 1]. Namely, for $\tau = T/N$ being the time step, this algorithm reads as follows :

$$\begin{cases} U^0 = u_0 \\ U^n + \tau A\beta(U^n) = U^{n-1} + \tau f(\beta(U^n)), \quad 1 \leq n \leq N. \end{cases} \tag{1.2}$$

This scheme corresponds to simple backward differences in time, which were introduced long time ago and successfully used for theoretical purposes ; we refer to [13, 17] and the references given therein. After discretization in the space variables, the scheme was studied from a numerical viewpoint in many papers, e.g. in [6, 19, 32], and more completely in [14, 26, 20, 21, 22, 23, 11, 24] where the accuracy of these methods have been studied by means of variational techniques. Other algorithms are suggested by the so-called nonlinear Chernoff formula which was studied in an abstract and general setting in [5]. This fact was first observed by Berger, Brézis & Rogers [2] who proved the convergence of the schemes provided that $u_0 \in L^\infty(\Omega)$ and $f = 0$. Among these algorithms the simplest one is the following :

$$\begin{cases} U^0 = u_0 \\ \Theta^n + \dfrac{\tau}{\mu} A\Theta^n = \beta(U^{n-1}) + \dfrac{\tau}{\mu} f(\beta(U^{n-1})) \\ U^n = U^{n-1} + \mu[\Theta^n - \beta(U^{n-1})], \quad \text{for} \quad 1 \leq n \leq N, \end{cases} \tag{1.3}$$

where $\mu > 0$, the *relaxation parameter*, satisfies the stability condition $\mu \leq L_\beta^{-1}$ ($L_\beta$ Lipschitz constant of $\beta$). Since the P.D.E. in (1.3) is linear in the unknown $\Theta^n$ and the further correction to calculate $U^n$ only requires the evaluation of a given nonlinear function, this algorithm is clearly expected to be very *efficient* from a computational viewpoint. At this stage, the question of how accurate the method is arises quite naturally.

The purpose of this paper is to present a rather complete answer to this question. To this aim we restrict ourself to analyze a model problem ; namely $A$ will be $-\Delta$, the Laplace operator, and the boundary conditions will be of homogeneous Dirichlet type. However the necessary modifications to handle more general second order elliptic operators and linear mixed boundary conditions are straightforward and, consequently, they are omitted here. Our results can be summarized as follows. First let us set $\vartheta := \beta(u)$ (for the Stefan problem, $\vartheta$ is the temperature whereas $u$ is the enthalpy) and denote by $e_\vartheta$ and $e_u$ the error in each unknown :

$$e_\vartheta(t) := \vartheta(t) - \Theta^n , \quad e_u(t) := u(t) - U^n \qquad \text{for} \quad (n-1)\tau < t \leqslant n\tau , \quad 1 \leqslant n \leqslant N .$$

Next let us set $Q := \Omega \times ]0, T[$ and denote by $E_\tau$ the global error defined by :

$$E_\tau := \|e_\vartheta\|_{L^2(Q)} + \left\| \int_0^t e_\vartheta \, ds \right\|_{L^\infty(0, T; H^1(\Omega))} + \|e_u\|_{L^\infty(0, T; H^{-1}(\Omega))} . \tag{1.4}$$

We then have the following *energy error estimates* (see Theorems 1, 2, 3, 4) :

I. DEGENERATE CASE ($\beta$ non-decreasing)

I.1. Let $u_0 \in L^2(\Omega)$. Then $E_\tau = O(\tau^{1/4})$.

I.2. Let $u_0 \in L^\infty(\Omega)$ and $\Delta\beta(u_0) \in L^1(\Omega)$. Then $E_\tau = O(\tau^{1/2})$.

II. NON-DEGENERATE CASE ($\beta' \geqslant l_\beta =$ positive constant)

II.1. Let $u_0 \in L^2(\Omega)$. Then $E_\tau + \|e_u\|_{L^2(Q)} = O(\tau^{1/2})$.

II.2. Let $\beta(u_0) \in H_0^1(\Omega)$. Then

$$E_\tau + \|e_u\|_{L^2(Q)} = O(\tau) , \quad \|e_\vartheta\|_{L^2(0, T; H^1(\Omega))} = O(\tau^{1/2}).$$

Their proof rely essentially upon the next three ideas which are better explained in section 4 :

i) the use of a variational technique first applied by Nochetto [20, 21, 22] ;

ii) the possibility of dealing with minimal regularity properties, say $u_0 \in L^2(\Omega)$, as in Nochetto & Verdi [24] ;

iii) the relationship between the scheme (1.3) and the discrete-time phase relaxation scheme introduced by Verdi & Visintin [29].

The result I.1 reproduces for (1.3) the one proved in [29] for the phase relaxation technique. The result I.2 is new and quasioptimal and the rates of convergence stated in II.1 and II.2 are both sharp. This is so on account of the time regularity of the continuous solution. The results II.1 and II.2 above extend and improve the ones known for other linearized schemes (see [8, 31, 16, 10, 12, 27] and the references given therein), and they are obtained under *minimal regularity* of the data occurring in (1.1) (see section 4.2). Our scheme is in the spirit of the Laplace-modified forward Galerkin method of Douglas & Dupont [9] for non-degenerate problems and of the alternating-phase truncation method of Rogers, Berger & Ciment [25] for degenerate problems.

The algorithm considered here may give rise to an effective numerical scheme after discretizing in space. This should be done by using numerical integration as in [11, 24] in order to get an algorithm easy to implement on a computer. The best space discretization for singular problems seems to be that one used by Verdi & Visintin [29], namely a $C^0$-piecewise linear approximation for $\vartheta$ and a piecewise constant approximation for $u$. However this point deserves a further analysis. We only add that the numerical experimentation already done [28, 29] confirms a very competitive performance of our scheme.

The outline of the paper is as follows. Section 2 is devoted to state the assumptions and the differential problems precisely as well as to introduce the notation. The stability of the discrete scheme (1.3) is shown in section 3. The energy error estimates I and II are demonstrated in section 4. The proof follows the same lines regardless the degeneracy while the concrete results are obtained at the end according to the regularity assumptions. The paper concludes with some further remarks about an open problem.

## 2. FORMULATION OF THE PROBLEM

In this section we shall establish the hypotheses upon the data and state precisely the continuous problem as well as the nonlinear Chernoff formula.

### 2.1. Basic assumptions and notation

Along the work we shall always assume the following hypotheses :

$(H_\Omega)$ $\Omega \subset R^d$ $(d \geq 1)$ is a bounded domain with sufficiently regular boundary. Set $Q := \Omega \times ]0, T[$, where $0 < T < +\infty$ is fixed.

$(H_\beta)$ $\beta : R \to R$ is a non-decreasing and Lipschitz continuous function, more precisely

$$0 \leq l_\beta \leq \beta'(s) \leq L_\beta < +\infty \quad \text{for} \quad \text{a.e. } s \in R$$

and $\beta(0) = 0$.

$(H_f)$ $f : R \to R$ is a Lipschitz continuous function, namely

$$|f(s_1) - f(s_2)| \leqslant L_f |s_1 - s_2| \ , \quad \forall s_1, s_2 \in R \ .$$

In order to simplify the exposition we are assuming that the functions $\beta$ and $f$ are independent of the space and time variables $(x, t)$. However, a complete treatment is still possible.

$(Hu_0)$ $$u_0 \in L^2(\Omega) \ .$$

We denote by $\langle .,. \rangle$ either the inner product on $L^2(\Omega)$ or the pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Now we introduce some notation concerning the time discretization. Let $\tau := T/N$ be the time step ($N$ positive integer) and set $t^n := n\tau$, $I^n := ]t^{n-1}, t^n]$ for $1 \leqslant n \leqslant N$. We also set

$$z^n := z(\,.\,, t^n) \ , \qquad \bar{z}^n := \frac{1}{\tau} \int_{I^n} z(\,.\,, t)\, dt \qquad (\bar{z}^0 := z^0)$$

for any continuous (resp. integrable) function in time defined in $Q$, and

$$\partial z^n := \frac{z^n - z^{n-1}}{\tau} \ , \qquad 1 \leqslant n \leqslant N$$

for any given family $\{z^n\}_{n=0}^N$.

### 2.2. The continuous problem

Let us now state the variational formulation of the differential problem (1.1) we shall work with.

PROBLEM $(P)$ : *Find* $\{u, \vartheta\}$ *such that*

$$u \in L^\infty(0, T ; L^2(\Omega)) \cap H^1(0, T ; H^{-1}(\Omega)) \ , \vartheta \in L^2(0, T ; H_0^1(\Omega)) \ , \tag{2.1}$$

$$\vartheta(x, t) = \beta(u(x, t)) \quad \textit{for} \quad \text{a.e. } (x, t) \in Q \ , \tag{2.2}$$

$$u(\,.\,, 0) = u_0 \tag{2.3}$$

*and for* a.e. $t \in \, ]0, T[$ *and for all* $\varphi \in H_0^1(\Omega)$ *the following equation holds*

$$\left\langle \frac{\partial u}{\partial t} , \varphi \right\rangle + \langle \nabla\vartheta, \nabla\varphi \rangle = \langle f(\vartheta), \varphi \rangle \ . \tag{2.4}$$

Existence and uniqueness are well known for $(P)$ (see, e.g. [13, 17] and the references given therein). Now, we recall the further global regularity results we will use in the sequel. Assuming that the assumptions $(H_\Omega)$, $(H_\beta)$, $(H_f)$ and $(Hu_0)$ hold, we have that :

*Degenerate case* $(l_\beta = 0)$

$(R1)$ Let $\beta(u_0) \in H_0^1(\Omega)$, then $\dfrac{\partial u}{\partial t} \in L^\infty(0, T; H^{-1}(\Omega))$ and

$$\vartheta \in L^\infty(0, T; H_0^1(\Omega)) \cap H^1(0, T; L^2(\Omega)).$$

$(R2)$ Let $\Delta\beta(u_0) \in L^1(\Omega)$, then $\dfrac{\partial u}{\partial t} \in L^\infty(0, T; M(\Omega))$, where $M(\Omega)$ denotes the set of regular Baire measures [14, 26].

*Remark* 1 : The assumption upon the initial datum may be weakened somewhat by taking $\Delta\beta(u_0) = l + m$ where $l \in L^1(\Omega)$ and $m \in M(\Omega)$ provided that the set $F_0 := \{x \in \Omega : \beta(u_0(x)) = 0\}$ is sufficiently regular and $\operatorname{supp} m \subset F_0$. The proof proceeds as in [14] after a convenient regularization. ∎

*Non-degenerate case* $(l_\beta > 0)$

$(R3)$ $u, \vartheta \in H^{1/2}(0, T; L^2(\Omega))$.

$(R4)$ Let $\beta(u_0) \in H_0^1(\Omega)$, then $u, \vartheta \in H^1(0, T; L^2(\Omega))$,

$$\vartheta \in H^{1/2}(0, T; H^1(\Omega)).$$

The last two regularity results follow by interpolation theory [15] : indeed one has $u, \vartheta \in H^1(0, T; H^{-1}(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ (hence $(R3)$) and if $\beta(u_0) \in H_0^1(\Omega)$ one has $u, \vartheta \in H^1(0, T; L^2(\Omega))$ and $\vartheta \in L^2(0, T; H^2(\Omega))$ (hence $(R4)$).

*Remark* 2 : All the previous results, except $\vartheta \in H^{1/2}(0, T; H^1(\Omega))$ in $(R4)$, hold for linear mixed boundary conditions. For non-homogeneous Dirichlet data, we need in addition either $u_0 \in L^\infty(\Omega)$ or the further assumption that $\beta$ grows at least linearly at infinity, namely

$$\exists C_1, C_2 > 0 : \forall s \in R, \quad |s| \leq C_1 + C_2|\beta(s)|. \quad ∎ \tag{2.5}$$

### 2.3. The nonlinear Chernoff formula

Finally we state the precise meaning of algorithm (1.3) by writing the differential equation in variational form. Let $0 < \mu \leq L_\beta^{-1}$ be a fixed number (the so-called *relaxation parameter*).

PROBLEM $(P_\tau)$ : *For any* $1 \leq n \leq N$, *find* $\{U^n, \Theta^n\}$ *such that* $U^n \in L^2(\Omega)$, $\Theta^n \in H_0^1(\Omega)$ *and, setting*

$$U^0 := u_0 \tag{2.6}$$

*we have*

$$\langle \Theta^n, \varphi \rangle + \frac{\tau}{\mu} \langle \nabla \Theta^n, \nabla \varphi \rangle = \frac{\tau}{\mu} \langle f(\beta(U^{n-1})), \varphi \rangle + \langle \beta(U^{n-1}), \varphi \rangle , \quad (2.7)$$

*for all* $\varphi \in H_0^1(\Omega)$, *and*

$$U^n = U^{n-1} + \mu[\Theta^n - \beta(U^{n-1})] \quad \text{a.e.} \quad in \quad \Omega. \quad (2.8)$$

Since the P.D.E. (2.7) is linear and coercive in the unknown $\Theta^n$ the solution of $(P_\tau)$ exists and is unique.

## 3. A PRIORI ESTIMATES

In order to show the stability of the discrete solutions, we combine the equations (2.7) and (2.8) of the nonlinear Chernoff formula and rewrite (2.7) as follows :

$$\langle \partial U^n, \varphi \rangle + \langle \nabla \Theta^n, \nabla \varphi \rangle = \langle f(\beta(U^{n-1})), \varphi \rangle , \quad \forall \varphi \in H_0^1(\Omega) . \quad (3.1)$$

The relaxation parameter constraints $0 < \mu \leq L_\beta^{-1}$ imply that the function

$$\alpha := I - \mu\beta \quad \text{satisfies} \quad O \leq \alpha'(s) \leq 1 , \quad \text{for} \quad \text{a.e.} \quad s \in R . \quad (3.2)$$

### 3.1. Stability in energy norms

Here we shall prove a priori estimates in energy norms under suitable assumptions upon the data. Before doing this, let us introduce some notation. Given an absolutely continuous function $\lambda : R \to R$ so that $\lambda(0) = 0$ and $0 \leq \lambda' \leq \Lambda < \infty$, we denote by $\Phi_\lambda$ the convex function

$$\Phi_\lambda(s) := \int_0^s \lambda(z)\, dz , \quad \text{for} \quad s \in R .$$

Then $\Phi_\lambda$ obviously satisfies

$$\frac{1}{2\Lambda} \lambda^2(s) \leq \Phi_\lambda(s) \leq \frac{\Lambda}{2} s^2 , \quad \text{for} \quad s \in R . \quad (3.3)$$

The following elementary relations will be used in the sequel

$$2ab \leq \eta a^2 + b^2/\eta , \quad \text{for} \quad a, b \in R , \quad \eta > 0 , \quad (3.4)$$

$$2a(a-b) = a^2 - b^2 + (a-b)^2 , \quad \text{for} \quad a, b \in R . \quad (3.5)$$

LEMMA 1 : *Assume that* $(H_\Omega)$, $(H_\beta)$, $(H_f)$ *and* $(Hu_0)$ *hold. Then there exists a constant* $C > 0$ *depending only on* $T$, $L_\beta$, $L_f$, $f(0)$, $\mu$ *and* $\|u_0\|_{L^2(\Omega)}$ *such that, for any* $N$

$$\max_{1 \leq n \leq N} \|\beta(U^n)\|_{L^2(\Omega)} + \sum_{n=1}^{N} \|U^n - U^{n-1}\|_{L^2(\Omega)}^2 + \sum_{n=1}^{N} \tau \|\nabla \Theta^n\|_{L^2(\Omega)}^2 \leq C . \tag{3.6}$$

The norm in the middle may be regarded as a discrete $H^{1/2}(0, T ; L^2(\Omega))$ norm. Moreover the same estimate holds for the unknows $\{\Theta^n\}$ because $\Theta^n = \mu^{-1}[U^n - \alpha(U^{n-1})]$.

*Proof*: We take $\varphi = \tau\Theta^n \in H_0^1(\Omega)$ as a test function in (3.1) and sum over $n$ for $n = 1, \dots, m \leq N$. We now proceed to estimate each resulting term separately. To begin with, notice that

$$\begin{aligned} \Theta^n &= \frac{1}{\mu}[U^n - U^{n-1}] + \beta(U^{n-1}) = \\ &= \frac{1}{2}\beta(U^n) + \frac{1}{2\mu}[\alpha(U^n) - \alpha(U^{n-1})] + \frac{1}{2\mu}U^n - \frac{1}{2\mu}\alpha(U^{n-1}). \end{aligned} \tag{3.7}$$

So using the convexity of $\Phi_\beta$ and $\Phi_\alpha$ and the elementary identity (3.5) we can write

$$\begin{aligned} 2\sum_{n=1}^{m} \langle U^n - U^{n-1}, \Theta^n \rangle &\geq \\ &\geq \sum_{n=1}^{m} \int_\Omega \left\{ [\Phi_\beta(U^n) - \Phi_\beta(U^{n-1})] + \frac{1}{\mu}[\Phi_\alpha(U^{n-1}) - \Phi_\alpha(U^n)] \right\} dx + \\ &\quad + \frac{1}{2\mu}\left[ \|U^m\|_{L^2(\Omega)}^2 - \|U^0\|_{L^2(\Omega)}^2 + \sum_{n=1}^{m} \|U^n - U^{n-1}\|_{L^2(\Omega)}^2 \right], \end{aligned}$$

where $U^0 = u_0$. This expression is further bounded by means of (3.2), (3.3) and $(Hu_0)$ ; namely

$$\begin{aligned} \sum_{n=1}^{m} \langle U^n - U^{n-1}, \Theta^n \rangle &\geq - C + C\|\beta(U^m)\|_{L^2(\Omega)}^2 + \\ &\quad + \frac{1}{4\mu}\sum_{n=1}^{m} \|U^n - U^{n-1}\|_{L^2(\Omega)}^2 . \end{aligned}$$

The next term on the left hand side of (3.1) provides the $H_0^1$-estimate. It only remains to analyze the source term. We use $(H_f)$, the first line in (3.7) and (3.4) to obtain

$$\left|\sum_{n=1}^{m} \tau\langle f(\beta(U^{n-1})), \Theta^n\rangle\right| \leq C + C \sum_{n=1}^{m} \tau\|\beta(U^{n-1})\|_{L^2(\Omega)}^2 + \\ + \frac{1}{8\mu}\sum_{n=1}^{m} \|U^n - U^{n-1}\|_{L^2(\Omega)}^2 .$$

Thus the last term can be hidden into the left hand side. Finally the desired estimate (3.6) follows after applying the discrete Gronwall inequality. ■

*Remark* 3 : From (3.6) and the first line in (3.7), it is easily seen that

$$\max_{1 \leq n \leq N} \|\Theta^n\|_{L^2(\Omega)} \leq C . \qquad ■$$

Let us now improve the previous a priori estimates under stronger assumptions upon the initial datum.

LEMMA 2 : *Assume that* $(H_\Omega)$, $(H_\beta)$, $(H_f)$ *and* $(Hu_0)$ *hold, and in addition that, for any* $N$

$$\beta(u_0) \in H_0^1(\Omega) . \tag{3.8}$$

*Then there exists a constant* $C > 0$ *depending only on* $T$, $L_f$, $L_\beta$, $f(0)$, $\mu$, $\|u_0\|_{L^2(\Omega)}$ *and* $\|\beta(u_0)\|_{H^1(\Omega)}$ *such that*

$$\sum_{n=1}^{N} \tau\|\partial\beta(U^n)\|_{L^2(\Omega)}^2 + \max_{1 \leq n \leq N} \|\nabla\Theta^n\|_{L^2(\Omega)} + \sum_{n=1}^{N} \|\nabla[\Theta^n - \Theta^{n-1}]\|_{L^2(\Omega)}^2 \leq C . \tag{3.9}$$

The last norm may be regarded as a discrete $H^{1/2}(0, T ; H_0^1(\Omega))$ norm. Note also that (3.8) together with a growth of $\beta$ at infinity at least linear (see (2.5)) yield $(Hu_0)$.

*Proof*: We take $\varphi := \tau\,\partial\Theta^n \in H_0^1(\Omega)$ as a test function in (3.1), and sum over $n$ for $n = 1, \ldots, m \leq N$ (we set $\Theta^0 := \beta(u_0)$). Then the assertion follows from estimating separately each resulting term. Let us rewrite (3.7) in a suitable way ; namely

$$\Theta^n = \frac{1}{2}\beta(U^n) + \frac{1}{2\mu}U^n + \frac{1}{2\mu}\alpha(U^n) - \frac{1}{\mu}\alpha(U^{n-1}) ,$$

for $0 \leq n \leq N$, where $U^{-1} := U^0$. The first term on the left hand side of

(3.1) is successively splitted and bounded as follows. By $(H_\beta)$ and (3.2) we get

$$\sum_{n=1}^{m} \tau \left\langle \partial U^n, \frac{1}{2}\, \partial\beta(U^n) + \frac{1}{2\,\mu}\, \partial U^n + \frac{1}{2\,\mu}\, \partial\alpha(U^n) \right\rangle \geqslant$$

$$\geqslant \sum_{n=1}^{m} \tau \left[ \frac{1}{2\,L_\beta} \left\| \partial\beta(U^n) \right\|^2_{L^2(\Omega)} + \frac{1}{2\,\mu} \left\| \partial U^n \right\|^2_{L^2(\Omega)} + \frac{1}{2\,\mu} \left\| \partial\alpha(U^n) \right\|^2_{L^2(\Omega)} \right].$$

We now use the Cauchy-Schwarz inequality and (3.4) (with $\eta = 1$) to obain

$$\left| \sum_{n=2}^{m} \tau \langle \partial U^n, \partial\alpha(U^{n-1}) \rangle \right| \leqslant \frac{1}{2} \sum_{n=1}^{m} \tau \| \partial U^n \|^2_{L^2(\Omega)} + \frac{1}{2} \sum_{n=1}^{m} \tau \| \partial\alpha(U^n) \|^2_{L^2(\Omega)}.$$

Collecting these two estimates yields

$$\sum_{n=1}^{m} \tau \langle \partial U^n, \partial\Theta^n \rangle \geqslant \frac{1}{2\,L_\beta} \sum_{n=1}^{m} \tau \| \partial\beta(U^n) \|^2_{L^2(\Omega)}.$$

The second term on the left hand side of (3.1) is handled by means of (3.5) ; namely

$$2 \sum_{n=1}^{m} \tau \langle \nabla\Theta^n, \partial\nabla\,\Theta^n \rangle =$$

$$= \|\nabla\Theta^m\|^2_{L^2(\Omega)} - \|\nabla\Theta^0\|^2_{L^2(\Omega)} + \sum_{n=1}^{m} \|\nabla[\Theta^n - \Theta^{n-1}]\|^2_{L^2(\Omega)} \geqslant$$

$$\geqslant \|\nabla\Theta^m\|^2_{L^2(\Omega)} + \sum_{n=1}^{m} \|\nabla[\Theta^n - \Theta^{n-1}]\|^2_{L^2(\Omega)} - C\,,$$

because $\Theta^0 := \beta(u_0) \in H^1_0(\Omega)$. In order to analyze the source term on the right of (3.1), notice that the following discrete summation by parts formula holds

$$\sum_{n=1}^{m} a_n [b_n - b_{n-1}] = a_m\, b_m - a_0\, b_0 - \sum_{n=1}^{m} b_{n-1} [a_n - a_{n-1}] =$$

$$= a_m\, b_m - a_1\, b_0 - \sum_{n=2}^{m} b_{n-1} [a_n - a_{n-1}] \quad (3.10)$$

where $a_n, b_n \in R$ for all $n = 1, \dots, m \leqslant N$. Then we can write

$$\left| \sum_{n=1}^{m} \tau \langle f(\beta(U^{n-1})), \partial\Theta^n \rangle \right| = \left| \langle f(\beta(U^{m-1})), \Theta^m \rangle - \langle f(\beta(U^0)), \Theta^0 \rangle - \right.$$

$$\left. - \sum_{n=2}^{m} \tau \langle \partial f(\beta(U^{n-1})), \Theta^{n-1} \rangle \right| \leqslant \frac{C}{\eta} + \eta \sum_{n=1}^{m} \tau \| \partial\beta(U^n) \|^2_{L^2(\Omega)},$$

where we have used $(H_f)$, $(Hu_0)$, Lemma 1 and Remark 3. Here $\eta > 0$ stands for a constant small enough so that the corresponding norm can be absorbed into the left hand side. This completes the proof. ■

*Remark* 4 : Assume $\beta$ to be non-degenerate, i.e. $l_\beta > 0$. Then from (3.9) it easily follows that

$$\sum_{n=1}^{N} \tau \| \partial U^n \|^2_{L^2(\Omega)} \leq C . \qquad ■ \quad (3.11)$$

### 3.2. Stability in maximum norm

Our aim here is to prove an a priori estimate in $L^\infty$ for the discrete solutions ; we refer to [2] for the case $f_- = 0$.

LEMMA 3 : *Assume that $(H_\Omega)$, $(H_\beta)$ and $(H_f)$ hold and that*

$$u_0 \in L^\infty(\Omega) . \qquad (3.12)$$

*Then there exists a constant $C > 0$ depending only on $T$, $L_\beta$, $L_f$, $f(0)$ and $\| u_0 \|_{L^\infty(\Omega)}$ such that, for any $N$*

$$\max_{1 \leq n \leq N} \| U^n \|_{L^\infty(\Omega)} \leq C . \qquad (3.13)$$

*Proof*: Let $c_0$ be a positive constant such that $-c_0 \leq U^0 := u_0 \leq c_0$ a.e. in $\Omega$ and let $f_0$ denote $|f(0)|$. We want to prove the following estimate

$$\| U^n \|_{L^\infty(\Omega)} \leq c_n := c_0 \, e^{n\tau L_f L_\beta} + \frac{f_0}{L_f L_\beta} [e^{n\tau L_f L_\beta} - 1] , \quad 1 \leq n \leq N \quad (3.14)$$

which obviously implies the assertion (3.13). The proof is carried out by induction. On account of $(H_\beta)$, $(H_f)$ and (3.2) the following inequalities hold a.e. in $\Omega$ :

$$\beta(-c_{n-1}) \leq \beta(U^{n-1}) \leq \beta(c_{n-1}) , \qquad (3.15)$$

$$-f_0 + L_f \, \beta(-c_{n-1}) \leq f(\beta(U^{n-1})) \leq f_0 + L_f \, \beta(c_{n-1}) , \quad (3.16)$$

$$\alpha(-c_{n-1}) \leq \alpha(U^{n-1}) \leq \alpha(c_{n-1}) . \qquad (3.17)$$

By a well known maximum principle for the elliptic operator $\vartheta - \frac{\tau}{\mu} \Delta \vartheta$ and in view of (3.15) and (3.16) the solution $\Theta^n$ of equation (2.7) satisfies a.e. in $\Omega$ the inequalities

$$\beta(-c_{n-1}) - \frac{\tau}{\mu}[f_0 - L_f\,\beta(-c_{n-1})] \leq \Theta^n \leq$$
$$\leq \beta(c_{n-1}) + \frac{\tau}{\mu}[f_0 + L_f\,\beta(c_{n-1})]\,. \quad (3.18)$$

Now, since

$$U^n = U^{n-1} + \mu[\Theta^n - \beta(U^{n-1})] = \alpha(U^{n-1}) + \mu\Theta^n\,,$$

(3.17) and (3.18) yield

$$-c_{n-1} - \tau[f_0 - L_f\,\beta(-c_{n-1})] \leq U^n \leq c_{n-1} + \tau[f_0 + L_f\,\beta(c_{n-1})] \quad \text{a.e. in } \Omega\,.$$

In other words, by $(H_\beta)$ we get

$$\|U^n\|_{L^\infty(\Omega)} \leq c_{n-1} + \tau[f_0 + L_f \max\,(\beta(c_{n-1}), -\beta(-c_{n-1}))] \leq$$
$$\leq \tau f_0 + c_{n-1}[1 + \tau L_f\,L_\beta]\,.$$

Finally, since $1 + \tau L_f\,L_\beta \leq e^{\tau L_f L_\beta}$ straightforward calculations lead to

$$\tau f_0 + c_{n-1}[1 + \tau L_f\,L_\beta] \leq c_n\,,$$

where $c_n$ was defined in (3.14). This completes the argument. ∎

*Remark* 5 : As a consequence of Lemma 3, the function $\beta$ may be assumed to be only locally Lipschitz continuous, provided that $f = 0$ (see also [2]). ∎

*Remark* 6 : Notice that Lemmas 1, 2, 3 hold for (linear) mixed boundary conditions on $\partial\Omega$, as well. For non-homogeneous Dirichlet boundary data one has to assume either $u_0 \in L^\infty(\Omega)$ or that $\beta$ grows at least linearly at infinity, as in Remark 2. ∎

## 4. ENERGY ERROR ESTIMATES

In this section we analyze the accuracy of the nonlinear Chernoff formula (2.6)-(2.8) in approximating the physical unknowns $\vartheta$ and $u$. So our aim is to derive bounds in energy norms for the errors $e_\vartheta$ and $e_u$, which are defined by

$$e_\vartheta(t) := \vartheta(t) - \Theta^n\,,\quad e_u(t) := u(t) - U^n \quad \text{for} \quad t \in I^n\,,\quad 1 \leq n \leq N\,. \quad (4.1)$$

The key argument is a combination of the following three features :

i) the use of a variational technique first applied by Nochetto [20, 21, 22] ;

ii) the possibility of dealing with minimal regularity properties, say $u_0 \in L^2(\Omega)$, as shown by Nochetto & Verdi [24] ;

iii) the relationship between the nonlinear Chernoff formula considered here and the discrete-time phase relaxation scheme studied by Verdi & Visintin [29].

Let us briefly explain how this scheme looks like for $f = 0$. Set $w := u - \mu\vartheta$ and write $(P)$ as follows

$$\frac{\partial[\mu\vartheta + w]}{\partial t} - \Delta\vartheta = 0\,, \quad w \in \Lambda(\vartheta)\,.$$

Here $\Lambda$ stands for the maximal monotone graph $\beta^{-1} - \mu I$ (recall that $\mu \leqslant L_\beta^{-1}$). For any $\varepsilon > 0$ we replace the constitutive relation with the phase relaxation equation introduced by Visintin [30]

$$\varepsilon\,\frac{\partial w}{\partial t} + \Lambda^{-1}(w) \ni \vartheta\,.$$

After coupling this equation with the P.D.E. and discretizing in time [29] we get the following algorithm :

$$\begin{cases} \mu[\Theta^n - \Theta^{n-1}] + [W^n - W^{n-1}] - \tau\,\Delta\Theta^n = 0 \\ \dfrac{\varepsilon}{\tau}\,[W^n - W^{n-1}] + \Lambda^{-1}(W^n) \ni \Theta^{n-1}\,, \end{cases}$$

for any $1 \leqslant n \leqslant N$. The stability constraint here is $\tau/\varepsilon \leqslant \mu$ [29]. Now it is not difficult to see that this scheme reduces to (1.3) if we choose $\varepsilon := \dfrac{\tau}{\mu}$ and set $U^n := \mu\Theta^n + W^n$.

With the tools above we are able to answer rather completely the question of how accurate the nonlinear Chernoff formula is for both degenerate $(l_\beta = 0)$ and non-degenerate $(l_\beta > 0)$ parabolic problems. In particular, some of the present ideas may be used to improve the rates of convergence obtained in [29] (see Theorem 2). The results below will be explained in detail so as to render the paper as selfcontained as possible. We first analyze the general strategy and next we distinguish between the various cases according to the regularity requirements upon $u_0$ and $\beta$.

Let us start by writing the set of discrete equations satisfied by the continuous solution ; namely

$$\langle \partial u^n, \varphi\rangle + \left\langle \nabla\bar{\vartheta}^n, \nabla\varphi\right\rangle = \left\langle \bar{f}^n, \varphi\right\rangle\,, \quad \forall\varphi \in H_0^1(\Omega)\,, \quad 1 \leqslant n \leqslant N\,, \tag{4.2}$$

which is obtained after integrating (2.4) on $I^n$. We now take the difference between (4.2) and (3.1), sum over $n$ from 1 to $i \leqslant N$ and multiply the resulting expression by $\tau$. Hence

$$\langle u^i - U^i, \varphi\rangle + \left\langle \nabla \sum_{n=1}^{i} \tau[\bar{\vartheta}^n - \Theta^n], \nabla\varphi\right\rangle = \left\langle \sum_{n=1}^{i} \tau[\bar{f}^n - f(\beta(U^{n-1}))], \varphi\right\rangle\,. \tag{4.3}$$

The next step is to choose a suitable test function $\varphi$. Let us take $\varphi = \tau[\bar{\vartheta}^i - \Theta^i]$ and sum over $i$ from 1 to $m \leq N$. We easily obtain the equality

$$\sum_{i=1}^{m} \int_{I^i} \langle e_u(t), e_\vartheta(t)\rangle \, dt + \sum_{i=1}^{m} \int_{I^i} \langle u^i - u(t), e_\vartheta(t)\rangle \, dt +$$
$$+ \sum_{i=1}^{m} \tau^2 \left\langle \nabla \sum_{n=1}^{i} [\bar{\vartheta}^n - \Theta^n], \nabla [\bar{\vartheta}^i - \Theta^i] \right\rangle =$$
$$= \sum_{i=1}^{m} \int_{I^i} \left\langle \sum_{n=1}^{i} \tau[\bar{f}^n - f(\beta(U^{n-1}))], e_\vartheta(t)\right\rangle dt \,, \quad (4.4)$$

which is written as (I) + (II) + (III) = (IV). The rest of the proof consists in estimating separately each term in the previous expression. To this aim notice first that (3.2) and (2.2), (2.8) yield

$$u - \mu\vartheta = \alpha(u) \quad \text{and} \quad U^n - \mu\Theta^n = \alpha(U^{n-1})\,. \quad (4.5)$$

This is the connection with the discrete-time phase relaxation scheme [29]. Moreover we have

$$e_\vartheta(t) = [\beta(u(t)) - \beta(U^{n-1})] - \frac{1}{\mu}[U^n - U^{n-1}]\,, \quad \text{for} \quad t \in I^n\,. \quad (4.6)$$

According to (4.6) we can split (I) into three terms as follows :

$$(\mathrm{I}) = \mu \|e_\vartheta\|^2_{L^2(0,t^m;L^2(\Omega))} + \sum_{i=1}^{m} \int_{I^i} \langle e_u - \mu e_\vartheta, \beta(u) - \beta(U^{i-1})\rangle \, dt -$$
$$- \frac{1}{\mu} \sum_{i=1}^{m} \int_{I^i} \langle e_u - \mu e_\vartheta, U^i - U^{i-1}\rangle \, dt \quad (\mathrm{I})_1^m + (\mathrm{I})_2 + (\mathrm{I})_3\,. \quad (4.7)$$

The middle term is further bounded by means of (4.5), $(H_\beta)$ and (3.2) ; namely

$$(\mathrm{I})_2 = \sum_{i=1}^{m} \int_{I^i} \langle \alpha(u) - \alpha(U^{i-1}), \beta(u) - \beta(U^{i-1})\rangle \, dt \geq$$
$$\geq l_\beta \sum_{i=1}^{m} \int_{I^i} \|\alpha(u) - \alpha(U^{i-1})\|^2_{L^2(\Omega)} \, dt =$$
$$= l_\beta \|e_u - \mu e_\vartheta\|^2_{L^2(0,t^m;L^2(\Omega))} \quad (\mathrm{I})_2^m\,. \quad (4.8)$$

The last term will be handled differently by distinguishing between the cases $l_\beta = 0$ and $l_\beta > 0$. Let us postpone this discussion and consider now the term (II). It is easy to check that

$$|(\mathrm{II})| = \left| \sum_{i=1}^{m} \int_{I^i} \left\langle \int_t^{t^i} \frac{\partial u}{\partial s}\, ds, e_\vartheta(t) \right\rangle dt \right| \leqslant$$
$$\leqslant \tau \left\| \frac{\partial u}{\partial t} \right\|_{L^2(0,t^m;H^{-s}(\Omega))} \| e_\vartheta \|_{L^2(0,t^m;H^s(\Omega))}, \quad (4.9)$$

where $s$ will be equal to 0 or 1 according to the assumptions of Theorem 4 or Theorems 1, 2, 3, respectively. In order to treat (III) note that from (3.5) the following elementary identity holds

$$2 \sum_{i=1}^{m} a_i \left[ \sum_{n=1}^{i} a_n \right] = \left[ \sum_{i=1}^{m} a_i \right]^2 + \sum_{i=1}^{m} a_i^2, \quad \text{for} \quad a_i \in R^d, \quad 1 \leqslant i \leqslant N.$$

Thus

$$2(\mathrm{III}) = \left\| \nabla \int_0^{t^m} e_\vartheta(t)\, dt \right\|_{L^2(\Omega)}^2 + \tau^2 \sum_{i=1}^{m} \left\| \nabla [\bar\vartheta^i - \Theta^i] \right\|_{L^2(\Omega)}^2. \quad (4.10)$$

It remains to estimate the source term (IV). In view of $(H_f)$, (4.6) and (3.4) we can write

$$|(\mathrm{IV})| \leqslant \frac{1}{2} (\mathrm{I})_1^m + C \sum_{i=1}^{m} \tau \sum_{n=1}^{i} \tau \left\| \bar f^n - f(\beta(U^{n-1})) \right\|_{L^2(\Omega)}^2 \leqslant$$
$$\leqslant \frac{1}{2} (\mathrm{I})_1^m + C \sum_{i=1}^{m} \tau (\mathrm{I})_1^i + C\tau \sum_{i=1}^{m} \| U^i - U^{i-1} \|_{L^2(\Omega)}^2, \quad (4.11)$$

where $C > 0$ is a constant depending only on $T$, $L_f$ and $\mu$. Collecting all the previous bounds yields

$$\| e_\vartheta \|_{L^2(0,t^m;L^2(\Omega))}^2 + l_\beta \| e_u - \mu e_\vartheta \|_{L^2(0,t^m;L^2(\Omega))}^2 + \left\| \nabla \int_0^{t^m} e_\vartheta(t)\, dt \right\|_{L^2(\Omega)}^2 +$$
$$+ \tau \| \nabla \bar e_\vartheta \|_{L^2(0,t^m;L^2(\Omega))}^2 \leqslant C \sum_{i=1}^{m} \int_{I^i} \langle e_u - \mu e_\vartheta, U^i - U^{i-1} \rangle\, dt +$$
$$+ C\tau \left\| \frac{\partial u}{\partial t} \right\|_{L^2(0,T;H^{-s}(\Omega))} \| e_\vartheta \|_{L^2(0,t^m;H^s(\Omega))} +$$
$$+ C\tau \sum_{i=1}^{N} \| U^i - U^{i-1} \|_{L^2(\Omega)}^2 + C \sum_{i=1}^{m} \tau \| e_\vartheta \|_{L^2(0,t^i;L^2(\Omega))}^2, \quad (4.12)$$

where $\bar e_\vartheta(t) := \bar\vartheta^n - \Theta^n$ for $t \in I^n$. After applying the discrete Gronwall inequality, the left hand side of this expression will be shown to be

$O(\tau^{2\nu})$, where $0 < \nu \leq 1$ depends on the regularity of $u_0$, $\beta(u_0)$ and $\beta$ (see the Theorems below). By virtue of the time regularity of $\vartheta$ we get in particular

$$\|e_\vartheta\|_{L^2(Q)} + \left\|\int_0^t e_\vartheta\, ds\right\|_{L^\infty(0,T;H^1(\Omega))} = O(\tau^\nu). \tag{4.13}$$

These estimates lead to the following error bound in $H^{-1}(\Omega)$ for the unknown $u$ :

$$\|e_u\|_{L^\infty(0,T;H^{-1}(\Omega))} = O(\tau^\nu). \tag{4.14}$$

The proof is very simple. Let us introduce the Green operator $G : H^{-1}(\Omega) \to H_0^1(\Omega)$ defined by

$$\langle \nabla G\,\varphi, \nabla\chi\rangle = \langle \varphi, \chi\rangle\ , \quad \forall \chi \in H_0^1(\Omega)\,, \quad \varphi \in H^{-1}(\Omega)\,. \tag{4.15}$$

It is well known (and easy to check !) that

$$\|\varphi\|^2_{H^{-1}(\Omega)} = \|\nabla G\,\varphi\|^2_{L^2(\Lambda)} = \langle \varphi, G\varphi\rangle\ . \tag{4.16}$$

Then, taking $\varphi := G[u^i - U^i]$ as a test function in (4.3) and using (4.15) and (4.16) results in

$$\|u^i - U^i\|^2_{H^{-1}(\Omega)} + \left\langle \nabla \sum_{n=1}^{i} \tau[\bar\vartheta^n - \Theta^n], \nabla G[u^i - U^i]\right\rangle =: (\mathrm{V}) + (\mathrm{VI}) =$$
$$= \left\langle \sum_{n=1}^{i} \tau[\bar f^n - f(\beta(U^{n-1}))], G[u^i - U^i]\right\rangle =: (\mathrm{VII})\,. \tag{4.17}$$

By (4.13) and (4.16) we obviously have

$$|(\mathrm{VI})| \leq \|u^i - U^i\|_{H^{-1}(\Omega)} \left\|\nabla \int_0^{t^i} e_\vartheta\right\|_{L^2(\Omega)} \leq \frac{1}{4}(\mathrm{V}) + C\tau^{2\nu}\,.$$

For (VII) we proceed as in (4.11) and use the Poincaré inequality and the discrete a priori estimates as before to arrive at

$$|(\mathrm{VII})| \leq \frac{1}{4}(\mathrm{V}) + C\tau^{2\nu}\,.$$

Inserting the last two estimates into (4.17) yields the error bound

$$\max_{1 \leq i \leq N} \|u^i - U^i\|^2_{H^{-1}(\Omega)} = O(\tau^{2\nu})\,.$$

The assertion (4.14) follows from the fact that either

$$\frac{\partial u}{\partial t} \in L^2(0, T ; H^{-1}(\Omega)) \quad \text{or} \quad \frac{\partial u}{\partial t} \in L^\infty(0, T ; H^{-1}(\Omega))$$

(since, as we shall see, $\nu \leqslant \frac{1}{2}$ and $\nu \leqslant 1$ respectively).

Consequently it only remains to estimate the right hand side of (4.12) to complete the argument. This will be done by considering various regularity assumptions upon $\beta$, $u_0$ and $\beta(u_0)$. Let us first denote by $(E_i)$ for $i = 1, 2, 3$ the first three terms on the right hand side of (4.12).

### 4.1. The degenerate case ($l_\beta = 0$)

This case corresponds to the so-called *singular parabolic problems*. The main examples are the two-phase Stefan problem (i.e. $\beta(s) = 0$ for $0 \leqslant s \leqslant 1$) and the porous medium equation (i.e. $\beta(s) = s|s|^{m-1}$, $m > 1$). We have the following error estimates.

THEOREM 1 : *Under the assumptions* $(H_\Omega)$, $(H_f)$, $(Hu_0)$ and $(H_\beta)$ *with* $l_\beta = 0$, *we have*

$$\|e_\vartheta\|_{L^2(Q)} + \left\| \int_0^t e_\vartheta \, ds \right\|_{L^\infty(0, T ; H^1(\Omega))} + \|e_u\|_{L^\infty(0, T ; H^{-1}(\Omega))} = O(\tau^{1/4}) . \tag{4.18}$$

This rate is not sharp and reproduces for the nonlinear Chernoff formula the one proved by Verdi & Visintin [29] for the phase relaxation approach.

*Proof*: Let us split the first term $(E_1)$ on the right hand side of (4.12) as follows

$$(E_1) = C \sum_{i=1}^m \int_{I^i} \langle u, U^i - U^{i-1} \rangle - C \sum_{i=1}^m \tau \langle U^i, U^i - U^{i-1} \rangle - \\ - C \sum_{i=1}^m \int_{I^i} \langle e_\vartheta, U^i - U^{i-1} \rangle .$$

Since $u_0 \in L^2(\Omega)$, we know by (2.1) that $u \in L^2(Q)$. Thus, Lemma 1 implies

$$\left| \sum_{i=1}^m \int_{I^i} \langle u, U^i - U^{i-1} \rangle \right| \leqslant \|u\|_{L^2(Q)} \left[ \tau \sum_{i=1}^m \|U^i - U^{i-1}\|^2_{L^2(\Omega)} \right]^{1/2} \leqslant C\tau^{1/2} .$$

At the same time, the middle term is handled by means of (3.5) and $(Hu_0)$ because

$$- \sum_{i=1}^{m} \tau \langle U^i, U^i - U^{i-1} \rangle \leq \frac{\tau}{2} \| U^0 \|_{L^2(\Omega)}^2 \leq C\tau .$$

For the last term we use again Lemma 1 combined with the Cauchy-Schwarz inequality and (3.4) to write

$$\left| \sum_{i=1}^{m} \int_{I^i} \langle e_\vartheta, U^i - U^{i-1} \rangle \right| \leq$$

$$\leq \frac{1}{2} \| e_\vartheta \|_{L^2(0,t^m; L^2(\Omega))}^2 + C\tau \sum_{i=1}^{m} \| U^i - U^{i-1} \|_{L^2(\Omega)}^2 \leq$$

$$\leq \frac{1}{2} \| e_\vartheta \|_{L^2(0,t^m; L^2(\Omega))}^2 + C\tau .$$

Therefore we have $(E_1) \leq C\tau^{1/2} + \frac{1}{2} \| e_\vartheta \|_{L^2(0,t^m; L^2(\Omega))}^2$. In order to bound the second term $(E_2)$ on the right hand side of (4.12) we choose $s = 1$ and use the a priori estimates (2.1) and (3.6) ; hence $(E_2) \leq C\tau$. The remaining term $(E_3)$ in (4.12) is $O(\tau)$ by virtue of (3.6). We then have

$$\| e_\vartheta \|_{L^2(0,t^m; L^2(\Omega))}^2 + \left\| \nabla \int_0^{t^m} e_\vartheta \right\|_{L^2(\Omega)}^2 \leq C\tau^{1/2} + C \sum_{i=1}^{m} \tau \| e_\vartheta \|_{L^2(0,t^i; L^2(\Omega))}^2 .$$

After applying the discrete Gronwall inequality we get (4.13) for $\nu = 1/4$. The complete assertion follows now from (4.14). ■

We can improve the previous rate of convergence under slight stronger assumptions, but still quite reasonable for singular parabolic problems.

THEOREM 2 : *Let $(H_\Omega)$, $(H_f)$ and $(H_\beta)$ with $l_\beta = 0$ hold. Assume in addition that*

$$u_0 \in L^\infty(\Omega) , \qquad \Delta\beta(u_0) \in L^1(\Omega) . \tag{4.19}$$

*Then we have*

$$\| e_\vartheta \|_{L^2(Q)} + \left\| \int_0^t e_\vartheta \, ds \right\|_{L^\infty(0,T; H^1(\Omega))} + \| e_u \|_{L^\infty(0,T; H^{-1}(\Omega))} = O(\tau^{1/2}) . \tag{4.20}$$

This order is sharp for both unknowns $u$ and $\vartheta$. Indeed, they satisfy $u \in H^1(0, T; H^{-1}(\Omega)) \subset C^{0,1/2}([0, T]; H^{-1}(\Omega))$ and

$$\vartheta \in H^{1/2-\delta}(0, T; L^2(\Omega)) \quad \text{for all} \quad \delta > 0 .$$

*Proof*: We proceed along the same lines as before except for the first term on the right hand side of $(E_1)$. Using the summation by parts formula (3.10) we can rewrite this term as follows

$$\sum_{i=1}^{m} \int_{I^i} \langle u, U^i - U^{i-1} \rangle = \sum_{i=1}^{m} \tau \langle \bar{u}^i, U^i - U^{i-1} \rangle =$$

$$= \tau \langle \bar{u}^m, U^m \rangle - \tau \langle u_0, U^0 \rangle - \sum_{i=1}^{m} \tau \langle \bar{u}^i - \bar{u}^{i-1}, U^{i-1} \rangle ,$$

since $\bar{u}^0 := u_0$. The assumption $u_0 \in L^\infty(\Omega)$ together with (2.1) and (3.13) implies that the first and middle term are $O(\tau)$. So it only remains to estimate the last term. Since

$$\bar{u}^i - \bar{u}^{i-1} = \frac{1}{\tau} \int_{I^i} [u(t) - u(t-\tau)]\, dt = \frac{1}{\tau} \int_{I^i} \int_{t-\tau}^{t} \frac{\partial u}{\partial s}(s)\, ds\, dt \tag{4.21}$$

and $\Delta\beta(u_0) \in L^1(\Omega)$ leads to $\dfrac{\partial u}{\partial t} \in L^\infty(0, T; M(\Omega))$ (see $(R2)$), what we have in mind is to use a duality argument between $M(\Omega)$ and $C^0(\bar{\Omega})$. Unfortunately this is not possible because $U^n \notin C^0(\bar{\Omega})$ in general. Therefore we turn momentarily our attention to the effect of a smoothing procedure. Namely, let $u_{0,\delta}$ be a continuous approximation of $u_0 \in L^\infty(\Omega)$ such that the bound $\|u_{0,\delta}\|_{L^\infty(\Omega)} \leq C$ holds uniformly in $\delta > 0$ (the smoothing parameter). Denote by $\{U_\delta^n, \Theta_\delta^n\}$ the solutions of the nonlinear Chernoff formula (2.7)-(2.8) with $U_\delta^0 := u_{0,\delta}$. Then by well known results on (linear) elliptic equations we easily get

$$U_\delta^n \in C^0(\bar{\Omega}) \quad \text{and} \quad U_\delta^n \to U^n \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad \delta \to 0\,, \quad \text{for} \quad 1 \leq n \leq N\,. \tag{4.22}$$

Moreover, the a priori $L^\infty$ estimates (3.13) clearly holds uniformly in ~~$\delta > 0$. From (4.21) and $(R2)$ we have $\|\bar{u}^i - \bar{u}^{i-1}\|_{M(\Omega)} = O(\tau)$. Thus~~

$$\left| \sum_{i=1}^{m} \tau \langle \bar{u}^i - \bar{u}^{i-1}, U_\delta^{i-1} \rangle \right| \leq \sum_{i=1}^{m} \tau \|\bar{u}^i - \bar{u}^{i-1}\|_{M(\Omega)} \|U_\delta^{i-1}\|_{C^0(\bar{\Omega})} \leq C\tau$$

holds uniformly in $\delta > 0$, as well. Now taking the limit in the previous expression as $\delta \to 0$ and using (4.22) results in $\left| \sum_{i=1}^{m} \tau \langle \bar{u}^i - \bar{u}^{i-1}, U^{i-1} \rangle \right| \leq C\tau$. After applying the discrete Gronwall inequality, (4.12) yields the desired estimate (4.20). ■

*Remark* 7 : The error estimate (4.20) and the corresponding proof hold also under the slight different assumption upon $u_0$ stated in Remark 1. Moreover, in view of Remark 5 the function $\beta$ may be assumed to be only locally Lipschitz continuous provided that $f = 0$. ∎

### 4.2. The non-degenerate case ($l_\beta > 0$)

Within this class one finds *mildly nonlinear heat equations*. Since $l_\beta > 0$ we can bound the first term $(E_1)$ on the right hand side of (4.12) as follows

$$|(E_1)| \leq \frac{l_\beta}{2} \|e_u - \mu e_\vartheta\|^2_{L^2(0, t^m; L^2(\Omega))} + C\tau \sum_{i=1}^{m} \|U^i - U^{i-1}\|^2_{L^2(\Omega)} .$$

Therefore the first term can be absorbed into the left hand side of (4.12). It only remains to estimate the second and third terms $(E_2)$ and $(E_3)$, respectively. Before doing this note that

$$\|e_u\|_{L^2(0, t^m; L^2(\Omega))} \leq \mu \|e_\vartheta\|_{L^2(0, t^m; L^2(\Omega))} + \|e_u - \mu e_\vartheta\|_{L^2(0, t^m; L^2(\Omega))} ,$$

so we will get an $L^2$ error estimate also for $u$.

THEOREM 3 : *Under the assumptions* $(H_\Omega)$, $(H_f)$, $(Hu_0)$ *and* $(H_\beta)$ *with* $l_\beta > 0$, *we have*

$$\|e_u\|_{L^2(Q)} + \|e_u\|_{L^\infty(0, T; H^{-1}(\Omega))} + \|e_\vartheta\|_{L^2(Q)} + \\ + \left\| \int_0^t e_\vartheta \, ds \right\|_{L^\infty(0, T; H^1(\Omega))} = O(\tau^{1/2}) . \quad (4.23)$$

This rate is sharp on account of the global in time regularity of $\vartheta$ and $u$ ; namely $\vartheta, u \in H^{1/2}(0, T; L^2(\Omega))$ and

$$u \in H^1(0, T; H^{-1}(\Omega)) \subset C^{0, 1/2}(0, T; H^{-1}(\Omega)) \qquad \text{(see } (R3)) .$$

*Proof*: To bound $(E_2)$ in (4.12) we simply take $s = 1$ and use the time regularity of $u$ recalled just above and the $H_0^1$ a priori estimates (2.1) and (3.6) for $\vartheta$. This results in $(E_2) = O(\tau)$. The remaining term $(E_3)$ is handled by means of the discrete $H^{1/2}(0, T; L^2(\Omega))$ a priori estimate in (3.6) ; hence $(E_3) = O(\tau)$. Finally the assertion (4.23) follows after applying the discrete Gronwall inequality to the resulting expression in (4.12). ∎

Let us now consider $\beta(u_0) \in H_0^1(\Omega)$, which yields $\vartheta \in H^1(0, T; L^2(\Omega))$. So we can expect an error $O(\tau)$ in $L^2$ to hold for any good time discretization of our P.D.E.. Results of this type were first proved by Wheeler [31], who also analyzed the $L^2$-accuracy of some linearization techniques like the extrapolation method of Douglas & Dupont [8] (see also [10, 12, 16, 27] and the references given therein). The nonlinear Chernoff formula (1.3) can be regarded as a linearization procedure with the advantage that, after space discretization, the stiffness matrix remains unchanged in each time step. The purpose of the next theorem is to show that this does not deteriorate the good approximating properties of the scheme or, in other words, that the rate is still $O(\tau)$ in $L^2$. Moreover, the regularity properties assumed for $\beta$ and $f$, i.e. global Lipschitz continuity, are the *minimal* ones compatible with the asserted order.

THEOREM 4 : *Under the assumptions* $(H_\Omega)$, $(H_f)$, $\beta(u_0) \in H_0^1(\Omega)$ *and* $(H_\beta)$ *with* $l_\beta > 0$, *we have*

$$\|e_u\|_{L^2(Q)} + \|e_u\|_{L^\infty(0,T;H^{-1}(\Omega))} + \|e_\vartheta\|_{L^2(Q)} + \left\| \int_0^t e_\vartheta \, ds \right\|_{L^\infty(0,T;H^1(\Omega))} = O(\tau) \quad (4.24)$$

*and*

$$\|e_\vartheta\|_{L^2(0,T;H^1(\Omega))} = O(\tau^{1/2}) . \quad (4.25)$$

Again these rates are sharp in view of the global regularity in time satisfied by $u$ and $\vartheta$ ; namely $u, \vartheta \in H^1(0,T;L^2(\Omega))$ and $\vartheta \in H^{1/2}(0,T;H^1(\Omega))$ (see $(R4)$).

*Proof*: As usual, the proof consists in analyzing the terms $(E_2)$ and $(E_3)$ in (4.12). By choosing $s = 0$ and using the time regularity of $u$ stated above, we can write

$$(E_2) \leq C\tau^2 + \frac{1}{2} \|e_\vartheta\|^2_{L^2(0,t^m;L^2(\Omega))} .$$

Instead for the other term we employ (3.11) to get

$$(E_3) = C\tau^2 \sum_{i=1}^{m} \tau \|\partial U^n\|^2_{L^2(\Omega)} \leq C\tau^2 .$$

Inserting these estimates into (4.12) and using the discrete Gronwall inequality yields the error bound (4.24) and the further information

$$\|\nabla \bar{e}_\vartheta\|_{L^2(Q)} = O(\tau^{1/2}) . \quad (4.26)$$

Moreover we have

$$\begin{aligned}\|\nabla \bar{e}_{\vartheta}\|^2_{L^2(Q)} &= \sum_{n=1}^{N} \int_{I^n} \langle \nabla[\vartheta - \Theta^n], \nabla[\vartheta - \Theta^n]\rangle + \\ &+ \sum_{n=1}^{N} \int_{I^n} \left\langle \nabla[\vartheta - \Theta^n], \nabla[\bar{\vartheta}^n - \vartheta]\right\rangle \geqslant \\ &\geqslant \frac{1}{2} \|\nabla e_{\vartheta}\|^2_{L^2(Q)} - C \sum_{n=1}^{N} \int_{I^n} \left\|\nabla[\bar{\vartheta}^n - \vartheta]\right\|^2_{L^2(\Omega)} .\end{aligned}$$

Since $\vartheta \in H^{1/2}(0, T ; H^1(\Omega))$ interpolation arguments between

$$L^2(0, T ; H^1(\Omega)) \quad \text{and} \quad H^1(0, T ; H^1(\Omega))$$

for the operator $\vartheta \to \vartheta - \bar{\vartheta}$ [15] (where $\bar{\vartheta}$ is defined by $\bar{\vartheta}(t) := \bar{\vartheta}^n$ for $t \in I^n$) imply

$$\sum_{n=1}^{N} \int_{I^n} \left\|\nabla[\bar{\vartheta}^n - \vartheta]\right\|^2_{L^2(\Omega)} = O(\tau) .$$

This completes the proof. ∎

*Remark* 8 : The error estimates (4.18), (4.20), (4.23), (4.24) and (4.26) hold also for (linear) mixed boundary conditions and second order uniformly elliptic operator

$$Au := - \sum_{i,j=1}^{d} \partial_i (a_{ij}(x)\, \partial_j u) + \sum_{j=1}^{d} \partial_j (b_j(x)\, u) + c(x)\, u$$

with sufficiently regular coefficients. For nonhomogeneous Dirichlet boundary data one has to impose either a linear growth of $\beta$ at infinity (see (2.5)) or $u_0 \in L^\infty(\Omega)$. The extensions are straightforward proceeding as in [24]. ∎

*Final remarks*

Let us consider the equation (1.1) with $l_\beta = 0$ and a nonlinear flux condition prescribed on $\partial\Omega$, namely

$$\frac{\partial \beta(u)}{\partial \nu} + g(\beta(u)) = 0 .$$

Error estimates for the associated Crandall-Liggett formula were proved by Nochetto [21, 22]. A related nonlinear Chernoff formula may be obtained by replacing the elliptic P.D.E. in (1.3) with homogeneous Dirichlet data by

$$\begin{cases} \Theta^n + \dfrac{\tau}{\mu} A\Theta^n = \beta(U^{n-1}) + \dfrac{\tau}{\mu} f(\beta(U^{n-1})) \\ \dfrac{\partial \Theta^n}{\partial \nu} + g(\Theta^n) = 0 . \end{cases} \tag{4.27}$$

This scheme was shown to be convergent by Verdi [28] (see also [18]). The error analysis is still an open question.

Another interesting problem is to study a completely linear algorithm obtained, for instance, by replacing the boundary condition in (4.27) by the following one

$$\frac{\partial \Theta^n}{\partial \nu} + g(\Theta^{n-1}) = 0 . \qquad \blacksquare$$

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