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SHAPE OPTIMIZATION IN TWO-DIMENSIONAL ELASTICITY BY THE DUAL FINITE ELEMENT METHOD (*)

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Résumé — *On considère deux problèmes de minimisation de fonction coût par rapport à la partie de la frontière, où un corps elastique est fixé Le critère correspond à* (i) *la fonction de Mises ou Tresca,* (ii) *une norme des forces de réaction sur la partie inconnue de la frontière Le problème élastostatique est résolu par le principe de Castigliano et par des elements finis equilibre On demontre l'existence d'une frontière optimale et la convergence des approximations dans un certain sens*

Abstract — *Two problems of minimization of a cost functional with respect to a part of the boundary, where the elastic body is fixed, are considered The criterion corresponds with* (i) *von Mises or Tresca yield function,* (ii) *a norm of reaction forces on the unknown part of the boundary The elastostatic problem is solved by means of Castigliano principle and equilibrium finite element model The existence of an optimal boundary and some convergence results are proven*

INTRODUCTION

If a part of the boundary of a two-dimensional elastic body is to be determined in such a way that a cost functional of stresses attains its minimum, one can employ the principle of Castigliano (minimum of complementary energy). Thus the approximate cost functional is evaluated directly by means of a piecewise linear stress field, if e.g. the equilibrium finite element model of Watwood and Hartz [12] is used. We consider the case, when zero displacements are prescribed on the part of the boundary, which plays the role of the design variable.

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A convergence of approximations will be studied for two optimization problems, the cost functional of which represents :

(i) a generalization of the well-known Mises or Tresca criterion,

(ii) a suitable norm of the reaction forces on the unknown part of the boundary.

In Section 1 we formulate the two optimization problems. Section 2 contains an equivalent reformulation of the state problem in terms of stresses. We prove the existence of an optimal solution in Section 3. Finite element approximations of the dual state problem are introduced in Section 4 and a continuous dependence of the approximate stress functions on the approximate control is proved. In Sections 5 and 6 we prove that a subsequence of approximate controls exists, which converges to an optimal control function of the first and second optimization problem, respectively.

1. FORMULATION OF THE OPTIMIZATION PROBLEMS

First let us recall the basic relations of linear two-dimensional elastostatics and define a mixed boundary value problem.

Let a body occupy a bounded domain $\Omega \subset R^2$ with a Lipschitz boundary $\partial\Omega$. Assume that

$$\partial\Omega = \bar{\Gamma} \cup \bar{\Gamma}_g , \qquad \Gamma \cap \Gamma_g = \varnothing ,$$

each of the parts Γ and Γ_g being open in $\partial\Omega$.

Henceforth $H^k(\Omega)$ denotes the Sobolev space $W_2^{(k)}(\Omega)$, $k = 0, 1, \ldots, (H^0 = L^2)$, with the norm $\| \cdot \|_{k,\Omega}$ and the inner product $(.,\ .)_{k,\Omega}$. For vector and tensor functions, the Euclidean norm is used with the same symbol for norms and inner products, respectively. We also use the summation convention : a repeated Latin index implies summation over the range 1, 2. The « dot product » of two vectors $u \in R^2$, $v \in R^2$ is defined as follows : $u.v = u_i v_i$.

We introduce the subspace of virtual displacements

$$V(\Omega) = \{ u \in [H^1(\Omega)]^2 : \gamma u = 0 \text{ on } \Gamma \} , \tag{1}$$

where γ denotes the trace operator, strain-displacement relations

$$e_{ij}(u) = \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i) , \tag{2}$$

stress-strain relations

$$\sigma_{ij} = c_{ijkm}\, e_{km} , \tag{3}$$

where

$$c_{ijkm} \in L^\infty(\Omega)\,, \quad c_{ijkm} = c_{kmij} = c_{jikm}\,, \tag{4}$$

$$c_{ijkm}\, t_{ij}\, t_{km} \geqq c_0\, t_{ij}\, t_{ij} \tag{5}$$

holds for almost all $x \in \Omega$, all symmetric 2×2 matrices and some positive constant c_0.

We define the following bilinear form

$$a(y, w) = \int_\Omega c_{ijkm}\, e_{ij}(y)\, e_{km}(w)\, dx \quad \forall y, w \in [H^1(\Omega)]^2 \tag{6}$$

and the functional

$$\mathscr{F}(w) = \int_\Omega F . w \, dx + \int_{\Gamma_g} P . \gamma w \, ds \quad \forall w \in [H^1(\Omega)]^2\,, \tag{7}$$

where $F \in [L^2(\Omega)]^2$ and $P \in [L^2(\Gamma_g)]^2$ are given body and surface forces, respectively.

A function $y \in V(\Omega)$ will be called a weak solution of the *primal state problem*, if

$$a(y, w) = \mathscr{F}(w) \quad \forall w \in V(\Omega)\,. \tag{8}$$

Using (5) and the Korn's inequality ~~(see e.g. [11]), one can prove the~~ existence and uniqueness of the weak solution.

From (4), (5) we deduce that an inverse matrix b exists such that

$$e_{ij} = b_{ijkm}\, \sigma_{km}\,,$$

where $b_{ijkm} \in L^\infty(\Omega)$ satisfy the same symmetry and uniform positive definiteness conditions as c_{ijkm} do. Consequently, the bilinear form

$$(\sigma, \tau)_{B,\Omega} = \int_\Omega b_{ijkm}\, \sigma_{ij}\, \tau_{km}\, dx$$

represents an inner product in the space of symmetric stress tensors

$$S(\Omega) = \{ \sigma = (\sigma_{ij})_{i,j=1}^2 \in [L^2(\Omega)]^4, \sigma_{12} = \sigma_{21} \}$$

and the associated norm $\| \sigma \|_{B,\Omega} = (\sigma, \sigma)_{B,\Omega}^{1/2}$ is equivalent with the norm in $[L^2(\Omega)]^4$. Note that $a(y, w) = (\sigma(y), \sigma(w))_{B,\Omega}$.

We shall consider a class of domains $\Omega = \Omega(v) \subset R^2$ (see *fig.* 1), where

$$\Omega(v) = \{ (x_1, x_2) : 0 < x_1 < v(x_2), 0 < x_2 < 1 \},$$

$$v \in \mathcal{U}_{ad} = \left\{ v \in C^{(0),1}([0, 1]) \text{ (i.e. Lipschitz functions),} \right.$$

$$\left. 0 < \alpha \leqq v \leqq \beta, \, | dv/dx_2 | \leqq C_1 \text{ a.e.}, \int_0^1 v \, dx_2 = C_2 \right\}$$

with given constants α, β, C_1, C_2.

For any $v \in \mathcal{U}_{ad}$, the graph $\Gamma(v)$ of v will coincide with the part $\bar{\Gamma}$ of the boundary, where the body is fixed.

The function $v \in \mathcal{U}_{ad}$ has to be determined from one of the following two optimization problems

$$\mathfrak{J}_i(y(v)) = \min \tag{9}$$

over the set of $v \in \mathcal{U}_{ad}$, $i = 1, 2$.

Here $y(v)$ is the weak solution of the primal state problem (8) on the domain $\Omega(v)$,

$$\mathfrak{J}_1(y(v)) = \int_{\Omega(v)} f(I_1, I_2) \, dx ,$$

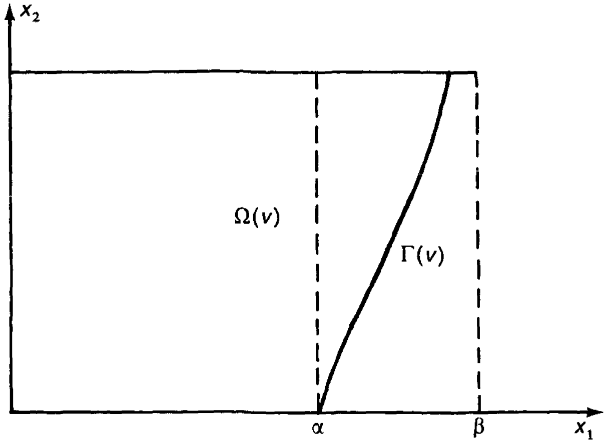

Figure 1.

I_i are the stress tensor invariants

$$I_1 = \sigma_{11} + \sigma_{22}, \qquad I_2 = \sigma_{12}^2 - \sigma_{11}\sigma_{22},$$

$\sigma_{ij} = \sigma_{ij}(y(v))$ are related to $y(v)$ by means of (3) and (2), where $\Omega = \Omega(v)$,

$$f(I_1, I_2) = a_1 I_1^2 + a_2 I_2, \quad a_1, a_2 \in R.$$

Setting $a_1 = 1$, $a_2 = 3$, we obtain a correspondence with the Mises criterion; the choice $a_1 = 1$, $a_2 = 4$ corresponds to the Tresca's criterion (see e.g. [2], chap. 4).

The second cost functional should express a minimization of reaction forces on the unknown part $\Gamma(v)$. The reaction forces belong to the surface tractions, so that we have first of all to establish a suitable definition of surface tractions on a part $\Gamma_0 \subset \partial\Omega$, extending the well-known formula

$$T_i = \sigma_{ij} \nu_j,$$

which is valid for smooth stress tensor components σ_{ij} and the unit outward normal ν to Γ_0.

DEFINITION 1.1 : *Let $\Gamma_0 \subset \partial\Omega$ be a subset of positive length. We introduce a subspace of " complementary " test functions*

$$V^c(\Gamma_0, \Omega) = \{ w \in [H^1(\Omega)]^2 : \gamma w = 0 \text{ on } \partial\Omega - \Gamma_0 \}.$$

~~*Denote for* $u, w \in [H^1(\Omega)]^2$~~

$$[u, w]_\Omega = (e_{ij}(u), e_{ij}(w))_{0,\Omega} + (u, w)_{0,\Omega}$$

$$|||\, u \,|||_\Omega = [u, u]_\Omega^{1/2},$$

$$\mathbb{H}^{1/2}(\Gamma_0) = [H^{1/2}(\Gamma_0)]^2 = \{ \varphi = (\gamma w_1, \gamma w_2) \equiv \gamma w, w \in V^c(\Gamma_0, \Omega) \}.$$

We introduce the following norm of $\varphi \in \mathbb{H}^{1/2}(\Gamma_0)$

$$\| \varphi \|_{1/2, \Gamma_0} = \inf_{\substack{u \in V^c(\Gamma_0, \Omega) \\ \gamma u = \varphi}} |||\, u \,|||. \tag{10}$$

The space of linear continuous functionals on $\mathbb{H}^{1/2}(\Gamma_0)$ will be denoted by $\mathbb{H}^{-1/2}(\Gamma_0)$.

For a subset of $S(\Omega)$ we introduce the operator div (in the sense of distributions) as follows. We say that $\sigma \in S(\Omega)$ belongs to the subspace $H(\mathrm{div}, \Omega)$, if a function $\psi \in [L^2(\Omega)]^2$ exists such that

$$(\sigma, e(w))_{0,\Omega} = -(\psi, w)_{0,\Omega} \quad \forall w \in [C_0^\infty(\Omega)]^2. \tag{11}$$

Then we set

$$\psi = \operatorname{div} \sigma .$$

For σ smooth enough, we have

$$\operatorname{div} \sigma = (\partial \sigma_{1j}/\partial x_j, \partial \sigma_{2k}/\partial x_k) .$$

For $\sigma \in H(\operatorname{div}, \Omega)$ the " norm of graph " is introduced

$$\| \sigma \|_{H(\operatorname{div},\Omega)} = (\| \sigma \|_{0,\Omega}^2 + \| \operatorname{div} \sigma \|_{0,\Omega}^2)^{1/2} .$$

A Green's formula holds for any $\tau \in H(\operatorname{div}, \Omega)$ and $w \in V^c(\Gamma_0, \Omega)$ (see e.g. [1]). Namely, there exists a unique mapping

$$T_{\Gamma_0} \in \mathscr{L}(H(\operatorname{div}, \Omega), \mathbb{H}^{-1/2}(\Gamma_0))$$

such that

$$(\tau, e(w))_{0,\Omega} + (\operatorname{div} \tau, w)_{0,\Omega} = \langle T_{\Gamma_0}(\tau), \gamma w \rangle_{\mathbb{H}^{-1/2}(\Gamma_0), \mathbb{H}^{1/2}(\Gamma_0)} . \tag{12}$$

If $\tau \in [H^1(\Omega)]^4$, $\tau_{12} = \tau_{21}$, then

$$T_{\Gamma_0}(\tau) = (\gamma \tau_{1j} \, \nu_j, \gamma \tau_{2k} \, \nu_k) \in [L^2(\Omega)]^2$$

and $\langle \cdot, \cdot \rangle$ coincides with the scalar product in $[L^2(\Gamma_0)]^2$.

1.1 : *T_{Γ_0} maps $H(\operatorname{div}, \Omega)$ onto $\mathbb{H}^{-1/2}(\Gamma_0)$.*

Proof : Let a $g \in \mathbb{H}^{-1/2}(\Gamma_0)$ be given. Then the problem : find $u \in V^c(\Gamma_0, \Omega)$ such that

$$[u, w]_\Omega = \langle g, \gamma w \rangle \quad \forall w \in V^c(\Gamma_0, \Omega) \tag{13}$$

has precisely one solution $u = u(g)$ (*cf.* e.g. [11]). It is easy to see that $\sigma = e(u(g)) \in H(\operatorname{div}, \Omega)$. By comparison of (11) and (13) we obtain $g = T_{\Gamma_0}(\sigma)$. Q.E.D.

Let us recall the standard norm in $\mathbb{H}^{-1/2}(\Gamma_0)$:

$$\| g \|_{-1/2,\Gamma_0} = \sup_{\substack{\varphi \in \mathbb{H}^{1/2}(\Gamma_0) \\ \varphi \neq 0}} \frac{\langle g, \varphi \rangle}{\| \varphi \|_{1/2,\Gamma_0}} . \tag{14}$$

LEMMA 1.2 : *For any $g \in \mathbb{H}^{-1/2}(\Gamma_0)$ it holds*

$$\| g \|_{-1/2,\Gamma_0} = |\!|\!| u(g) |\!|\!|_\Omega .$$

where $u(g)$ is the solution of the problem (13).

Proof: Let $\sigma = e(u(g))$. Then $T_{\Gamma_0}(\sigma) = g$ and for every $z \in \mathbb{H}^{1/2}(\Gamma_0)$ we have by the formula (12)

$$\langle g, z \rangle = \langle T_{\Gamma_0}(\sigma), z \rangle = (\sigma, e(w))_{0,\Omega} + (\operatorname{div} \sigma, w)_{0,\Omega}$$
$$\forall w \in V^c(\Gamma_0, \Omega),\ \gamma w = z .$$

Hence we may write

$$\langle g, z \rangle \leqq \| \sigma \|_{H(\operatorname{div},\Omega)} \||| w |||_{\Omega} \qquad \forall w \in V^c(\Gamma_0, \Omega), \qquad \gamma w = z ,$$

so that

$$\| g \|_{-1/2,\Gamma_0} \leqq \| \sigma \|_{H(\operatorname{div},\Omega)} \tag{15}$$

holds by virtue of (10) and (14).

Obviously, we have

$$\| \sigma \|_{H(\operatorname{div},\Omega)} = \| e(u(g)) \|_{H(\operatorname{div},\Omega)} = ||| u(g) |||_{\Omega} . \tag{16}$$

Inserting $w = u$ in (13) and using (16), (10), we obtain

$$\langle g, \gamma u(g) \rangle = ||| u(g) |||_{\Omega}^2 = \| \sigma \|_{H(\operatorname{div},\Omega)} ||| u(g) |||_{\Omega}$$
$$\geqq \| \sigma \|_{H(\operatorname{div},\Omega)} \| \gamma u(g) \|_{1/2,\Gamma_0} .$$

Hence

$$\| g \|_{-1/2,\Gamma_0} \geqq \| \sigma \|_{H(\operatorname{div},\Omega)} .$$

Combining this result with (15), (16), the assertion follows. Q.E.D.

From (6), (7), (8) and (11) we conclude that the stress tensor $\sigma(y)$ (calculated from the solution y of (8) on the basis of the relations (3) and (2)) belongs to $H(\operatorname{div}, \Omega)$, $\operatorname{div} \sigma(y) = -F$. The functional $T_{\Gamma_0}(\sigma(y)) \in \mathbb{H}^{-1/2}(\Gamma_0)$ is defined for any part $\Gamma_0 \subset \partial\Omega$ of positive length and Lemma 1.2 enables us to calculate the norm of $T_{\Gamma_0}(\sigma(y))$.

If we choose $\mathfrak{J}_2(\sigma(y))$ to be the norm of $T_{\Gamma}(\sigma(y))$, where $\overline{\Gamma} = \Gamma(v)$, the " values of surface tractions at the end-points of $\Gamma(v)$ " would not be taken into account. This fact would be more apparent if we introduce the finite element approximations (*cf.* Sections 4 and 6). To remove this deficiency, we choose Γ_0 to be an extension of $\Gamma(v)$, i.e.

$$\Gamma(v) \subset \Gamma_0 \subset \partial\Omega , \quad \Gamma_0 \neq \Gamma(v) .$$

Obviously, Γ_0 depends on the control variable v. For simplicity, however, the set

$$\partial\Omega(v) - \Gamma_0(v) = \Gamma_1$$

will be chosen *independent of* v.

Then we set

$$\mathfrak{J}_2(y(v)) = \| T_{\Gamma_0}(\sigma(y(v)) \|^2_{-1/2,\Gamma_0} .$$

In the next Section, we shall employ Lemma 1.2 to the evaluation of the second cost functional.

2. DUAL VARIATIONAL FORMULATION OF THE STATE PROBLEM

Since both the cost functionals are expressed in terms of the stress tensor $\sigma(y)$ and not of the displacement vector y, it seems to be advantageous to employ a suitable dual variational formulation. Thus the stress tensor can be calculated directly on the basis of the principle of minimum complementary energy (Castigliano-Menabrea, see e.g. [11], [5]). To this end we introduce the space of selfequilibriated tensor functions

$$Q_0(\Omega) = \{ \tau \in H(\mathrm{div}, \Omega) : \quad \mathrm{div}\, \tau = 0 \text{ in } \Omega , \quad T_{\Gamma_g}(\tau) = 0 \} .$$

Using the Green's formula (12), we find that

$$Q_0(\Omega) = \{ \tau \in S(\Omega) : \quad (\tau, e(w))_{0,\Omega} = 0 \quad \forall w \in V(\Omega) \} .$$

Assume that a particular tensor field $\sigma^0 \in S(\Omega)$ is available such that

$$(\sigma^0, e(w))_{0,\Omega} = \mathscr{F}(w) \quad \forall w \in V(\Omega) . \tag{17}$$

For the construction of σ^0-see Remark 4.1 in Section 4.

Then a suitable dual formulation of the state problem is : find $\sigma \in Q_0(\Omega)$ such that

$$\Phi(\sigma) \leqq \Phi(\tau) \quad \forall \tau \in Q_0(\Omega) , \tag{18}$$

where

$$\Phi(\tau) = \frac{1}{2} \| \tau \|^2_{B,\Omega} + (\tau, \sigma^0)_{B,\Omega} . \tag{19}$$

One easily obtains an equivalent formulation, which yields the following *Dual state problems* : Find $\sigma \in Q_0(\Omega)$ such that

$$(\sigma, \tau)_{B,\Omega} = -(\sigma^0, \tau)_{B,\Omega} \quad \forall \tau \in Q_0(\Omega)\,. \tag{20}$$

There exists a unique solution σ of (20) and

$$\sigma^0 + \sigma = \sigma(y) \tag{21}$$

holds, where y is the solution of the primal problem (8).

The first cost functional is expressed directly by means of the stress tensor $\sigma(y)$, so that we may write

$$\mathfrak{J}_1(y) = \mathfrak{J}^*(\sigma(y))\,.$$

To simplify the second cost functional, in accordance with Lemma 1.2 we introduce the following *auxiliary problem* : find $u \in V^c(\Gamma_0, \Omega)$ such that

$$[u, w]_\Omega = (\sigma^0 + \sigma, e(w))_{0,\Omega} - (F, w)_{0,\Omega} \quad \forall w \in V^c(\Gamma_0, \Omega)\,. \tag{22}$$

In fact, we have

$$\langle T_{\Gamma_0}(\sigma(y)), \gamma w \rangle = (\sigma(y), e(w))_{0,\Omega} - (F, w)_{0,\Omega}\,, \tag{23}$$

where the formula (12) has been used together with the relation

$$\operatorname{div} \sigma(y) = -F \quad \text{in} \quad \Omega\,. \tag{24}$$

Then substituting from (21) leads to the right-hand side of (22). Note that the problem (22) has a unique solution $u = u(\sigma(y))$. Making use of Lemma 1.2, we may write

$$\begin{aligned}\mathfrak{J}_2(y(v)) &= \| T_{\Gamma_0}(\sigma(y(v))) \|^2_{-1/2,\Gamma_0} = ||| u(\sigma(y(v))) |||^2_{\Omega(v)} \\ &= (\sigma^0 + \sigma(v), e(u(v)))_{0,\Omega(v)} - (F, u(v))_{0,\Omega(v)} \\ &= \mathfrak{J}_2^*(\sigma(v), u(v))\,,\end{aligned} \tag{25}$$

where $\sigma(v)$ and $u(v)$ is the solution of (20) and (22) on the domain $\Omega(v)$, respectively.

Hence we are led to the *optimization problems*

$$\mathfrak{J}_1^*(\sigma(v)) = \min\,, \tag{26}$$

$$\mathfrak{J}_2^*(\sigma(v), u(v)) = \min \tag{27}$$

over the set of $v \in \mathcal{U}_{ad}$. (Here we set $\mathfrak{J}^*(\sigma^0 + \sigma(v)) = \mathfrak{J}_1^*(\sigma(v))$.)

Assume that the body forces F_i are constant everywhere, being represented by the gravitational forces only. Moreover, denoting

$$\Omega_\delta = (0, \delta) \times (0, 1),$$

where $\delta > \beta$, $\delta = \mathrm{Cte}$, $\Gamma_\delta = \{ (x_1, x_2) : x_1 = \delta, 0 < x_2 < 1 \}$, assume that the prescribed surface loading P is defined on the whole $\partial\Omega_\delta - \Gamma_\delta$ and P is piecewise linear, being independent of v.

Finally, assume that a constant $b_0 > 0$ exists such that

$$b_{ijkm}\, t_{ij}\, t_{km} \geqq b_0\, t_{ij}\, t_{ij} \tag{28}$$

holds for almost all $x \in \Omega_\delta$ and all symmetric (2×2) matrices t, $b_{ijkm} \in L^\infty(\Omega_\delta)$.

3. EXISTENCE OF AN OPTIMAL BOUNDARY

We shall prove that at least one solution of the optimization problem (26) or (27) exists. The proof will be based (i) on the compactness of the set $\mathcal{U}_{ad}$ and (ii) on the continuity of the cost functionals with respect to the control function v. To prove (ii), we first verify the continuity of the solution σ of the dual state problem (Proposition 1) and then the continuity of the solution u of the auxiliary problem (Proposition 2).

Proposition 1 : *Let a sequence $\{ v_n \}$, $v_n \in \mathcal{U}_{ad}$, converge to a function v in $C([0, 1])$. Then*

$$\tilde{\sigma}(v_n) \to \tilde{\sigma}(v) \quad in \quad [L^2(\Omega_\delta)]^4 \quad for \quad n \to \infty,$$

where $\tilde{\sigma}(v_n)$ is the solution of (20) *on $\Omega(v_n)$, extended by zero to $\Omega_\delta - \Omega(v_n)$ and $\tilde{\sigma}(v)$ is the solution of* (20) *on $\Omega(v)$, extended by zero to $\Omega_\delta - \Omega(v)$.*

Proof :

1° Henceforth we shall denote

$$\sigma(v_n) = \sigma_n, \quad \Omega(v_n) = \Omega_n, \quad \Omega(v) = \Omega.$$

It follows from (20) and (28)

$$\| \sigma_n \|^2_{B,\Omega_n} \leqq \| \sigma^0 \|_{B,\Omega_n} \| \sigma_n \|_{B,\Omega_n}.$$

Thus we obtain

$$b_0^{1/2} \| \sigma_n \|_{0,\Omega_n} \leqq \| \sigma_n \|_{B,\Omega_n} \leqq \| \sigma^0 \|_{B,\Omega_\delta}$$

and

$$\| \tilde{\sigma}_n \|_{0,\Omega_\delta} = \| \sigma_n \|_{0,\Omega_n} \leqq C \quad \forall n . \tag{29}$$

Therefore a function $\tilde{\sigma} \in [L^2(\Omega_\delta)]^4$ and a subsequence of $\{ \tilde{\sigma}_n \}$ (which will be denoted by the same symbol) exist such that

$$\tilde{\sigma}_n \to \tilde{\sigma} \text{ (weakly) in } [L^2(\Omega_\delta)]^4 . \tag{30}$$

2° We can show that

$$\tilde{\sigma} |_\Omega \in Q_0(\Omega) . \tag{31}$$

In fact, let us consider an arbitrary $w \in V(\Omega)$ and denote its extension to Ω_δ by means of zero by $\tilde{w}$. A sequence $\{ w_\kappa \}$, $\kappa \to 0$, exists such that

$$\begin{gathered} w_\kappa \in [C^\infty(\overline{\Omega}_\delta)]^2 , \quad w_\kappa = 0 \quad \text{on} \quad \overline{\Omega}_\delta - \Omega , \\ \operatorname{supp} w_\kappa \cap \Gamma(v) = \varnothing , \\ w_\kappa \to \tilde{w} \quad \text{in} \quad [H^1(\Omega_\delta)]^2 . \end{gathered} \tag{32}$$

There exists a $n_0(\kappa)$ such that w_κ vanishes on $\Gamma(v_n)$ for $n > n_0(\kappa)$, so that $w_\kappa |_{\Omega_n} \in V(\Omega_n)$ for $n > n_0(\kappa)$.

Since $\sigma_n \in Q_0(\Omega_n)$, we have

$$(\sigma_n, e(w_\kappa))_{0,\Omega_n} = 0 .$$

Using (30), we obtain for $n \to \infty$

$$0 = (\tilde{\sigma}_n, e(w_\kappa))_{0,\Omega_\delta} \to (\tilde{\sigma}, e(w_\kappa))_{0,\Omega_\delta} .$$

Passing to the limit with $\kappa \to 0$ and using (32), we arrive at

$$(\tilde{\sigma}, e(w))_{0,\Omega} = (\tilde{\sigma}, e(\tilde{w}))_{0,\Omega_\delta} = 0 .$$

Consequently, (31) holds.

3° Next we show that

$$\tilde{\sigma} = 0 \text{ a.e. in } \Omega_\delta - \Omega . \tag{33}$$

In fact, let $\tilde{\sigma} \neq 0$ on a set $E \subset \Omega_\delta - \Omega$, mes $E > 0$. Let χ_E be the characteristic function of the set E. We deduce from (30) that

$$(\tilde{\sigma}_n, \chi_E \, \tilde{\sigma})_{0,\Omega_\delta} \to (\tilde{\sigma}, \chi_E \, \tilde{\sigma})_{0,\Omega_\delta} = \| \tilde{\sigma} \|_{0,E}^2 > 0 .$$

On the other hand, we may write

$$(\tilde{\sigma}_n, \chi_E\, \tilde{\sigma})_{0,\Omega_\delta} = (\tilde{\sigma}_n, \tilde{\sigma})_{0,E\cap\Omega_n}$$
$$\leqq \| \tilde{\sigma}_n \|_{0,\Omega_\delta} \| \tilde{\sigma} \|_{0,E\cap\Omega_n} \to 0$$

by virtue of (29) and of the fact that

$$\text{mes}\,(E \cap \Omega_n) \to 0\,.$$

Thus we come to a contradiction.

4° Let us show that the restriction of $\tilde{\sigma}$ to Ω solves the dual problem (20) on Ω.

Let us consider a $\tau \in Q_0(\Omega)$. From Theorem 4.3 of the paper [4] and from its proof we deduce that a sequence $\{ \tau^\kappa \}$, $\kappa \to 0$, exists such that

$$\tau^\kappa \in [C^\infty(\bar{\Omega}_\delta)]^4 \cap S(\Omega_\delta)\,, \tag{34}$$

$$\tau^\kappa |_\Omega \in Q_0(\Omega)\,, \tag{35}$$

$$\text{supp}\, \tau^\kappa \cap (\partial\Omega - \Gamma(v)) = \varnothing\,, \tag{36}$$

$$\| \tau^\kappa - \tau \|_{0,\Omega} \to 0 \quad \text{for} \quad \kappa \to 0\,. \tag{37}$$

In fact, we may first extend τ to a $\tilde{\tau} \in \tilde{Q}_0(\Omega_\delta)$, where

$$\tilde{Q}_0(\Omega_\delta) = \{ \tau \in S(\Omega_\delta) : \text{div}\, \tau = 0 \text{ in } \Omega_\delta,\ T_{\Gamma_g}(\tau) = 0 \}\,,$$

where

$$\Gamma_g = \partial\Omega - \Gamma(v)$$

and then apply the proof of Theorem 4.3 (Case I).

From (36) we easily deduce that

$$\tau^\kappa \in Q_0(\Omega_n) \quad \forall n \geqq n_0(\kappa)\,. \tag{38}$$

By definition, we have

$$(\sigma_n, \tau^\kappa)_{B,\Omega_n} = -(\sigma^0; \tau^\kappa)_{B,\Omega_n}\,. \tag{39}$$

From the weak convergence (30) and (33) we conclude that

$$(\tilde{\sigma}_n, \tau^\kappa)_{B,\Omega_\delta} \to (\sigma, \tau^\kappa)_{B,\Omega_\delta} = (\sigma, \tau^\kappa)_{B,\Omega}\,.$$

Consequently,

$$(\sigma_n, \tau^\kappa)_{B,\Omega_n} \to (\sigma, \tau^\kappa)_{B,\Omega}\,. \tag{40}$$

On the other hand, we obviously have

$$(\sigma^0, \tau^\kappa)_{B,\Omega_n} \to (\sigma^0, \tau^\kappa)_{B,\Omega}\,.$$

Combining this with (39), (40), we obtain

$$(\tilde{\sigma}, \tau^\kappa)_{B,\Omega} = -\,(\sigma^0, \tau^\kappa)_{B,\Omega}\,.$$

Passing to the limit with $\kappa \to 0$ and using (37), we are led to the equation (20). Since the solution of (20) is unique,

$$\tilde{\sigma}\,|_{\Omega} = \sigma(v)$$

follows and the weak convergence (30) holds for the whole sequence $\{\,\tilde{\sigma}_n\,\}$.

5° To prove the strong convergence, we first deduce from (30), (20) and (33) that

$$\|\,\tilde{\sigma}_n\,\|^2_{B,\Omega_\delta} = -\,(\sigma^0, \tilde{\sigma}_n)_{B,\Omega_\delta} \to -\,(\sigma^0, \tilde{\sigma})_{B,\Omega} = \|\,\tilde{\sigma}\,\|^2_{B,\Omega} = \|\,\tilde{\sigma}\,\|^2_{B,\Omega_\delta}\,.$$

Combining the weak convergence with the convergence of norms, we obtain the strong convergence

$$\|\,\tilde{\sigma}_n - \tilde{\sigma}\,\|_{B,\Omega_\delta} \to 0\,.$$

Since the norms $\|\,.\,\|_{0,\Omega_\delta}$ and $\|\,.\,\|_{B,\Omega_\delta}$ are equivalent, the assertion of the Proposition 1 follows. Q.E.D.

PROPOSITION 2 : *Let a sequence* $\{\,v_n\,\}$, $v_n \in \mathcal{U}_{ad}$, *converge to a function* v *in* ~~$C([0,1])$. *Let* $u(v_n)$ *be the solution of* (22) *on the domain* $\Omega(v_n)$ *and with* $\sigma(v_n)$ *in*~~ *the right-hand side.*

Then

$$u(v_n) \rightharpoonup u(v)\ (\textit{weakly})\ \textit{in}\ [H^1(G_m)]^2 \quad \forall m\,,$$

where m is a positive integer and

$$G_m = \{\,(x_1, x_2) : 0 < x_1 < v(x_2) - 1/m,\ 0 < x_2 < 1\,\}\,,$$

$u(v)$ *is the solution of* (22) *on* $\Omega(v)$.

Proof :

1° Denote $\Omega(v) = \Omega$, $u(v_n) = u_n$, $\sigma(v_n) = \sigma_n$ in what follows. Inserting $w = u_n$ in (22) and using (29), we obtain

$$|||\,u_n\,|||^2_{\Omega_n} \leqq \|\,\sigma^0 + \sigma_n\,\|_{0,\Omega_n}\,\|\,e(u_n)\,\|_{0,\Omega_n} + \|\,F\,\|_{0,\Omega_n}\,\|\,u_n\,\|_{0,\Omega_n} \leqq$$
$$\leqq C\,|||\,u_n\,|||_{\Omega_n}\,.$$

Consequently,

$$||| u_n |||_{\Omega_n} \leqq C \quad \forall n .$$

Let us consider a fixed domain G_m. There exists $n_0(m)$ such that

$$G_m \subset \Omega_n \quad \forall n > n_0(m) .$$

Then

$$||| u_n |||_{G_m} \leqq ||| u_n |||_{\Omega_n} \leqq C \quad \forall n > n_0(m) . \tag{41}$$

Consequently, a subsequence $\{ u_{n_1} \}$ exists such that

$$u_{n_1} \rightharpoonup u^{(m)} \text{ (weakly) in } [H^1(G_m)]^2 , \quad n_1 \to \infty , \tag{42}$$

where $u^{(m)} \in [H^1(G_m)]^2$.

For G_{m+1} we obtain a similar assertion, if we choose a proper subsequence $\{ u_{n_2} \}$ of the sequence $\{ u_{n_1} \}$, converging to $u^{(m+1)}$, etc. Let us consider the diagonal subsequence $\{ u_{n_D} \}$ of all subsequences $\{ u_{n_1} \}$, $\{ u_{n_2} \}$, ...

We can prove that a function $u \in [H^1(\Omega)]^2$ exists such that

$$u_{n_D} \to u |_{G_m} \quad \text{(weakly) in } [H^1(G_m)]^2 \tag{43}$$

holds for any m if $n_D \to \infty$.

First we show that

$$u^{(m+k)} |_{G_m} = u^{(m)} \quad \text{a.e. in } G_m \tag{44}$$

for any positive integer k. In fact, let us denote

$$u^{(m+k)} |_{G_m} - u^{(m)} = \psi$$

and let $\tilde{\psi}$ be an extension of ψ by zero to $G_{m+k} - G_m$. Consider the equation

$$(\psi, u_{n_D})_{0,G_m} = (\tilde{\psi}, u_{n_D})_{0,G_{m+k}}$$

and pass to the limit with $n_D \to \infty$ on both sides. Then (42) implies

$$(\psi, u^{(m)})_{0,G_m} = (\tilde{\psi}, u^{(m+k)})_{0,G_{m+k}} = (\psi, u^{(m+k)})_{0,G_m} ,$$

so that

$$\| \psi \|^2_{0,G_m} = (\psi, u^{(m+k)} |_{G_m} - u^{(m)})_{0,G_m} = 0$$

and (44) is proved.

Consequently, we may define

$$u|_{G_m} = u^{(m)} \quad \forall m\,. \tag{45}$$

Since any closed convex set in $[H^1(G_m)]^2$ is weakly closed, (41) and (42) imply

$$|||\, u^{(m)} \,|||_{G_m} \leqq C \quad \forall m\,, \tag{46}$$

so that

$$|||\, u \,|||_{\Omega}^2 = \lim_{m \to \infty} |||\, u^{(m)} \,|||_{G_m}^2 \leqq C$$

follows from (45) and (46). Hence u defined by (45) belongs to $[H^1(\Omega)]^2$ and (43) holds.

2° Let us show that $u = u(\sigma(v))$, i.e. u is a solution of the problem (22). Let a $w \in V^c(\Gamma_0, \Omega)$ be given. There exists a sequence $\{w_\kappa\}$, $\kappa \to 0$, $w_\kappa \in [C^\infty(\overline{\Omega}_\delta)]^2$ such that

$$\Gamma_1 \cap \operatorname{supp} w_\kappa = \emptyset \quad (\Gamma_1 = \partial\Omega_n - \Gamma_{0n},\ \Gamma_{0n} = \text{extension of } \Gamma(v_n)) \tag{47}$$

$$w_\kappa \to w \quad \text{in } [H^1(\Omega)]^2\,. \tag{48}$$

From (47) we deduce that $w_\kappa|_{\Omega_n} \in V^c(\Gamma_{0n}, \Omega_n)$ for all n, κ.

Substituting into (22), we obtain

$$[u_{n_D}, w_\kappa]_{\Omega_{n_D}} = (\sigma^0 + \sigma_{n_D}, e(w_\kappa))_{0,\Omega_{n_D}} - (F, w_\kappa)_{0,\Omega_{n_D}}\,. \tag{49}$$

Let κ be fixed for the time being. For simplicity, we shall write n instead of n_D in what follows. We have

$$|[u_n, w_\kappa]_{\Omega_n} - [u, w_\kappa]_{G_m}| \leqq |[u_n - u, w_\kappa]_{G_m}| + |[u_n, w_\kappa]_{\Omega_n - G_m}| = I_1 + I_2\,.$$

From (43) we obtain $I_1 \to 0$ for any fixed m and $n \to \infty$. From (41) we deduce that

$$I_2 \leqq |||\, u_n \,|||_{\Omega_n}\, |||\, w_\kappa \,|||_{\Omega_n - G_m} \leqq C \,\|\, w_\kappa \,\|_{1,\Omega_n - G_m}$$

holds for $n > n_0(m)$. Therefore, we may write

$$|[u_n, w_\kappa]_{\Omega_n} - [u, w_\kappa]_{\Omega}| \leqq |[u_n, w_\kappa]_{\Omega_n} - [u, w_\kappa]_{G_m}| + |[u, w_\kappa]_{\Omega - G_m}| \leqq$$
$$\leqq I_1 + C \,\|\, w_\kappa \,\|_{1,\Omega_n - G_m} + \tilde{C} \,\|\, w_\kappa \,\|_{1,\Omega - G_m}$$

and conclude that

$$[u_n, w_\kappa]_{\Omega_n} \to [u, w_\kappa]_{\Omega} \quad \text{for } n \to \infty\,. \tag{50}$$

Furthermore, we have

$$\begin{aligned}|(\sigma^0 + \sigma_n, e(w_\kappa))_{0,\Omega_n} - (\sigma^0 + \sigma(v), e(w_\kappa))_{0,\Omega}| \leqq \\ \leqq |(\sigma^0 + \sigma_n, e(w_\kappa))_{0,\Omega_n - G_m}| + |(\sigma^0 + \sigma_n, e(w_\kappa))_{0,G_m} - \\ - (\sigma^0 + \sigma(v), e(w_\kappa))_{0,\Omega}| = I_3 + I_4 . \quad (51)\end{aligned}$$

Using (29), we obtain

$$I_3 \leqq (\| \sigma^0 \|_{0,\Omega_\beta} + \| \sigma_n \|_{0,\Omega_n}) \| e(w_\kappa) \|_{0,\Omega_n - G_m} \to 0 \quad (52)$$

for $m \to \infty$, $n > n_0(m)$, $n \to \infty$.

Making use of Proposition 1, we may write

$$\begin{aligned}I_4 \leqq |(\sigma^0 + \sigma_n, e(w_\kappa))_{0,G_m} - (\sigma^0 + \sigma(v), e(w_\kappa))_{0,G_m}| + \\ + |(\sigma^0 + \sigma(v), e(w_\kappa))_{0,\Omega - G_m}| = \\ = |(\sigma_n - \sigma(v), e(w_\kappa))_{0,G_m}| + |(\sigma^0 + \sigma(v), e(w_\kappa))_{0,\Omega - G_m}| \to 0 \quad (53)\end{aligned}$$

for $m \to \infty$, $n > n_1(m)$, $n \to \infty$.

Combining (50)-(53), we conclude that

$$(\sigma^0 + \sigma_n, e(w_\kappa))_{0,\Omega_n} \to (\sigma^0 + \sigma(v), e(w_\kappa))_{0,\Omega}, \quad n \to \infty . \quad (54)$$

It is readily seen that

$$(F, w_\kappa)_{0,\Omega_n} \to (F, w_\kappa)_{0,\Omega} . \quad (55)$$

Passing to the limit with $n \to \infty$ in (49) and using (50), (54), (55), we arrive at

$$[u, w_\kappa]_\Omega = (\sigma^0 + \sigma(v), e(w_\kappa))_{0,\Omega} - (F, w_\kappa)_{0,\Omega} .$$

Passing to the limit with $\kappa \to 0$ and using (48), we obtain (22) with $\sigma = \sigma(v)$ in the right-hand side.

3° It remains to verify that $\gamma u = 0$ on Γ_1. Since Γ_1 and $\Gamma(v)$ are disjoint by assumption for any $v \in \mathcal{U}_{ad}$, there exists m_0 such that $\Gamma_1 \subset \partial G_{m_0}$. The subspace

$$V^c_{m_0} = \{ w \in [H^1(G_{m_0})]^2 : \gamma w = 0 \quad \text{on } \Gamma_1 \}$$

is weakly closed in $[H^1(G_{m_0})]^2$. Consequently, using (43), we conclude that the weak limit

$$u|_{G_{m_0}} \in V^c_{m_0}$$

and therefore $\gamma u = 0$ on Γ_1, $u \in V^c(\Gamma_0, \Omega)$.

4° By virtue of the uniqueness of the solution of the problem (22), we obtain $u = u(v)$ and the whole sequence $\{ u_n \}$ converges in the sense mentioned in Proposition 2.

PROPOSITION 3 : *Let a sequence $\{ v_n \}$, $v_n \in \mathcal{U}_{ad}$, converge to a function v in $C([0, 1])$. Let $\sigma(v_n)$ and $u(v_n)$ be the solutions of the problem (20) and (22) on the domain $\Omega(v_n)$, respectively. Then*

$$\mathfrak{J}_1^*(\sigma(v_n)) \to \mathfrak{J}_1^*(\sigma(v)), \tag{56}$$

$$\mathfrak{J}_2^*(\sigma(v_n), u(v_n)) \to \mathfrak{J}_2^*(\sigma(v), u(v)) \tag{57}$$

holds for $n \to \infty$.

Proof : Case $i = 1$ is easy, since the function f is a quadratic homogeneous form of the stress tensor components with constant coefficients. In fact, it suffices to verify that

$$(\sigma_{ij}(v_n), \sigma_{km}(v_n))_{0,\Omega(v_n)} = (\tilde{\sigma}_{ij}(v_n), \tilde{\sigma}_{km}(v_n))_{0,\Omega_\delta} \to$$
$$\to (\tilde{\sigma}_{ij}(v), \tilde{\sigma}_{km}(v))_{0,\Omega_\delta} = (\sigma_{ij}(v), \sigma_{km}(v))_{0,\Omega(v)}$$

for any two couples (ij), (km) and $n \to \infty$. The latter convergence, however, follows immediately from Proposition 1.

Case $i = 2$. Let us denote again $u(v_n) = u_n$, $\sigma(v_n) = \sigma_n$, $\Omega(v_n) = \Omega_n$, $\Omega(v) = \Omega$, $u(v) = u$, $\sigma(v) = \sigma$.

On the basis of (25), we may write

$$\mathfrak{J}_2^*(\sigma_n, u_n) = |||\, u_n \,|||_{\Omega_n}^2 = (\sigma^0 + \sigma_n, e(u_n))_{0,\Omega_n} - (F, u_n)_{0,\Omega_n} . \tag{58}$$

First we have

$$| (\sigma^0, e(u_n))_{0,\Omega_n} - (\sigma^0, e(u))_{0,\Omega} | \leqq$$
$$\leqq | (\sigma^0, e(u_n) - e(u))_{0,G_m} | + | (\sigma^0, e(u_n))_{0,\Omega_n - G_m} | +$$
$$+ | (\sigma^0, e(u))_{0,\Omega - G_m} | = I_1 + I_2 + I_3 .$$

Proposition 2 implies that I_1 tends to zero for $n \to \infty$. Making use of (41), we may write

$$I_2 \leqq \| \sigma^0 \|_{0,\Omega_n - G_m} |||\, u_n \,|||_{\Omega_n} \to 0$$

for $m \to \infty$, $n > n_0(m)$, $n \to \infty$.

Obviously, $I_3 \to 0$ for $m \to \infty$. Altogether, we obtain

$$(\sigma^0, e(u_n))_{0,\Omega_n} \to (\sigma^0, e(u))_{0,\Omega} \quad \text{for } n \to \infty . \tag{59}$$

By a parallel way, we deduce

$$(F, u_n)_{0,\Omega_n} \to (F, u)_{0,\Omega}\,. \tag{60}$$

Next we can estimate

$$\begin{aligned}|(\sigma_n, e(u_n))_{0,\Omega_n} - (\sigma, e(u))_{0,\Omega}| &\leqq \\ &\leqq |(\sigma_n, e(u_n))_{0,\Omega_n} - (\sigma, e(u_n))_{0,G_n}| + \\ &\quad + |(\sigma, e(u_n))_{0,\Omega} - (\sigma, e(u))_{0,\Omega}| = I_4 + I_5\,.\end{aligned}$$

By virtue of Proposition 1 and (41), we may write

$$I_4 \leqq \| \tilde{\sigma}_n - \tilde{\sigma} \|_{0,\Omega_\delta} \; |||u_n|||_{\Omega_n} \to 0\,.$$

For I_5 we apply (59), replacing only σ^0 by σ, so that $I_5 \to 0$, as well. Thus we obtain

$$(\sigma_n, e(u_n))_{0,\Omega_n} \to (\sigma, e(u))_{0,\Omega}\,. \tag{61}$$

Passing to the limit with $n \to \infty$ in (58) and using (59), (60) and (61), we arrive at

$$\mathfrak{J}_2^*(\sigma_n, u_n) \to (\sigma^0 + \sigma, e(u))_{0,\Omega} - (F, u)_{0,\Omega} = \mathfrak{J}_2^*(\sigma(v), u(v))\,. \qquad \text{Q.E.D.}$$

THEOREM 1 : *There exists at least one solution of the first or second optimization problem* (26) *and* (27), *respectively.*

Proof: Using Arzelà theorem, we can easily prove that the set $\mathcal{U}_{ad}$ is compact in $C([0, 1])$. Then the existence of a minimizer of the function $v \mapsto \mathfrak{J}_1^*(\sigma(v))$ or $v \mapsto \mathfrak{J}_2^*(\sigma(v), u(v))$ follows immediately from its continuity, which has been verified in Proposition 3. Q.E.D.

4. APPROXIMATIONS OF THE DUAL STATE PROBLEM

We introduce a discretization of the interval [0, 1] in the x_2-coordinate, approximate piecewise linear boundaries and moving meshes of the variable domain. For the solution of approximate dual state problem we shall employ piecewise linear finite element subspaces of the space $Q_0(\Omega_h)$ of self-equilibriated stress fields. Finally, an analogue of Proposition 1 will be proved for the approximate solutions of the dual state problem.

Let N be a positive integer and $h = 1/N$. We denote by Δ_j, $j = 1, 2, \ldots, N$, the subintervals $[(j - 1)\,h, jh]$ of the interval [0, 1] on the x_2-axis and introduce the set

$$\mathcal{U}_{ad}^h = \{ v_h \in \mathcal{U}_{ad} : v_h|_{\Delta_j} \in P_1(\Delta_j) \quad \forall j \},$$

where P_1 denotes the set of linear polynomials. Let Ω_h denote the domain $\Omega(v_h)$, bounded by the graph Γ_h of the function $v_h \in \mathcal{U}_{ad}^h$. The domain Ω_h will be carved into triangles by the following way (see *fig.* 2).

We choose $\alpha_0 \in (0, \alpha)$ and introduce a uniform triangulation of the rectangle $\mathcal{R} = (0, \alpha_0) \times (0, 1)$, independent of v_h, if h is fixed.

In the remaining part $\Omega_h - \mathcal{R}$ let the nodal points divide the intervals $[\alpha_0, v_h(jh)]$ into M equal segments, where

$$M = 1 + [(\beta - \alpha_0) N]$$

and the square brackets denote the integer part of the number inside. One can find easily, that the segments parallel with the x_1-axis are not longer than h and shorter than $h(\alpha - \alpha_0)/(\beta - \alpha_0)$. One also deduces the following estimate for the interior angles of the triangulation

$$\operatorname{tg} \omega \geqq \frac{\alpha - \alpha_0}{\beta - \alpha_0} (1 + C_1 + C_1^2)^{-1}.$$

Consequently, one obtains a regular family $\{ \mathcal{T}_h(v_h) \}$, $h \to 0$, $v_h \in \mathcal{U}_{ad}^h$, of triangulations.

Note that for any $v_h \in \mathcal{U}_{ad}^h$ we construct a unique triangulation $\mathcal{T}_h(v_h)$.

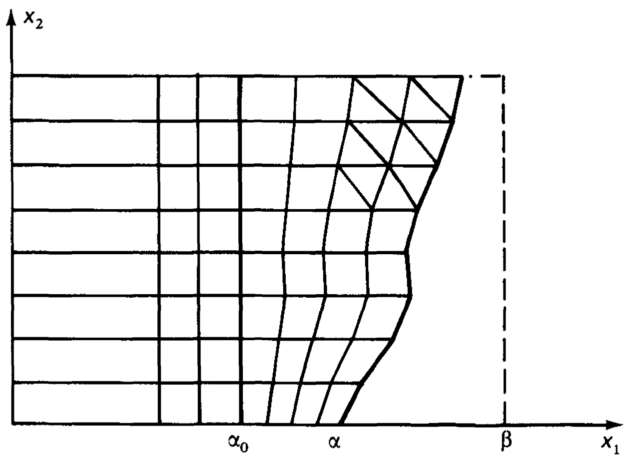

Figure 2.

We shall employ the spaces $\mathcal{N}_h(\Omega_h)$ of piecewise linear triangular block-elements, which have been proposed by Watwood and Hartz [12]. Each triangle $K \in \mathcal{T}_h(v_h)$ is divided into 3 subtriangles K_i, $i = 1, 2, 3$, by connecting the vertices a_i with the center of gravity O. We define the following subspaces

$$\mathcal{M}(K_i) = \{ \tau \in S(K_i) \cap [P_1(K_i)]^4 , \quad \operatorname{div} \tau = 0 \} .$$

Then the set

$$\mathcal{N}(K) = \{ \tau = (\tau^1, \tau^2, \tau^3), \tau |_{K_i} = \tau^i \in \mathcal{M}(K_i) ,$$
$$T_{\Gamma_{0i}}(\tau^i) + T_{\Gamma_{0i}}(\tau^{i-1}) = 0,\ i = 1, 2, 3 \} ,$$

where $$\Gamma_{0i} = Oa_i,\ \tau^0 \equiv \tau^3 ,$$

consists of three linear divergence-free tensor fields τ^i, the surface tractions of which are continuous when crossing the common boundary of any two subtriangles. Let us define

$$\mathcal{N}_h(\Omega_h) = \{ \tau \in S(\Omega_h) : \tau |_K \in \mathcal{N}(K) \quad \forall K \in \mathcal{T}_h(v_h) ,$$
$$T_{K \cap K'}(\tau |_K) + T_{K \cap K'}(\tau |_{K'}) = 0 \quad \forall K, K' \in \mathcal{T}_h(v_h) \} .$$

Thus $\mathcal{N}_h(\Omega_h)$ is the space composed of tensor fields from $\mathcal{N}(K)$, the surface tractions of which are continuous on any interelement boundary.

Finally, we define

$$Q_h(\Omega_h) = \mathcal{N}_h(\Omega_h) \cap Q_0(\Omega_h) =$$
$$= \{ \sigma^h \in \mathcal{N}_h(\Omega_h) : T_{\Gamma_{gh}}(\sigma^h) = 0 \} , \quad \Gamma_{gh} = \partial\Omega_h - \Gamma_h .$$

In proving the following analogue of Proposition 1, we shall apply a density result concerning the space $Q_0(\Omega)$ and some approximability properties of the spaces $\mathcal{N}_h(\Omega_h)$, which were proved in the paper [4].

Remark 4.1 : Now we are able to suggest a construction of a particular field $\sigma^0 \in S(\Omega_\delta)$, which satisfies the condition (17) on any domain $\Omega(v_h)$, where $v_h \in \mathcal{U}_{ad}^h$.

Assume for simplicity that the gravitational forces act in the direction of x_2-axis. Then $F_1 = 0$, $F_2 = - \rho g = \text{Cte}$ and the function

$$\tau^0 = \begin{bmatrix} 0, & 0 \\ 0, & \rho g x_2 \end{bmatrix}$$

satisfies the homogeneous equilibrium equations in Ω_δ. Setting

$$\sigma^0 = \tau^0 + \lambda ,$$

we are led to the following problem : find $\lambda \in H(\operatorname{div}, \Omega_\delta)$ such that

$$\operatorname{div} \lambda = 0 \text{ in } \Omega_\delta , \quad T_{\Gamma_{0\delta}}(\lambda) = P_i - \tau_{ij}^0 \nu_j \equiv P_i^0 ,$$

where $\Gamma_{0\delta} = \partial\Omega_\delta - \Gamma_\delta$. Since P_i are piecewise linear by assumption and the term $\tau_{ij}^0 \nu_j$ is linear on every side of $\partial\Omega_\delta$, P_i^0 is piecewise linear. Consequently, a suitable fixed triangulation $\mathcal{T}_{h_0}$ of Ω_δ exists such that λ can be found in the space $\mathcal{N}_{h_0}(\Omega_\delta)$. One easily verifies that σ^0 satisfies (17) for any $\Omega(v)$ with $v \in \mathcal{U}_{ad}$.

Instead of the problem (20) we shall solve the following *approximate state problem* : find $\sigma^h \in Q_h(\Omega_h)$ such that

$$(\sigma^h, \tau^h)_{B,\Omega_h} = -(\sigma^0, \tau^h)_{B,\Omega_h} \quad \forall \tau^h \in Q_h(\Omega_h) . \tag{62}$$

There exists a unique solution of the problem (62) for any h and $v_h \in \mathcal{U}_{ad}^h$. Moreover, one can prove the following.

LEMMA 4.1 : *Let $\{ v_h \}$, $h \to 0$, be a sequence of $v_h \in \mathcal{U}_{ad}^h$, converging uniformly to a function v. Then*

$$\tilde{\sigma}^h \to \tilde{\sigma}(v) \quad \text{in} \quad [L^2(\Omega_\delta)]^4 \quad \text{for} \quad h \to 0 ,$$

where $\tilde{\sigma}^h$ is the solution of (62), *extended by zero to the domain $\Omega_\delta - \Omega_h$ and $\tilde{\sigma}(v)$ is the solution of* (20) *on the domain $\Omega(v)$, extended by zero to $\Omega_\delta - \Omega(v)$.*

Proof : Following the argument used in the proof of Proposition 1, we show that a subsequence of $\{ \tilde{\sigma}^h \}$ exists such that

$$\tilde{\sigma}^h \rightharpoonup \tilde{\sigma} \text{ (weakly)} \quad \text{in} \quad [L^2(\Omega_\delta)]^4 , \tag{63}$$

$$\tilde{\sigma} \,|_{\Omega(v)} \in Q_0(\Omega(v)) , \tag{64}$$

$$\tilde{\sigma} = 0 \quad \text{a.e.} \quad \text{in} \quad \Omega_\delta - \Omega(v) . \tag{65}$$

Given any $\tau \in Q_0(\Omega(v))$, we consider a sequence $\{ \tau^\kappa \}$, $\kappa \to 0$, satisfying (34), (35), (36), (37) and such that $\tau^\kappa \in Q_0(\Omega_h)$ holds for all $h < h_1(\kappa)$.

In the paper [4] a projection operator

$$r_h : [C^\infty(\bar{\Omega}_\delta)]^4 \cap Q_0(\Omega_h) \to \mathcal{N}_h(\Omega_h)$$

has been introduced. The properties of τ^κ and r_h imply that

$$r_h \tau^\kappa \in Q_h(\Omega_h) \quad \forall h < h_1(\kappa)$$

(*cf.* the proof of Theorem 3.1 in [4]), and the following estimate holds (*cf.* [4], Theorem 2.5)

$$\| r_h \tau^\kappa - \tau^\kappa \|_{0,\Omega_h} \leqq Ch^2 \| \tau^\kappa \|_{2,\Omega_\delta} \tag{66}$$

with C independent of h, Ω_h and τ^κ.

By virtue of (62) we have

$$(\sigma^h, r_h \tau^\kappa)_{B,\Omega_h} = -(\sigma^0, r_h \tau^\kappa)_{B,\Omega_h}. \tag{67}$$

Let us extend $r_h \tau^\kappa$ to $\Omega_\delta - \Omega_h$ by zero and denote the extension by the same symbol. We may write

$$\begin{aligned} &|(\tilde{\sigma}^h, r_h \tau^\kappa)_{B,\Omega_\delta} - (\tilde{\sigma}, \tau^\kappa)_{B,\Omega_\delta}| \leqq \\ &\qquad \leqq |(\tilde{\sigma}^h, r_h \tau^\kappa - \tau^\kappa)_{B,\Omega_\delta}| + |(\tilde{\sigma}^h - \tilde{\sigma}, \tau^\kappa)_{B,\Omega_\delta}|. \end{aligned} \tag{68}$$

The last term tends to zero if $h \to 0$, by virtue of (63). The first term can be estimated as follows

$$|(\tilde{\sigma}^h, r_h \tau^\kappa - \tau^\kappa)_{B,\Omega_\delta}| \leqq C \| \tilde{\sigma}^h \|_{0,\Omega_\delta} \| r_h \tau^\kappa - \tau^\kappa \|_{0,\Omega_h} \to 0,$$

where (66) and the boundedness of $\{ \tilde{\sigma}^h \}$ has been used.

Consequently, using also (65), we arrive at

$$(\sigma^h, r_h \tau^\kappa)_{B,\Omega_h} = (\tilde{\sigma}^h, r_h \tau^\kappa)_{B,\Omega} \to (\sigma, \tau^\kappa)_{B,\Omega(v)} \tag{69}$$

for $h \to 0$. Furthermore, we may write

$$\begin{aligned} &|(\sigma^0, r_h \tau^\kappa)_{B,\Omega_h} - (\sigma^0, \tau^\kappa)_{B,\Omega(v)}| \leqq \\ &\qquad \leqq |(\sigma^0, r_h \tau^\kappa - \tau^\kappa)_{B,\Omega_h}| + |(\sigma^0, \tau^\kappa)_{B,\Omega_h} - (\sigma^0, \tau^\kappa)_{B,\Omega(v)}| \to 0 \end{aligned} \tag{70}$$

if we use (66) and the convergence of v_h to v.

Passing to the limit with $h \to 0$ in the equation (67) and using (69), (70), we arrive at

$$(\sigma, \tau^\kappa)_{B,\Omega(v)} = -(\sigma^0, \tau^\kappa)_{B,\Omega(v)}.$$

Passing to the limit with $\kappa \to 0$ and using (37), we obtain the equation (20) on $\Omega(v)$. Since the solution of the problem (20) is unique, $\sigma = \sigma(v)$ follows and the weak convergence (63) holds for the whole sequence $\{ \tilde{\sigma}^h \}$.

The strong convergence can be proved by a way parallel to that of proof of Proposition 1 (5^0).

5. APPROXIMATIONS OF THE FIRST OPTIMIZATION PROBLEM

First we shall prove the continuity of the cost functional with respect to the control function. Then a convergence of approximate solutions can be proven by a standard way.

LEMMA 5.1 : *Let $\{ v_h \}$, $h \to 0$, be a sequence of $v_h \in \mathcal{U}^h_{ad}$, converging uniformly to a function v. Let $\sigma^h(v_h)$ be the solutions of the approximate state problems* (62). *Then*

$$\mathcal{J}_1^*(\sigma^h(v_h)) \to \mathcal{J}_1^*(\sigma(v)), \quad h \to 0,$$

where $\sigma(v)$ is the solution of (20) *on the domain $\Omega(v)$.*

Proof : is analogous to that of Proposition 3, $i = 1$.

THEOREM 5.1 : *Let $\{ \omega_h \}$, $h \to 0$, $\omega_h \in \mathcal{U}^h_{ad}$, be a sequence of solutions of the following approximate problem*

$$\mathcal{J}_1^*(\sigma^h(\omega_h)) \leqq \mathcal{J}_1^*(\sigma^h(v_h)) \quad \forall v_h \in \mathcal{U}^h_{ad}\,. \tag{71}$$

Then a subsequence $\{ \omega_{\tilde{h}} \}$ exists such that

$$\begin{aligned} \omega_{\tilde{h}} &\to \omega \quad \text{in} \quad C([0,1])\,, \\ \tilde{\sigma}^{\tilde{h}}(\omega_{\tilde{h}}) &\to \tilde{\sigma}(\omega) \quad \text{in} \quad [L^2(\Omega_\delta)]^4 \end{aligned} \tag{72}$$

holds for $\tilde{h} \to 0$, where $\tilde{\sigma}^{\tilde{h}}$ is the solution of the approximate state problem (62) *on the domain $\Omega_{\tilde{h}} = \Omega(\omega_{\tilde{h}})$, extended by zero, $\tilde{\sigma}(\omega)$ is the solution of* (20) *on $\Omega(\omega)$, extended by zero and ω is a solution of the first optimization problem* (26).

Any uniformly convergent subsequence of $\{ \omega_h \}$ tends to a solution of (26) *and* (72) *holds.*

Proof : Let us consider a $v \in \mathcal{U}_{ad}$. There exists a sequence $\{ v_h \}$, $h \to 0$, such that $v_h \in \mathcal{U}^h_{ad}$, $v_h \to v$ in $C([0, 1])$ (see [3], Lemma 7.1).

Since $\mathcal{U}_{ad}$ is compact in $C([0, 1])$, a subsequence $\{ \omega_{\tilde{h}} \}$ and $\omega \in \mathcal{U}_{ad}$ exist such that $\omega_{\tilde{h}} \to \omega$ in $C([0, 1])$ for $\tilde{h} \to 0$.

By definition (71) we have

$$\mathfrak{J}_1^*(\tilde{\sigma}^{\tilde{h}}(\omega_{\tilde{h}})) \leqq \mathfrak{J}_1^*(\tilde{\sigma}^{\tilde{h}}(v_{\tilde{h}}))\,.$$

Applying Lemma 5.1 to both the sequences $\{\omega_{\tilde{h}}\}$ and $\{v_{\tilde{h}}\}$, we obtain

$$\mathfrak{J}_1^*(\sigma(\omega)) \leqq \mathfrak{J}_1^*(\sigma(v))\,.$$

Consequently, ω is a solution of the optimization problem (26). The convergence (72) follows from Lemma 4.1 and the rest of the assertion is obvious.

LEMMA 5.2 : *The problem* (71) *has at least one solution for any h.*

Proof : Denoting by $a \in R^{N+1}$ the vector of nodal values

$$v_h(ih) = a_i, \quad i = 0, 1, \dots, N\,,$$

one can prove that

(i) $$v_h \in \mathcal{U}_{ad}^h \Leftrightarrow a \in \mathcal{A}$$

where $\mathcal{A}$ is compact and

(ii) the function $a \mapsto \mathfrak{J}_1^*(\sigma^h(a))$ is continuous on $\mathcal{A}$. Consequently, a minimizing vector exists.

6. APPROXIMATIONS OF THE SECOND OPTIMIZATION PROBLEM

As we have seen in Section 2, the second cost functional can be written in terms of the solution u of the auxiliary problem (22). Instead of the latter, however, we shall solve an approximate problem, using the subspace

$$V_h^c \subset [C(\bar{\Omega}_h)]^2 \cap V^c(\Gamma_{0h}, \Omega_h)$$

of standard piecewise linear finite elements on the triangulation $\mathcal{T}_h(v_h)$. Let us choose

$$\Gamma_1 = \partial\Omega_h - \Gamma_{0h} = \partial\mathcal{R} \cap \partial\Omega_h \tag{73}$$

so that Γ_1 is independent of h and v_h.

We define the following problem : find $u_h \in V_h^c$ such that

$$[u_h, w_h]_{\Omega_h} = \int_{\Gamma_h} (\sigma^0 + \sigma^h).\,\nu w_h\,ds + \int_{\Gamma_{0h}-\Gamma_h} P.w_h\,ds \quad \forall w_h \in V_h^c\,. \tag{74}$$

Note that replacing $\Omega(v)$ by Ω_h, $\sigma(v)$ by $\sigma^h \in Q_h(\Omega_h)$ and w by $w_h \in V_h^c$, the right-hand side of (22) can be transformed into that of (74) by means of integration by parts. In fact, making use of Remark 4.1, we may write

$$(\sigma^0 + \sigma^h, e(w_h))_{0,\Omega_h} - (F, w_h)_{0,\Omega_h} =$$
$$= \int_{\partial\Omega_h} (\sigma^0 + \sigma^h).\nu w_h \, ds = \int_{\Gamma_h} (\sigma^0 + \sigma^h).\nu w_h \, ds +$$
$$+ \int_{\Gamma_{0h} - \Gamma_h} P.w_h \, ds . \tag{75}$$

By means of (74) we can define the approximate second cost functional

$$J_2^h(v_h) = \mathfrak{J}_{2h}^*(\sigma^h(v_h), u_h) = |||\, u_h \,|||_{\Omega_h}^2 =$$
$$= \int_{\Gamma_h} (\sigma^0 + \sigma^h(v_h)).\nu u_h \, ds + \int_{\Gamma_{0h} - \Gamma_h} P.u_h \, ds , \tag{76}$$

where $u_h = u_h(v_h)$ and $\sigma^h(v_h)$ is the solution of (74) and (62), respectively. Then the second optimization problem (27) can be replaced by the following approximate problem

$$J_2^h(v_h) = \min , \quad v_h \in \mathcal{U}_{ad}^h . \tag{77}$$

LEMMA 6.1 : *The problem* (77) *has at least one solution for any h.*

Proof : Is analogous to that of Lemma 5.2.

LEMMA 6.2 : *Let* $\{ v_h \}$, $h \to 0$, *be a sequence of* $v_h \in \mathcal{U}_{ad}^h$, *converging uniformly to a function v. Let* u_h *be the solution of* (74). *Then*

$$u_h \rightharpoonup u \,(\text{weakly}) \quad \text{in} \quad [H^1(G_m)]^2 \quad \forall m ,$$

and $u = u(\sigma(v))$ is the solution of (22) on $\Omega(v)$.

Proof : Following the argument used in the proof of Proposition 2, we show that a function $u \in [H^1(\Omega(v))]^2$ exists such that a diagonal subsequence

$$u_{h_D} \rightharpoonup u\,|_{G_m} \quad \text{in} \quad [H^1(G_m)]^2 \quad (\text{weakly}) .$$

Given any $w \in V^c(\Gamma_0, \Omega)$, we consider a sequence $\{ w_\kappa \}$, $\kappa \to 0$, $w_\kappa \in [C^\infty(\overline{\Omega}_\delta)]^2$

such that

$$\Gamma_1 \cap \operatorname{supp} w_\kappa = \emptyset\,, \tag{78}$$

$$w_\kappa \to w \quad \text{in} \quad [H^1(\Omega)]^2\,. \tag{79}$$

From (78) $w_\kappa|_\Omega \in V^c(\Gamma_0, \Omega)$ follows.

Let $\pi_h w_\kappa$ denote the Lagrange linear interpolate of w_κ on the triangulation $\mathcal{T}_h(v_h)$. Consequently,

$$\pi_h w_\kappa \in V_h^c \quad \forall h\,, \quad \forall \kappa\,.$$

Let κ be fixed for the time being. Obviously, we may insert $\pi_h w_\kappa$ into (74) and use (75) to obtain

$$[u_h, \pi_h w_\kappa]_{\Omega_h} = (\sigma^0 + \sigma^h, e(\pi_h w_\kappa))_{0,\Omega_h} - (F, \pi_h w_\kappa)_{0,\Omega_h}\,. \tag{80}$$

We shall consider (80) for the subsequence $\{ h_D \}$ and pass to the limit with $h_D \to 0$. For simplicity, however, we shall write indeces h instead of h_D. Thus we may write

$$\begin{aligned} &| [u_h, \pi_h w_\kappa]_{\Omega_h} - [u, w_\kappa]_{G_m} | = \\ &= | [u_h, w_\kappa]_{G_m} + [u_h, \pi_h w_\kappa - w_\kappa]_{G_m} + [u_h, \pi_h w_\kappa]_{\Omega_h - G_m} - [u, w_\kappa]_{G_m} | \\ &\leqq | [u_h - u, w_\kappa]_{G_m} | + | [u_h, \pi_h w_\kappa - w_\kappa]_{G_m} | + | [u_h, \pi_h w_\kappa]_{\Omega_h - G_m} | \\ &= I_1 + I_2 + I_3\,. \end{aligned} \tag{81}$$

Consider a positive ε. From the weak convergence of $\{ u_h \}$ in $[H^1(G_m)]^2$ we conclude that

$$I_1 < \varepsilon/6 \quad \forall h < h_1(\varepsilon, m)\,. \tag{82}$$

To estimate I_2, we employ the well-known result

$$\| w_\kappa - \pi_h w_\kappa \|_{1,\Omega_h} \leqq Ch \| w_\kappa \|_{2,\Omega_h} \leqq Ch \| w_\kappa \|_{2,\Omega_\delta}\,. \tag{83}$$

Combining this with the boundedness of the norms $||| u_h |||_{G_m}$, we obtain

$$I_2 \leqq ||| u_h |||_{G_m} ||| \pi_h w_\kappa - w_\kappa |||_{G_m} \leqq Ch \| w_\kappa \|_{2,\Omega_\delta} < \varepsilon/6\,, \quad h < h_2\,. \tag{84}$$

It remains to estimate I_3. For all triangles $K \in \mathcal{T}_h(v_h)$ and h we have

$$\| \pi_h w_\kappa \|_{1,K} \leqq C \| w_\kappa \|_{2,K}\,. \tag{85}$$

Let G_m^h be the smallest union U of triangles $K \in \mathcal{T}_h(v_h)$ such that $U \supset \Omega_h - G_m$. Obviously, we may write

$$\text{meas}\,(G_m^h) \leqq m^{-1} + 2\,h + \|\, v_h - v\,\|_\infty \tag{86}$$

where the last terms stands for the norm in $C([0, 1])$.

Consequently, (85) yields

$$\|\,\pi_h\, w_\kappa\,\|_{1,\Omega_h - G_m} \leqq \|\,\pi_h\, w_\kappa\,\|_{1,G_m^h} \leqq C\,\|\,w_\kappa\,\|_{2,G_m^h}\,. \tag{87}$$

Making use of the boundedness of norms $|||\,u_h\,|||_{\Omega_h}$ and of (87), we obtain

$$I_3 \leqq |||\,u_h\,|||_{\Omega_h}\ |||\,\pi_h\, w_\kappa\,|||_{\Omega_h - G_m} \leqq C\,\|\,w_\kappa\,\|_{2,G_m^h}\,. \tag{88}$$

Combining (81), (82) and (88), we arrive at

$$\begin{aligned} |\,[u_h, \pi_h\, w_\kappa]_{\Omega_h} - [u, w_\kappa]_\Omega\,| &\leqq |\,[u_h, \pi_h\, w_\kappa]_{\Omega_h} - [u, w_\kappa]_{G_m}\,| + |\,[u, w_\kappa]_{\Omega - G_m}\,| \leqq \\ &\leqq \varepsilon/3 + C\,\|\,w_\kappa\,\|_{2,G_m^h} + |||\,u\,|||_{\Omega(v)}\ |||\,w_\kappa\,|||_{\Omega(v) - G_m}\,. \end{aligned}$$

By virtue of (86) we conclude that

$$[u_h, \pi_h\, w_\kappa]_{\Omega_h} \to [u, w_\kappa]_{\Omega(v)}\,, \quad h \to 0\,.$$

Furthermore, we have

$$\begin{aligned} &|\,(\sigma^0 + \sigma^h, e(\pi_h\, w_\kappa))_{0,\Omega_h} - (\sigma^0 + \sigma(v), e(w_\kappa))_{0,\Omega}\,| = \\ &\qquad = |\,(\sigma^0 + \sigma^h, e(\pi_h\, w_\kappa) - e(w_\kappa))_{0,\Omega_h} + (\sigma^0 + \sigma^h, e(w_\kappa))_{0,\Omega_h} \\ &- (\sigma^0 + \sigma^h, e(w_\kappa))_{0,G_m} + (\sigma^0 + \sigma^h, e(w_\kappa))_{0,G_m} \\ &- (\sigma^0 + \sigma(v), e(w_\kappa))_{0,\Omega}\,| \leqq \\ &\leqq |\,(\sigma^0 + \sigma^h, e(\pi_h\, w_\kappa - w_\kappa))_{0,\Omega_h} + \\ &+ |\,(\sigma^0 + \sigma^h, e(w_\kappa))_{0,\Omega_h - G_m}\,| + |\,(\sigma^0 + \sigma^h, e(w_\kappa))_{0,G_m} \\ &- (\sigma^0 + \sigma(v), e(w_\kappa))_{0,\Omega}\,| = I_{1h} + I_{2h} + I_{3h}\,. \end{aligned} \tag{90}$$

Using the boundedness of norms of σ^h and (83), we may write

$$I_{1h} \leqq (\|\,\sigma^0\,\|_{0,\Omega_\beta} + \|\,\sigma^h\,\|_{0,\Omega_h})\,\|\,\pi_h\, w_\kappa - w_\kappa\,\|_{1,\Omega_h} \to 0\,, \quad h \to 0\,, \tag{91}$$

$$I_{2h} \leqq (\|\,\sigma^0\,\|_{0,\Omega_\beta} + \|\,\sigma^h\,\|_{0,\Omega_h})\,\|\,e(w_\kappa)\,\|_{0,\Omega_h - G_m} \to 0 \tag{92}$$

for $m \to \infty$, $h < h_0(m)$, $h \to 0$.

Finally, making use of Lemma 4.1, we obtain

$$I_{3h} \leqq \left| (\sigma^0 + \sigma^h, e(w_\kappa))_{0,G_m} - (\sigma^0 + \sigma(v), e(w_\kappa))_{0,G_m} \right| + \\ + \left| (\sigma^0 + \sigma(v), e(w_\kappa))_{0,\Omega - G_m} \right| = \left| (\sigma^h - \sigma(v), e(w_\kappa))_{0,G_m} \right| \\ + \left| (\sigma^0 + \sigma(v), e(w_\kappa))_{0,\Omega - G_m} \right| \to 0 \qquad (93)$$

for $m \to \infty$, $h < h_1(m)$, $h \to 0$.

Combining (90)-(93), we deduce that

$$(\sigma^0 + \sigma^h, e(\pi_h w_\kappa))_{0,\Omega_h} \to (\sigma^0 + \sigma(v), e(w_\kappa))_{0,\Omega(v)}, \; h \to 0 . \qquad (94)$$

We also have

$$\left| (F, \pi_h w_\kappa)_{0,\Omega_h} - (F, w_\kappa)_{0,\Omega} \right| \leqq \\ \leqq \left| (F, \pi_h w_\kappa - w_\kappa)_{0,\Omega_h} \right| + \left| (F, w_\kappa)_{0,\Omega_h} - (F, w_\kappa)_{0,\Omega} \right| \\ \leqq \| F \|_{0,\Omega_\beta} \| \pi_h w_\kappa - w_\kappa \|_{0,\Omega_h} + \left| (F, w_\kappa)_{0,\Delta(\Omega_h,\Omega)} \right| \to 0 \qquad (95)$$

by virtue of (83). (Here $\Delta(\Omega_h, \Omega) = (\Omega_h - \Omega) \cup (\Omega - \Omega_h)$.)

Passing to the limit with $h \to 0$ and using (89), (94), (95), we arrive at the equation

$$[u, w_\kappa]_\Omega = (\sigma^0 + \sigma(v), e(w_\kappa))_{0,\Omega} - (F, w_\kappa)_{0,\Omega} .$$

The rest of the proof is parallel to that of Proposition 2 (*cf.* points 3° and 4°).

LEMMA 6.3 : *Let the assumptions of Lemma 6.2 be satisfied. Then*

$$\mathfrak{J}^*_{2h}(\sigma^h(v_h), u_h(v_h)) \to \mathfrak{J}^*_2(\sigma(v), u(v)), \quad h \to 0 .$$

Proof : The argument is parallel to that of Proposition 3, Case $i = 2$, where we replace u_n by u_h, Ω_n by Ω_h, Proposition 2 by Lemma 6.2 and Proposition 1 by Lemma 4.1.

THEOREM 6.1 : *Let* $\{ v_h \}$, $h \to 0$, *be a sequence of solutions of the approximate problems* (77).

Then a subsequence $\{ v_{\tilde h} \}$, $\tilde h \to 0$, *exists such that*

$$v_{\tilde h} \to v \quad in \quad C([0, 1]) ,$$

where v *is a solution of the second optimization problem* (27). *The solutions* $\sigma^{\tilde h}(v_{\tilde h})$ *of the approximate state problem* (62) *and the solutions* $u_{\tilde h}$ *of the auxiliary problem* (74) *converge in the sense of Lemma* 4.1 *and Lemma* 6.2, *respectively.*

Any uniformly convergent subsequence of $\{ v_h \}$ *has the properties mentioned above.*

Proof is parallel to that of Theorem 5.1. Instead of Lemma 5.1 we employ Lemma 6.3.

List of notations

$\mathcal{U}_{ad}$	: set of admissible functions
α, β	: bounds of the admissible functions
$\Gamma(v) \equiv \Gamma$	: unknown part of the boundary
$\sigma(v)$	: solution of the dual state problem (20)
$u(v) = u(\sigma(v))$	: solution of the auxiliary problem (22)
$\sigma^h = \sigma^h(v_h)$	: solution of the problem (62)
$u_h = u_h(v_h)$	: solution of the problem (74)
$\mathcal{R} = [0, \alpha_0] \times [0, 1]$	: fixed rectangle
$\Gamma_1 = \partial\Omega(v) - \Gamma_0$	: part of the boundary, independent of v
$\Omega_h = \Omega(v_h)$	: domain bounded by the graph of v_h
$\mathcal{N}_h(\Omega_h)$	: space of piecewise linear self-equilibriated triangular block-elements
$Q_0(\Omega)$	: space of self-equilibriated stress tensor functions
$V^c(\Gamma_0, \Omega)$	: space of « complementary test functions ».

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