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THE LIMITING AMPLITUDE PRINCIPLE APPLIED TO THE MOTION OF FLOATING BODIES (*)

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Communique par E SANCHEZ PALENCIA

Résumé — On considère un corps rigide flottant à la surface d'un océan infiniment profond, soumis à un mouvement forcé de faible amplitude, harmonique en temps. Au moyen de techniques d'analyse spectrale, on démontre le résultat suivant ou « Principe d'Amplitude Limite » lorsque le temps tend vers l'infini, le mouvement transitoire du fluide tend vers un mouvement harmonique en temps de même période que celle du mouvement imposé au corps.

Abstract — Consider a rigid body, floating on an ocean of infinite depth, subjected to time-harmonic motions of small amplitude. Using spectral analysis techniques, a proof of the following result is given as time tends to infinity, the transient motion of the fluid tends to a time-harmonic motion, with the same period as that of the prescribed motion of the body. This asymptotic behaviour is known as the Limiting Amplitude Principle.

1. INTRODUCTION

Consider a ship or an off-shore structure subjected to time-harmonic forced motions of small amplitude. It is expected that after a certain time, the swell diffracted by the ship is also time-harmonic, with the same period as that of the prescribed motions. This kind of behaviour, known as the Limiting Amplitude Principle, is due to the gravity waves which propagate energy toward infinity and therefore induce a dissipative mechanism.

This paper is devoted to a study of the linear evolution equations satisfied by the fluid motion and to a proof of the Limiting Amplitude Principle (L.A.P.).

The mathematical formulation of the problem, along with the assumptions and notations used in this paper, is briefly described first.

Let a rigid body, floating on an ocean of infinite depth, be subjected to time-harmonic forced motions. The motions of the fluid and the body are supposed to be of small amplitude around the equilibrium position, which allows to use

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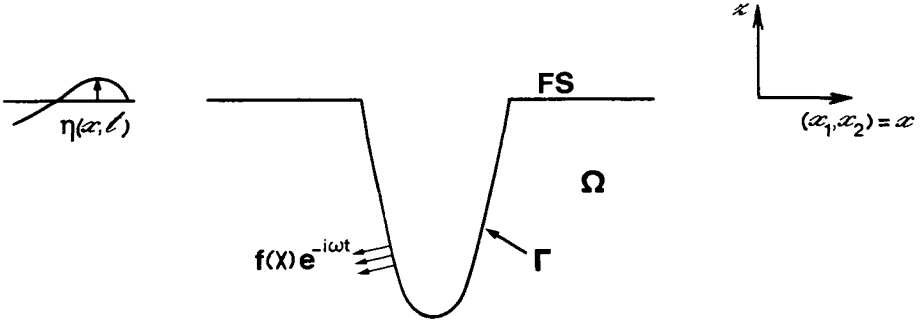


Figure 1.

a linear theory. Viscosity and surface tension are neglected and the fluid motion is assumed to be irrotational and incompressible, so that the velocity field in the fluid domain is the gradient of a harmonic potential φ . The body and the fluid respectively occupy the domains B and $\Omega = \{ X = (x, z) \in \mathbb{R}^3 \mid x = (x_1, x_2) \in \mathbb{R}^2, z < 0 \} \setminus \bar{B}$ at rest. The fluid-structure interface at equilibrium is a smooth surface $\Gamma = \bar{\Omega} \cap \bar{B}$ and $FS = \{ X \in \mathbb{R}^3 \mid z = 0 \} \setminus \bar{B}$ denotes the free surface equilibrium position. Therefore, if $\eta(x, t)$ denotes the vertical displacement of the free surface, the linearized equations of this *evolution problem* are :

$$\Delta\varphi(X, t) = 0 \text{ in } \Omega, \tag{1.1}$$

$$\partial_t\varphi(X, t) = -g\eta(x, t) \text{ on } FS, \tag{1.2}$$

$$\partial_t\eta(x, t) = \partial_z\varphi(X, t) \text{ on } FS, \tag{1.3}$$

$$\partial_n\varphi(X, t) = f(X) e^{-i\omega t} \text{ on } \Gamma. \tag{1.4}$$

where n denotes the outer unit normal to Γ , g is the acceleration of gravity and the circular frequency ω is a positive real number. To make this set of equations complete, the initial values of φ and η must be prescribed :

$$\varphi(X, 0) = \varphi_0(X), \tag{1.5}$$

$$\eta(x, 0) = \eta_0(x), \tag{1.6}$$

where φ_0 is given satisfying conditions (1.1) and (1.4) with $t = 0$.

As a preliminary result, it is proved here that provided these initial data are chosen in an appropriate Hilbert space, the problem (1.1)-(1.6) has a unique solution.

The associated *steady-state problem*, also known as the “sea-keeping problem without forward speed”, consists in finding a velocity potential $\tilde{\varphi}$

independent of time, such that :

$$\Delta \tilde{\varphi}(X) = 0 \quad \text{in } \Omega, \tag{1.7}$$

$$\partial_z \tilde{\varphi}(X) = (\omega^2/g) \tilde{\varphi}(X) \quad \text{on } FS, \tag{1.8}$$

$$\partial_n \tilde{\varphi}(X) = f(X) \quad \text{on } \Gamma, \tag{1.9}$$

$$\lim_{z \rightarrow -\infty} \partial_z \tilde{\varphi}(X) = 0, \tag{1.10}$$

$$\lim_{R \rightarrow +\infty} \int_{-\infty}^0 \int_0^{2\pi} R \left| \partial_R \tilde{\varphi} - i(\omega^2/g) \tilde{\varphi} \right|^2 d\theta dz = 0, \tag{1.11}$$

(R, θ, z) denoting the cylindrical coordinates in \mathbb{R}^3 . In order to provide the well-posedness of this last problem (F. John [1]), the following geometrical assumption will be made on the shape of the body :

$$\text{No point of } \Gamma \text{ lies below a point of } FS. \tag{1.12}$$

Then, the main result proved in this paper (L.A.P.) is that $\tilde{\varphi}(X)$ is the limiting amplitude of $\varphi(X, t)$:

$$\forall X \in \Omega, \quad \lim_{t \rightarrow +\infty} \left| \varphi(X, t) - \tilde{\varphi}(X) e^{-i\omega t} \right| = 0. \tag{1.13}$$

In the past, the L.A.P. has been mostly studied in the case of the three dimensional wave equation in the exterior of an obstacle, using several different methods. Ladyzenskaya [2] constructs a truncation of the solution of the steady-state wave equation, having finite energy, and then uses the local energy decay property satisfied by the solutions of the homogeneous wave equation (see also Sanchez-Palencia [3], chap. 16, sect. 3). The proofs of Morawetz [4] and Buchal [5], are based upon estimates of the spread of energy toward infinity of the solutions of the wave equation; when the body is star-shaped, this method allows furthermore to derive a rate of convergence in (1.13) (Morawetz [6]). P. D. Lax and R. S. Phillips [7] obtain the L.A.P. as a consequence of the theory they develop to construct the scattering matrix (especially the representation of the wave operator as a multiplication operator). However, all these proofs use specific properties of the wave equation such as the finite speed of wave propagation, and are therefore not suitable for the equations (1.1)-(1.6) of Linear Naval Hydrodynamics. An alternative proof of the L.A.P. applied to the wave equation, which does not use the finite speed of propagation is given by Eidus [8]; it is based upon the spectral analysis of the reduced wave equation and the spectral representation of the solution of the initial value problem. In the present work, using a technique similar to that of [8], the L.A.P. is derived in the case of Linear Naval Hydrodynamics.

The remaining sections are organized as follows. Section 2 is devoted to the study of the evolution problem (1.1)-(1.6); the underlying idea is to rewrite it under the form :

$$\begin{cases} \partial_t u(t) = Au(t) + Fe^{-i\omega t}, \\ u(0) = u_0, \end{cases} \quad (1.14)$$

where $u(t) = (\varphi(t), \eta(t))$ and A is a skew-selfadjoint operator acting on a Hilbert space \mathcal{H} of data having finite energy. Then, (1.14) (and therefore (1.1)-(1.6)) has a unique solution which may be written :

$$u(t) = \int_{-\infty}^{\infty} e^{-i\sigma t} dE(\sigma) u_0 + i \int_{-\infty}^{\infty} \frac{e^{-i\sigma t} - e^{-i\omega t}}{\sigma - \omega} dE(\sigma) F, \quad (1.15)$$

$\{ E(\sigma) \}$ denoting the spectral family of the self-adjoint operator iA . This extends the results of Beale [9] to the case of infinite depth. Then, in order to get an expression of the spectral density $dE(\sigma)$, it is necessary to study steady-state problems analogous to (1.7)-(1.11), in which the harmonic forced motion $fe^{-i\omega t}$ on Γ is replaced by a harmonic source on the free surface; this study is carried out in section 3 and an expression of $dE(\sigma)$ is derived in section 4. Last, in section 5, using the results of section 4 together with (1.15) allows to prove the Limiting Amplitude Principle.

It should be stressed that, to a large extent, the results obtained in section 4 and 5 are based upon spectral representations of the functions involved and upon elliptic *interior* regularity estimates for Laplace's operator. Although these estimates can be extended up to the boundaries, the restrictions of some functions to the free surface are not regular enough, neither to extend (1.13) to points X lying on the free surface, nor to allow deriving a result similar to (1.13) for the free surface elevation η . For that reason, the result proved in this paper deals with the asymptotic behaviour of the first component of u , at points X located *inside the fluid domain* Ω .

Notations

- n denotes the outer unit normal to surfaces.
- The partial derivative $\partial a / \partial b$ of a function a with respect to a variable b will be denoted $\partial_b a$.
- C denotes different constants.
- κ being a subset of \mathbb{R}^3 , $\overset{\circ}{\kappa}$ denotes its interior and $\bar{\kappa}$ its closure in the usual topology of \mathbb{R}^3 .
- A subset κ of \mathbb{R}^3 is said to be an interior subset of Ω if $\bar{\kappa} \subset \Omega$.

— $\mathcal{C}(\kappa)$ denotes the space of functions continuous on κ , equipped with the uniform convergence norm.

— $\mathcal{C}_c^\infty(\kappa)$ denotes the space of functions indefinitely differentiable on κ , having compact support in κ .

— If κ is an open subset 0 of \mathbb{R}^3 , $\mathcal{D}'(0)$ denotes the dual of $\mathcal{C}_c^\infty(0)$, the space of distributions in 0 .

$$\begin{aligned} - L^1(0) &= \left\{ h \in \mathcal{D}'(0) : \int_0 |h(X)| dX < \infty \right\}, \\ L^2(0) &= \{ h \in \mathcal{D}'(0) : h^2 \in L^1(0) \}, \\ L_c^2(0) &= \{ h \in L^2(0) : h \text{ has compact support in } 0 \}, \\ H^1(0) &= \{ h \in L^2(0) : \partial_{x_1} h, \partial_{x_2} h, \partial_z h \in L^2(0) \}. \end{aligned}$$

These spaces are equipped with their usual norms and scalar products.

$$- H_{loc}^1(0) = \{ h \in \mathcal{D}'(0) : h \in H^1(U \cap 0), \forall U \text{ bounded open set in } \mathbb{R}^3 \}.$$

Analogous definitions and notations are used when 0 is an open set of \mathbb{R}^2 or \mathbb{R} .

— Depending on the context, the arguments of the functions under consideration will sometimes be either partially or totally omitted (e.g. $\varphi(X, t) = \varphi(X) = \varphi(t) = \varphi$).

2. THE EVOLUTION PROBLEM

2.1. The homogeneous evolution problem.

In this subsection, it is assumed that $f = 0$ in (1.4). Let $u(t) = (\varphi(t), \eta(t))$ satisfy (1.1)-(1.4); note first that multiplying (1.1) by $\overline{\varphi}_p$, formally integrating over Ω , using Green's formula and combining the result with (1.2), (1.3) and (1.4) shows the conservation of the energy form :

$$E(u(t)) = \frac{1}{2} \left\{ \int_{\Omega} |\nabla\varphi(X, t)|^2 dX + \int_{FS} |\eta(x, t)|^2 dx \right\},$$

where the acceleration of gravity has been set to unity in order to simplify the calculations. In particular, the energy of a solution $u(t)$ is equal to the energy of the initial datum :

$$E(u_0) = \frac{1}{2} \left\{ \int_{\Omega} |\nabla\varphi_0(X)|^2 dX + \int_{FS} |\eta_0(x)|^2 dx \right\}, \quad (2.1)$$

$$u_0 = (\varphi_0, \eta_0).$$

The purpose of this subsection is to rewrite the system of equations (1.1)-(1.4)

under the form $\partial_t u = Au$, A being an operator acting on a Hilbert space \mathcal{H} whose norm is given by (2.1). The energy expression suggests to introduce the space defined as being the closure of $\mathcal{C}_c^\infty(\bar{\Omega})$ in norm :

$$\|\psi\|_{1,\Omega}^2 = \int_{\Omega} |\nabla\psi|^2 dX. \quad (2.2)$$

This space coincides with the weighted Sobolev space :

$$W_0^1(\Omega) = \{ \psi \in \mathcal{D}'(\Omega) : (1 + r^2)^{-1/2} \psi \in L^2(\Omega), \nabla\psi \in [L^2(\Omega)]^3 \},$$

where r denotes the radial distance in \mathbb{R}^3 (cf. Hanouzet [10]); in the following, $W_0^1(\Omega)$ will be equipped with the norm (2.2), which is equivalent to the graph norm.

Functions in $W_0^1(\Omega)$ have restrictions to Γ and FS , and more precisely :

LEMMA 2.1 : *There exist trace operators*

$$W_0^1(\Omega) \rightarrow H^{1/2}(\Gamma)$$

$$\psi \rightarrow \psi|_{\Gamma},$$

and

$$W_0^1(\Omega) \rightarrow W_0^{1/2}(FS)$$

$$\psi \rightarrow \psi|_{FS},$$

which are linear, continuous and surjective applications.

The space $W_0^{1/2}(FS)$ satisfies

$$W_0^{1/2}(FS) \cap L^2(FS) = H^{1/2}(FS). \quad (2.3)$$

The first part of the result is classical ; a proof of the second part as well as the structure of $W_0^{1/2}(FS)$ can be found in [10]. ■

Since conditions (1.1) and (1.4) do not involve time derivatives, they need to be treated as side conditions (conditions satisfied by all elements in \mathcal{H}), which motivates the following definition.

DEFINITION 2.2 : *A function ψ in $W_0^1(\Omega)$ is said to satisfy $\Delta\psi = 0$ in Ω and $\partial_n\psi = 0$ on Γ if and only if :*

$$\int_{\Omega} \nabla\psi \cdot \overline{\nabla\chi} dX = 0,$$

for all $\chi \in W_0^1(\Omega)$, such that $\chi|_{FS} = 0$.

According to the energy expression (2.1), $L^2(FS)$ is a natural space for the free surface elevation η , and \mathcal{H} will therefore be defined as :

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_1 \times L^2(FS), \\ \mathcal{H}_1 &= \{ \psi \in W_0^1(\Omega) : \Delta\psi = 0 \text{ in } \Omega \text{ and } \partial_n\psi = 0 \text{ on } \Gamma \}. \end{aligned}$$

Equipped with the norm

$$\| v \|_{\mathcal{H}} = \| (\psi, \tau) \|_{\mathcal{H}} = (\| \psi \|_{L^2(\Omega)}^2 + \| \tau \|_{L^2(FS)}^2)^{1/2}$$

and with the associated scalar product $(\cdot, \cdot)_{\mathcal{H}}$, \mathcal{H} is a Hilbert space, since it is the orthogonal space of $\{ \psi : \psi \in W_0^1(\Omega), \psi|_{FS} = 0 \} \times \{ 0 \}$ in $W_0^1(\Omega) \times L^2(FS)$.

The equation satisfied by $\partial_t\eta$ is given by (1.3). In order to derive an equation for $\partial_t\varphi$, differentiate formally (1.1) and (1.4) with respect to t ; combining the result with (1.2) shows that φ' = $\partial_t\varphi$ satisfies :

$$\begin{cases} \Delta\varphi' = 0 \text{ in } \Omega, \\ \partial_n\varphi' = 0 \text{ on } \Gamma, \\ \varphi' = -\eta \text{ on } FS. \end{cases} \tag{2.4}$$

Now, for $\eta \in H^{1/2}(FS)$, (2.4) has a unique solution $B(-\eta)$ in $W_0^1(\Omega)$ (cf. Hamdache [11]) and the operator B is linear and continuous from $H^{1/2}(FS)$ into $W_0^1(\Omega)$. The operator A is therefore defined by :

$$\begin{aligned} D(A) &= \{ v = (\psi, \tau) \in \mathcal{H} : \partial_z\psi|_{FS} \in L^2(FS), \tau \in H^{1/2}(FS) \} \\ A \begin{bmatrix} \psi \\ \tau \end{bmatrix} &= \begin{bmatrix} 0 & -B \\ \partial_z|_{FS} & 0 \end{bmatrix} \begin{bmatrix} \psi \\ \tau \end{bmatrix} = \begin{bmatrix} -B\tau \\ \partial_z\psi|_{FS} \end{bmatrix}. \end{aligned}$$

With these definitions, when $f = 0$, equations (1.1)-(1.6) can be rewritten :

$$\begin{cases} \partial_t u(t) = Au(t), \\ u(0) = u_0. \end{cases}$$

THEOREM 2.3 : *A is a skew-selfadjoint operator on \mathcal{H} .*

Proof : The proof proceeds in several steps.

(i) *D(A) is dense in \mathcal{H} .*

Since $\mathcal{C}_c^\infty(FS) \subset H^{1/2}(FS)$ and $\mathcal{C}_c^\infty(FS)$ is dense in $L^2(FS)$, $H^{1/2}(FS)$ is dense in $L^2(FS)$. In order to prove that :

$$\mathcal{H}_2 = \{ \chi \in \mathcal{H}_1 : \partial_z\chi|_{FS} \in L^2(FS) \}$$

is dense in \mathcal{H}_1 , it is enough to show that if $\psi \in \mathcal{H}_1$ is orthogonal to all $\chi \in \mathcal{H}_2$, then necessarily $\psi = 0$. Given $g \in \mathcal{C}_c^\infty(FS)$, a variational formulation straightforwardly shows that there exists a unique $\chi(g)$ in \mathcal{H}_2 such that $\partial_z \chi(g)|_{FS} = g$. Then, integrating by parts the orthogonality relation between $\chi(g)$ and ψ yields :

$$\int_{FS} g \bar{\psi} \, dx = 0,$$

for all $g \in \mathcal{C}_c^\infty(FS)$. This implies $\psi|_{FS} = 0$ and ψ may thus be extended by antisymmetry with respect to $z = 0$. But this extension satisfies Laplace's equation in an exterior domain and a homogeneous Neumann boundary condition; therefore it is identically zero (cf. Nedelec [12]), which in turn implies that $\psi = 0$.

(ii) *A is closed.*

Let $(v_p) = (\psi_p, \tau_p) \in D(A)$ be such that :

$$v_p \rightarrow v = (\psi, \tau) \text{ in } \mathcal{H}, \tag{2.5}$$

$$Av_p \rightarrow w = (\chi, \mu) \text{ in } \mathcal{H}. \tag{2.6}$$

According to (2.5) (resp. (2.6)), (τ_p) converges to τ in $L^2(FS)$ (resp. converges to $-\chi|_{FS}$ in $W_0^{1/2}(FS)$). Therefore, it follows from (2.3) that the convergence holds in $H^{1/2}(FS)$, that $\tau \in H^{1/2}(FS)$ and $\chi|_{FS} = -\tau$. Then, χ and $-B\tau$ are two elements of $W_0^1(\Omega)$ which both satisfy (2.4); the uniqueness property thus implies :

$$\chi = -B\tau. \tag{2.7}$$

Next, if ζ in $W_0^1(\Omega)$ is such that $\zeta|_{FS} \in L^2(FS)$:

$$\int_{\Omega} \nabla \psi_p \cdot \bar{\nabla} \zeta \, dX - \int_{FS} (\partial_z \psi_p) \bar{\zeta} \, dx = \int_{\Omega} (\Delta \psi_p) \bar{\zeta} \, dX = 0. \tag{2.8}$$

Using (2.5) and (2.6), limits can be taken in equation (2.8) and an integration by parts of the result yields :

$$\partial_z \psi|_{FS} = \mu. \tag{2.9}$$

Eventually, (2.7) together with (2.9) proves that $v \in D(A)$ and $Av = w$, and therefore that A is closed.

Since A is closed, it is enough to prove that A is antisymmetric and that the deficiency indices of $I \pm A$ are zero to show that A is skew-selfadjoint. The

antisymmetry of A resulting from standard calculations, the second property alone will be proved :

(iii) $\text{Im}(I \pm A) = \mathcal{H}$.

If $v = (\psi, \tau) \in D(A)$ and $G = (g_1, g_2) \in \mathcal{H}$, the relation $(I \pm A)u = G$ implies

$$\begin{cases} \Delta\psi = 0 \text{ in } \Omega, \\ \partial_z\psi + \psi = g_1 \pm g_2 \text{ on } FS, \\ \partial_n\psi = 0 \text{ on } \Gamma, \\ \tau = \pm (\psi|_{FS} - g_1|_{FS}). \end{cases} \tag{2.10}$$

If $g_1|_{FS} \in L^2(FS)$, this problem has a unique solution ψ which belongs to $\{\chi \in W_0^1(\Omega) : \chi|_{FS} \in L^2(FS)\}$ (Lenoir & Martin [13]). Then, according to the last equation of (2.10) and to property (2.3), τ belongs to $H^{1/2}(FS)$. Therefore, $\text{Im}(I \pm A) \supset \mathcal{H}$ where $\mathcal{H} = \{\chi \in \mathcal{H}_1 : \chi|_{FS} \in L^2(FS)\} \times L^2(FS)$; since A is closed and antisymmetric, it is enough to show that \mathcal{H} is dense in \mathcal{H} to prove (iii). The proof of this last result is a consequence of lemma 4.9. ■

From a theorem of Stone, it follows that A generates a strongly continuous group of unitary operators $W(t) = \exp(tA)$ on \mathcal{H} . Therefore, the homogeneous Cauchy problem (1.1)-(1.6) has a unique solution given by $u(t) = W(t)u_0$.

2.2. The non-homogeneous evolution problem

In order to use the previous formalism in the case where f is non zero, a datum F in \mathcal{H} is associated with f in the following way.

Let ν be a positive real number, for all f in $H^{-1/2}(\Gamma)$ the problem :

$$\begin{cases} \Delta\psi = 0 \text{ in } \Omega, \\ \partial_z\psi + \nu\psi = 0 \text{ on } FS, \\ \partial_n\psi = f \text{ on } \Gamma, \end{cases} \tag{2.11}$$

has a unique solution Tf in $V = \{\psi \in W_0^1(\Omega) : \psi|_{FS} \in L^2(FS)\}$ [13]. Then, if φ_1 and η_1 are defined by :

$$\begin{aligned} \varphi_1(X, t) &= Tf(X) e^{-i\omega t}, \\ \eta_1(x, t) &= i\omega Tf|_{FS}(X) e^{-i\omega t} \quad (\text{with } X = (x, 0)), \end{aligned}$$

and if $u(X, t) = (\varphi(X, t), \eta(x, t))$ satisfies the equations (1.1)-(1.6), then

$$\begin{aligned} \varphi_2(X, t) &= \varphi(X, t) - \varphi_1(X, t), \\ \eta_2(x, t) &= \eta(x, t) - \eta_1(x, t), \end{aligned}$$

satisfy

$$\left\{ \begin{array}{l} \Delta\varphi_2(X, t) = 0 \text{ in } \Omega, \\ \partial_t\varphi_2(X, t) = -\eta_2(x, t) \text{ on } FS, \\ \partial_t\eta_2(x, t) = \partial_z\varphi_2(X, t) - (v + \omega^2) Tf|_{FS}(X) e^{-i\omega t} \text{ on } FS, \\ \partial_n\varphi_2(X, t) = 0 \text{ on } \Gamma, \\ \varphi_2(X, 0) = \varphi_0(X) - Tf(X), \\ \eta_2(x, 0) = \eta_0(x) - i\omega Tf(X), \quad X = (x, 0). \end{array} \right.$$

Thus, $u = u_1 + u_2$, where :

$$u_1(t) = (\varphi_1, \eta_1) = (Tf, i\omega Tf|_{FS}) e^{-i\omega t}, \quad (2.12)$$

and $u_2 = (\varphi_2, \eta_2)$ is a solution of

$$\left\{ \begin{array}{l} \partial_t u_2 = Au_2 + Fe^{-i\omega t}, \\ u_2(0) = u_{20}, \end{array} \right. \quad (2.13)$$

with $F = (0, -(v + \omega^2) Tf|_{FS})$ and $u_{20} = u_0 - (Tf, i\omega Tf|_{FS})$. Since A is skew-selfadjoint, the system (2.13) has a unique solution which reads :

$$u_2(t) = W(t) u_{20} + \int_0^t e^{-i\omega s} W(t-s) F ds. \quad (2.14)$$

Therefore, the Cauchy problem (1.1)-(1.6) also has a unique solution, obtained by adding up expressions (2.12) and (2.14).

Remark : Another proof of the well-posedness of (1.1)-(1.6) can also be found in [11], as a consequence of a more general study on the motion of submerged bodies.

3. THE STEADY-STATE PROBLEM

3.1. Introduction.

For the various problems considered in this section, the following terminology will be used :

— A problem P is said to have the *uniqueness property* if, when the datum is set to zero, the only solution is the trivial one.

— A problem P is said to have the *existence property for a class* \mathcal{A} of data if, given any datum in \mathcal{A} , P has at least one solution.

If solutions of (1.1)-(1.4) are sought under the form

$$\begin{aligned} \varphi(X, t) &= \tilde{\varphi}(X) e^{-i\omega t}, \\ \eta(x, t) &= \tilde{\eta}(x) e^{-i\omega t}, \quad \omega > 0, \end{aligned}$$

the time dependence can be eliminated, the unknown function $\tilde{\eta}$ reads :

$$\tilde{\eta} = i\omega\tilde{\varphi}|_{FS}.$$

and one gets the system of equations (1.7)-(1.9) that only involves the unknown $\tilde{\varphi}$. It is interesting to note that if Tf is again the solution of (2.11) and $\tilde{u} = (\tilde{\varphi} - Tf, -i\omega(\tilde{\varphi} - Tf)|_{FS})$, then (1.7)-(1.9) formally amounts to solve

$$(A - i\omega)\tilde{u} = F, \tag{3.1}$$

where $F = (0, (\nu + \omega^2)Tf|_{FS})$. Since A is skew-selfadjoint, its spectrum is included in $i\mathbb{R}$, and it is therefore necessary to look for solutions \tilde{u} of (3.1) in a larger space than \mathcal{H} . This is why in the steady-state problem, the unknown function $\tilde{\varphi}$ is a priori sought in $H^1_{loc}(\Omega)$; yet, in order to ensure uniqueness, $\tilde{\varphi}$ is subjected to the radiation conditions (1.10) and (1.11). The condition (1.10) ensures that the fluid is at rest infinitely deep in the z -direction, and the outgoing Rellich radiation condition (1.11) expresses that energy radiates toward infinity in the x -plane. To emphasize the dependence on the frequency parameter, the system of equations (1.7)-(1.11) will be denoted Q_σ^+ , with $\sigma = \omega^2$.

The steady-state problem Q_σ^+ , $\sigma > 0$, has been studied by F. John [1], who proved under the geometrical assumption (1.12) that provided $f \in H^{-1/2}(\Gamma)$, there is a unique solution for all values of σ . Note that once the uniqueness property is known, the fact that Q_σ^+ has the existence property for $H^{-1/2}(\Gamma)$ can be proved either by means of the limiting absorption principle [13] or else by means of Fredholm operators techniques (cf. Vullierme-Ledard [14]).

The same results can be derived for Q_σ^- , problem obtained when (1.11) is replaced by the incoming Rellich radiation condition :

$$\lim_{R \rightarrow +\infty} \int_{-\infty}^0 \int_0^{2\pi} R |\partial_R \tilde{\varphi} + i\sigma \tilde{\varphi}|^2 d\theta dz = 0.$$

In order to obtain an expression for the spectral family of iA , a slightly different type of steady-state problem needs to be studied; σ being a positive

real number and g in $L_c^2(FS)$, P_σ^\pm is defined by :

$$P_\sigma^\pm \left\{ \begin{array}{l} \text{Find } \check{\varphi} \in H_{loc}^1(\Omega) \text{ s.t.} \\ \Delta \check{\varphi} = 0 \text{ in } \Omega, \\ \partial_z \check{\varphi} = \sigma \check{\varphi} + g \text{ on } FS, \\ \partial_n \check{\varphi} = 0 \text{ on } \Gamma, \\ \lim_{z \rightarrow -\infty} \partial_z \check{\varphi} = 0, \\ \lim_{R \rightarrow +\infty} \int_{-\infty}^0 \int_0^{2\pi} R | \partial_R \check{\varphi} \mp i\sigma \check{\varphi} |^2 d\theta dz = 0. \end{array} \right.$$

The goal of this section is to prove the well-posedness of P_σ^\pm and to study the dependence of its solution on the parameter σ . Since the homogeneous problem Q_σ^\pm and the homogeneous problem P_σ^\pm coincide, it follows that P_σ^\pm has the uniqueness property. By techniques similar to those of [14], it is proved in subsection 3.4, that P_σ^\pm has the existence property for $L_c^2(FS)$. The next two subsections collect some preliminary results which will be needed in the proof.

3.2. Green functions

Let X and X' be two points in \mathbb{R}^3 and δ_X be the Dirac measure at point X , the Green function $G_+(\sigma, X, X')$ of the problem P_σ^\pm is the unique solution (cf. [13]) of :

$$\left\{ \begin{array}{l} \Delta G_+(\sigma, X, X') = \delta_X(X'), \\ \partial_{z'} G_+(\sigma, X, X') = \sigma G_+(\sigma, X, X') \text{ on } \{z' = 0\}, \end{array} \right. \quad (3.2)$$

and of the radiation conditions

$$\lim_{z' \rightarrow -\infty} \partial_{z'} G_+(\sigma, X, X') = 0, \quad (3.3)$$

$$\lim_{R' \rightarrow +\infty} \int_{-\infty}^0 \int_0^{2\pi} R' | \partial_{R'} G_+(\sigma, X, X') - i\sigma G_+(\sigma, X, X') |^2 d\theta' dz' = 0. \quad (3.4)$$

The Green function $G_+(\sigma)$ is initially defined for $\sigma > 0$; it can be proved ([14] and also [15] appendix) that $G_+(\sigma)$ has an analytic continuation for $\sigma \in \mathbb{C}/\mathbb{R}_-$, this continuation showing a discontinuity along \mathbb{R}_- .

Similarly, there exists a unique Green function $G_-(\sigma)$ of the problem P_σ^- (it satisfies the same equations as $G_+(\sigma)$ except for the plus sign in condition

(3.4)); in the same fashion, $G_-(\sigma)$ has an analytic continuation for $\sigma \in \mathbb{C} \setminus \mathbb{R}_-$, continuation which also shows a discontinuity along \mathbb{R}_- .

THEOREM 3.2 : *Let σ be a positive real number ; when σ tends to zero, $G_+(\sigma, X, X')$ and $G_-(\sigma, X, X')$ tend to*

$$G_0(X, X') = - (1/4 \pi) (1/\|XX'\| + 1/\|X^*X'\|),$$

the convergence being uniform if X and X' belong to disjoint compact subsets of $\bar{\Omega}$ (here X^ denotes the symmetric of X with respect to $z = 0$).*

A proof of this result, as well as an expression of the Green functions $G_+(\sigma)$ and $G_-(\sigma)$, is given in the appendix. ■

The properties above mentioned are valid, with appropriate changes, for the first and second partial derivatives of G_+ , G_- and G_0 , with respect to any coordinate of X or X' .

3.3. An auxiliary problem

The study of P_σ^\pm , $\sigma \in \mathbb{R}$, will use the properties of the following problem :

$$P_\lambda \begin{cases} \text{Find } \psi \in V \text{ such that} \\ \Delta\psi = 0 \text{ in } \Omega, \\ \partial_z\psi = \lambda\psi + g \text{ on } FS, \\ \partial_n\psi = 0 \text{ on } \Gamma, \end{cases}$$

where λ is a complex number with non-zero imaginary part, $g \in L_c^2(FS)$ and V denotes the Hilbert space $\{ \psi \in W_\delta^1(\Omega) : \psi|_{FS} \in L^2(FS) \}$ equipped with the graph norm.

THEOREM 3.3 : *For all g in $L_c^2(FS)$, P_λ has a unique solution $T(\lambda)g$ in V .*

Proof : Let $\sqrt{-\lambda}$ be one of the two complex square roots of $-\lambda$ and consider the equation

$$(A - \sqrt{-\lambda}I)v = G, \tag{3.5}$$

where $v = (\psi, \tau)$, $G = (0, g)$ and I is the identity operator on \mathcal{H} ; a straightforward calculation shows that (3.5) is equivalent to

$$\begin{cases} \psi \text{ is a solution of } P_\lambda, \\ \tau = -\sqrt{-\lambda}\psi|_{FS}. \end{cases}$$

Since λ has a non-zero imaginary part, $\sqrt{-\lambda}$ does not belong to the spectrum

of A and (3.5) has a unique solution for all $G \in \mathcal{H}$, which implies the result of the theorem. ■

THEOREM 3.4 : *If $\text{Im } \lambda > 0$ (resp. $\text{Im } \lambda < 0$), the function $G_+(\lambda)$ (resp. $G_-(\lambda)$), defined by analytic continuation in section 3.2, belongs to V and is the unique Green function of the problem P_λ .*

The proof of this theorem is given in [13].

The last results suggest that in the half complex plane $\text{Im } \lambda > 0$ (resp. $\text{Im } \lambda < 0$), P_λ is “an analytic continuation” of P_σ^+ (resp. P_σ^-). As a consequence, in the following, the distinction between P_σ^+ (σ real number) and P_λ (λ complex with $\text{Im } \lambda > 0$) is dropped and the notation P_σ^+ is used for both problems. Similarly, P_σ^- is used for $\sigma \in \mathbb{R}_+$ as well as for $\text{Im } \sigma < 0$, and denotes P_σ if $\text{Im } \sigma < 0$.

3.4. Study of the problems P_σ^\pm

Throughout the remainder of this paper, ρ denoting a positive real number, the following notations will be used :

$$\begin{aligned} B(0, \rho) &= \{ X \in \mathbb{R}^3 : |X| < \rho \}, \\ \Omega_\rho &= \Omega \cap B(0, \rho), \\ FS_\rho &= FS \cap B(0, \rho), \\ \Sigma_\rho &= \overline{\Omega}_\rho \cap \{ X \in \mathbb{R}^3 : |X| = \rho \}, \\ L^2_\rho(FS) &= \{ g \in L^2(FS) : \text{supp}(g) \subset FS_\rho \}, \end{aligned}$$

and the positive real number ρ_0 is chosen such that $B \subset \Omega_{\rho_0}$.

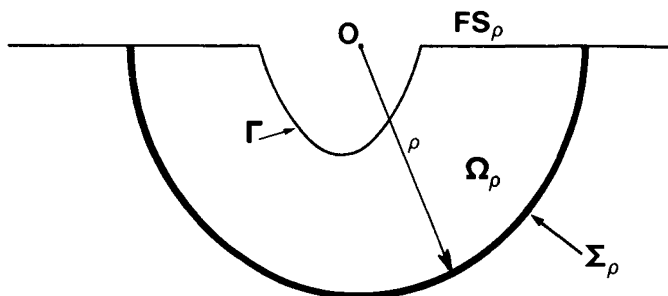


Figure 2.

The difficulty in studying P_σ^\pm lies in the fact that the domain Ω is unbounded. In order to overcome this difficulty, a problem similar to P_σ^\pm but set in a bounded domain is now introduced. For g in $L_c^2(FS)$, let $\rho > \rho_0$ be such that $\text{supp}(g) \subset FS_\rho$ and consider \hat{P}_σ^+ defined by :

$$\hat{P}_\sigma^+ \left\{ \begin{array}{l} \text{Find } \hat{\phi} \in H^1(\Omega_\rho) \text{ such that} \\ \Delta \hat{\phi} = 0 \text{ in } \Omega_\rho, \\ \partial_z \hat{\phi} = \sigma \hat{\phi} + g \text{ on } FS_\rho, \\ \partial_n \hat{\phi} = 0 \text{ on } \Gamma, \\ \forall X \in \Sigma_\rho, \quad D\hat{\phi}(X) = \int_\Gamma \hat{\phi}(X') \partial_{n'} DG_+(\sigma, X, X') ds' - \\ \qquad \qquad \qquad - \int_{FS_\rho} g(X') DG_+(\sigma, X, X') dx', \end{array} \right.$$

where D is the boundary operator defined by :

$$D\hat{\phi} = \partial_n \hat{\phi} |_{\Sigma_\rho} - i\hat{\phi} |_{\Sigma_\rho},$$

and the superscript' indicates that the object under consideration refers to X' (e.g. n' is the outer unit normal to the surface which X' belongs to). According to subsection 3.2, the problem \hat{P}_σ^+ is unambiguously defined for $\sigma \in \mathbb{C} \setminus \mathbb{R}_-$. Existence and uniqueness properties of \hat{P}_σ^+ and P_σ^+ are linked in the following :

LEMMA 3.5 : *Let $\sigma \in \mathbb{C} \setminus \mathbb{R}_-$ be such that $\text{Im } \sigma \geq 0$, the uniqueness property (resp. the existence property for $L_\rho^2(FS)$) holds for \hat{P}_σ^+ if and only if it holds for P_σ^+ .*

Moreover, when σ is such that the problems \hat{P}_σ^+ and P_σ^+ are well-posed, the solution of \hat{P}_σ^+ is nothing but the restriction to Ω_ρ of the solution of P_σ^+ .

Proof : (i) Suppose that \hat{P}_σ^+ has the existence property for $L_\rho^2(FS)$, let g be in $L_\rho^2(FS)$ and $\hat{\phi}$ be a solution of \hat{P}_σ^+ with datum g ; for $X \in \Omega$ define :

$$\psi(X) = \int_\Gamma \hat{\phi}(X') \partial_{n'} G_+(\sigma, X, X') ds' - \int_{FS_\rho} g(X') G_+(\sigma, X, X') dx'. \tag{3.6}$$

If $X \in \Omega_\rho$, $\hat{\phi}$ satisfies the integral representation :

$$\begin{aligned} \hat{\phi}(X) &= \int_{\Gamma} \hat{\phi}(X') \partial_n G_+(\sigma, X, X') ds' - \int_{FS_\rho} g(X') G_+(\sigma, X, X') dx' + \\ &+ \int_{\Sigma_\rho} [\hat{\phi}(X') \partial_n G_+(\sigma, X, X') - \partial_n \hat{\phi}(X') G_+(\sigma, X, X')] ds', \\ &= \psi|_{\Omega_\rho}(X) + \chi(X). \end{aligned}$$

It is readily seen that the second integral in (3.7) actually defines the function χ in the whole $B(0, \rho)$ and that it satisfies :

$$\Delta \chi = 0 \quad \text{in } B(0, \rho), \quad (3.8)$$

$$\partial_z \chi = \sigma \chi \quad \text{on } B(0, \rho) \cap \{z = 0\}, \quad (3.9)$$

$$D\chi = 0 \quad \text{on } \Sigma_\rho, \quad (3.10)$$

the last equality resulting from the fact that :

$$D\chi = D\hat{\phi} - D\psi = 0.$$

Multiplying (3.8) by $\bar{\chi}$, integrating by parts over $B(0, \rho)$ and using the boundary conditions (3.9) and (3.10) gives :

$$\int_{B(0, \rho)} |\nabla \chi|^2 dX = \sigma \int_{B(0, \rho) \cap \{z=0\}} |\chi|^2 dx + i \int_{\Sigma_\rho} |\chi|^2 ds.$$

Then, taking the imaginary part of both sides yields :

$$(\text{Im } \sigma) \int_{B(0, \rho) \cap \{z=0\}} |\chi|^2 ds + \int_{\Sigma_\rho} |\chi|^2 ds = 0.$$

Therefore, since $D\chi = 0$,

$$\partial_n \chi = \chi = 0 \quad \text{on } \Sigma_\rho,$$

which implies $\chi = 0$ in $B(0, \rho)$ and thus

$$\psi|_{\Omega_\rho} = \hat{\phi}.$$

In particular,

$$\partial_n \psi = \partial_n \hat{\phi} = 0 \quad \text{on } \Gamma. \quad (3.11)$$

From (3.6) which defines ψ and (3.11), it follows that ψ is a solution of P_σ^+ with datum g . Therefore, P_σ^+ has the existence property for $L_\rho^2(FS)$.

(ii) If P_σ^+ enjoys the existence property, so does \hat{P}_σ^+ , for the restriction to Ω_ρ of a solution of P_σ^+ is obviously a solution of \hat{P}_σ^+ .

The equivalence of P_σ^+ and \hat{P}_σ^+ with respect to the uniqueness property follows exactly the same lines and will therefore not be repeated; the second part of the lemma follows straightforwardly. ■

Combining the results of this lemma with the results of Theorem 3.3, one deduces :

- if $\text{Im } \sigma > 0$, \hat{P}_σ^+ and P_σ^+ are well-posed,
- if $\sigma > 0$, \hat{P}_σ^+ and P_σ^+ have the uniqueness property.

Now, the following lemma shows how \hat{P}_σ^+ can be rewritten in form of a Fredholm equation for a compact operator.

LEMMA 3.6 : \hat{P}_σ^+ is equivalent to

$$(I + K(\sigma)) \hat{\phi} = \gamma(\sigma, g), \tag{3.12}$$

where I is the identity operator on $H^1(\Omega_\rho)$, $\gamma(\sigma, g) \in H^1(\Omega_\rho)$ and $K(\sigma)$ is a compact operator from $H^1(\Omega_\rho)$ into $H^1(\Omega_\rho)$; both $\gamma(\sigma, g)$ and $K(\sigma)$ are holomorphic functions of the variable $\sigma \in \mathbb{C} \setminus \mathbb{R}_-$, with values in $H^1(\Omega_\rho)$ and $\mathcal{L}(H^1(\Omega_\rho), H^1(\Omega_\rho))$ (space of linear continuous applications from $H^1(\Omega_\rho)$ into itself) respectively.

Proof : Let $\hat{\psi}$ be in $H^1(\Omega_\rho)$, multiplying the first equation of \hat{P}_σ^+ by $\hat{\psi}$ and integrating the result by parts over Ω_ρ yields :

$$\int_{\Omega_\rho} \nabla \hat{\phi} \cdot \overline{\nabla \hat{\psi}} \, dX = \sigma \int_{FS_\rho} \hat{\phi} \overline{\hat{\psi}} \, dx + \int_{FS_\rho} g \overline{\hat{\psi}} \, dx + \int_{\Sigma_\rho} \overline{\hat{\psi}} \, \partial_n \hat{\phi} \, ds. \tag{3.13}$$

If $H^1(\Omega_\rho)$ is equipped with the usual scalar product

$$[\hat{\phi}, \hat{\psi}] = \int_{\Omega_\rho} \nabla \hat{\phi} \cdot \overline{\nabla \hat{\psi}} \, dX + \int_{\Omega_\rho} \hat{\phi} \overline{\hat{\psi}} \, dX,$$

(3.13) can be rewritten under the form :

$$[\hat{\phi}, \hat{\psi}] + [K(\sigma) \hat{\phi}, \hat{\psi}] = [\gamma(\sigma, g), \hat{\psi}],$$

where $K(\sigma)$ and $\gamma(\sigma, g)$ are defined by

$$[K(\sigma) \hat{\phi}, \hat{\psi}] = -\sigma \int_{FS_\rho} \hat{\phi} \bar{\hat{\psi}} dx - \int_{\Sigma_\rho} \bar{\hat{\psi}} \left(\int_{\Gamma} \hat{\phi} \partial_n DG_+(\sigma) ds \right) ds - \\ - i \int_{\Sigma_\rho} \hat{\phi} \bar{\hat{\psi}} ds - \int_{\Omega_\rho} \hat{\phi} \bar{\hat{\psi}} dX, \quad (3.14)$$

$$[\gamma(\sigma, g), \hat{\psi}] = \int_{FS_\rho} g \bar{\hat{\psi}} dx - \int_{\Sigma_\rho} \bar{\hat{\psi}} \left(\int_{FS_\rho} g DG_+(\sigma) dx \right) ds. \quad (3.15)$$

Note that the double surface integral in (3.15) is not singular since $(\text{supp } g) \cap \Sigma_\rho = \emptyset$. Clearly, $\gamma(\sigma, g)$ belongs to $H^1(\Omega_\rho)$ with

$$\| \gamma(\sigma, g) \|_{H^1(\Omega_\rho)} \leq C \| g \|_{L^2(FS_\rho)} + C \left\| \int_{FS_\rho} g DG_+(\sigma) dx \right\|_{L^2(\Sigma_\rho)}.$$

Similarly,

$$\| K(\sigma) \hat{\phi} \|_{H^1(\Omega_\rho)} \leq C \sigma \| \hat{\phi} \|_{L^2(FS_\rho)} + C \left\| \int_{\Gamma} \hat{\phi} \partial_n DG_+(\sigma) ds \right\|_{L^2(\Sigma_\rho)} + \\ + C \| \hat{\phi} \|_{L^2(\Sigma_\rho)} + C \| \hat{\phi} \|_{L^2(\Omega_\rho)},$$

with

$$\left\| \int_{\Gamma} \hat{\phi} \partial_n DG_+(\sigma) ds \right\|_{L^2(\Sigma_\rho)}^2 = \int_{\Sigma_\rho} \left| \int_{\Gamma} \hat{\phi}(X') \partial_n DG_+(\sigma, X, X') ds' \right|^2 ds, \\ \leq \int_{\Sigma_\rho} \| \hat{\phi} \|_{L^2(\Gamma)}^2 \| \partial_n DG_+(\sigma, X, \cdot) \|_{L^2(\Gamma)}^2 ds, \\ \leq \| \hat{\phi} \|_{L^2(\Gamma)} \| \partial_n DG_+(\sigma) \|_{L^2(\Gamma) \times L^2(\Sigma_\rho)}.$$

Therefore,

$$\| K(\sigma) \hat{\phi} \|_{H^1(\Omega_\rho)} \leq C(\sigma \| \hat{\phi} \|_{L^2(FS_\rho)} + \| \hat{\phi} \|_{L^2(\Omega_\rho)} + \| \hat{\phi} \|_{L^2(\Sigma_\rho)}) + \\ + C \| \hat{\phi} \|_{L^2(\Gamma)} \| \partial_n DG_+(\sigma) \|_{L^2(\Gamma) \times L^2(\Sigma_\rho)}.$$

Let $(\hat{\phi}_n)$ be a weakly convergent sequence in $H^1(\Omega_\rho)$, then $\hat{\phi}_n$ converges strongly in $L^2(\Omega_\rho)$ and $\hat{\phi}_n|_{\Gamma}$, $\hat{\phi}_n|_{FS_\rho}$, $\hat{\phi}_n|_{\Sigma_\rho}$ are strongly convergent sequences in $L^2(\Gamma)$, $L^2(FS_\rho)$ and $L^2(\Sigma_\rho)$ respectively; this implies that $K(\sigma) \hat{\phi}_n$ is strongly convergent in $H^1(\Omega_\rho)$, and thus that $K(\sigma)$ is a compact operator on $H^1(\Omega_\rho)$.

The holomorphy of $\gamma(\sigma, g)$ and $K(\sigma)$ with respect to the variable σ is then a straightforward consequence of the definitions (3.14) and (3.15) and of the properties of $G_+(\sigma)$. ■

Remark : although it does not appear explicitly in the notations used, $K(\sigma)$ and $\gamma(\sigma, g)$ depend on the particular choice of ρ . Yet, given g in $L_c^2(FS)$, it is always possible to choose ρ such that $\text{supp}(g) \subset FS_\rho$.

According to lemma 3.5, if $\sigma \in \mathbb{C} \setminus \mathbb{R}_-$ is such that $\text{Im } \sigma \geq 0$, the only solution of the equation

$$(I + K(\sigma)) \phi = 0,$$

is the zero solution. Since $K(\sigma)$ is compact, the Fredholm operator $(I + K(\sigma))$ is invertible in $\{\sigma \in \mathbb{C} \setminus \mathbb{R}_- \mid \text{Im } \sigma \geq 0\}$. Therefore, it follows from a theorem of Steinberg ([17] p. 370) that for $\sigma \in \mathbb{C} \setminus \mathbb{R}_-$, $(I + K(\sigma))^{-1}$ is a meromorphic family of operators with values in $\mathcal{L}(H^1(\Omega_\rho), H^1(\Omega_\rho))$, its poles being located in $\{\sigma \in \mathbb{C} \mid \text{Im } \sigma < 0\}$. Combining this result with those of lemma 3.5, the following theorem can be deduced :

THEOREM 3.7 : *If $\sigma > 0$ and $g \in L_c^2(FS)$, P_σ^+ has a unique solution denoted $S_+(\sigma)g$. The map $\sigma \rightarrow S_+(\sigma)g$ has a meromorphic continuation in $\mathbb{C} \setminus \mathbb{R}_-$; the continued function, still denoted $S_+(\sigma)g$, coincides with $T(\sigma)g$ when $\text{Im } \sigma > 0$ and its possible poles are located in $\{\sigma \in \mathbb{C} \mid \text{Im } \sigma < 0\}$.*

The same arguments can be repeated for the problem P_σ^- , yielding :

THEOREM 3.8 : *If $\sigma > 0$ and $g \in L_c^2(FS)$, P_σ^- has a unique solution denoted $S_-(\sigma)g$. The map $\sigma \rightarrow S_-(\sigma)g$ has a meromorphic continuation in $\mathbb{C} \setminus \mathbb{R}_-$; the continued function, still denoted $S_-(\sigma)g$, coincides with $T(\sigma)g$ when $\text{Im } \sigma < 0$ and its possible poles are located in $\{\sigma \in \mathbb{C} \mid \text{Im } \sigma > 0\}$.*

Remark 3.9 : Theorem 3.7 and 3.8 hold if, instead of being an element of $L_c^2(FS)$, g is of the form Th , $h \in H^{-1/2}(\Gamma)$ (see section 2.2 of this paper). In that case, σ being a positive number, the solution $\tilde{\phi}$ of P_σ^\pm satisfies

$$\left\{ \begin{array}{l} \Delta \tilde{\phi} = 0 \quad \text{in } \Omega, \\ \partial_z \tilde{\phi} = \sigma \tilde{\phi} + Th \quad \text{on } FS, \\ \partial_n \tilde{\phi} = 0 \quad \text{on } \Gamma, \\ \lim_{z \rightarrow -\infty} \partial_z \tilde{\phi} = 0, \\ \lim_{R \rightarrow +\infty} \int_{-\infty}^0 \int_0^{2\pi} R |\partial_R \tilde{\phi} \mp i\sigma \tilde{\phi}|^2 d\theta dz = 0, \end{array} \right.$$

so that $\chi = \tilde{\phi} - (\nu + \sigma)^{-1} Th$ satisfies Q_σ^\pm with $(\nu + \sigma)^{-1} h$ as a Neumann datum on Γ . According to the results obtained for Q_σ^\pm (see [14] and [15] sect. 3), χ exists, is unique and depends meromorphically on σ in the same way as $S_\pm(\sigma)g$ does ($g \in L_c^2(FS)$). Therefore, the same results are valid for $\tilde{\phi} = \chi + (\nu + \sigma)^{-1} Th$.

4. SPECTRAL DENSITY OF iA

4.1. Introduction

Since A is a skew-selfadjoint operator, iA is a selfadjoint operator which spectrum $\Sigma(iA)$ is entirely contained in the real axis. The spectral family and the resolvent of iA will respectively be denoted $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ and $R(\alpha)$, $\alpha \in \mathbb{C} \setminus \mathbb{R}$, so that

$$R(\alpha) = (iA - \alpha)^{-1},$$

$$iA = \int_{-\infty}^{\infty} \lambda dE(\lambda).$$

Following the notations of [16] (chap. 10, sect. 1), for $G \in \mathcal{H}$, $m_G(S)$ will denote the nonnegative measure associated with the spectral family of iA and defined on the Borel sets S of \mathbb{R} by

$$m_G(S) = (E(S)G, G),$$

with

$$E(S) = E(b) - E(a),$$

if S is any interval $(a, b]$ of \mathbb{R} . The subspace of all $G \in \mathcal{H}$ such that m_G is absolutely continuous with respect to the Lebesgue measure (subspace of absolute continuity) will be denoted \mathcal{H}_{ac} . Recall that \mathcal{H}_{ac} is a closed linear manifold of \mathcal{H} .

In fact, it is proved later in this section that the spectrum of iA is absolutely continuous ($\mathcal{H}_{ac} = \mathcal{H}$) and is the whole real line. However, the result needed in the derivation of the spectral family of iA is the following :

THEOREM 4.1 : *Under the assumption (1.12), iA has no eigenvalue.*

The proof, being rather technical, is given in the appendix. ■

The technique used to derive the Limiting Amplitude Principle relies upon an expression of the spectral density $dE(\sigma)$ in terms of the outgoing and incoming solutions $S_+(\sigma)$ and $S_-(\sigma)$ introduced in section 3. If $[a, b] \subset \mathbb{R}$ and $G, H \in \mathcal{H}$, the starting point is Stone's formula :

$$([E(b) + E(b-) - E(a) - E(a-)] G, H)_{\mathcal{H}} =$$

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{i\pi} \int_a^b ([R(\sigma + i\varepsilon) - R(\sigma - i\varepsilon)] G, H)_{\mathcal{H}} d\sigma.$$

In this case, since A has no eigenvalue, $\alpha \rightarrow (E(\alpha)G, H)_{\mathcal{H}}$ is continuous and

Stone's formula reduces to :

$$([E(b) - E(a)] G, H)_{\mathcal{H}} = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2 i\pi} \int_a^b ([R(\sigma + i\varepsilon) - R(\sigma - i\varepsilon)] G, H)_{\mathcal{H}} d\sigma . \tag{4.1}$$

The idea is then first to derive from (4.1) a convergence result holding weakly in \mathcal{H} :

$$[E(b) - E(a)] G = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2 i\pi} \int_a^b [R(\sigma + i\varepsilon) - R(\sigma - i\varepsilon)] G d\sigma , \text{ weakly} . \tag{4.2}$$

Recall that

$$\begin{aligned} E(\lambda) G &= ([E(\lambda) G]_1, [E(\lambda) G]_2) , \\ R(\alpha) G &= ([R(\alpha) G]_1, [R(\alpha) G]_2) , \end{aligned}$$

where the first (resp. second) components are functions in $W_0^1(\Omega)$ (resp. $L^2(FS)$). Using elliptic regularity theorems, (4.2) is in turn transformed into a pointwise convergence :

$$([E(b) - E(a)] G)_1(X) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2 i\pi} \int_a^b ([R(\sigma + i\varepsilon) - R(\sigma - i\varepsilon)] G)_1(X) d\sigma , \tag{4.3}$$

for all X in Ω . Next, the results of section 3 are used to compute the limit in the right hand side of (4.3) and finally an expression of $d[E(\lambda) G]_1(X)$ is obtained.

The first step of the argument will use some basic definitions and results of measure theory for vector-valued functions; those are recalled in the next subsection (see for example [17] or [18] for details and proofs).

4.2. Integral of vector-valued functions

DEFINITION 4.2 : *Let E be a Hilbert space, equipped with the scalar product (\cdot, \cdot) , and (X, μ) a measured space ; a function h*

$$\begin{aligned} h &: X \rightarrow E \\ x &\rightarrow h(x) \end{aligned}$$

is said to be integrable with respect to the measure μ if for all $z \in E$, $x \rightarrow (h(x), z)$ is integrable with respect to the measure μ .

THEOREM 4.3 : *With the same notations as in definition 4.2, let h be integrable with respect to the measure μ ; if $x \rightarrow \|h(x)\|$ is integrable with respect to the measure μ , there exists a unique element in E , denoted $\int h(x) d\mu(x)$ and called weak integral of h , such that for all $z \in E$:*

$$\left(\int h(x) d\mu(x), z \right) = \int (h(x), z) d\mu(x).$$

COROLLARY 4.4 : *If $[a, b] \subset \mathbb{R}$, any continuous function from $[a, b]$ into E is integrable with respect to the Lebesgue measure, and its weak integral exists.*

COROLLARY 4.5 : *The weak integral commutes with linear continuous functionals on E .*

4.3. An expression of the spectral density of iA

LEMMA 4.6 : $\varepsilon > 0$ and $G \in \mathcal{H}$ being fixed, the map

$$\begin{aligned} [a, b] &\rightarrow \mathcal{H} \\ \sigma &\rightarrow R(\sigma + i\varepsilon) G \end{aligned}$$

is continuous from $[a, b]$ into \mathcal{H} .

Proof : If $\sigma, \sigma' \in [a, b]$, the resolvent equation yields :

$$\| R(\sigma + i\varepsilon) G - R(\sigma' + i\varepsilon) G \|_{\mathcal{H}} = |\sigma - \sigma'| \| R(\sigma + i\varepsilon) R(\sigma' + i\varepsilon) G \|_{\mathcal{H}},$$

and since

$$\| R(\xi) \|_{\mathcal{H}} \leq 1/\text{dist}(\xi, \Sigma(iA)),$$

it follows that :

$$\| R(\sigma + i\varepsilon) G - R(\sigma' + i\varepsilon) G \|_{\mathcal{H}} \leq \frac{|\sigma - \sigma'|}{\varepsilon^2} \| G \|_{\mathcal{H}},$$

which implies the result of the lemma. ■

Corollary 4.4 shows that $\sigma \rightarrow R(\sigma + i\varepsilon) G$ is integrable on $[a, b]$ with respect to the Lebesgue measure, and from theorem 4.3 :

$$\int_a^b (R(\sigma + i\varepsilon) G, H)_{\mathcal{H}} d\sigma = \left(\int_a^b R(\sigma + i\varepsilon) G d\sigma, H \right)_{\mathcal{H}}.$$

The same result being valid for $\sigma \rightarrow R(\sigma - i\varepsilon) G$, (4.1) can be rewritten :

$$([E(b) - E(a)] G, H)_{\mathcal{H}} = \lim_{\varepsilon \rightarrow 0^+} \left(\frac{1}{2i\pi} \int_a^b (R(\sigma + i\varepsilon) - R(\sigma - i\varepsilon)) G d\sigma, H \right)_{\mathcal{H}}. \tag{4.4}$$

Denoting

$$w_\varepsilon = (\psi_\varepsilon, \tau_\varepsilon) = \frac{1}{2i\pi} \int_a^b (R(\sigma + i\varepsilon) - R(\sigma - i\varepsilon)) G d\sigma, \\ w = (\psi, \tau) = [E(b) - E(a)] G,$$

(4.4) also reads :

$$w_\varepsilon \rightarrow w, \text{ weakly in } \mathcal{H},$$

or else,

$$\psi_\varepsilon \rightarrow \psi \text{ weakly in } W_0^1(\Omega), \\ \tau_\varepsilon \rightarrow \tau \text{ weakly in } L^2(FS). \tag{4.5}$$

LEMMA 4.7 : *Let U be a bounded interior open subset of Ω ; then, ψ_ε and ψ belong to $\mathcal{C}(\bar{U})$ and ψ_ε converges to ψ , uniformly for $X \in \bar{U}$.*

Proof : Let U' be a bounded open set such that $\bar{U} \subset U' \subset \Omega$, (4.5) implies

$$\psi_\varepsilon \rightarrow \psi \text{ weakly in } H^1(U'), \\ \psi_\varepsilon \rightarrow \psi \text{ strongly in } L^2(U').$$

Since $\Delta\psi_\varepsilon = \Delta\psi = 0$, it follows from the interior regularity theorems for elliptic operators [19] that

$$\| \psi_\varepsilon - \psi \|_{H^2(U)} \leq C \| \psi_\varepsilon - \psi \|_{L^2(U')},$$

which implies that ψ_ε converges to ψ strongly in $H^2(U)$. To complete the proof, it is enough to note that because $U \subset \Omega \subset \mathbb{R}^3$, $H^2(U)$ has continuous embedding into $\mathcal{C}(\bar{U})$. ■

In particular, Lemma 4.7 implies

$$\forall X \in \Omega, ([E(b) - E(a)] G)_1(X) = \\ = \lim_{\varepsilon \rightarrow 0^+} \left(\frac{1}{2i\pi} \int_a^b (R(\sigma + i\varepsilon) - R(\sigma - i\varepsilon)) G d\sigma \right)_1(X). \tag{4.6}$$

LEMMA 4.8 : The map $\mathcal{H} \rightarrow \mathbb{C}$

$$G = (g_1, g_2) \rightarrow g_1(X),$$

is linear continuous for all $X \in \Omega$.

Proof : Let X be in Ω , U be an open bounded interior subset of Ω such that $X \in U$, and U' as in Lemma 4.7,

$$\begin{aligned} |g_1(X)| &\leq \|g_1\|_{\mathcal{C}(\bar{U})} \leq C \|g_1\|_{H^2(U)} \leq C \|g_1\|_{L^2(U')}, \\ &\leq C \|g_1\|_{W_0^1(\Omega)} \leq C \|G\|_{\mathcal{H}}. \end{aligned} \quad \blacksquare$$

From Corollaries 4.5 and 4.8, it follows that the order of the parenthesis can be reversed in (4.6), yielding (4.3).

Up to the end of this subsection, in order to be able to apply the results of section 3, the function G will be an element of either class of functions :

$$\begin{aligned} G \in \mathcal{H}_I &= \{ (g_1, g_2) \in \mathcal{H} : g_1|_{FS} \in \mathcal{C}_c^\infty(FS) \text{ and } g_2 \in \mathcal{C}_c^\infty(FS) \} \\ G \in \mathcal{H}_{II} &= \{ (0, Th) \in \mathcal{H} : h \in H^{-1/2}(\Gamma), Th \text{ solution of (2.11)} \}. \end{aligned}$$

LEMMA 4.9 : \mathcal{H}_I is dense in \mathcal{H} .

Proof : $\mathcal{C}_c^\infty(FS)$ being dense in $L^2(FS)$, it is enough to show that

$$\mathcal{H}_3 = \{ g \in \mathcal{H}_I : g|_{FS} \in \mathcal{C}_c^\infty(FS) \},$$

is dense in \mathcal{H}_I to prove the result. Given g in \mathcal{H}_I and $\varepsilon > 0$, there exists $\varphi \in \mathcal{C}_0^\infty(\bar{\Omega})$ such that

$$\|g - \varphi\|_{W_0^1(\Omega)} < \varepsilon.$$

Now, \mathcal{H}_I being a closed subspace of $W_0^1(\Omega)$, let P be the orthogonal projection on \mathcal{H}_I in $W_0^1(\Omega)$; because $Pg = g$,

$$\|g - P\varphi\|_{W_0^1(\Omega)} = \|Pg - P\varphi\|_{W_0^1(\Omega)} \leq \|g - \varphi\|_{W_0^1(\Omega)} < \varepsilon,$$

and since $P\varphi|_{FS} = \varphi|_{FS}$, $P\varphi \in \mathcal{H}_3$, and the result is proved. \blacksquare

THEOREM 4.10 : *If $X \in \Omega$, $G \in \mathcal{X}_1$ and $\sigma \in [a, b]$, $b > a > 0$, then*

$$\lim_{\varepsilon \rightarrow 0^+} [R(\sigma + i\varepsilon) G]_1(X) = S_+(\sigma^2) (\sigma g_1 - i g_2) (X), \tag{4.7}$$

$$\lim_{\varepsilon \rightarrow 0^+} [R(\sigma - i\varepsilon) G]_1(X) = S_-(\sigma^2) (\sigma g_1 - i g_2) (X), \tag{4.8}$$

the convergence being uniform with respect to $\sigma \in [a, b]$.

Proof : $R(\sigma + i\varepsilon) G = v$ with $v = (\psi, \tau)$ is equivalent to

$$\left\{ \begin{array}{l} \psi \in W_0^1(\Omega), \quad \psi|_{FS} \in H^{1/2}(FS), \\ \Delta\psi = 0 \text{ in } \Omega, \\ \partial_z\psi = (\sigma + i\varepsilon)^2 \psi + (\sigma + i\varepsilon) g_1|_{FS} - i g_2, \\ \partial_n\psi = 0 \text{ on } \Gamma, \\ \tau = i(\sigma + i\varepsilon) \psi|_{FS} + i g_1|_{FS}. \end{array} \right.$$

Thus,

$$\psi = [R(\sigma + i\varepsilon) G]_1 = S_+((\sigma + i\varepsilon)^2) ((\sigma + i\varepsilon) g_1|_{FS} - i g_2).$$

Similarly,

$$[R(\sigma - i\varepsilon) G]_1 = S_-((\sigma - i\varepsilon)^2) ((\sigma - i\varepsilon) g_1|_{FS} - i g_2). \tag{4.9}$$

Choose $\rho > \rho_0$ such that $X \in \Omega_\rho$, $\text{supp}(g_1)$ and $\text{supp}(g_2) \subset FS_\rho$. From section 3,

$$[R(\sigma + i\varepsilon) G]_1(X) = (I + K((\sigma + i\varepsilon)^2))^{-1} \gamma((\sigma + i\varepsilon)^2, (\sigma + i\varepsilon) g_1|_{FS} - i g_2),$$

where $K((\sigma + i\varepsilon)^2)$ and $\gamma((\sigma + i\varepsilon)^2, (\sigma + i\varepsilon) g_1|_{FS} - i g_2)$ are defined by (3.14) and (3.15) respectively. Therefore,

$$\begin{aligned} & \| [R(\sigma + i\varepsilon) G]_1 - S_+(\sigma^2) (\sigma g_1|_{FS} - i g_2) \|_{H^1(\Omega_\rho)} = \\ & = \| (I + K((\sigma + i\varepsilon)^2))^{-1} \gamma((\sigma + i\varepsilon)^2, (\sigma + i\varepsilon) g_1|_{FS} - i g_2) \\ & \quad - (I + K(\sigma^2))^{-1} \gamma(\sigma^2, \sigma g_1|_{FS} - i g_2) \|_{H^1(\Omega_\rho)}, \\ & \leq \| (I + K((\sigma + i\varepsilon)^2))^{-1} \|_{\mathcal{L}(H^1(\Omega_\rho), H^1(\Omega_\rho))} \\ & \quad \times \| \gamma((\sigma + i\varepsilon)^2, (\sigma + i\varepsilon) g_1|_{FS} - i g_2) - \gamma(\sigma^2, \sigma g_1|_{FS} - i g_2) \|_{H^1(\Omega_\rho)} \\ & \quad + \| (I + K((\sigma + i\varepsilon)^2))^{-1} - (I + K(\sigma^2))^{-1} \|_{\mathcal{L}(H^1(\Omega_\rho), H^1(\Omega_\rho))} \\ & \quad \times \| \gamma(\sigma^2, \sigma g_1|_{FS} - i g_2) \|_{H^1(\Omega_\rho)}. \end{aligned} \tag{4.10}$$

Since $(I + K(v))^{-1}$ depends meromorphically on v , $v \in \mathbb{C} \setminus \mathbb{R}_-$, $\alpha \rightarrow (I + K(\alpha^2))^{-1}$ depends continuously on α in a neighbourhood of $[a, b]$. Let $\varepsilon_0 > 0$ be such that $\{\alpha \in \mathbb{C} \mid a \leq \operatorname{Re} \alpha \leq b, -\varepsilon_0 \leq \operatorname{Im} \alpha \leq \varepsilon_0\}$ is included in this neighbourhood,

$$\| (I + K((\sigma + i\varepsilon)^2))^{-1} \|_{\mathcal{L}(H^1(\Omega_\rho), H^1(\Omega_\rho))} \leq C, \sigma \in [a, b], 0 \leq \varepsilon \leq \varepsilon_0,$$

and

$$\| (I + K((\sigma + i\varepsilon)^2))^{-1} - (I + K(\sigma^2))^{-1} \|_{\mathcal{L}(H^1(\Omega_\rho), H^1(\Omega_\rho))}$$

tends to zero as $\varepsilon \rightarrow 0$. Besides, if $\chi \in H^1(\Omega_\rho)$,

$$\begin{aligned} & (\gamma((\sigma + i\varepsilon)^2, (\sigma + i\varepsilon) g_1|_{FS} - ig_2) - \gamma(\sigma^2, \sigma g_1|_{FS} - ig_2), \chi)_{H^1(\Omega_\rho)} = \\ & = i\varepsilon \int_{FS_\rho} \bar{\chi} g_1|_{FS} dx - \int_{\Sigma_\rho} \bar{\chi} \left[\int_{FS_\rho} ((\sigma + i\varepsilon) g_1|_{FS} - ig_2) DG_+(\sigma + i\varepsilon)^2 \right. \\ & \quad \left. - [\sigma g_1|_{FS} - ig_2] DG_+(\sigma^2) dx \right] ds, \\ & = i\varepsilon \int_{FS_\rho} \bar{\chi} g_1|_{FS} dx - \int_{\Sigma_\rho} \bar{\chi} \left[\int_{FS_\rho} (i\varepsilon g_1|_{FS} DG_+(\sigma + i\varepsilon)^2 \right. \\ & \quad \left. + [DG_+(\sigma + i\varepsilon)^2 - DG_+(\sigma^2)] [\sigma g_1|_{FS} - ig_2] dx \right] ds. \end{aligned}$$

Therefore,

$$\begin{aligned} & \| \gamma((\sigma + i\varepsilon)^2, (\sigma + i\varepsilon) g_1|_{FS} - ig_2) - \gamma(\sigma^2, \sigma g_1|_{FS} - ig_2) \|_{H^1(\Omega_\rho)} \leq \\ & \leq \varepsilon \| g_1|_{FS} \|_{L^2(FS_\rho)} + \varepsilon \| g_1|_{FS} \|_{L^2(FS_\rho)} \| DG_+(\sigma + i\varepsilon)^2 \|_{L^2(FS_\rho) \times L^2(\Sigma_\rho)} \\ & + \| \sigma g_1|_{FS} - ig_2 \|_{L^2(\theta)} \| DG_+(\sigma + i\varepsilon)^2 - DG_+(\sigma^2) \|_{L^2(\theta) \times L^2(\Sigma)}, \end{aligned} \tag{4.11}$$

where θ is an open set which contains the supports of both $g_1|_{FS}$ and g_2 , such that $\bar{\theta} \subset FS_\rho$. According to the properties of $G_+(\sigma)$ with respect to σ , as $\varepsilon \rightarrow 0^+$, each term on the right hand side of (4.11) tends to zero uniformly with respect to $\sigma \in [a, b]$. Then, it follows that the left hand side of (4.10) tends to zero uniformly with respect to $\sigma \in [a, b]$, as $\varepsilon \rightarrow 0^+$. Finally, the same argument as in Lemma 4.7 allows to derive a pointwise convergence and shows that

$$[R(\sigma + i\varepsilon) G]_1(X) \rightarrow S_+(\sigma^2) (\sigma g_1|_{FS} - ig_2)(X)$$

as $\varepsilon \rightarrow 0^+$, uniformly with respect to $\sigma \in [a, b]$, which gives (4.7). The derivation of (4.8) follows exactly the same lines, starting from (4.9). ■

LEMMA 4.11 : If $X \in \Omega$

$$\begin{aligned} \text{for } \sigma > 0, \quad \frac{d}{d\sigma} [E(\sigma) G]_1(X) &= \\ &= \frac{1}{2 i\pi} [S_+(\sigma^2) - S_-(\sigma^2)] (\sigma g_1 |_{FS} - ig_2)(X) \end{aligned}$$

$$\begin{aligned} \text{for } \sigma < 0, \quad \frac{d}{d\sigma} [E(\sigma) G]_1(X) &= \\ &= -\frac{1}{2 i\pi} [S_+(\sigma^2) - S_-(\sigma^2)] (\sigma g_1 |_{FS} - ig_2)(X). \end{aligned}$$

Proof : First consider $\sigma > 0$; since the convergence in (4.7) and (4.8) holds uniformly with respect to $\sigma \in [a, b]$, the respective orders of limit and integration can be reversed in the right hand side of (4.3), yielding :

$$([E(b) - E(a)] G)_1(X) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2 i\pi} \int_a^b [S_+(\sigma^2) - S_-(\sigma^2)] (\sigma g_1 |_{FS} - ig_2)(X) d\sigma.$$

Then, it is enough to take alternatively $a = \sigma, b = \sigma + h$ and $a = \sigma - h, b = \sigma$ with $0 \leq h \leq \sigma/2$, divide both sides by h and let h tend to zero to obtain the result. Now, if $\sigma < 0$,

$$(\sigma + i\varepsilon)^2 = \sigma^2 - \varepsilon^2 + 2 i\sigma\varepsilon,$$

has a negative imaginary part. Therefore, if $a < b < 0$ and $\sigma \in [a, b]$,

$$\lim_{\varepsilon \rightarrow 0^+} [R(\sigma + i\varepsilon) G]_1(X) = S_-(\sigma^2) (\sigma g_1 |_{FS} - ig_2)(X),$$

$$\lim_{\varepsilon \rightarrow 0^+} [R(\sigma - i\varepsilon) G]_1(X) = S_+(\sigma^2) (\sigma g_1 |_{FS} - ig_2)(X),$$

and the same argument as before gives the result for $\sigma < 0$. ■

The next lemma is used to show that the expressions of the spectral density given in Lemma 4.11 are also valid for $\sigma = 0$; its proof is given in the appendix.

LEMMA 4.12 : If $\rho > \rho_0$ and $g \in L^2(FS_\rho)$, then $[S_+(\sigma^2) - S_-(\sigma^2)] g$ converges to zero strongly in $L^2(\Omega_\rho)$ as σ tends to zero.

Since this result is valid for any ρ such that $\text{supp}(g) \subset FS_\rho$, combining it with the argument of Lemma 4.7 again, shows that

$$\forall X \in \Omega, \quad \lim_{\sigma \rightarrow 0} [S_+(\sigma^2) - S_-(\sigma^2)] (\sigma g_1 |_{FS} - ig_2)(X) = 0.$$

Therefore, $\sigma \rightarrow [E(\sigma) G]_1(X)$ is continuous at $\sigma = 0$, differentiable on $\mathbb{R}_+ - \{0\}$ and on $\mathbb{R}_- - \{0\}$, and its derivative can be extended continuously by zero at $\sigma = 0$. It follows from elementary calculus that $\sigma \rightarrow [E(\sigma) G]_1(X)$ is differentiable at $\sigma = 0$ and that its derivative is zero. Finally, the following theorem sums up the results obtained in this section on the spectral density of iA .

THEOREM 4.13 : *If $\sigma \in \mathbb{R}$, $X \in \Omega$ and $G \in \mathcal{K}_I$,*

$$d[E(\sigma) G]_1(X) = \frac{1}{2i\pi} \operatorname{sgn}(\sigma) [S_+(\sigma^2) - S_-(\sigma^2)] (\sigma g_1|_{FS} - ig_2)(X) d\sigma . \quad (4.12)$$

The results obtained from Theorem 4.10 on are valid, with appropriate changes, if G belongs to \mathcal{K}_{II} . The corresponding expression of the spectral density is given in the following theorem.

THEOREM 4.14 : *If $\sigma \in \mathbb{R}$, $X \in \Omega$ and $G = (0, Th) \in \mathcal{K}_{II}$,*

$$d[E(\sigma) G]_1(X) = \frac{1}{2i\pi} \operatorname{sgn}(\sigma) [S_+(\sigma^2) - S_-(\sigma^2)] (-iTh|_{FS})(X) d\sigma . \quad (4.13)$$

Notation : For the sake of brevity, $\Theta(\sigma, X, G)$ will be used to denote, according to the context, either the right hand side of (4.12) or that of (4.13).

Although it will not explicitly be used in this paper, note that the expression of the spectral density of iA derived above implies in particular :

THEOREM 4.15 : *The spectrum of iA is absolutely continuous on \mathcal{H} .*

Proof : If $G \in \mathcal{K}_I$ and S is a Borel set of \mathbb{R} ,

$$\forall X \in \Omega, \quad [E(S) G](X) = \int_S \Theta(\sigma, X, G) d\sigma .$$

Therefore, if S is such that its Lebesgue measure is zero

$$\forall X \in \Omega, \quad [E(S) G](X) = 0 ,$$

which implies $E(S) G = 0$ and thus $m_G(S) = (E(S) G, G) = 0$. Consequently, $\mathcal{K}_I \subset \mathcal{H}_{ac}$ and since \mathcal{H}_{ac} is closed, $\mathcal{H} = \mathcal{K}_I = \mathcal{H}_{ac}$ which implies the result of the theorem. ■

5. THE LIMITING AMPLITUDE PRINCIPLE

Returning to the notations of subsection 2.2,

$$e^{i\omega t} u(t) = (Tf, i\omega Tf|_{FS}) + u_2(t), \tag{5.1}$$

where, recalling that

$$F = (0, -(\nu + \omega^2) Tf|_{FS}) \quad \text{and} \quad u_{20} = u_0 - (Tf, i\omega Tf|_{FS}),$$

$$u_2(t) = W(t) u_{20} + \int_0^t e^{-i\omega s} W(t-s) F ds =$$

$$= u_{21}(t) + u_{22}(t) = (\varphi_{21}(t), \eta_{21}(t)) + (\varphi_{22}(t), \eta_{22}(t)). \tag{5.2}$$

The function $u_{21}(t)$ is solution of the homogeneous problem

$$\begin{cases} \partial_t u_{21}(t) = Au_{21}(t), \\ u_{21}(0) = u_{20}. \end{cases}$$

When A and \mathcal{H} are respectively the operator and the space associated with the wave equation in three dimensions, the fact that $u_{21}(t)$ tends to zero strongly in \mathcal{H} is a classical consequence of the absolute continuity of the spectrum of A (see for example [3], [16]). The following theorem gives an analogous result in terms of pointwise convergence for the equations of the problem under consideration.

THEOREM 5.1 :

$$\forall X \in \Omega, \quad \lim_{t \rightarrow +\infty} |\varphi_{21}(X, t)| = 0, \tag{5.3}$$

uniformly with respect to X in a bounded interior subset of Ω .

The proof requires two steps.

LEMMA 5.2 : *The convergence result (5.3) holds if $u_{20} = (\varphi_{20}, \eta_{20}) \in \mathcal{H}_1$.*

Proof of lemma 5.2 :

$$u_{21}(t) = \exp(tA) u_{20} = \int_{-\infty}^{\infty} e^{-i\sigma t} dE(\sigma) u_{20},$$

reads

$$u_{21}(t) = \int_{-N}^N e^{-i\sigma t} dE(\sigma) u_{20} + \left(\int_{-\infty}^{-N} + \int_N^{\infty} \right) e^{-i\sigma t} dE(\sigma) u_{20},$$

$$= v_1(t) + v_2(t) = (\psi_1(t), \rho_1(t)) + (\psi_2(t), \rho_2(t)).$$

Since $v_2(t)$ belongs to \mathcal{H} ,

$$\Delta\psi_2(t) = 0.$$

Therefore, if κ is a bounded interior subset of Ω , a Sobolev imbedding theorem yields

$$\|\psi_2(t)\|_{\mathcal{G}(\kappa)} \leq C \|\psi_2(t)\|_{H^2(\mathbb{R})}.$$

Moreover, if κ_1 is a bounded open set such that $\bar{\kappa} \subset \kappa_1 \subset \bar{\kappa}_1 \subset \Omega$, the previous inequality combined with elliptic interior regularity gives :

$$\|\psi_2(t)\|_{\mathcal{G}(\kappa)} \leq C \|\psi_2(t)\|_{L^2(\kappa_1)} \leq C \|\psi_2(t)\|_{W_0^1(\Omega)} \leq C \|v_2(t)\|_{\mathcal{H}}. \quad (5.4)$$

The positive real number ε being fixed, N is then chosen such that

$$\|v_2(t)\|_{\mathcal{H}} = \left(\int_{-N}^{-\infty} + \int_N^{\infty} \right) d(E(\sigma) u_{20}, u_{20}) \leq \varepsilon/2C. \quad (5.5)$$

Now,

$$\psi_1(t, X) = \int_{-N}^N e^{-i\sigma t} \Theta(\sigma, u_{20}, X) d\sigma,$$

where

$$\Theta(\sigma, u_{20}, X) = \frac{1}{2} \frac{1}{i\pi} \operatorname{sgn}(\sigma) [S_+(\sigma^2) - S_-(\sigma^2)] (\sigma\varphi_{20}|_{FS} - i\eta_{20})(X).$$

Since $\sigma \rightarrow \Theta(\sigma, u_{20}, X)|_{[-N, N]}$ belongs to $L^1(\mathbb{R})$, its Fourier transform $\psi_1(t, X)$ tends to zero when $t \rightarrow +\infty$, and this result holds uniformly with respect to $X \in \kappa$. Therefore, t can be chosen large enough such that

$$\forall X \in \kappa, \quad |\psi_1(t, X)| \leq \varepsilon/2. \quad (5.6)$$

Combining (5.4), (5.5) and (5.6) gives

$$\forall X \in \kappa, \quad |\varphi_{21}(t, X)| \leq \varepsilon,$$

which ends the proof of lemma 5.2. ■

Proof of theorem 5.1 : the proof is based upon the fact that

$$\mathcal{H}_1 = \{ (\varphi, \eta) \in \mathcal{H}, \varphi|_{FS} \in \mathcal{C}_c^\infty(FS), \eta \in \mathcal{C}_c^\infty(FS) \}$$

is dense in \mathcal{H} . Since $u_{20} \in \mathcal{H}$, there exists $(u_n) \in \mathcal{H}_1$ such that

$$\lim_{n \rightarrow +\infty} \|u_n - u_{20}\|_{\mathcal{H}} = 0.$$

Let κ be as in lemma 5.2 and $X \in \kappa$,

$$|\varphi_{21}(X, t)| = |[W(t) u_{20}]_1(X)| \leq |[W(t) u_{20}]_1(X) - [W(t) u_n]_1(X)| + |[W(t) u_n]_1(X)|, \tag{5.7}$$

and

$$\begin{aligned} |[W(t) u_{20}]_1(X) - [W(t) u_n]_1(X)| &\leq \| [W(t) u_{20}]_1 - [W(t) u_n]_1 \|_{\mathcal{C}(\bar{\kappa})}, \\ &\leq C \| [W(t) u_{20}]_1 - [W(t) u_n]_1 \|_{H^2(\mathbb{R})}, \\ &\leq C \| [W(t) u_{20}]_1 - [W(t) u_n]_1 \|_{L^2(\kappa_1)}, \\ &\leq C \| [W(t) u_{20}]_1 - [W(t) u_n]_1 \|_{W_0^1(\Omega)}, \\ &\leq C \| W(t) u_{20} - W(t) u_n \|_{\mathcal{X}} = \\ &= C \| u_{20} - u_n \|_{\mathcal{X}}. \end{aligned} \tag{5.8}$$

Therefore, $\varepsilon > 0$ being given, n is first chosen large enough such that

$$C \| u_{20} - u_n \|_{\mathcal{X}} < \varepsilon/2, \tag{5.9}$$

and then, using lemma 5.2, it is possible to choose t large enough such that

$$|[W(t) u_n]_1(X)| < \varepsilon/2, \tag{5.10}$$

uniformly with respect to $X \in \kappa$. Combining the estimations (5.7) to (5.10) yields the result of theorem 5.1. ■

THEOREM 5.3 :

$$\lim_{t \rightarrow +\infty} |e^{i\omega t} \varphi_{22}(X, t) - (\tilde{\varphi}(X) - Tf(X))| = 0$$

uniformly with respect to X in a bounded interior subset of Ω .

The proof of Theorem 5.3 will use the following lemma :

LEMMA 5.4 :

$$\forall X \in \Omega, \quad \tilde{\varphi}(X) - Tf(X) = iPv \int_{-\infty}^{\infty} \frac{e^{-i\sigma t}}{\sigma - \omega} \Theta(\sigma, F, X) d\sigma - \pi \Theta(\omega, F, X).$$

Proof : With the notations of section 3, Theorem 3.7 and Remark 3.9 imply

$$\begin{aligned} \tilde{\varphi} - Tf &= S_+(\omega^2) ((v + \omega^2) Tf|_{FS}), \\ &= \lim_{\varepsilon \rightarrow 0^+} S_+(\omega^2 + i\varepsilon) ((v + \omega^2) Tf|_{FS}), \\ &= i \lim_{\varepsilon \rightarrow 0^+} [R((\omega^2 + i\varepsilon)^{1/2}) F]_1, \end{aligned}$$

where $(\omega^2 + i\varepsilon)^{1/2}$ is the complex square root of $\omega^2 + i\varepsilon$ with positive real part. Since

$$[R((\omega^2 + i\varepsilon)^{1/2}) F]_1(X) = \int_{-\infty}^{\infty} \frac{\Theta(\sigma, F, X)}{\sigma - (\omega^2 + i\varepsilon)^{1/2}} d\sigma,$$

using the well-known equality

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{x + i\varepsilon} = Pv \frac{1}{x} - i\pi\delta \quad \text{in } \mathcal{D}'(\mathbb{R}),$$

yields

$$\lim_{\varepsilon \rightarrow 0^+} [R((\omega^2 + i\varepsilon)^{1/2}) F]_1(X) = Pv \int_{-\infty}^{\infty} \frac{\Theta(\sigma, F, X)}{\sigma - \omega} d\sigma + i\pi\Theta(\omega, F, X)$$

and the result follows. ■

Proof of theorem 5.3 : it can be shown (cf. appendix) that $u_{22}(t)$ also reads

$$u_{22}(t) = i \int_{-\infty}^{\infty} \frac{e^{-i\sigma t} - e^{-i\omega t}}{\sigma - \omega} dE(\sigma) F.$$

If N is such that $0 < \omega < N - 1$, then $u_{22}(t)$ may be rewritten :

$$\begin{aligned} u_{22}(t) &= i \left[\int_{-\infty}^{-N} + \int_N^{\infty} \right] \left(\frac{e^{-i\sigma t}}{\sigma - \omega} dE(\sigma) F \right) - i \left[\int_{-\infty}^{-N} + \int_N^{\infty} \right] \left(\frac{e^{-i\omega t}}{\sigma - \omega} dE(\sigma) F \right) \\ &+ i \int_{-N}^N \frac{e^{-i\sigma t} - e^{-i\omega t}}{\sigma - \omega} dE(\sigma) F. \\ &= w_1(t) + w_2(t) + w_3(t), \\ &= (\chi_1(t), \tau_1(t)) + (\chi_2(t), \tau_2(t)) + (\chi_3(t), \tau_3(t)). \end{aligned}$$

Now,

$$\begin{aligned} \|w_1(t)\|_{\mathcal{H}}^2 &= \left[\int_{-\infty}^{-N} + \int_N^{\infty} \right] \left(\frac{1}{(\sigma - \omega)^2} d(E(\sigma) F, F) \right), \\ &\leq \left[\int_{-\infty}^{-N} + \int_N^{\infty} \right] (d(E(\sigma) F, F)). \end{aligned}$$

Therefore, using the same argument as for the function $v_2(t)$ in Lemma 5.2, if κ is a bounded interior subset of Ω , it is possible to choose N large enough such that

$$\|\chi_1(t)\|_{\mathcal{H}(\kappa)} \leq C \|w_1(t)\|_{\mathcal{H}} \leq \varepsilon/2, \quad (5.11)$$

$\varepsilon > 0$ being an arbitrary number.

Then, using the results of section 4, $\chi_3(X, t)$ reads

$$\begin{aligned} \chi_3(X, t) &= i \int_{-N}^N \frac{e^{-i\sigma t} - e^{-i\omega t}}{\sigma - \omega} \Theta(\sigma, F, X) d\sigma, \\ &= iPv \int_{-N}^N \frac{e^{-i\sigma t}}{\sigma - \omega} \Theta(\sigma, F, X) d\sigma - ie^{-i\omega t} Pv \int_{-N}^N \frac{\Theta(\sigma, F, X)}{\sigma - \omega} d\sigma. \end{aligned}$$

where Pv denotes the principal value of the integral under consideration. Thus,

$$\begin{aligned} \chi_2(t) + \chi_3(t) &= iPv \int_{-N}^N \frac{e^{-i\sigma t}}{\sigma - \omega} \Theta(\sigma, F, X) d\sigma - \\ &\quad - ie^{-i\omega t} Pv \int_{-\infty}^{\infty} \frac{\Theta(\sigma, F, X)}{\sigma - \omega} d\sigma, \end{aligned}$$

Now, $\sigma \rightarrow \Theta(\sigma, F, X)$ being holomorphic in the neighbourhood of $\omega > 0$,

$$\sigma \rightarrow \frac{\Theta(\sigma, F, X) - \Theta(\omega, F, X)}{\sigma - \omega} \times 1 \Big|_{[-N, N]},$$

belongs to $L^1(\mathbb{R})$, and its Fourier transform tends to zero at infinity; this also reads,

$$\begin{aligned} \lim_{t \rightarrow \infty} ie^{i\omega t} Pv \int_{-N}^N \frac{e^{-i\sigma t}}{\sigma - \omega} \Theta(\sigma, F, X) d\sigma \\ &= \lim_{t \rightarrow \infty} ie^{i\omega t} \Theta(\omega, F, X) Pv \int_{-N}^N \frac{e^{-i\sigma t}}{\sigma - \omega} d\sigma \\ &= \pi \Theta(\omega, F, X). \end{aligned}$$

Let t be chosen large enough such that

$$\left| ie^{i\omega t} Pv \int_{-N}^N \frac{e^{-i\sigma t}}{\sigma - \omega} \Theta(\sigma, F, X) d\sigma - \pi \Theta(\omega, F, X) \right| \leq \varepsilon/2. \quad (5.12)$$

Since,

$$\begin{aligned} |e^{i\omega t} \varphi_{22}(X, t) - (\tilde{\varphi}(X) - Tf(X))| &\leq |\chi_1(t, X)| + \\ &+ \left| ie^{i\omega t} Pv \int_{-N}^N \frac{e^{-i\sigma t}}{\sigma - \omega} \Theta(\sigma, F, X) d\sigma - \pi \Theta(\omega, F, X) \right| \\ &+ \left| \tilde{\varphi}(X) - Tf(X) - iPv \int_{-\infty}^{\infty} \frac{\Theta(\sigma, F, X)}{\sigma - \omega} d\sigma + \pi \Theta(\omega, F, X) \right|. \end{aligned}$$

the result of the theorem follows from Lemma 5.4 and inequalities (5.11) and (5.12). ■

The results of Theorem 5.1 and Theorem 5.3, together with equations (5.1) and (5.2) finally yield :

THEOREM 5.4 : (*Limiting Amplitude Principle*) :

$$\lim_{t \rightarrow +\infty} |e^{i\omega t} \varphi(X, t) - \tilde{\varphi}(X)| = 0,$$

uniformly with respect to X in a bounded interior subset of Ω .

6. APPENDIX

6.1. The Green function

Let X and X' be two points in \mathbb{R}_-^3 with respective coordinates (x_1, x_2, z) and (x'_1, x'_2, z') ; the Green function $G_+(\sigma)$, solution of (3.2)-(4.4), reads

$$G_+(\sigma, X, X') = G_0(X, X') + H_+(\sigma, X, X'),$$

with the following expressions for $H_+(\sigma, X, X')$:

— If $\sigma \in \mathbb{R}_+^*$

$$H_+(\sigma, X, X') = -\sigma Pv \left[\int_0^\infty \frac{\exp[2\pi t(z+z')] J_0(2\pi tR)}{2\pi t - \sigma} dt - \right. \\ \left. - i(\sigma/2) \exp[\sigma(z+z')] J_0(\sigma R) \right]$$

where $R = [(x_1 - x'_1)^2 - (x_2 - x'_2)^2]^{1/2}$ and Pv is Cauchy's principal value of the integral.

— If $\text{Im } \sigma > 0$

$$H_+(\sigma, X, X') = -\sigma \int_0^\infty \frac{\exp[2\pi t(z+z')] J_0(2\pi tR)}{2\pi t - \sigma} dt$$

— If $\text{Im } \sigma < 0$

$$H_+(\sigma, X, X') = -\sigma \int_0^\infty \frac{\exp[2\pi t(z+z')] J_0(2\pi tR)}{2\pi t - \sigma} dt - \\ - i\sigma \exp[\sigma(z+z')] J_0(\sigma R).$$

More details about the derivation of this function when $\sigma > 0$ can be found in [1] and in [13]; the analytic continuation on σ is explained in [15] on a very similar case.

As defined above, $G_+(\sigma, X, X')$ depends holomorphically on $\sigma, \sigma \in \mathbb{C} \setminus \mathbb{R}_-,$ and shows a cut along $\mathbb{R}_-.$ More precisely, for $\sigma \in \mathbb{R}_+^* :$

$$\begin{aligned} &\text{if } (\lambda_m) \rightarrow \sigma, \text{ with } \text{Im } \lambda_m > 0, \\ &(\mu_p) \rightarrow \sigma, \text{ with } \text{Im } \mu_p < 0, \end{aligned}$$

and if $C_+(\sigma, X, X')$ is defined by :

$$C_+(\sigma, X, X') = 2 i \sigma \exp[\sigma(z + z')] J_0(\sigma R),$$

then

$$\| G_+(\lambda_m, X, X') - G_+(\mu_p, X, X') - C_+(\sigma, X, X') \|_{L^\infty(K \times K')} \rightarrow 0$$

as $m, p \rightarrow + \infty, K$ and K' being two compact sets of \mathbb{R}_-^3 such that $K \cap K' = \emptyset,$ which X and X' respectively belong to.

Similarly, the Green function $G_-(\sigma, X, X')$ reads

$$G_-(\sigma, X, X') = G_0(X, X') + H_-(\sigma, X, X'),$$

with

$$H_-(\sigma, X, X') = H_+(\sigma, X, X') + \frac{1}{2} C_+(\sigma, X, X').$$

Remark : The properties above mentioned about the dependence of $G_\pm(\sigma)$ on $\sigma,$ remain valid when one studies the dependence of $G_\pm(\sigma^2)$ on $\sigma.$

6.2. The spectrum of iA

THEOREM 4.1 : *Under the assumption (1.12), iA has no eigenvalues.*

Proof : (i) 0 is not an eigenvalue.

Indeed, $Au = 0$ with $u = (\varphi, \eta)$ implies that $\eta = -B\eta = 0$ on $FS,$ and that φ satisfies :

$$\begin{aligned} \Delta\varphi &= 0 \text{ in } \Omega, \\ \partial_x\varphi &= 0 \text{ on } FS, \\ \partial_n\varphi &= 0 \text{ on } \Gamma. \end{aligned}$$

Since $\varphi \in W_0^1(\Omega),$ this implies $\varphi = 0$ (cf. [12]).

(ii) $\alpha \neq 0$ is not an eigenvalue.

Suppose that there exists $u = (\varphi, \eta)$ in $D(A)$ such that

$$(iA - \alpha)u = 0. \quad (6.1)$$

Equation (6.1) is equivalent to :

$$\begin{cases} -iB\eta - \alpha\varphi = 0, \\ i\partial_z\varphi|_{FS} - \alpha\eta = 0, \end{cases}$$

which implies that φ belongs to $W_0^1(\Omega)$, $\varphi|_{FS}$ belongs to $H^{1/2}(FS)$ and

$$\Delta\varphi = 0 \quad \text{in } \Omega, \quad (6.2)$$

$$\partial_z\varphi = \alpha^2\varphi \quad \text{on } FS, \quad (6.3)$$

$$\partial_n\varphi = 0 \quad \text{on } \Gamma, \quad (6.4)$$

$$\eta = i\alpha\varphi \quad \text{on } FS. \quad (6.5)$$

In order to prove that (6.2)-(6.5) imply $u = 0$, several steps are required. First, some estimates of the L^2 -norm of φ on horizontal planes are derived and the following notations are used :

$$\begin{aligned} \Pi_h &= \Omega \cap \{(x, z) \mid z = -h\} \\ \Omega_h &= \Omega \cap \{(x, z) \mid z > -h\}, \\ \Omega_h^k &= \Omega \cap \{(x, z) \mid -h < z < -k\}, \\ \omega_h &= \{x \mid (x, -h) \in \Pi_h\}, \end{aligned}$$

where h, k are positive real numbers such that $k < h$; h_0 is any positive real number such that $\omega_h = \mathbb{R}^2$.

Recall that since φ belongs to $W_0^1(\Omega)$ and $\varphi|_{FS} \in L^2(FS)$, φ belongs to $H^1(\Omega_h)$ for any value of h (cf. [11]). Hence the restrictions of φ to Π_h , $h > 0$, belong to $L^2(\omega_h)$, which justifies the notation :

$$\lambda(h) = \int_{\omega_h} \varphi^2(x, h) dx.$$

LEMMA 6.1 : *The following inequality holds*

$$\forall h \geq h_0, \quad \lambda(h) \leq 2\lambda(h_0) + 3(h - h_0) \|\nabla\varphi\|_{L^2(\Omega)}^2. \quad (6.6)$$

Proof of lemma 6.1 : if $z \leq -h_0$ and $x \in \mathbb{R}^2$:

$$|\varphi(x, z)|^2 = |\varphi(x, -h_0)|^2 + 2 \int_z^{-h_0} \operatorname{Re} \{ \varphi(x, \zeta) \overline{\partial_z \varphi(x, \zeta)} \} d\zeta. \quad (6.7)$$

Once integrated over \mathbb{R}^2 with respect to the x variable, (6.7) yields for $z = -h$ and $h > h_0$.

$$\lambda(h) \leq \lambda(h_0) + 2 \|\varphi\|_{L^2(\Omega_h^{h_0})} \|\nabla\varphi\|_{L^2(\Omega_h^{h_0})},$$

and thus

$$\lambda(h) \leq \lambda(h_0) + 2 \|\varphi\|_{L^2(\Omega)} \|\nabla\varphi\|_{L^2(\Omega)}. \tag{6.8}$$

Another integration of (6.7) over $\Omega_h^{h_0}$ gives :

$$\begin{aligned} \|\varphi\|_{L^2(\Omega_h^{h_0})}^2 &= \int_{\mathbb{R}^2} \int_{-h}^{-h_0} |\varphi(x, z)|^2 dz dx, \\ &\leq (h - h_0) \lambda(h_0) + 2 \int_{-h}^{-h_0} \int_{\mathbb{R}^2} \int_{-h}^{-h_0} |\varphi(x, \zeta)| |\partial_z \varphi(x, \zeta)| d\zeta dx dz. \end{aligned}$$

But for all $\beta > 0$,

$$|\partial_z \varphi(x, \zeta)| |\varphi(x, \zeta)| \leq (1/2 \beta) |\partial_z \varphi(x, \zeta)|^2 + (\beta/2) |\varphi(x, \zeta)|^2,$$

thus,

$$\begin{aligned} \|\varphi\|_{L^2(\Omega_h^{h_0})}^2 &\leq (h - h_0) \lambda(h_0) + 2(h - h_0) [(1/2 \beta) \|\nabla\varphi\|_{L^2(\Omega_h^{h_0})}^2 + \\ &\quad + (\beta/2) \|\varphi\|_{L^2(\Omega_h^{h_0})}^2]. \end{aligned}$$

Then, choosing $\beta = 1/2(h - h_0)$ yields :

$$\|\varphi\|_{L^2(\Omega_h^{h_0})}^2 \leq 2(h - h_0) \lambda(h_0) + 4(h - h_0)^2 \|\nabla\varphi\|_{L^2(\Omega)}^2. \tag{6.9}$$

Substituting (6.9) into (6.8) gives :

$$\begin{aligned} \lambda(h) &\leq \lambda(h_0) + 2 \|\nabla\varphi\|_{L^2(\Omega)} [2(h - h_0) \lambda(h_0) + 4(h - h_0)^2 \|\nabla\varphi\|_{L^2(\Omega)}^2]^{1/2}, \\ &\leq \lambda(h_0) + (1/\gamma) \|\nabla\varphi\|_{L^2(\Omega)}^2 + \gamma(h - h_0) \lambda(h_0) + \\ &\quad + 2 \gamma(h - h_0)^2 \|\nabla\varphi\|_{L^2(\Omega)}^2, \end{aligned}$$

for all $\gamma > 0$; finally, choosing $\gamma = 1/(h - h_0)$, one obtains :

$$\lambda(h) \leq 2 \lambda(h_0) + 3(h - h_0) \|\nabla\varphi\|_{L^2(\Omega)}^2. \blacksquare$$

For any positive real number v , define formally :

$$a_h^v(x) = \int_{-h}^0 e^{vz} \varphi(x, z) dz, \tag{6.10}$$

$$a^v(x) = \int_{-\infty}^0 e^{vz} \varphi(x, z) dz. \tag{6.11}$$

LEMMA 6.2 : a_h^v and a^v belong to $L^2(\omega_0)$.

Proof of lemma 6.2 : Note first that since $\varphi \in W_0^1(\Omega)$, $(1 + r^2)^{-1/2} \varphi$ belongs to $L^2(\Omega)$ and it follows from Fubini theorem that

$$z \rightarrow (1 + r^2)^{-1/2} \varphi(x, z) \in L_z^2(\mathbb{R}), \quad \text{a.e. for } x \in \omega_0,$$

where $L_z^2(\mathbb{R})$ denotes the space of functions in the z -variable which belong to $L^2(\mathbb{R})$. Now,

$$e^{vz} \varphi(x, z) = e^{vz}(1 + r^2)^{1/2} \times (1 + r^2)^{-1/2} \varphi(x, z)$$

being the product of two functions of $L_z^2(\mathbb{R})$, belongs to $L_z^1(\mathbb{R})$, which shows that the definitions (6.10) and (6.11) make sense a.e. (almost everywhere) for $x \in \omega_0$. Then

$$\begin{aligned} \| a^v \|_{L^2(\omega_0)} &= \int_{\omega_0} \left(\int_{-\infty}^0 e^{vz} \varphi(x, z) dz \right)^2 dx, \\ &\leq \int_{\omega_0} \left(\int_{-\infty}^0 e^{vz} dz \right) \left(\int_{-\infty}^0 e^{vz} \varphi^2(x, z) dz \right) dx, \\ &\leq (1/v) \int_{-\infty}^0 e^{vz} \left(\int_{\omega_0} \varphi^2(x, z) dx \right) dz, \end{aligned}$$

Since $\omega_0 \subset \omega_h$,

$$\int_{\omega_0} \varphi^2(x, z) dx \leq \int_{\omega_h} \varphi^2(x, z) dx = \lambda(z),$$

and thus

$$\begin{aligned} \| a^v \|_{L^2(\omega_0)} &\leq (1/v) \left[\int_{-h_0}^0 e^{vz} \lambda(z) dz + \right. \\ &\quad \left. + \int_{-\infty}^{-h_0} e^{vz} (2 \lambda(h_0) + 3(z - h_0) \| \nabla \varphi \|_{L^2(\Omega)}^2) dz \right]. \end{aligned}$$

The right-hand side being bounded, the result of the lemma is proved for a^v . The proof of the result for a_h^v is exactly the same. ■

LEMMA 6.3 : When h tends to infinity, a_h^v converges to a^v in $L^2(\omega_0)$.

Proof of lemma 6.3 :

$$\begin{aligned} \| a_h^v - a^v \|_{L^2(\omega_0)} &= \int_{\omega_0} \left(\int_{-\infty}^{-h} e^{vz} \varphi(x, z) dz \right)^2 dx, \\ &\leq \int_{\omega_0} \left(\int_{-\infty}^{-h} e^{vz} dz \right) \left(\int_{-\infty}^{-h} e^{vz} \varphi^2(x, z) dz \right) dx, \\ &\leq \frac{e^{-vh}}{v} \int_{-\infty}^{-h} e^{vz} \lambda(z) dz. \end{aligned}$$

According to Lemma 6.1, $e^{vz} \lambda(z)$ belongs to $L^1(\mathbb{R})$; therefore, when $h \rightarrow +\infty$, the right-hand side in the inequality above tends to zero, which proves the result. ■

LEMMA 6.4 : For all $v > 0$, $a^v = 0$.

Proof of lemma 6.4 : Since a_h^v belongs to $L^2(\omega_0)$, its two-dimensional Laplacian Δa_h^v is defined as an element of $\mathcal{D}'(\omega_0)$. Let v be in $\mathcal{C}_c^\infty(\omega_0)$; if $\langle \cdot, \cdot \rangle$ denotes the duality product between $\mathcal{D}'(\omega_0)$ and $\mathcal{C}_c^\infty(\omega_0)$,

$$\begin{aligned} \langle \Delta a_h^v, v \rangle &= \int_{\omega_0} a_h^v \Delta v dx, \\ &= \int_{-h}^0 e^{vz} \langle (\partial_{x_1 x_1}^2 + \partial_{x_2 x_2}^2) \varphi(\cdot, z), v \rangle dz, \\ &= - \int_{-h}^0 e^{vz} \partial_{zz}^2 \langle \varphi(\cdot, z), v \rangle dz. \end{aligned}$$

Integrating by parts twice with respect to z yields :

$$\begin{aligned} \langle (\Delta + v^2) a_h^v, v \rangle &= \langle v\varphi|_{FS} - \partial_z \varphi|_{FS}, v \rangle + \\ &\quad + e^{-vh} \langle \partial_z \varphi|_{z=-h} - v\varphi|_{z=-h}, v \rangle. \end{aligned} \tag{6.12}$$

Note that $\varphi|_{FS}$ and $\partial_z \varphi|_{FS}$ being in $L^2(FS)$, the first duality product on the right-hand side of (6.12) is also a scalar product in $L^2(\omega_0)$ (if $\varphi|_{FS}$ and $\partial_z \varphi|_{FS}$ are considered as functions of x). Similarly, since $\Delta \varphi = 0$, φ belongs to $C^\infty(\Omega)$ so that $\varphi|_{z=-h}$ and $\partial_z \varphi|_{z=-h}$ belong to $C^\infty(\omega_h)$; since $\omega_h \subset \omega_0$, the second duality product is again a scalar product in $L^2(\omega_0)$. In order to take the limits

in (6.12), notice that integrating by parts the expression for $\langle a_h^v, v \rangle$ yields

$$\langle a_h^v, v \rangle = - (1/v) \left\langle \int_{-h}^0 e^{vz} \partial_z \varphi(\cdot, z) dz, v \right\rangle + (1/v) \langle \varphi|_{FS}, v \rangle - (1/v) e^{-vh} \langle \varphi|_{z=-h}, v \rangle. \tag{6.13}$$

As h tends to infinity, a_h^v tends to a^v in $L^2(\omega_0)$ so that $\langle a_h^v, v \rangle$ tends to $\langle a^v, v \rangle$. Moreover, $\partial_z \varphi$ belongs to $L^2(\Omega)$ so that a.e. for $x \in \omega_0$, $\partial_z \varphi(x, \cdot)$ belongs to $L_z^2(\mathbb{R})$, and since e^{vz} also belongs to $L^2(\mathbb{R})$, the product $e^{vz} \partial_z \varphi(x, \cdot)$ belongs to $L^1(\mathbb{R})$. As a consequence, when h tends to infinity, the first term on the right-hand side of (6.13) tends to

$$- (1/v) \left\langle \int_{-\infty}^0 e^{vz} \partial_z \varphi(\cdot, z) dz, v \right\rangle,$$

and therefore by difference

$$e^{-vh} \langle \varphi|_{z=-h}, v \rangle \tag{6.14}$$

has a limit when h tends to infinity. Then, equation (6.12) shows that

$$e^{-vh} \langle \partial_z \varphi|_{z=-h}, v \rangle \tag{6.15}$$

also has a limit when h tends to infinity. Since this is true for all values of the parameter v , the limits in (6.14) and (6.15) must be zero. Now, choosing $v = \alpha^2$ in (6.12), using the boundary condition (6.3) and letting h tend to infinity gives :

$$\Delta a + \alpha^4 a = 0 \quad \text{in } \mathcal{D}'(\omega_0), \tag{6.16}$$

where $a = a^{\alpha^2}$. But a belongs to $L^2(\omega_0)$; therefore according to Rellich uniqueness theorem for the Helmholtz equation [20], (6.16) implies that $a = 0$. ■

Proof of theorem 4.1 : Integrating by parts the expression for a yields :

$$\varphi(x, 0) = \int_{-\infty}^0 e^{\alpha^2 z} \partial_z \varphi(x, z) dz, \quad x \in \omega_0.$$

Thus,

$$|\varphi(x, 0)|^2 \leq \left(\int_{-\infty}^0 e^{2\alpha^2 z} dz \right) \left(\int_{-\infty}^0 |\partial_z \varphi(x, z)|^2 dz \right), \tag{6.17}$$

and integrating (6.17) over ω_0 gives

$$2 \alpha^2 \|\varphi\|_{L^2(FS)}^2 \leq \|\partial_z \varphi\|_{L^2(\Omega)}^2 \leq \|\nabla \varphi\|_{L^2(\Omega)}^2. \tag{6.18}$$

Similarly, multiplying (6.2) by $\overline{\varphi}$, integrating by parts over Ω and using (6.3) yields

$$\| \nabla \varphi \|_{L^2(\Omega)}^2 = \int_{FS} \varphi(\overline{\partial_z \varphi}) \, dx = \alpha^2 \| \varphi \|_{L^2(FS)}^2. \tag{6.19}$$

Comparing (6.18) and (6.19) shows that

$$2 \alpha^2 \| \varphi \|_{L^2(FS)}^2 \leq \| \nabla \varphi \|_{L^2(\Omega)}^2 = \alpha^2 \| \varphi \|_{L^2(FS)}^2,$$

which in turn implies $\nabla \varphi = 0$ in Ω and $\varphi = 0$ on FS . Finally, $(\varphi, \eta) = (0, 0)$ and the result of Theorem 4.1 is proved. ■

6.3. Technical lemmas.

LEMMA 4.12 : *If $\rho > \rho_0$ and $g \in L^2(FS_\rho)$, when σ tends to zero, $[S_+(\sigma^2) - S_-(\sigma^2)] g$ converges to zero strongly in $L^2(\Omega_\rho)$.*

Proof : The first step consists in proving that $\| S_+(\sigma^2) g \|_{H^1(\Omega_\rho)}$ is bounded when $\sigma \rightarrow 0$. Suppose ab absurdo that

$$\forall p \in \mathbb{N}, \exists \sigma_p \in \mathbb{R}, \| S_+(\sigma_p^2) g \|_{H^1(\Omega_\rho)} > p.$$

Note that σ_p tends necessarily to zero when $p \rightarrow + \infty$, and define

$$\psi_p = S_+(\sigma_p^2) g / \| S_+(\sigma_p^2) g \|_{H^1(\Omega_\rho)}.$$

Then

$$\| \psi_p \|_{H^1(\Omega_\rho)} = 1, \tag{6.20}$$

and

$$\left\{ \begin{array}{l} \Delta \psi_p = 0 \text{ in } \Omega, \\ \partial_z \psi_p = \sigma_p^2 \psi_p + g_p \text{ on } FS, \\ \partial_n \psi_p = 0 \text{ on } \Gamma, \\ \psi_p \text{ satisfies the outgoing Rellich radiation condition,} \end{array} \right.$$

with $g_p = g / \| S_+(\sigma_p^2) g \|_{H^1(\Omega_\rho)}$. Passing to a subsequence if necessary, it is possible to assume that there exists χ such that

$$\begin{aligned} \psi_p &\rightarrow \chi \text{ weakly in } H^1(\Omega_\rho), \\ \psi_p &\rightarrow \chi \text{ strongly in } L^2(\Omega_\rho). \end{aligned}$$

Standard estimations using the ellipticity of Laplace’s operator give

$$\| \Psi_p - \chi \|_{H^2(U)} \leq C \| \Psi_p - \chi \|_{L^2(\Omega_p)},$$

where U is a bounded interior open subset of Ω_p ; thus it can be deduced that

$$\Delta \chi = 0 \quad \text{in } \Omega_p.$$

Moreover, for $\psi \in H^1(\Omega_p)$,

$$\int_{\Omega_p} \nabla \Psi_p \overline{\nabla \psi} dX = \sigma \int_{FS_p} \Psi_p \overline{\psi} dx + \int_{\Sigma_p} \partial_n \Psi_p \overline{\psi} ds + \int_{FS_p} g_p \overline{\psi} dx. \quad (6.21)$$

Noticing that g_p tends to zero strongly in $L^2(FS)$ as $p \rightarrow +\infty$, allows to take the limit $p \rightarrow +\infty$ in (6.21), which gives

$$\forall \psi \in H^1(\Omega_p) \text{ such that } \psi|_{\Sigma_p} = 0, \int_{\Omega_p} \nabla \chi \overline{\nabla \psi} dX = 0.$$

Integrating by parts this last result yields

$$\partial_n \chi = 0 \quad \text{on } \Gamma, \quad \partial_z \chi = 0 \quad \text{on } FS_p.$$

Moreover, the functions ψ_p satisfy the integral representation :

$$\psi_p(X) = \int_{\Gamma} \psi_p(X') \partial_n G_+(\sigma_p^2, X, X') ds' - \int_{FS} g_p(x') G_+(\sigma_p^2, X, X') dx'.$$

When $p \rightarrow +\infty$, $G_+(\sigma_p^2, X, X')$ tends to $G_0(X, X')$ uniformly for X and X' belonging to two disjoint compact subsets of $\overline{\Omega}$, ψ_p tends to zero strongly in $L^2(\Gamma)$ and g_p tends to zero strongly in $L^2(FS)$, so that the limits can be taken in the previous integral representation, yielding

$$\forall X \in \Omega, \quad \lim_{p \rightarrow +\infty} \psi_p(X) = \tilde{\chi}(X) = \int_{\Gamma} \chi(X') \partial_n G_0(X, X') ds'. \quad (6.22)$$

Therefore, $\tilde{\chi}$ defined by (6.22) is an extension of χ to the whole Ω which satisfies

$$\begin{cases} \Delta \tilde{\chi} = 0 & \text{in } \Omega, \\ \partial_z \tilde{\chi} = 0 & \text{on } FS, \\ \partial_n \tilde{\chi} = 0 & \text{on } \Gamma. \end{cases} \quad (6.23)$$

System (6.23) together with the behaviour of $\tilde{\chi}$ at infinity given by (6.22) implies

$\tilde{\chi} = 0$ [12], and thus

$$\lim_{p \rightarrow +\infty} \|\psi_p\|_{L^2(\Omega_p)} = 0. \tag{6.24}$$

Now, choosing $\psi = \psi_p$ in (6.21) and taking the limit $p \rightarrow +\infty$ gives

$$\lim_{p \rightarrow +\infty} \int_{\Omega_p} |\nabla \psi_p|^2 dX = 0, \tag{6.25}$$

and (6.24) together with (6.25) contradicts (6.20).

Therefore, $\|S_+(\sigma^2)g\|_{H^1(\Omega_p)}$, and similarly $\|S_-(\sigma^2)g\|_{H^1(\Omega_p)}$ are bounded as $\sigma \rightarrow 0$. Now, the function defined by

$$\xi_\sigma = [S_+(\sigma^2) - S_-(\sigma^2)]g,$$

satisfies

$$\begin{cases} \Delta \xi_\sigma = 0 & \text{in } \Omega, \\ \partial_x \xi_\sigma = \sigma^2 \xi_\sigma & \text{on } FS, \\ \partial_n \xi_\sigma = 0 & \text{on } \Gamma. \end{cases} \tag{6.26}$$

Passing to a subsequence if necessary, there exists ξ in $H^1(\Omega_p)$ such that

$$\begin{aligned} \xi_\sigma &\rightharpoonup \xi \text{ weakly in } H^1(\Omega_p), \\ \xi_\sigma &\rightarrow \xi \text{ strongly in } L^2(\Omega_p), \end{aligned}$$

and using the elliptic interior regularity for Laplace's operator, it is easy to see that

$$\Delta \xi = 0 \text{ in } \Omega_p.$$

Then, ψ being an element of $H^1(\Omega_p)$, a variational formulation of (6.26) reads

$$\int_{\Omega_p} \nabla \xi_\sigma \overline{\nabla \psi} dX = \sigma^2 \int_{FS_p} \xi_\sigma \overline{\psi} dx + \int_{\Sigma_p} \partial_n \xi_\sigma \overline{\psi} ds. \tag{6.27}$$

Moreover, subtracting the integral representations of $S_+(\sigma)$ and $S_-(\sigma)$ implies

$$\begin{aligned} \xi_\sigma &= \int_\Gamma \xi_\sigma \partial_n G_+(\sigma^2) ds + \int_\Gamma S_+(\sigma^2) \partial_n (G_+(\sigma^2) - G_-(\sigma^2)) ds + \\ &\quad + \int_{FS_p} g(G_+(\sigma^2) - G_-(\sigma^2)) dx. \end{aligned} \tag{6.28}$$

In particular, taking the normal derivatives on Σ_ρ of both sides gives

$$\begin{aligned} \partial_n \xi_\sigma |_{\Sigma_\rho} &= \int_\Gamma \xi_\sigma \partial_{n_{\Sigma_\rho}} \partial_{n_\Gamma} [G_+(\sigma)] ds + \\ &+ \int_\Gamma S_+(\sigma) \partial_{n_{\Sigma_\rho}} \partial_{n_\Gamma} (G_+(\sigma) - G_-(\sigma)) ds + \int_{FS} g \partial_{n_{\Sigma_\rho}} (G_+(\sigma) - G_-(\sigma)) dx. \end{aligned} \tag{6.29}$$

Thus, from subsection 6.1, the limit $\sigma \rightarrow 0$ can be taken in (6.28) and (6.29) yielding

$$\xi(X) = \int_\Gamma \xi(X') \partial_{n_\Gamma} G_0(X, X') ds, \quad \text{a.e. } X \in \Omega_\rho, \tag{6.30}$$

$$\lim_{\sigma \rightarrow 0} \partial_n \xi_\sigma |_{\Sigma_\rho}(X) = \int_\Gamma \xi(X') \partial_{n_{\Sigma_\rho}} \partial_{n_\Gamma} G_0(X, X') ds, \quad \text{a.e. } X \in \Sigma_\rho. \tag{6.31}$$

Note that the double derivative involved in formula (6.31) is not singular, since the normal derivative on Γ refers to the integration variable X' , whereas the normal derivative on Σ_ρ refers to the variable X .

Comparing (6.30) and (6.31) shows that $\partial_n \xi_\sigma |_{\Sigma_\rho}$ tends to $\partial_n \xi |_{\Sigma_\rho}$ as $\sigma \rightarrow 0$; therefore, letting $\sigma \rightarrow 0$ in (6.27), one gets after an integration by parts

$$\int_{FS_\rho} (\partial_z \xi) \bar{\Psi} dx + \int_\Gamma (\partial_n \xi) \bar{\Psi} ds = 0,$$

which in turn implies

$$\begin{cases} \partial_z \xi = 0 & \text{on } FS_\rho, \\ \partial_n \xi = 0 & \text{on } \Gamma. \end{cases} \tag{6.32}$$

Finally, it is enough to notice that formula (6.30) actually defines an extension $\tilde{\xi}$ of ξ to the whole Ω and that $\tilde{\xi}$ satisfies (cf. (6.32))

$$\begin{aligned} \Delta \tilde{\xi} &= 0 & \text{in } \Omega, \\ \partial_z \tilde{\xi} &= 0 & \text{on } FS, \\ \partial_n \tilde{\xi} &= 0 & \text{on } \Gamma, \end{aligned}$$

which, together with the behaviour of $\tilde{\xi}$ at infinity given by (6.30), implies $\tilde{\xi} = 0$ and proves Lemma 4.12. ■

LEMMA 6.5 :

$$u_{22}(t) = i \int_{-\infty}^{\infty} \frac{e^{-i\sigma t} - e^{-i\omega t}}{\sigma - \omega} dE(\sigma) F .$$

Proof : From (5.2)

$$u_{22}(t) = \int_0^t e^{-i\omega s} W(t-s) F ds = \int_0^t e^{-i\omega s} e^{(t-s)A} F ds .$$

According to the spectral representation,

$$e^{(t-s)A} F = \int_{-\infty}^{\infty} e^{-i(t-s)\sigma} dE(\sigma) F ,$$

and therefore

$$u_{22}(t) = \int_0^t e^{-i\omega s} \left(\int_{-\infty}^{\infty} e^{-i(t-s)\sigma} dE(\sigma) F \right) ds .$$

It is easy to see that $(s, \sigma) \rightarrow e^{-i\omega s} e^{-i(t-s)\sigma}$ belongs to $L^1([0, t] \times \mathbb{R})$ and Fubini's theorem can be applied, yielding

$$\begin{aligned} u_{22}(t) &= \int_{-\infty}^{\infty} e^{-i\sigma t} \left[\int_0^t e^{i(\sigma-\omega)s} ds \right] dE(\sigma) F , \\ &= i \int_{-\infty}^{\infty} \frac{e^{-i\sigma t} - e^{-i\omega t}}{\sigma - \omega} dE(\sigma) F . \quad \blacksquare \end{aligned}$$

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