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MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
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A MIXED-LAGRANGE MULTIPLIER FINITE ELEMENT METHOD FOR THE POLYHARMONIC EQUATION (*)

by James H. BRAMBLE [1] and Richard S. FALK [2]

Abstract. — *A finite element method requiring only C^0 elements is developed for the approximation of the first boundary value problem for the polyharmonic equation, based on the reformulation of this problem as a system of second order equations. The resulting linear system of equations can be easily preconditioned and efficiently solved by the conjugate-gradient method.*

Keywords : Mixed methods, Error estimates, Polyharmonic equation.

Résumé. — *Une méthode d'éléments finis qui exige seulement des éléments C^0, est développée ici, en vue de l'approximation du premier problème aux limites pour l'équation polyharmonique ; cette méthode est basée sur une reformulation du problème comme un système d'équations du second ordre. Le système linéaire, ainsi obtenu peut être facilement préconditionné et efficacement résolu par la méthode du gradient conjugué.*

1. INTRODUCTION

In this paper we wish to consider the approximation by finite elements of the first boundary value problem for the polyharmonic equation, i.e., for an integer $p \geqslant 2$, we seek a solution of

$$\Delta^p u = f \quad \text{in} \quad \Omega \tag{1.1}$$

$$\Delta^j u = 0 \quad \text{on} \quad \Gamma, \qquad j = 0, 1, \ldots, [(p-1)/2] \tag{1.2}$$

$$\partial \Delta^j u/\partial n = 0 \quad \text{on} \quad \Gamma, \qquad j = 0, 1, \ldots, [(p-2)/2], \tag{1.3}$$

where Ω is a bounded domain in R^2 with smooth boundary Γ and $[s]$ denotes the greatest integer contained in s.

The approach we take will involve a combination of the techniques of

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Lagrange multiplier and mixed methods. In following this approach we will use many of the ideas formulated in Bramble [4] for analyzing the Lagrange multiplier method for the second order Dirichlet problem introduced by Babuška [2]. In terms of the use of mixed methods, the formulation will most closely resemble that given in Falk [11] for the biharmonic problem, although the method of proof will be that of [4].

For the case of the biharmonic problem (i.e., $p = 2$) we also note the work of Ciarlet and Raviart [9] and Ciarlet and Glowinski [10]. Further ideas in this direction for the Stokes and biharmonic problems can be found in Glowinski and Pirroneau [12]. The techniques of this paper have also been used previously by the present authors to analyze two mixed finite element methods for the simply supported plate problem [5] and by Bramble and Pasciak [7] to study a method for computing the linearized scalar potential for the magnetostatic field problem.

The basic ideas in the approach taken can be described as follows. By the mixed method technique of introducing new dependent variables for appropriate derivatives, we are able to reformulate the problem as a lower order system of equations so that a conforming finite element method can be used with only continuous finite elements. Following [4] we show how the linear system of equations resulting from the Galerkin method applied to the reformulated problem can be easily preconditioned and efficiently solved by the conjugate gradient method. In order to do this, we first introduce appropriate Lagrange multipliers so that the iteration scheme produced will involve only a sequence of second order boundary value problems with natural boundary conditions. Thus a code which implements the Lagrange multiplier method for the second order Dirichlet problem will, with minor modifications, be able to solve the first boundary value problem for the polyharmonic equation.

The approximation scheme we develop is based on the following variational formulation of (1.1)-(1.3).

Let $\vec{w} = (w_0, \dots, w_{p-1})$ and $\vec{\sigma} = (\sigma_1, \dots, \sigma_{[(p+1)/2]})$.

Problem (P) : Find $\vec{w}, \vec{\sigma} \in (H^1(\Omega))^p \times (H^{-1/2}(\Gamma))^{[(p+1)/2]}$ such that

$$A_\alpha(w_{p-1}, v) = -(f, v) + \langle \sigma_1, v \rangle \qquad \forall v \in H^1(\Omega), \tag{1.4}$$

$$A_\alpha(w_{p-j}, v) = -(w_{p+1-j}, v) + \langle \sigma_j, v \rangle \qquad \forall v \in H^1(\Omega), \quad j = 2, 3, \dots, [(p+1)/2], \tag{1.5}$$

$$A_\alpha(w_{j-1}, v) = -(w_j, v) \qquad \forall v \in H^1(\Omega), \quad j = 2, 3, \dots, [p/2], \tag{1.6}$$

and

$$\langle w_{j-1}, \theta \rangle = 0 \qquad \forall \theta \in H^{-1/2}(\Gamma), \quad j = 1, 2, \ldots, [(p+1)/2], \quad (1.7)$$

where $(.,.)$ and $\langle .,. \rangle$ denote the L_2 inner product on Ω and Γ respectively, and $A_\alpha(u, v) = (\nabla u, \nabla u) + \alpha \langle u, v \rangle$ with $\alpha > 0$ constant.

To understand the relation between Problem (P) and the polyharmonic problem (1.1)-(1.3) observe that equations (1.5)-(1.6) imply that

$$w_j = \Delta w_{j-1}, \qquad j = 1, 2, \ldots, p-1$$

and (1.4) implies $\Delta w_{j-1} = f$. Hence $w_j = \Delta^j w_0$ and so

$$\Delta^p w_0 = \Delta \Delta^{p-1} w_0 = \Delta w_{p-1} = f.$$

Condition (1.7) then implies $\Delta^j w_0 = 0$ on Γ for $j = 0, 1, \ldots, [(p-1)/2]$. From (1.6) we also have that

$$\partial w_{j-1}/\partial n + \alpha w_{j-1} = 0 \quad \text{on} \quad \Gamma, \qquad j = 1, 2, \ldots, [p/2]$$

and the preceeding easy implies

$$\partial \Delta^j w_0/\partial n = 0, \qquad j = 0, 1, \ldots, [(p-2)/2].$$

The purpose of this paper is to introduce and analyze a finite element method based on this variational formulation. This will include both the derivation of error estimates for such a scheme (Section 5) and a discussion of an efficient iteration method for its implementation (Section 6). Preceding these results we shall, in Section 2, present some notation and state some a priori estimates for the continuous problem. In Section 3 we define the approximating subspaces to be used in the finite element method and collect some results on the approximation properties of these spaces. Section 4 then contains the description of the approximation scheme along with some additional estimates, the discrete analogues of those in Section 2.

2. NOTATION AND PRELIMINARY RESULTS

For $s \geqslant 0$ let $H^s(\Omega)$ and $H^s(\Gamma)$ denote the Sobolev spaces of order s of functions on Ω and Γ respectively, with associated norms $\| \cdot \|_s$ and $| \cdot |_s$ respectively (*cf.* [13]). For $s < 0$, let $H^s(\Omega)$ and $H^s(\Gamma)$ be the respective duals of $H^{-s}(\Omega)$ and $H^{-s}(\Gamma)$, with the usual dual norm.

To simplify the exposition of this paper we shall also use the norm $||| \cdot |||_s$ defined on functions in $H^s(\Omega)$ with traces in $H^{s-1/2}(\Gamma)$ *by* $||| \phi |||_s = \| \phi \|_s + | \phi |_{s-1/2}$ and the vector norm

$$| \vec{\sigma} |_s = \sum_{j=1}^{\left[\frac{p+1}{2}\right]} | \sigma_j |_{s+2(j-1)}$$

defined for

$$\vec{\sigma} = (\sigma_1, \dots, \sigma_{[(p+1)/2]}) \in \vec{H}^s(\Gamma)$$

where

$$\vec{H}^s(\Gamma) = H^s(\Gamma) \times \cdots \times H^{s+2[(p+1)/2]-2}(\Gamma).$$

In order to analyze Problem (P) and its finite element approximation, it will also be convenient to introduce the following notation.

Define operators

$$T : H^s(\Omega) \to H^{s+2}(\Omega)$$

and

$$G : H^s(\Gamma) \to H^{s+3/2}(\Omega)$$

by

$$A_\alpha(Tf, v) = (f, v) \quad \text{for all} \quad v \in C^\infty(\bar{\Omega})$$

and

$$A_\alpha(G\sigma, v) = \langle \sigma, v \rangle \quad \text{for all} \quad v \in C^\infty(\bar{\Omega}).$$

We then have the following estimates (*cf.* [5] and [14]).

LEMMA 2.1 : *There exists a constant C independent of σ and f such that for all real s*

$$| Tf |_{s-1/2} + \| Tf \|_s \leqslant C \| f \|_{s-2}, \tag{2.1}$$

and

$$| G\sigma |_{s-1/2} + \| G\sigma \|_s \leqslant C | \sigma |_{s-3/2}. \tag{2.2}$$

Using the operators T and G we may write

$$w_{p-1} = -Tf + G\sigma_1, \tag{2.3}$$

$$w_{p-j} = -Tw_{p+1-j} + G\sigma_j, \qquad j = 2, 3, \dots, [(p+1)/2], \tag{2.4}$$

and

$$w_{j-1} = -Tw_j, \qquad j = 1, 2, \dots, [p/2]. \tag{2.5}$$

Hence

$$u = w_0 = (-1)^p T^p f + \sum_{j=1}^{[\frac{p+1}{2}]} (-1)^{p-j} T^{p-j} G\sigma_j. \tag{2.6}$$

Let us now define

$$w_i(\vec{\sigma}) = \sum_{j=1}^{m} (-1)^{p-i-j} T^{p-i-j} G\sigma_j, \qquad i = 0, 1, \dots, p-1, \tag{2.7}$$

where $m = \min(p - i, [(p+1)/2])$.

Then

$$w_i = w_i(\vec{\sigma}) + (-1)^{p-i}\, T^{p-i} f, \qquad i = 0, 1, \ldots, p-1. \tag{2.8}$$

Since we will be working mostly with the bilinear forms we note that $\{ w_i(\vec{\sigma}) \}$ satisfy for all $v \in H^1(\Omega)$ the following variational equations :

$$A_\alpha(w_{p-1}(\vec{\sigma}), v) = \langle \sigma_1, v \rangle, \tag{2.9}$$

$$A_\alpha(w_{p-j}(\vec{\sigma}), v) = -(w_{p+1-j}(\vec{\sigma}), v) + \langle \sigma_j, v \rangle, \qquad j = 2, 3, \ldots, [(p+1)/2], \tag{2.10}$$

and

$$A_\alpha(w_{j-1}(\vec{\sigma}), v) = -(w_j(\vec{\sigma}), v), \qquad j = 1, 2, \ldots, [p/2]. \tag{2.11}$$

We further note that (2.9)-(2.11) imply that

$$w_j(\vec{\sigma}) = \Delta^j w_0(\vec{\sigma}), \qquad j = 1, \ldots, p-1, \tag{2.12}$$

$$\sigma_j = \left(\frac{\partial}{\partial n} + \alpha\right) w_{p-j}(\vec{\sigma}) = \left(\frac{\partial}{\partial n} + \alpha\right) \Delta^{p-j} w_0(\vec{\sigma}), \qquad j = 1, 2, \ldots, [(p+1)/2], \tag{2.13}$$

and

$$0 = \left(\frac{\partial}{\partial n} + \alpha\right) w_{j-1}(\vec{\sigma}) = \left(\frac{\partial}{\partial n} + \alpha\right) \Delta^{j-1} w_0(\vec{\sigma}), \qquad j = 1, 2, \ldots, [p/2]. \tag{2.14}$$

In terms of this notation it is then possible to restate Problem P in the form :

Problem (Q) : Find $\vec{\sigma} = (\sigma_1, \ldots, \sigma_{[(p+1)/2]}) \in (H^{-1/2}(\Gamma))^{[(p+1)/2]}$ such that

$$w_i(\vec{\sigma}) = (-1)^{p-i+1}\, T^{p-i} f \ \text{ on } \ \Gamma, \qquad i = 0, 1, \ldots, [(p-1)/2], \tag{2.15}$$

where $w_i(\vec{\sigma})$ is defined by (2.4).

As in [4] and [5], it will be from this point of view that we will approximate u, i.e. we will approximate G, T, and $\vec{\sigma}$ to obtain an approximation to u.

In this section we wish to prove several a priori estimates relating the functions $\vec{\sigma}$ and $w_i(\vec{\sigma})$. To do so we first state some Green's formulas and standard a priori estimates for solutions of the polyharmonic equation $\Delta^p z = 0$.

Using the Green's formula

$$(\Delta^p z, v) = \left\langle \frac{\partial}{\partial n} \Delta^{p-1} z, v \right\rangle - \left\langle \Delta^{p-1} z, \frac{\partial v}{\partial n} \right\rangle + (\Delta^{p-1} z, \Delta v)$$

we easily obtain for any $1 \leqslant j \leqslant p$ that

$$(\Delta^p z, v) = \sum_{i=1}^{j} \left\{ \left\langle \frac{\partial}{\partial n} \Delta^{p-i} z, \Delta^{i-1} v \right\rangle - \left\langle \Delta^{p-i} z, \frac{\partial}{\partial n} \Delta^{i-1} v \right\rangle \right\} + (\Delta^{p-j} z, \Delta^j v).$$

Hence for any $1 \leqslant j \leqslant p$

$$(\Delta^p z, v) = (\Delta^{p-j} z, \Delta^j v) + \\ + \sum_{i=1}^{j} \left\{ \left\langle \left(\frac{\partial}{\partial n} + \alpha \right) \Delta^{p-i} z, \Delta^{i-1} v \right\rangle - \left\langle \Delta^{p-i} z, \left(\frac{\partial}{\partial n} + \alpha \right) \Delta^{i-1} v \right\rangle \right\}. \tag{2.16}$$

Using (2.16) we can immediately obtain the following equalities.

Choosing $j = p/2$ we get for p even and all $z \in H^p(\Omega)$ with $\Delta^p z = 0$ that

$$(\Delta^{p/2} z, \Delta^{p/2} v) = \sum_{i=1}^{p/2} \left\{ \left\langle \Delta^{p-i} z, \left(\frac{\partial}{\partial n} + \alpha \right) \Delta^{i-1} v \right\rangle - \left\langle \left(\frac{\partial}{\partial n} + \alpha \right) \Delta^{p-i} z, \Delta^{i-1} v \right\rangle \right\}. \tag{2.17}$$

Choosing $j = (p-1)/2$ we get for p odd and all $z \in H^{p+1}(\Omega)$ with $\Delta^p z = 0$ that

$$(\Delta^{(p+1)/2} z, \Delta^{(p-1)/2} v) = \sum_{i=1}^{(p-1)/2} \left\{ \left\langle \Delta^{p-i} z, \left(\frac{\partial}{\partial n} + \alpha \right) \Delta^{i-1} v \right\rangle - \\ - \left\langle \left(\frac{\partial}{\partial n} + \alpha \right) \Delta^{p-i} z, \Delta^{i-1} v \right\rangle \right\}.$$

Since

$$A_\alpha(\Delta^{(p-1)/2} z, \Delta^{(p-1)/2} v) = \\ = (\nabla \Delta^{(p-1)/2} z, \nabla \Delta^{(p-1)/2} v) + \alpha \langle \Delta^{(p-1)/2} z, \Delta^{(p-1)/2} v \rangle \\ = -(\Delta^{(p+1)/2} z, \Delta^{(p-1)/2} v) + \left\langle \left(\frac{\partial}{\partial n} + \alpha \right) \Delta^{(p-1)/2} z, \Delta^{(p-1)/2} v \right\rangle,$$

we further obtain for p odd and all $z \in H^p(\Omega)$ with $\Delta^p z = 0$ that

$$A_\alpha(\Delta^{(p-1)/2} z, \Delta^{(p-1)/2} v) = \\ = \left\langle \left(\frac{\partial}{\partial n} + \alpha \right) \Delta^{(p-1)/2} z, \Delta^{(p-1)/2} v \right\rangle + \sum_{i=1}^{(p-1)/2} \left\{ \left\langle \left(\frac{\partial}{\partial n} + \alpha \right) \Delta^{p-i} z, \Delta^{i-1} v \right\rangle \\ - \left\langle \Delta^{p-i} z, \left(\frac{\partial}{\partial n} + \alpha \right) \Delta^{i-1} v \right\rangle \right\}. \tag{2.18}$$

Choosing $j = p$ we get for all z satisfying $\Delta^p z = 0$ and all v satisfying $\Delta^p v = 0$ that

$$\sum_{i=1}^{p} \left\langle \left(\frac{\partial}{\partial n} + \alpha\right) \Delta^{p-i} z, \Delta^{i-1} v \right\rangle = \sum_{i=1}^{p} \left\langle \Delta^{p-i} z, \left(\frac{\partial}{\partial n} + \alpha\right) \Delta^{i-1} v \right\rangle . \quad (2.19)$$

In particular, choosing $z = w_0(\vec{\sigma})$ and recalling from (2.10) and (2.11) that

$$\left(\frac{\partial}{\partial n} + \alpha\right) \Delta^{p-j} w_0(\vec{\sigma}) = \sigma_j , \qquad j = 1, 2, \ldots, [(p+1)/2]$$

and
$$\left(\frac{\partial}{\partial n} + \alpha\right) \Delta^{j-1} w_0(\vec{\sigma}) = 0 , \qquad j = 1, 2, \ldots, [p/2] ,$$

we get for all v satisfying $\Delta^p v = 0$ that

$$\sum_{j=1}^{\left[\frac{p+1}{2}\right]} \langle \sigma_i, \Delta^{i-1} v \rangle = \sum_{i=1}^{p} \left\langle \Delta^{p-i} w_0(\vec{\sigma}), \left(\frac{\partial}{\partial n} + \alpha\right) \Delta^{i-1} v \right\rangle .$$

Replacing $p - i$ by $i - 1$ in the right hand sum, we get for all v satisfying $\Delta^p v = 0$ that

$$\sum_{i=1}^{\left[\frac{p+1}{2}\right]} \langle \sigma_i, \Delta^{i-1} v \rangle = \sum_{i=1}^{p} \left\langle \Delta^{i-1} w_0(\vec{\sigma}), \left(\frac{\partial}{\partial n} + \alpha\right) \Delta^{p-i} v \right\rangle . \quad (2.20)$$

We now derive another result relating $\vec{\sigma}$ and $w_i(\vec{\sigma})$.

LEMMA 2.2 :

$$\sum_{i=1}^{\left[\frac{p+1}{2}\right]} \langle \sigma_i, w_{i-1}(\vec{\sigma}) \rangle = \begin{cases} -(\Delta^{p/2} w_0(\vec{\sigma}), \Delta^{p/2} w_0(\vec{\sigma})) , & p \text{ even} \\ A_\alpha(\Delta^{(p-1)/2} w_0(\vec{\sigma}), \Delta^{(p-1)/2} w_0(\vec{\sigma})) , & p \text{ odd} . \end{cases}$$

Proof : From (2.9) and (2.10) we have

$$\sum_{i=1}^{\left[\frac{p+1}{2}\right]} \langle \sigma_i, w_{i-1}(\vec{\sigma}) \rangle$$
$$= \sum_{i=1}^{\left[\frac{p+1}{2}\right]} A_\alpha(w_{p-i}(\vec{\sigma}), w_{i-1}(\vec{\sigma})) + \sum_{i=2}^{\left[\frac{p+1}{2}\right]} (w_{p+1-i}(\vec{\sigma}), w_{i-1}(\vec{\sigma})) .$$

For p even, $[(p+1)/2] = [p/2]$ and so by (2.11)

$$\sum_{i=1}^{[\frac{p+1}{2}]} \langle \sigma_i, w_{i-1}(\vec{\sigma}) \rangle = \sum_{i=1}^{p/2} - (w_i(\vec{\sigma}), w_{p-i}(\vec{\sigma})) + \sum_{i=2}^{p/2} (w_{p+1-i}(\vec{\sigma}), w_{i-1}(\vec{\sigma}))$$
$$= - (w_{p/2}(\vec{\sigma}), w_{p/2}(\vec{\sigma})) .$$

For p odd,

$$\sum_{i=1}^{[\frac{p+1}{2}]} \langle \sigma_i, w_{i-1}(\vec{\sigma}) \rangle$$
$$= A_\alpha(w_{p-(p+1)/2}(\vec{\sigma}), w_{(p+1)/2-1}(\vec{\sigma})) - \sum_{i=1}^{\frac{p-1}{2}} (w_i(\vec{\sigma}), w_{p-i}(\vec{\sigma})) + \sum_{i=2}^{\frac{p+1}{2}} (w_{p+1-i}(\vec{\sigma}), w_{i-1}(\vec{\sigma}))$$
$$= A_\alpha(w_{(p-1)/2}(\vec{\sigma}), w_{(p-1)/2}(\vec{\sigma})) .$$

We shall also require the following a priori estimate satisfied by solutions of the polyharmonic equation (*cf.* [14]).

LEMMA 2.3 : *Let z be a solution of the polyharmonic equation $\Delta^p z = 0$ in Ω. Then there exists a constant C independent of z such that for all real s*

$$\| z \|_s \leqslant C\left(\sum_{j=0}^{[\frac{p-1}{2}]} | \Delta^j z |_{s-2j-1/2} + \sum_{j=1}^{[\frac{p-2}{2}]} \left| \frac{\partial}{\partial n} \Delta^j z \right|_{s-3/2-2j} \right).$$

Using these results we now establish two further a priori estimates which are the basis for our error analysis.

LEMMA 2.4 : *There exist positive constants C_0 and C_1 such that*

$$C_0 \, | \vec{\sigma} |_{1/2-p}^2 \leqslant \left| \sum_{i=1}^{[\frac{p+1}{2}]} \langle \sigma_i, w_{i-1}(\vec{\sigma}) \rangle \right| \leqslant C_1 \, | \vec{\sigma} |_{1/2-p}^2$$

Proof : For $1 \leqslant j \leqslant [(p+1)/2]$ with j fixed, let v be the solution of the boundary value problem $\Delta^p v = 0$ in Ω,

$$\left(\frac{\partial}{\partial n} + \alpha\right) \Delta^{i-1} v = 0 , \quad i = 1, ..., [p/2] ,$$
$$\Delta^{i-1} v = 0 , \quad i = 1, 2, ..., [(p+1)/2] , \quad i \neq j$$
$$\Delta^{j-1} v = \psi , \quad \text{on} \quad \Gamma .$$

Using (2.17) and (2.18) with $z = w_0(\vec{\sigma})$ and recalling (2.13) that $\sigma_j = \left(\frac{\partial}{\partial n} + \alpha\right)\Delta^{p-j}w_0(\vec{\sigma})$, $j = 1, 2, ..., [(p+1)/2]$, we get that

$$\langle \sigma_j, \psi \rangle = \begin{cases} -(\Delta^{p/2}w_0(\vec{\sigma}), \Delta^{p/2}v), & p \text{ even} \\ A_\alpha(\Delta^{(p-1)/2}w_0(\vec{\sigma}), \Delta^{(p-1)/2}v), & p \text{ odd}. \end{cases}$$

Hence

$$|\langle \sigma_j, \psi \rangle| \leqslant \begin{cases} \| \Delta^{p/2}w_0(\vec{\sigma}) \|_0 \, \| \Delta^{p/2}v \|_0 & p \text{ even} \\ A_\alpha^{1/2}(\Delta^{(p-1)/2}w_0(\vec{\sigma}), \Delta^{(p-1)/2}w_0(\vec{\sigma}))\, A_\alpha^{1/2}(\Delta^{(p-1)/2}v, \Delta^{(p-1)/2}v), & p \text{ odd}. \end{cases}$$

Applying Lemma (2.2) we get in either case that

$$\begin{aligned} |\langle \sigma_j, \psi \rangle| &\leqslant \| v \|_p \left| \sum_{i=1}^{\left[\frac{p+1}{2}\right]} \langle \sigma_i, w_{i-1}(\vec{\sigma}) \rangle \right|^{1/2} \\ &\leqslant C \left| \sum_{i=1}^{\left[\frac{p+1}{2}\right]} \langle \sigma_i, w_{i-1}(\vec{\sigma}) \rangle \right|^{1/2} |\psi|_{p-1/2-2(j-1)} \end{aligned}$$

(using Lemma (2.3)).

Hence

$$\begin{aligned} |\sigma_j|_{1/2-p+2(j-1)} &= \sup_{\psi \in H^{p-1/2-2(j-1)}(\Gamma)} \frac{|\langle \sigma_j, \psi \rangle|}{|\psi|_{p-1/2-2(j-1)}} \\ &\leqslant C \left| \sum_{i=1}^{\left[\frac{p+1}{2}\right]} \langle \sigma_i, w_{i-1}(\vec{\sigma}) \rangle \right|^{1/2}. \end{aligned}$$

Summing from $j = 1, ..., [(p+1)/2]$ gives the first inequality.

To get the second inequality, we use Lemma (2.2) and some simple estimates to see that

$$\left| \sum_{i=1}^{\left[\frac{p+1}{2}\right]} \langle \sigma_i, w_{i-1}(\vec{\sigma}) \rangle \right| \leqslant C \| w_0(\vec{\sigma}) \|_p^2 .$$

Now using (2.7) and Lemma (2.1) we have

$$\begin{aligned} \| w_0(\vec{\sigma}) \|_p &\leqslant \sum_{j=1}^{\left[\frac{p+1}{2}\right]} \| T^{p-j} G\sigma_j \|_p \leqslant C \sum_{j=1}^{\left[\frac{p+1}{2}\right]} |\sigma_j|_{p-2(p-j)-3/2} \\ &= C \sum_{j=1}^{\left[\frac{p+1}{2}\right]} |\sigma_j|_{1/2-p+2(j-1)} . \end{aligned}$$

LEMMA 2.5 : *There exist positive constants C_0 and C_1 such that*

$$C_0 \, | \vec{\sigma} |_{s+1/2-2p} \leqslant \sum_{j=1}^{\left[\frac{p-1}{2}\right]} | w_i(\vec{\sigma}) |_{s-1/2-2i} \leqslant C_1 \, | \vec{\sigma} |_{s+1/2-2p}$$

for all real s.

Proof : Since $\sigma_j = \left(\frac{\partial}{\partial n} + \alpha\right) \Delta^{p-j} w_0(\vec{\sigma})$ we have for $s > 0$ that

$$\begin{aligned} | \sigma_j |_{s+2(j-1)} &= \left| \left(\frac{\partial}{\partial n} + \alpha\right) \Delta^{p-j} w_0(\vec{\sigma}) \right|_{s+2(j-1)} \\ &\leqslant C \, \| \Delta^{p-j} w_0(\vec{\sigma}) \|_{s+3/2+2(j-1)} \quad \text{(by the trace theorem)} \\ &\leqslant C \, \| w_0(\vec{\sigma}) \|_{s+3/2+2(p-1)} \\ &\leqslant C \sum_{i=0}^{\left[\frac{p-1}{2}\right]} | w_i(\vec{\sigma}) |_{s+3/2+2(p-1)-1/2-2i} \quad \text{(by Lemma (2.3))} \\ &\leqslant C \sum_{i=0}^{\left[\frac{p-1}{2}\right]} | w_i(\vec{\sigma}) |_{s+2(p-i)-1} \,. \end{aligned}$$

Hence

$$| \vec{\sigma} |_s \leqslant C \sum_{i=0}^{\left[\frac{p-1}{2}\right]} | w_i(\vec{\sigma}) |_{s+2(p-i)-1} \,. \tag{2.21}$$

For $s = 1/2 - p$ we have by Lemma (2.4) that

$$\begin{aligned} | \vec{\sigma} |^2_{1/2-p} &\leqslant C \sum_{i=1}^{\left[\frac{p+1}{2}\right]} \langle \sigma_i, w_{i-1}(\vec{\sigma}) \rangle \\ &\leqslant C \sum_{i=1}^{\left[\frac{p+1}{2}\right]} | \sigma_i |_{1/2-p+2(i-1)} \, | w_{i-1}(\vec{\sigma}) |_{p-1/2-2(i-1)} \\ &\leqslant C \, | \vec{\sigma} |_{1/2-p} \sum_{i=0}^{\left[\frac{p-1}{2}\right]} | w_i(\vec{\sigma}) |_{p-1/2-2i} \\ &\leqslant C \, | \vec{\sigma} |_{1/2-p} \sum_{i=0}^{\left[\frac{p-1}{2}\right]} | w_i(\vec{\sigma}) |_{1/2-p+2(p-i)-1} \,. \end{aligned}$$

Hence (2.21) also holds for $s = 1/2 - p$. By interpolation, (2.21) holds for $s \geqslant 1/2 - p$.

For $s < 1/2 - p$ we observe that $s + 2(j-1) < -1/2$, $j = 1, 2, \ldots,$

$[(p+1)/2]$. Using Green's identity (2.20) we get for all v with $\Delta^p v = 0$ that

$$\sum_{i=1}^{\left[\frac{p+1}{2}\right]} \langle \sigma_i, \Delta^{i-1} v \rangle = \sum_{i=1}^{p} \left\langle \Delta^{i-1} w_0(\vec{\sigma}), \left(\frac{\partial}{\partial n} + \alpha\right) \Delta^{p-i} v \right\rangle. \tag{2.22}$$

Now let v satisfy the boundary conditions

$$\begin{aligned} \left(\frac{\partial}{\partial n} + \alpha\right) \Delta^{i-1} v &= 0, \qquad i = 1, 2, \ldots, [p/2], \\ \Delta^{i-1} v &= 0, \qquad i = 1, 2, \ldots, [(p+1)/2], \qquad i \neq j, \\ \Delta^{j-1} v &= \psi. \end{aligned}$$

Then (2.22) becomes

$$\begin{aligned} \langle \sigma_j, \psi \rangle &= \sum_{i=1}^{\left[\frac{p+1}{2}\right]} \left\langle \Delta^{i-1} w_0(\vec{\sigma}), \left(\frac{\partial}{\partial n} + \alpha\right) \Delta^{p-i} v \right\rangle \\ &\leqslant \sum_{i=1}^{\left[\frac{p+1}{2}\right]} | w_{i-1}(\vec{\sigma}) |_{s+2(p-i+1)-1} \left| \left(\frac{\partial}{\partial n} + \alpha\right) \Delta^{p-i} v \right|_{-s-2(p-i+1)+1}. \end{aligned}$$

Now for $s < 1/2 - p$

$$-s - 2(p-i+1) + 1 > p - 1/2 - 2p + 2(i-1) + 1 = 1/2 - p + 2(i-1).$$

Hence from the preceeding results we have that

$$\left| \left(\frac{\partial}{\partial n} + \alpha\right) \Delta^{p-i} v \right|_{-s-2(p-i+1)+1} = \left| \left(\frac{\partial}{\partial n} + \alpha\right) \Delta^{p-i} v \right|_{-s-2p+1+2(i-1)}$$

$$\leqslant C \sum_{i=0}^{\left[\frac{p-1}{2}\right]} | \Delta^i v |_{-s-2p+1+2(p-i)-1} = C \sum_{i=0}^{\left[\frac{p-1}{2}\right]} | \Delta^i v |_{-s-2i} \leqslant C | \psi |_{-s-2(j-1)}.$$

Then

$$\begin{aligned} | \sigma_j |_{s+2(j-1)} &= \sup_{\psi \in H^{-s-2(j-1)}(\Gamma)} \frac{\langle \sigma_j, \psi \rangle}{| \psi |_{-s-2(j-1)}} \\ &\leqslant C \sum_{i=1}^{\left[\frac{p+1}{2}\right]} | w_{i-1}(\vec{\sigma}) |_{s+2(p-i+1)-1} \\ &= C \sum_{i=0}^{\left[\frac{p-1}{2}\right]} | w_i(\vec{\sigma}) |_{s+2(p-i)-1}. \end{aligned}$$

Summing from $j = 1, \ldots, [(p+1)/2]$ gives (2.21) for $s < 1/2 - p$. The left hand inequality follows by replacing s by $s + 1/2 - 2p$.

To prove the right hand inequality we have from (2.7) that

$$| w_i(\vec{\sigma}) |_{s-1/2-2i} \leqslant \sum_{j=1}^{m} | T^{p-i-j} G\sigma_j |_{s-1/2-2i} .$$

Now for $i + j < p$, we have by Lemma (2.1) that

$$\begin{aligned} | T^{p-i-j} G\sigma_j |_{s-1/2-2i} &\leqslant C \| T^{p-i-j-1} G\sigma_j \|_{s-2-2i} \\ &\leqslant C \| G\sigma_j \|_{s-2-2i-2(p-i-j-1)} \\ &\leqslant C | \sigma_j |_{s-2(p-j)-3/2} \\ &= C | \sigma_j |_{s+1/2-2p+2(j-1)} \end{aligned}$$

and for $i + j = p$ that

$$\begin{aligned} | T^{p-i-j} G\sigma_j |_{s-1/2-2i} &\leqslant C | \sigma_j |_{s-1/2-2i-1} \\ &= C | \sigma_j |_{s+1/2-2p+2(j-1)} . \end{aligned}$$

Hence

$$\begin{aligned} \sum_{i=0}^{[\frac{p-1}{2}]} | w_i(\vec{\sigma}) |_{s-1/2-2i} &\leqslant C \sum_{j=1}^{m} | \sigma_j |_{s+1/2-2p+2(j-1)} \\ &\leqslant C | \vec{\sigma} |_{s+1/2-2p} . \end{aligned}$$

3. APPROXIMATING SPACES ON Ω AND Γ

For $0 < h < 1$, let $\{ S_h \}$ be a family of finite dimensional subspaces of $H^1(\Omega)$ and $V_h = \{ \vec{w}_h : w_{h,i} \in S_h, i = 0, 1, \ldots, p-1 \}$. Let $r \geqslant 2$ be an integer. We shall assume that for $\phi \in H^l(\Omega)$, with $1 \leqslant l \leqslant r$, there is a constant C such that

$$\inf_{\chi \in S_h} \| \phi - \chi \|_j \leqslant Ch^{l-j} \| \phi \|_l , \qquad j \leqslant 1 . \tag{3.1}$$

We now define operators $G_h : H^{-1/2}(\Gamma) \to S_h$ and $T_h : H^{-1}(\Omega) \to S_h$ by

$$A_\alpha(G_h \theta, \chi) = \langle \theta, \chi \rangle \qquad \forall \chi \in S_h$$

and

$$A_\alpha(T_h f, \chi) = (f, \chi) \qquad \forall \chi \in S_h .$$

These are just the standard Ritz-Galerkin approximations to G and T.

It follows from the approximation assumptions and standard duality arguments that we have the following well known results (*cf.* [3] and [6]).

LEMMA 3.1 : *There exists a constant C independent of σ, f, and h such that*

$$|(G - G_h)\sigma|_{j-1/2} + \|(G - G_h)\sigma\|_j \leqslant Ch^{l-j}\|G\sigma\|_l$$

and

$$|(T - T_h)f|_{j-1/2} + \|(T - T_h)f\|_j \leqslant Ch^{l-j}\|Tf\|_l$$

for $2 - r \leqslant j \leqslant 1 \leqslant l \leqslant r$, $\sigma \in H^{l-3/2}(\Gamma)$, *and* $f \in H^{l-2}(\Omega)$.

Note that the restriction to Ω of continuous piecewise polynomials of degree $r - 1$ on a quasi-uniform triangulation of R^2 or a rectangular mesh of width h are examples of spaces S_h satisfying Lemma (3.1)

In the analysis in the subsequent sections we shall require additional estimates for the approximation of the operators T and G by T_h and G_h. These are contained in the following theorems.

THEOREM 3.1 : *There exists a constant C independent of h and f such that for integer $l \geqslant 1$ and $f \in H^t(\Omega)$*

$$|(T^l - T_h^l)f|_{j-1/2} + \|(T^l - T_h^l)f\|_j \leqslant Ch^{t+2l+\min(r-2l,-j)}\|f\|_t$$

for all $-1 \leqslant t \leqslant \max(r - 2l, -j)$ *and all* $2 - r \leqslant j \leqslant 1$.

Proof : The case $l = 1$ is Lemma (3.1). We now prove the general result by induction on l. Assuming the result holds for some integer $l \geqslant 1$, we write

$$T^{l+1} - T_h^{l+1} = T^l(T - T_h) + (T^l - T_h^l)(T_h - T) + (T^l - T_h^l)T.$$

Then

$$\begin{aligned}|\|T^l(T - T_h)f\||_j &\leqslant C\|(T - T_h)f\|_{j-2l} \leqslant Ch^{\min(r-2,2l-j)+t+2}\|f\|_t \\ &\leqslant Ch^{t+2(l+1)+\min(r-2(l+1),-j)}\|f\|_t\end{aligned}$$

for all $-1 \leqslant t \leqslant r - 2$, by Lemma (3.1).

By the induction hypothesis, we have for all $-1 \leqslant \tilde{t} \leqslant \max(r - 2l, -j)$ that

$$\begin{aligned}|\|(T^l - T_h^l)Tf\||_j &\leqslant Ch^{\tilde{t}+2l+\min(r-2l,-j)}\|Tf\|_{\tilde{t}} \\ &\leqslant Ch^{\tilde{t}+2l+\min(r-2l,-j)}\|f\|_{\tilde{t}-2}.\end{aligned}$$

We now consider three cases :

(i) If $j \geqslant 2l + 2 - r$, we get for $-1 \leqslant t \leqslant \max(r - 2(l+1), -j) = r - 2l - 2$ and $\tilde{t} = t + 2$ that

$$\|\!|(T^l - T_h^l)\,Tf\|\!|_j \leqslant Ch^{t+2(l+1)+\min(r-2l,-j)}\,\|f\|_t$$
$$\leqslant Ch^{t+2(l+1)+\min(r-2(l+1),-j)}\,\|f\|_t\,.$$

(ii) If $2 - r \leqslant j \leqslant 2l - r$, we get for $-1 \leqslant t \leqslant \max(r - 2(l+1), -j) = -j$ and $\tilde{t} = t$ that

$$\|\!|(T^l - T_h^l)\,Tf\|\!|_j \leqslant Ch^{t+2l+\min(r-2l,-j)}\,\|f\|_{t-2}$$
$$\leqslant Ch^{t+2(l+1)+r-2(l+1)}\,\|f\|_{t-2}$$
$$\leqslant Ch^{t+2(l+1)+\min(r-2(l+1),-j)}\,\|f\|_t\,.$$

(iii) If $2l - r < j < 2l + 2 - r$, we get for $-1 \leqslant t \leqslant \max(r - 2(l+1), -j) = -j$ and $\tilde{t} = t + r - 2l + j$ that

$$\|\!|(T^l - T_h^l)\,Tf\|\!|_j \leqslant Ch^{t+r-2l+j+2l+\min(r-2l,-j)}\,\|f\|_{\tilde{t}-2}$$
$$\leqslant Ch^{t+r}\,\|f\|_t$$
$$\leqslant Ch^{t+2(l+1)+\min(r-2(l+1),-j)}\,\|f\|_t\,,$$

where we have observed that for t in the above range,

$$-1 + r - 2l + j \leqslant \tilde{t} \leqslant -j + r - 2l + j = r - 2l \leqslant \max(-j, r - 2l)\,.$$

To estimate the remaining term, we again use the induction hypothesis to see that for $-1 \leqslant t \leqslant r - 2$ and $r \geqslant 3$

$$\|\!|(T^l - T_h^l)(T_h - T)f\|\!|_j \leqslant Ch^{-1+2l+\min(r-2l,-j)}\,\|(T_h - T)f\|_{-1}$$
$$\leqslant Ch^{-1+2l+\min(r-2l,-j)+t+3}\,\|f\|_t$$
$$\leqslant Ch^{t+2(l+1)+\min(r-2(l+1),-j)}\,\|f\|_t\,.$$

To complete the lemma we need only consider this last term in the case $r = 2$ and $0 \leqslant j \leqslant 1$. Using the induction hypothesis we have for $-1 \leqslant t \leqslant 0$ that

$$\|\!|(T^l - T_h^l)(T_h - T)f\|\!|_j \leqslant Ch^{-j+2l+\min(2-2l,-j)}\,\|(T_h - T)f\|_{-j}$$
$$\leqslant Ch^{2l-j+\min(2-2l,-j)+t+2}\,\|f\|_t$$
$$\leqslant Ch^{t+2(l+1)+\min(2-2l,-j)-j}\,\|f\|_t$$
$$\leqslant Ch^{t+2(l+1)+\min(2-2(l+1),-j)}\,\|f\|_t\,.$$

The last inequality follows since $0 \leqslant j \leqslant 1$ and

$$\min(2 - 2l, -j) - j = \begin{cases} -2j, & l = 1 \\ 2 - 2l - j, & l \geqslant 2 \end{cases}$$

and

$$\min(2 - 2(l + 1), -j) = \begin{cases} -2, & l = 1 \\ 2 - 2l - 2, & l \geqslant 2. \end{cases}$$

The theorem now follows by combining these results.

The following is an important special case of the theorem which we shall frequently use.

COROLLARY 3.1 :

$$|(T^l - T_h^l) f|_{j-1/2} + \|(T^l - T_h^l) f\|_j \leqslant C h^{t+2l-j} \|f\|_t$$

for all $2l - r \leqslant j \leqslant 1$ *and* $-1 \leqslant t \leqslant r - 2l$.

In an exactly analogous manner one can show the following.

THEOREM 3.2 : *There exists a constant* C *independent of* h *and* σ *such that for integer* $l \geqslant 1$ *and* $\sigma \in H^{t+1/2}(\Gamma)$

$$|T^{l-1} G\sigma - T_h^{l-1} G_h \sigma|_{j-1/2} + \|T^{l-1} G\sigma - T_h^{l-1} G_h \sigma\|_j \leqslant C h^{t+2l+\min(r-2l,-j)} |\sigma|_{t+1/2}$$

for all $-1 \leqslant t \leqslant \max(r - 2l, -j)$ *and all* $2 - r \leqslant j \leqslant 1$.

COROLLARY 3.2 :

$$|T^{l-1} G\sigma - T_h^{l-1} G_h \sigma|_{j-1/2} + \|T^{l-1} G\sigma - T_h^{l-1} G_h \sigma\|_j \leqslant C h^{t+2l-j} |\sigma|_{t+1/2}$$

for $2l - r \leqslant j \leqslant 1$ *and* $-1 \leqslant t \leqslant r - 2l$.

For $0 < k < 1$, we shall let $\{\dot{S}_k\}$ be a family of finite dimensional subspaces of $H^n(\Gamma)$, $n \geqslant \max(0, 2[(p-1)/2] - 1/2)$, and let $\dot{V}_k = \{\vec{\sigma} : \sigma_i \in \dot{S}_k,\ i = 0, 1, \dots, [(p-1)/2]\}$. Let $\dot{r} \geqslant 1$ be an integer. We shall suppose that for $\phi \in H^l(\Gamma)$ with $j \leqslant n$ and $j \leqslant l \leqslant \dot{r}$, there is a constant C such that

$$\inf_{\chi \in \dot{S}_k} |\phi - \chi|_j \leqslant C k^{l-j} |\phi|_l. \tag{3.2}$$

We further assume that for $j \leqslant i \leqslant n$ there is a constant C such that

$$| \phi |_i \leqslant C k^{j-i} | \phi |_j , \tag{3.3}$$

for all $\phi \in \dot{S}_k$.

The conditions (3.2) and (3.3) together imply that for any given $j_0 \leqslant n$ there is an operator $\Pi_k : H^{j_0}(\Gamma) \to \dot{S}_k$ with

$$| \phi - \Pi_k \phi |_j \leqslant C_{j_0} k^{l-j} | \phi |_l , \tag{3.4}$$

uniformly in j and l for $j \leqslant n$ and $j_0 \leqslant j \leqslant l \leqslant \dot{r}$. This result can be found in [8]. Finally, we denote by P_0 the $L_2(\Gamma)$ orthogonal projection onto $\dot{S}_k$, i.e. for $\phi \in L_2(\Gamma) = H^0(\Gamma)$,

$$\langle P_0 \phi, \theta \rangle = \langle \phi, \theta \rangle \quad \text{for all} \quad \theta \in \dot{S}_k .$$

We now note for further reference the following properties satisfied by the projection operator P_0.

LEMMA 3.2 : *For* $-\dot{r} \leqslant j \leqslant n$ *and* $\max(-n, j) \leqslant l \leqslant \dot{r}$ *there is a constant* C *such that for* $\phi \in H^l(\Gamma)$

$$| (I - P_0) \phi |_j \leqslant C k^{l-j} | \phi |_l \tag{3.5}$$

and

$$| P_0 \phi |_j \leqslant C [| \phi |_j + k^{l-j} | \phi |_l] . \tag{3.6}$$

Finally we prove the following result which we shall need in Section 6.

LEMMA 3.3 : *For all* $\sigma, \theta \in \dot{S}_k$

$$\langle P_0 T_h^m G_h \sigma, \theta \rangle = \langle \sigma, P_0 T_h^m G_h \theta \rangle , \qquad m \geqslant 1 .$$

Proof : We first observe that for all $u_h, v_h \in S_h$,

$$A_\alpha(T_h u_h, v_h) = (u_h, v_h) = A_\alpha(u_h, T v_h) .$$

Hence it easily follows that

$$A_\alpha(T_h^m G_h \sigma, G_h \theta) = A_\alpha(G_h \sigma, T_h^m G_h \theta) .$$

But

$$\begin{aligned} \langle P_0 T_h^m G_h \sigma, \theta \rangle = \langle T_h^m G_h \sigma, \theta \rangle &= A_\alpha(T_h^m G_h \sigma, G_h \theta) \\ &= A_\alpha(G_h \sigma, T_h^m G_h \theta) = \langle \sigma, T_h^m G_h \theta \rangle \\ = \langle \sigma, P_0 T_h^m G_h \theta \rangle . \end{aligned}$$

4. THE APPROXIMATION SCHEME

We now are ready to define a finite element method for the approximation of the first boundary value problem for the polyharmonic equation, based on the variational formulation of Problem (P).

Let $\vec{w}_h = (w_{h,0}, \dots, w_{h,p-1})$ and $\vec{\sigma}_k = (\sigma_{k,1}, \dots, \sigma_{k,[(p+1)/2]})$.

Problem (P_h^k) : Find $\vec{w}_h, \vec{\sigma}_k \in V_h \times \dot{V}_k$ such that

$$A_\alpha(w_{h,p-1}, v_h) = -(f, v_h) + \langle \sigma_{k,1}, v_h \rangle \qquad \forall v_h \in S_h, \tag{4.1}$$

$$A_\alpha(w_{h,p-j}, v_h) = -(w_{h,p+1-j}, v_h) + \langle \sigma_{k,j}, v_h \rangle \qquad \forall v_h \in S_h, \quad j = 2, 3, \dots, [(p+1)/2], \tag{4.2}$$

$$A_\alpha(w_{h,j-1}, v_h) = -(w_{h,j}, v_h) \qquad \forall v_h \in S_h, \quad j = 1, 2, \dots, [p/2], \tag{4.3}$$

and

$$\langle w_{h,j-1}, \theta_k \rangle = 0 \qquad \forall \theta_k \in \dot{S}_k, \quad j = 1, 2, \dots, [(p+1)/2]. \tag{4.4}$$

Using the operators T_h and G_h we may also restate Problem (P_h^k) in a form analogous to Problem Q. From (4.1)-(4.3) we get

$$w_{h,p-1} = -T_h f + G_h \sigma_{k,1}, \tag{4.5}$$

$$w_{h,p-j} = -T_h w_{h,p+1-j} + G_h \sigma_{k,j}, \qquad j = 2, 3, \dots, [(p+1)/2], \tag{4.6}$$

and

$$w_{h,j-1} = -T_h w_{h,j}, \qquad j = 1, 2, \dots, [p/2]. \tag{4.7}$$

We will approximate the functions w_j by $w_{h,j}$ (and $u = w_0$ by $u_h = w_{h,0}$). From (4.5)-(4.7) we get that

$$u_h = w_{h,0} = (-1)^p T_h^p f + \sum_{j=1}^{\left[\frac{p+1}{2}\right]} (-1)^{p-j} T_h^{p-j} G_h \sigma_{k,j}. \tag{4.8}$$

In an analogous manner to (2.7) let us now define

$$w_{h,i}(\vec{\sigma}) = \sum_{j=1}^{m} (-1)^{p-i-j} T_h^{p-i-j} G_h \sigma_j, \qquad i = 0, 1, \dots, p-1 \tag{4.9}$$

where $m = \min(p - i, [(p+1)/2])$.

Then

$$w_{h,i} = w_{h,i}(\vec{\sigma}_k) + (-1)^{p-i} T_h^{p-i} f, \qquad i = 0, 1, \ldots, p-1 . \qquad (4.10)$$

In terms of the bilinear forms we note that $\{ w_{h,i}(\vec{\sigma}) \}$ satisfy for all $v_h \in S_h$ the following variational equations.

$$A_\alpha(w_{h,p-1}(\vec{\sigma}), v_h) = \langle \sigma_1, v_h \rangle \qquad (4.11)$$

$$A_\alpha(w_{h,p-j}(\vec{\sigma}), v_h) = -(w_{h,p+1-j}(\vec{\sigma}), v_h) \qquad j = 2, 3, \ldots, [(p+1)/2], \qquad (4.12)$$

and

$$A_\alpha(w_{h,j-1}(\vec{\sigma}), v_h) = -(w_{h,j}(\vec{\sigma}), v_h), \qquad j = 1, 2, \ldots, [p/2] . \qquad (4.13)$$

In terms of this notation it is then possible to restate Problem (P_h^k) in the form :

Problem (Q_h^k) : Find $\vec{\sigma}_k = (\sigma_{k,1}, \ldots, \sigma_{k,[(p+1)/2]}) \in \dot{V}_k$ such that

$$P_0\, w_{h,i}(\vec{\sigma}) = P_0 \{ (-1)^{p-i+1} T_h^{p-i} f \}, \qquad i = 0, 1, \ldots, [(p-1)/2], \qquad (4.14)$$

where $w_{h,i}(\vec{\sigma}_k)$ is defined by (4.9).

Our aim is now to study the functions $w_{h,i}(\vec{\sigma})$ and prove estimates analogous to those in Lemmas (2.4) and (2.5).

To simplify the analysis, it will be convenient to first prove the following result.

LEMMA 4.1 : *Let $w_i(\vec{\sigma})$ and $w_{h,i}(\vec{\sigma})$ be defined by (2.7) and (4.9) respectively. Then if $\vec{\sigma} \in \overrightarrow{H}^{t+1/2}(\Gamma)$ we have for $-1 \leqslant t \leqslant 2i + \max(r-2p, -s-2i)$ and $2 - r \leqslant s \leqslant 1$ that*

$$\| w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma}) \|_s + | w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma}) |_{s-1/2} \leqslant C h^{t+2p+\min(r-2p,-s-2i)} | \vec{\sigma} |_{t+1/2} .$$

Proof :

$$||| w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma}) |||_s \leqslant \sum_{j=1}^{m} ||| [T^{p-i-j} G - T_h^{p-i-j} G_h]\, \sigma_j |||_s .$$

We now consider three cases :

(i) $-2i + 2p - r \leqslant s \leqslant 1$, $-1 \leqslant t \leqslant 2i + r - 2p$. Then

$$-1 \leqslant t + 2(j-1) \leqslant 2i + r - 2p + 2(j-1) = \max(r - 2(p-j-i+1), -s)$$

holds for $i \geqslant 0$ and $j \geqslant 1$. Hence by Theorem (3.1)

$$\begin{aligned}
&||| w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma}) |||_s \\
&\quad \leqslant \sum_{j=1}^{m} C h^{t+2(j-1)+2(p-i-j+1)+\min(r-2(p-i-j+1),-s)} | \sigma_j |_{t+2(j-1)+1/2} \\
&\quad \leqslant \sum_{j=1}^{m} C h^{t+2(p-i)-s} | \sigma_j |_{t+2(j-1)+1/2} \\
&\quad \leqslant C h^{t+2(p-i)-s} | \vec{\sigma} |_{t+1/2} .
\end{aligned}$$

(ii) $-1 \leqslant t \leqslant -s$ and $2 - r \leqslant s \leqslant 2(p-i-m+1) - r$. Then $-s \geqslant r - 2(p-i-j+1)$ for $j = 1, \dots, m$ and we get using Theorem (3.1) that

$$\begin{aligned}
||| w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma}) |||_s &\leqslant \sum_{j=1}^{m} C h^{t+2(p-i-j+1)+r-2(p-i-j+1)} | \sigma_j |_{t+1/2} \\
&\leqslant \sum_{j=1}^{m} C h^{t+r} | \sigma_j |_{t+2(j-1)+1/2} \\
&\leqslant C h^{t+r} | \vec{\sigma} |_{t+1/2} .
\end{aligned}$$

(iii) $-1 \leqslant t \leqslant -s$ and $2(p-l-i) - r \leqslant s \leqslant 2(p-l-i+1) - r$ where $l = 1, \dots, m-1$. Then by Theorem (3.1), we get for

$$-1 \leqslant t_j \leqslant \max(r - 2(p-i-j+1), -s) \tag{4.15}$$

and $2 - r \leqslant s \leqslant 1$ that

$$\begin{aligned}
&||| w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma}) |||_s \\
&\quad \leqslant \sum_{j=1}^{m} C h^{t_j+2(p-i-j+1)+\min(r-2(p-i-j+1),-s)} | \sigma_j |_{t_j+1/2} \\
&\quad \leqslant \sum_{j=1}^{l} C h^{t_j+2(p-i-j+1)+r-2(p-i-j+1)} | \sigma_j |_{t+1/2} \\
&\qquad\qquad + \sum_{j=l+1}^{m} C h^{t_j+2(p-i-j+1)-s} | \sigma_j |_{t_j+1/2} \\
&\quad \leqslant \sum_{j=1}^{l} C h^{t_j+r} | \sigma_j |_{t_j+1/2} + \sum_{j=l+1}^{m} C h^{t_j+2(p-i-j+1)-s} | \sigma_j |_{t_j+1/2} .
\end{aligned}$$

Choose $t_j = t$ for $j = 1, \dots, l$ and $t_j = t + s + r - 2(p-i-j+1)$ for $j = l+1, \dots, m$.

Since

$$\max(r - 2(p-i-j+1), -s) = \begin{cases} -s, & j = 1, \dots, l \\ r - 2(p-i-j+1), & j = l+1, \dots, m \end{cases}$$

and $r + s - 2(p - j - i + 1) \geqslant 0$ for $j = l + 1, \ldots, m$, we observe that for $-1 \leqslant t \leqslant -s$ this choice of t_j does give values in the range required by (4.15).

Hence we have

$$|||\, w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma})\, |||_s \leqslant \sum_{j=1}^{l} Ch^{t+r} \,|\, \sigma_j \,|_{t+1/2} + \sum_{j=l+1}^{m} Ch^{t+r} \,|\, \sigma_j \,|_{t+r+s-2(p-i-j+1)+1/2}\,.$$

Now $s \leqslant 2(p - l - i + 1) - r$ implies

$$\begin{aligned} t + r + s - 2(p - i - j + 1) + 1/2 &\leqslant t + 2(j - 1) + 1/2 + 2(1 - l) \\ &\leqslant t + 2(j - 1) + 1/2\,. \end{aligned}$$

Hence $|||\, w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma})\, |||_s \leqslant Ch^{t+r} \,|\, \vec{\sigma} \,|_{t+1/2}$.

Combining these results we see that

$$|||\, w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma})\, |||_s \leqslant \begin{cases} Ch^{t+2(p-i)-s} \,|\, \vec{\sigma} \,|_{t+1/2} & \text{for} \quad \begin{array}{l} -2i + 2p - r \leqslant s \leqslant 1 \\ -1 \leqslant t \leqslant 2i + r - 2p \end{array} \\ Ch^{t+r} \,|\, \vec{\sigma} \,|_{t+1/2} & \text{for} \quad \begin{array}{l} 2 - r \leqslant s \leqslant 2p - r - 2i \\ -1 \leqslant t \leqslant -s\,. \end{array} \end{cases}$$

We can further simplify this to the form given in the statement of the lemma, i.e.

$$|||\, w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma})\, |||_s \leqslant Ch^{t+2p+\min(r-2p,\,-s-2i)} \,|\, \vec{\sigma} \,|_{t+1/2}$$

for $-1 \leqslant t \leqslant 2i + \max(r - 2p, -s - 2i)$ and $2 - r \leqslant s \leqslant 1$.

LEMMA 4.2 : *For* $-1 \leqslant t \leqslant 2i + r - 2p$ *and* $2p - r \leqslant \tilde{s} \leqslant 1 + 2i$

$$\|\, w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma})\, \|_{s-2i} + |\, w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma})\, |_{\tilde{s}-2i-1/2} \leqslant Ch^{t+2p-\tilde{s}} \,|\, \vec{\sigma} \,|_{t+1/2}\,, \qquad i = 0, 1, \ldots, p - 1\,.$$

Proof : Replace s in Lemma (4.1) by $\tilde{s} - 2i$ and observe that for

$$2p - r \leqslant \tilde{s} \leqslant 1\,, \qquad 2 - r \leqslant \tilde{s} - 2i \leqslant 1\,,$$

and that $\min(r - 2p, -\tilde{s}) = -\tilde{s}$.

LEMMA 4.3 : *Suppose that $r - 2p \geqslant -1$. Then for $h \leqslant \varepsilon k$ with ε sufficiently small, there exist positive constants C_0 and C_1 such that for all $\vec{\sigma} \in \dot{V}_k$,*

$$C_0 \, | \vec{\sigma} |^2_{1/2-p} \leqslant \left| \sum_{i=1}^{\left[\frac{p+1}{2}\right]} \langle \sigma_i, w_{h,i-1}(\vec{\sigma}) \rangle \right| \leqslant C_1 \, | \vec{\sigma} |^2_{1/2-p} .$$

Proof : By Lemma (2.4) and the triangle inequality, we get

$$\begin{aligned} C_0 \, | \vec{\sigma} |^2_{1/2-p} &- \left| \sum_{i=1}^{\left[\frac{p+1}{2}\right]} \langle \sigma_i, w_{i-1}(\vec{\sigma}) - w_{h,i-1}(\vec{\sigma}) \rangle \right| \\ &\leqslant \left| \sum_{i=1}^{\left[\frac{p+1}{2}\right]} \langle \sigma_i, w_{h,i-1}(\vec{\sigma}) \rangle \right. \\ &\leqslant C_1 \, | \vec{\sigma} |^2_{1/2-p} + \left| \sum_{i=1}^{\left[\frac{p+1}{2}\right]} \langle \sigma_i, w_{i-1}(\vec{\sigma}) - w_{h,i-1}(\vec{\sigma}) \rangle \right| . \end{aligned} \tag{4.16}$$

Now

$$\begin{aligned} | \langle \sigma_i, w_{i-1}(\vec{\sigma}) - w_{h,i-1}(\vec{\sigma}) \rangle | &\leqslant | \vec{\sigma}_i |_{-1/2} \, | w_{i-1}(\vec{\sigma}) - w_{h,i-1}(\vec{\sigma}) |_{1/2} \\ &\leqslant | \sigma_i |_{-1/2} \, C h^{-1+2p-1-2(i-1)} \, | \vec{\sigma} |_{-1/2} \end{aligned}$$

(by Lemma (4.1), assuming $r - 2p \geqslant -1$)

$$\leqslant C h^{2(p-i)} \, k^{1-p+2(i-1)} \, | \sigma_i |_{1/2-p+2(i-1)} \, k^{1-p} \, | \vec{\sigma} |_{1/2-p}$$

(by (3.3))

$$\leqslant C (h/k)^{2(p-i)} \, | \sigma_i |_{1/2-p+2(i-1)} \, | \vec{\sigma} |_{1/2-p} .$$

Hence

$$\left| \sum_{i=1}^{\left[\frac{p+1}{2}\right]} \langle \sigma_i, w_{i-1}(\vec{\sigma}) - w_{h,i-1}(\vec{\sigma}) \rangle \right| \leqslant C (h/k)^{2[p/2]} \, | \vec{\sigma} |^2_{1/2-p} .$$

The lemma follows for $p \geqslant 2$ and $h \leqslant \varepsilon k$ with ε sufficiently small.

LEMMA 4.4 : *There exist positive constants C_0 and C_1 independent of $\vec{\sigma}$ and k such that for all $-\dot{r} + 2[(p-1)/2] + 1/2 \leqslant s \leqslant \min(p, n + 1/2)$ and all $\vec{\sigma} \in \dot{V}_k$,*

$$C_0 \, | \vec{\sigma} |_{s+1/2-2p} \leqslant \sum_{i=0}^{\left[\frac{p-1}{2}\right]} | P_0 \, w_i(\vec{\sigma}) |_{s-1/2-2i} \leqslant C_1 \, | \vec{\sigma} |_{s+1/2-2p} .$$

Proof : From Lemma (2.5) and the triangle inequality we have

$$C_0 | \vec{\sigma} |_{s+1/2-2p} - \sum_{i=0}^{[\frac{p-1}{2}]} | (I - P_0) w_i(\vec{\sigma}) |_{s-1/2-2i}$$
$$\leqslant \sum_{i=0}^{[\frac{p-1}{2}]} | P_0 w_i(\vec{\sigma}) |_{s-1/2-2i}$$
$$\leqslant C_1 | \vec{\sigma} |_{s+1/2-2p} + \sum_{i=0}^{[\frac{p-1}{2}]} | (I - P_0) w_i(\vec{\sigma}) |_{s-1/2-2i} .$$

Now for $-\dot{r} + 2[(p-1)/2] + 1/2 \leqslant s \leqslant n + 1/2$, and $0 \leqslant i \leqslant [(p-1)/2]$ we have $-\dot{r} \leqslant s - 2i - 1/2 \leqslant n$ and so by Lemma (3.2),

$$\begin{aligned} | (I - P_0) w_i(\vec{\sigma}) |_{s-1/2-2i} &\leqslant Ck^{p-s} | w_i(\vec{\sigma}) |_{p-1/2-2i} \\ &\leqslant Ck^{p-s} \| w_i(\vec{\sigma}) \|_{p-2i} \\ &\leqslant Ck^{p-s} \| \Delta^i w_0(\vec{\sigma}) \|_{p-2i} \quad \text{(by 2.12)} \\ &\leqslant Ck^{p-s} \| w_0(\vec{\sigma}) \|_p . \end{aligned}$$

Now by (2.7) and Lemma (2.1),

$$\begin{aligned} \| w_0(\vec{\sigma}) \|_p &\leqslant \sum_{j=1}^{[\frac{p+1}{2}]} \| T^{p-j} G\sigma_j \|_p \\ &\leqslant C \sum_{j=1}^{[\frac{p+1}{2}]} \| G\sigma_j \|_{p-2(p-j)} \leqslant C \sum_{j=1}^{[\frac{p+1}{2}]} | \sigma_j |_{p-2(p-j)-3/2} \\ &\leqslant C \sum_{j=1}^{[\frac{p+1}{2}]} | \sigma_j |_{1/2-p+2(j-1)} \leqslant C | \vec{\sigma} |_{1/2-p} . \end{aligned}$$

Hence from Lemma (2.4) we get for

$$-\dot{r} + 2[(p-1)/2] + 1/2 \leqslant s \leqslant \min(p, n + 1/2)$$

$$\begin{aligned} | (I - P_0) w_i(\vec{\sigma}) |^2_{s-1/2-2i} &\leqslant Ck^{2(p-s)} \left| \sum_{i=1}^{[\frac{p+1}{2}]} \langle \sigma_i, w_{i-1}(\vec{\sigma}) \rangle \right| \\ &\leqslant Ck^{2(p-s)} \sum_{i=1}^{[\frac{p+1}{2}]} | \langle \sigma_i, P_0 w_{i-1}(\vec{\sigma}) \rangle | \\ &\leqslant Ck^{2(p-s)} \sum_{i=1}^{[\frac{p+1}{2}]} | \sigma_i |_{-\alpha_i} | P_0 w_{i-1}(\vec{\sigma}) |_{\alpha_i} \end{aligned}$$

where $\alpha_i = \min(n, p - 1/2) - 2(i-1)$. Note that since

$$n \geqslant \max(0, 2[(p-1)/2] - 1/2),\ -\alpha_i = 2(i-1) - \min(n, p-1/2) \leqslant \begin{cases} 1/2 & p \geqslant 3 \\ 0, & p = 2 \end{cases}$$

and hence $-\alpha_i \leqslant n$. From (3.3) we have that

$$| \sigma_i |_{-\alpha_i} \leqslant C k^{s+1/2-2p+\min(n,p-1/2)} | \sigma_i |_{s+1/2-2p+2(i-1)}$$

and

$$| P_0 w_{i-1}(\vec{\sigma}) |_{\alpha_i} \leqslant C k^{s-1/2-\min(n,p-1/2)} | P_0 w_{i-1}(\vec{\sigma}) |_{s-1/2-2(i-1)}$$

since for $s \leqslant \min(p, n+1/2)$, $s - 1/2 - \min(n, p-1/2) \leqslant 0$ and

$$s + 1/2 - 2p + \min(n, p - 1/2) \leqslant 2\{\min(n+1/2, p) - p\} \leqslant 0 .$$

Combining these results we get that

$$\begin{aligned} | (I - P_0) w_i(\vec{\sigma}) |^2_{s-1/2-2i} &\leqslant C \sum_{i=1}^{[\frac{p+1}{2}]} | \sigma_i |_{s+1/2-2p+2(i-1)} | P_0 w_{i-1}(\vec{\sigma}) |_{s-1/2-2(i-1)} \\ &\leqslant C | \vec{\sigma} |_{s+1/2-2p} \sum_{i=0}^{[\frac{p-1}{2}]} | P_0 w_{i-1}(\vec{\sigma}) |_{s-1/2-2i} . \end{aligned}$$

Applying the arithmetic-geometric mean inequality we easily obtain for any $\beta > 0$

$$\sum_{i=0}^{[\frac{p-1}{2}]} | (I - P_0) w_i(\vec{\sigma}) |_{s-1/2-2i} \leqslant \beta | \vec{\sigma} |_{s+1/2-2p} + \frac{C}{\beta} \sum_{i=0}^{[\frac{p-1}{2}]} | P_0 w_i(\vec{\sigma}) |_{s-1/2-2i} .$$

Choosing β sufficiently small, we obtain the left hand inequality and choosing β sufficiently large gives the right hand inequality.

THEOREM 4.1 : *Suppose $p \geqslant 2$ and that*

$$\max(2p - 1, 2p - n - 1/2) \leqslant r \leqslant \dot{r} - 2[(p-1)/2] - 1/2 + 2p .$$

Then for $h \leqslant \varepsilon k$ with ε sufficiently small, there exists positive constants C_0

and C_1 independent of $\vec{\sigma}$, h, and k such that for all $2p-r \leqslant s \leqslant \min(p, n+1/2)$ *and* $\vec{\sigma} \in \dot{V}_k$,

$$C_0 \,|\vec{\sigma}|_{s+1/2-2p} \leqslant \sum_{i=0}^{[\frac{p-1}{2}]} |P_0\, w_{h,i}(\vec{\sigma})|_{s-1/2-2i} \leqslant C_1\, |\vec{\sigma}|_{s+1/2-2p}\,.$$

Proof : Since for

$$r \leqslant \dot{r} - 2[(p-1)/2] + 2p - 1/2\,, \qquad 2p - r \geqslant -\dot{r} + 2[(p-1)/2] + 1/2\,,$$

we have using Lemma (4.4) and the triangle inequality that for

$$2p - r \leqslant s \leqslant \min(p, n + 1/2)$$

$$\begin{aligned} C_0\,|\vec{\sigma}|_{s+1/2-2p} &- \sum_{i=0}^{[\frac{p-1}{2}]} |P_0(w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma}))|_{s-1/2-2i} \\ &\leqslant \sum_{i=0}^{[\frac{p-1}{2}]} |P_0\, w_{h,i}(\vec{\sigma})|_{s-1/2-2i} \\ &\leqslant C_1\,|\vec{\sigma}|_{s+1/2-2p} + \sum_{i=0}^{[\frac{p-1}{2}]} |P_0(w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma}))|_{s-1/2-2i}\,. \end{aligned} \tag{4.17}$$

Now by Lemmas (3.2) and (3.3) we get for $2p-r \leqslant s \leqslant \min(p, n+1/2, 1+2i)$ that

$$|P_0(w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma}))|_{s-1/2-2i} \leqslant C\{\,|w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma})|_{s-1/2-2i} + k^{1+2i-s}\,|w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma})|_{1/2}\,\}\,.$$

For $2p - r \leqslant s \leqslant 1 + 2i$ we have from Lemma (4.2) that

$$|w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma})|_{s-1/2-2i} \leqslant C h^{-1+2p-s}\,|\vec{\sigma}|_{-1/2}$$

and from Lemma (4.1) that

$$|w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma})|_{1/2} \leqslant C h^{-1+2p-1-2i}\,|\vec{\sigma}|_{-1/2}\,.$$

Hence

$$\begin{aligned} |P_0(w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma}))|_{s-1/2-2i} &\leqslant C\{\,h^{2p-s-1} + k^{1+2i-s}\,h^{-2+2p-2i}\,\}\,|\vec{\sigma}|_{-1/2} \\ &\leqslant C\{\,h^{2p-s-1} + k^{1+2i-s}\,h^{-2+2p-2i}\,\}\,k^{s+1-2p}\,|\vec{\sigma}|_{s+1/2-2p} \end{aligned}$$

(using (3.3))

$$\leqslant C\{\,(h/k)^{2p-s-1} + (h/k)^{2p-2-2i}\,\}\,|\vec{\sigma}|_{s+1/2-2p}\,.$$

When $1 + 2i \leqslant s \leqslant \min(p, n + 1/2)$ we get using (3.3) and Lemmas (3.2) and (4.1) that

$$\begin{aligned} | P_0(w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma})) |_{s-1/2-2i} &\leqslant Ck^{1-s+2i} | P_0(w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma})) |_{1/2} \\ &\leqslant Ck^{1-s+2i} | w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma}) |_{1/2} \\ &\leqslant Ck^{1-s+2i} h^{2p-2+2i} | \vec{\sigma} |_{-1/2} \\ &\leqslant Ck^{1-s+2i} h^{2p-2+2i} k^{s+1-2p} | \vec{\sigma} |_{s+1/2-2p} \\ &\leqslant C(h/k)^{2p-2-2i} | \vec{\sigma} |_{s+1/2-2p} . \end{aligned}$$

Hence for $h \leqslant \varepsilon k$, $\varepsilon < 1$, $p \geqslant 2$

$$\sum_{i=0}^{[\frac{p-1}{2}]} | P_0(w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma})) |_{s-1/2-2i} \leqslant C\varepsilon | \vec{\sigma} |_{s+1/2-2p}$$

and the result follows immediately for ε sufficiently small.

LEMMA 4.5 : *Suppose that $r = 2$, $p \geqslant 2$ and $\dot{S}_k \subset H^n(\Gamma)$ with*

$$n \geqslant \max(0, -1/2 + 2[(p-1)/2]) .$$

Let $\alpha = -\min(0, n - 2[(p-1)/2] - 1/2)$. Then for $h \leqslant \varepsilon k^{2(p-\alpha)/(2-\alpha)}$ with ε sufficiently small, there exist positive constants C_0 and C_1 such that for all $\vec{\sigma} \in \dot{V}^k$,

$$C_0 | \vec{\sigma} |^2_{1/2-p} \leqslant \left| \sum_{i=1}^{[\frac{p+1}{2}]} \langle \sigma_i, w_{h,i-1}(\vec{\sigma}) \rangle \right| \leqslant C_1 | \vec{\sigma} |^2_{1/2-p} .$$

Proof : Let $\alpha = -\min(0, n - 2[(p-1)/2] - 1/2)$. Since

$$n \geqslant \max(0, 2[(p-1)/2] - 1/2)$$

we get that $0 \leqslant \alpha \leqslant 1$ and hence using Lemma (4.1) with $t = -\alpha$ we have

$$\begin{aligned} \langle \sigma_i, w_{i-1}(\vec{\sigma}) - w_{h,i-1}(\vec{\sigma}) \rangle &\leqslant | \sigma_i |_{-\alpha+1/2} | w_{i-1}(\vec{\sigma}) - w_{h,i-1}(\vec{\sigma}) |_{\alpha-1/2} \\ &\leqslant | \sigma_i |_{-\alpha+1/2} Ch^{2-\alpha} | \vec{\sigma} |_{1/2-\alpha} \\ &\leqslant Ch^{2-\alpha} k^{-p+2(i-1)+\alpha} | \sigma_i |_{1/2-p+2(i-1)} k^{\alpha-p} | \vec{\sigma} |_{1/2-p} \\ &\leqslant \frac{Ch^{2-\alpha}}{k^{2(p-\alpha)}} | \sigma_i |_{1/2-p+2(i-1)} | \vec{\sigma} |_{1/2-p} . \end{aligned}$$

Hence

$$\left| \sum_{i=1}^{[\frac{p+1}{2}]} \langle \sigma_i, w_{i-1}(\vec{\sigma}) - w_{h,i-1}(\vec{\sigma}) \rangle \right| \leqslant \frac{Ch^{2-\alpha}}{k^{2(p-\alpha)}} | \vec{\sigma} |^2_{1/2-p} .$$

The lemma follows from (4.16) for $p \geqslant 2$ and $h \leqslant \varepsilon k^{2(p-\alpha)/(2-\alpha)}$ with ε sufficiently small.

THEOREM 4.2 : *Suppose that* $r = 2$, $p \geqslant 2$ *and* $\dot{S}_k \subset H^n(\Gamma)$ *with*

$$n \geqslant \max(0, 2[(p-1)/2] - 1/2).$$

Let $\alpha = -\min(0, n - 1/2 - 2[(p-1)/2])$. *Then for* ε *sufficiently small there exist positive constants* C_0 *and* C_1 *such that for all* $\vec{\sigma} \in \dot{V}_k$,

$$C_0 \,|\vec{\sigma}|_{s+1/2-2p} \leqslant \sum_{i=0}^{[\frac{p-1}{2}]} |P_0\, w_{h,i}(\vec{\sigma})|_{s-1/2-2i} \leqslant C_1\, |\vec{\sigma}|_{s+1/2-2p}$$

holds for $0 \leqslant s \leqslant \alpha$ *when* $h \leqslant \varepsilon k^{(2p-\alpha-s)/(2-\alpha)}$ *and holds for*

$$\alpha \leqslant s \leqslant \min(p, n + 1/2)$$

when $h \leqslant \varepsilon k^{(2p-2\alpha)/(2-\alpha)}$.

Proof : Since $\dot{r} \geqslant n + 1$, we have from Lemma (4.4) and the triangle inequality that for $0 \leqslant s \leqslant \min(p, n + 1/2)$

$$\begin{aligned} C_0\, |\vec{\sigma}|_{s+1/2-2p} &- \sum_{i=0}^{[\frac{p-1}{2}]} |P_0(w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma}))|_{s-1/2-2i} \\ &\leqslant \sum_{i=0}^{[\frac{p-1}{2}]} |P_0\, w_{h,i}(\vec{\sigma})|_{s-1/2-2i} \\ &\leqslant C_1\, |\vec{\sigma}|_{s+1/2-2p} + \sum_{i=0}^{[\frac{p-1}{2}]} |P_0(w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma}))|_{s-1/2-2i}\,. \end{aligned} \tag{4.18}$$

Now for $0 \leqslant s \leqslant 2i + \alpha$, we have that

$$|P_0(w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma}))|_{s-1/2-2i} \leqslant |P_0(w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma}))|_{\alpha-1/2}\,.$$

But since $n \geqslant \max(0, 2[(p-1)/2] - 1/2)$,

$$\alpha - 1/2 = \max(-1/2, 2[(p-1)/2] - n) \leqslant \begin{cases} 1/2 & p \geqslant 3 \\ 0 & p = 2 \end{cases}$$

and hence $\alpha - 1/2 \leqslant n$ for $p \geqslant 2$. Since one easily sees that $\alpha - 1/2 \geqslant -n$, we have by Lemma (3.2) that

$$\begin{aligned} |P_0(w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma}))|_{\alpha-1/2} &\leqslant |w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma})|_{\alpha-1/2} \\ &\leqslant Ch^{2-\alpha}\, |\vec{\sigma}|_{1/2-\alpha} \qquad \text{(by Lemma (4.1))} \\ &\leqslant Ch^{2-\alpha} k^{s-2p+\alpha}\, |\vec{\sigma}|_{s+1/2-2p}\,. \end{aligned}$$

For $2i + \alpha \leqslant s \leqslant \min(p, n + 1/2)$,

$$\begin{aligned} | P_0(w_i(\vec\sigma) - w_{h,i}(\vec\sigma)) |_{s-1/2-2i} &\leqslant Ck^{\alpha+2i-s} | P_0(w_i(\vec\sigma) - w_{h,i}(\vec\sigma)) |_{\alpha-1/2} \\ &\leqslant Ck^{\alpha+2i-s} | w_i(\vec\sigma) - w_{h,i}(\vec\sigma) |_{\alpha-1/2} \\ &\leqslant Ck^{\alpha+2i-s} h^{2-\alpha} | \vec\sigma |_{1/2-\alpha} \\ &\leqslant Ck^{\alpha+2i-s} h^{2-\alpha} k^{s-2p+\alpha} | \vec\sigma |_{s+1/2-2p} \\ &\leqslant Ch^{2-\alpha} k^{2(\alpha-p)+2i} | \vec\sigma |_{s+1/2-2p} . \end{aligned}$$

Hence we get that for $0 \leqslant s \leqslant \min(p, n + 1/2)$

$$\begin{aligned} | P_0(w_i(\vec\sigma) - w_{h,i}(\vec\sigma)) |_{s-1/2-2i} &\leqslant Ch^{2-\alpha} k^{\alpha-2p+\min(s,2i+\alpha)} | \vec\sigma |_{s+1/2-2p} \\ &\leqslant \begin{cases} Ch^{2-\alpha} k^{\alpha-2p+s} | \vec\sigma |_{s+1/2-2p}, & 0 \leqslant s \leqslant \alpha \\ Ch^{2-\alpha} k^{2\alpha-2p} | \vec\sigma |_{s+1/2-2p}, & \alpha \leqslant s \leqslant \min(p, n+1/2) . \end{cases} \end{aligned}$$

Using the relationships between k and h given in the hypotheses of the theorem and inserting this result in (4.18) completes the proof.

5. ERROR ESTIMATES

We begin this section by proving a preliminary result.

LEMMA 5.1 : *Let $\vec\sigma$ and $\vec\sigma_k$ be the respective solutions of Problems Q and Q_h^k, let $\Pi_k \vec\sigma$ be an approximation to $\vec\sigma$ satisfying* (3.4), *and suppose $w_i(\vec\sigma)$ and $w_{h,i}(\vec\sigma)$ are defined by* (2.7) *and* (4.9) *respectively. Then for $\vec\sigma$ sufficiently smooth and $h \leqslant k$ we have that*

$$\begin{aligned} &| w_i(\vec\sigma) - w_{h,i}(\Pi_k \vec\sigma) |_{s-2i-1/2} + \| w_i(\vec\sigma) - w_{h,i}(\Pi_k \vec\sigma) \|_{s-2i} \\ &\quad \leqslant C \{ h^{r-s} | \vec\sigma |_{r-2p+1/2} + k^{\dot r-2[(p-1)/2]-s+2p-1/2} | \vec\sigma |_{\dot r-2[(p-1)/2]} \} \end{aligned} \quad (5.1)$$

and

$$\begin{aligned} &| w_i(\vec\sigma) - w_{h,i}(\vec\sigma_k) |_{s-2i-1/2} + \| w_i(\vec\sigma) - w_{h,i}(\vec\sigma_k) \|_{s-2i} \\ &\quad \leqslant C \{ h^{r-s} | \vec\sigma |_{r-2p+1/2} + k^{\dot r-2[(p-1)/2]-s+2p-1/2} | \vec\sigma |_{\dot r-2[(p-1)/2]} \\ &\qquad + | \vec\sigma - \vec\sigma_k |_{s-2p+1/2} \} \end{aligned} \quad (5.2)$$

for all $2p - r \leqslant s \leqslant 1 + 2i$.

Proof : By the triangle inequality,

$$\begin{aligned} ||| w_i(\vec\sigma) - w_{h,i}(\vec\beta) |||_{s-2i} &\leqslant ||| w_i(\vec\sigma) - w_{h,i}(\vec\sigma) |||_{s-2i} \\ &\quad + ||| w_{h,i}(\vec\sigma - \vec\beta) - w_i(\vec\sigma - \vec\beta) |||_{s-2i} + ||| w_i(\vec\sigma) - w_i(\vec\beta) |||_{s-2i} . \end{aligned}$$

From Lemma (4.2), we get for $2p - r \leqslant s \leqslant 1 + 2i$ that

$$||| w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma}) |||_{s-2i} \leqslant C h^{t+2p-s} | \vec{\sigma} |_{t+1/2}, \qquad -1 \leqslant t \leqslant 2i + r - 2p$$

and

$$||| w_{h,i}(\vec{\sigma} - \vec{\beta}) - w_i(\vec{\sigma} - \vec{\beta}) |||_{s-2i} \leqslant C h^{-1+2p-s} | \vec{\sigma} - \vec{\beta} |_{1/2} .$$

Using (2.7) and Lemma (2.1) we get that

$$\begin{aligned} ||| w_i(\vec{\sigma}) - w_i(\vec{\beta}) |||_{s-2i} &\leqslant \sum_{j=1}^{m} ||| T^{p-i-j} G(\sigma_j - \beta_j) |||_{s-2i} \\ &\leqslant C \sum_{j=1}^{m} \| G(\sigma_j - \beta_j) \|_{s-2i-2(p-i-j)} \\ &\leqslant C \sum_{j=1}^{m} | \sigma_j - \beta_j |_{s-2(p-j)-3/2} \\ &\leqslant C \sum_{j=1}^{m} | \sigma_j - \beta_j |_{s-2p+1/2+2(j-1)} \\ &\leqslant C | \vec{\sigma} - \vec{\beta} |_{s-2p+1/2} . \end{aligned} \tag{5.3}$$

Choosing $\vec{\beta} = \Pi_k \vec{\sigma}$, we get from (3.4) that

$$| \vec{\sigma} - \Pi_k \vec{\sigma} |_{-1/2} \leqslant C k^{\dot{r}+1/2-2[(p-1)/2]} | \vec{\sigma} |_{\dot{r}-2[(p-1)/2]}$$

and

$$| \vec{\sigma} - \Pi_k \vec{\sigma} |_{s-2p+1/2} \leqslant C k^{\dot{r}-2[(p-1)/2]-s+2p-1/2} | \vec{\sigma} |_{\dot{r}-2[(p-1)/2]} .$$

Combining results we have for $h \leqslant k$

$$\begin{aligned} &||| w_i(\vec{\sigma}) - w_{h,i}(\Pi_k \vec{\sigma}) |||_{s-2i} \\ &\qquad \leqslant C \{ h^{r-s} | \vec{\sigma} |_{r-2p+1/2} + k^{\dot{r}-2[(p-1)/2]-s+2p-1/2} | \vec{\sigma} |_{r-2[(p-1)/2]} \} . \end{aligned}$$

Choosing $\vec{\beta} = \vec{\sigma}_k$ we first use (3.3) to write

$$\begin{aligned} | \vec{\sigma} - \vec{\sigma}_k |_{-1/2} \leqslant | \vec{\sigma} - \Pi_k \vec{\sigma} |_{-1/2} + C k^{s-2p+1} \{ &| \vec{\sigma} - \vec{\sigma}_k |_{s-2p+1/2} \\ &+ | \vec{\sigma} - \Pi_k \vec{\sigma} |_{s-2p+1/2} \} . \end{aligned}$$

Again using (3.4) we get for $h \leqslant k$ that

$$\begin{aligned} ||| w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma}_k) |||_{s-2i} \leqslant C \{ &h^{r-s} | \vec{\sigma} |_{r-2p+1/2} \\ &+ k^{\dot{r}-2[(p-1)/2]+2p-s-1/2} | \vec{\sigma} |_{r-2[(p-1)/2]} + | \vec{\sigma} - \vec{\sigma}_k |_{s-2p+1/2} \} . \end{aligned}$$

THEOREM 5.1 : *Let $\vec{\sigma}$ and $\vec{\sigma}_k$ be the respective solutions of Problems Q and Q_k^h. If* $\max(2p-1, 2p-n-1/2) \leqslant r \leqslant \dot{r} - 2[(p-1)/2] - 1/2 + 2p$ *and* $h \leqslant \varepsilon k$ *with* ε *sufficiently small, we have for* $\vec{\sigma}$ *and f sufficiently smooth that*

$$|\vec{\sigma} - \vec{\sigma}_k|_{s+1/2+2p} \leqslant C\{h^{r-s}(1+(k/h)^{\alpha})(\|f\|_{r-2p} + |\vec{\sigma}|_{r-2p+1/2}) + k^{\dot{r}-2[(p-1)/2]-s+2p-1/2} |\vec{\sigma}|_{\dot{r}-2[(p-1)/2]}\}$$

for all $2p - r \leqslant s \leqslant 1$ *where* $\alpha = \max(0, -n-s+1/2+2[(p-1)/2])$, *and* $\dot{S}_k \subset H^n(\Gamma)$.

Proof : Let $\Pi_k \vec{\sigma} \in \dot{V}_k$ be an approximation to $\vec{\sigma}$ satisfying (3.4). By the linearity of $w_{h,i}(\vec{\sigma})$ and Theorem (4.1) we get for $2p - r \leqslant s \leqslant \min(1, n+1/2)$ that

$$C_0 |\vec{\sigma}_k - \Pi_k \vec{\sigma}|_{s+1/2-2p} \leqslant \sum_{i=0}^{[\frac{p-1}{2}]} |P_0 w_{h,i}(\vec{\sigma}_k - \Pi_k \vec{\sigma})|_{s-1/2-2i}.$$

Using (2.15) and (4.14) we have

$$P_0 w_{h,i}(\vec{\sigma}_k) - P_0 w_{h,i}(\Pi_k \vec{\sigma}) = P_0(-1)^{p-i+1} T_h^{p-i} f - P_0 w_{h,i}(\Pi_k \vec{\sigma})$$
$$= (-1)^{p-i+1} P_0(T_h^{p-i} f - T^{p-i} f) + P_0 w_i(\vec{\sigma}) - P_0 w_{h,i}(\Pi_k \vec{\sigma}).$$

Hence from Lemma (3.2), we get

$$C |\vec{\sigma}_k - \Pi_k \vec{\sigma}|_{s+1/2-2p} \leqslant \sum_{i=0}^{[\frac{p-1}{2}]} \{ |P_0(T^{p-i} - T_h^{p-i}) f|_{s-1/2-2i} + |P_0(w_i(\vec{\sigma}) - w_{h,i}(\Pi_k \vec{\sigma}))|_{s-1/2-2i} \}$$
$$\leqslant C \sum_{i=0}^{[\frac{p-1}{2}]} \{ |(T^{p-i} - T_h^{p-i}) f|_{s-1/2-2i} + k^{\alpha_i - s + 2i + 1/2} |(T^{p-i} - T_h^{p-i}) f|_{\alpha_i} + |w_i(\vec{\sigma}) - w_{h,i}(\Pi_k \vec{\sigma})|_{s-1/2-2i} + k^{\alpha_i - s + 2i + 1/2} |w_i(\vec{\sigma}) - w_{h,i}(\Pi_k \vec{\sigma})|_{\alpha_i} \}$$

where $\alpha_i = \max(-n, s - 2i - 1/2)$.

Using Corollary (3.1) with $l = p - i$, $j = s - 2i$, and $t = r - 2p$ we get

$$|(T^{p-i} - T_h^{p-i}) f|_{s-1/2-2i} \leqslant C h^{r-s} \|f\|_{r-2p}$$

and with $j = \alpha_i + 1/2$ and l and t as above we get

$$|(T^{p-i} - T_h^{p-i}) f|_{\alpha_i} \leqslant C h^{r-2i-\alpha_i-1/2} \| f \|_{r-2p},$$

where we have observed that $2(p-i) - r \leqslant \alpha_i + 1/2 \leqslant 1$.

From Lemma (5.1) we have

$$| w_i(\vec{\sigma}) - w_{h,i}(\Pi_k \vec{\sigma}) |_{s-2i-1/2} \leqslant C \{ h^{r-s} | \vec{\sigma} |_{r-2p+1/2} + k^{\dot{r}-2[(p-1)/2]-s+2p-1/2} | \vec{\sigma} |_{\dot{r}-2[(p-1)/2]} \}$$

and

$$| w_i(\vec{\sigma}) - w_{h,i}(\Pi_k \vec{\sigma}) |_{\alpha_i} \leqslant C \{ h^{r-\alpha_i-2i-1/2} | \vec{\sigma} |_{r-2p+1/2} + k^{\dot{r}-2[(p-1)/2]-\alpha_i-2i-2p-1} | \vec{\sigma} |_{\dot{r}-2[(p-1)/2]} \}$$

where we have observed that $1 + 2i \geqslant \alpha_i + 2i + 1/2 \geqslant 2p - r$ for $2p - r \leqslant s \leqslant 1$. Combining these results we obtain

$$\begin{aligned} | \vec{\sigma}_k - \Pi_k \vec{\sigma} |_{s+1/2-2p} \leqslant\ & C \sum_{i=0}^{[\frac{p-1}{2}]} (h^{r-s} \{ (1 + (k/h)^{\alpha_i - s + 1/2 + 2i}) \} \{ \| f \|_{r-2p} \\ & + | \vec{\sigma} |_{r-2p+1/2} \} + k^{\dot{r}-2[(p-1)/2]-s+2p-1/2} | \vec{\sigma} |_{\dot{r}-2[(p-1)/2]}) \\ \leqslant\ & C \{ h^{r-s} (1 + (k/h)^{\alpha}) (\| f \|_{r-2p} + | \vec{\sigma} |_{r-2p+1/2}) \\ & + k^{\dot{r}-2[(p-1)/2]-s+2p-1/2} | \vec{\sigma} |_{\dot{r}-2[(p-1)/2]} \} \end{aligned} \quad (5.4)$$

where $\alpha = \max(0, -n - s + 2[(p-1)/2] + 1/2)$.

For $\min(1, n + 1/2) < s \leqslant 1$, we have by (3.3) that

$$| \vec{\sigma}_k - \Pi_k \vec{\sigma} |_{s+1/2-2p} \leqslant C k^{n+1/2-s} | \vec{\sigma}_k - \Pi_k \vec{\sigma} |_{n+1-2p}.$$

Now using (5.4) with $s = n + 1/2$ and the fact that $h \leqslant k$ gives us (5.4) in the full range $2p - r \leqslant s \leqslant 1$. The theorem now follows from (3.4) and the triangle inequality.

THEOREM 5.2 : *Suppose the hypotheses of Theorem* 5.1 *are satisfied and that* w_i *and* $w_{h,i}$ *are defined by* (2.8) *and* (4.10) *respectively* (*in particular* $u = w_0$ *and* $u_h = w_{h,0}$). *Then for all* $2p - r \leqslant s \leqslant 1 + 2i$,

$$\| w_i - w_{h,i} \|_{s-2i} + | w_i - w_{h,i} |_{s-2i-1/2} \leqslant C \{ h^{r-s} (1 + (k/h)^{\alpha}) (\| f \|_{r-2p} + | \vec{\sigma} |_{r-2p+1/2}) + k^{\dot{r}-2[(p-1)/2]-s+2p-1/2} | \vec{\sigma} |_{\dot{r}-2[(p-1)/2]} \},$$

where $\alpha = \max(0, -n - s + 2[(p-1)/2] + 1/2)$.

Proof : From (2.8) and (4.10) we have that

$$||| w_i - w_{h,i} |||_{s-2i} \leqslant ||| w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma}_k) |||_{s-2i} + ||| T^{p-i} f - T_h^{p-i} f |||_{s-2i} .$$

Applying Corollary (3.1) with $t = r - 2p$, $l = p - i$, and $j = s - 2i$, we get

$$||| T^{p-i} f - T_h^{p-i} f |||_{s-2i} \leqslant C h^{r-s} \| f \|_{r-2p} .$$

The result now follows immediately from Lemma 5.1 and Theorem 5.1 in the case $2p - r \leqslant s \leqslant 1$. For $1 \leqslant s \leqslant 1 + 2i$, we have from (3.3), (3.4), and Theorem 5.1 that

$$\begin{aligned} | \vec{\sigma} - \vec{\sigma}_k |_{1/2+s-2p} &\leqslant | \vec{\sigma} - \Pi_k \vec{\sigma} |_{1/2+s-2p} \\ &\quad + k^{1-s} \{ | \vec{\sigma} - \Pi_k \vec{\sigma} |_{3/2-2p} + | \vec{\sigma} - \vec{\sigma}_k |_{3/2-2p} \} \\ &\leqslant C \{ h^{r-1} k^{1-s} (1 + (k/h)^{\alpha} (\| f \|_{r-2p} + | \vec{\sigma} |_{r-2p+1/2}) \\ &\quad + k^{\dot{r} - 2[(p-1)/2] - s + 2p - 1/2} | \vec{\sigma} |_{\dot{r} - 2[(p-1)/2]} \} . \end{aligned}$$

The result now follows from Lemma 5.1 since $h \leqslant \varepsilon k$.

Example 5.1 : $p = 2$, $r = 4$, $\dot{r} = 2$, $n = 1$, $s = 0$, $\alpha - 0$

$$\| w_i - w_{h,i} \|_{-2i} \leqslant C \{ h^4 (\| f \|_0 + | \vec{\sigma} |_{1/2}) + k^{11/2} | \vec{\sigma} |_2 \} .$$

Choosing $h = k^{11/8}$ we get that $h \leqslant \varepsilon k$ for k sufficiently small.

Example 5.2 : $p \geqslant 3$, $r = 2p - 1$, $\dot{r} = 2[(p-1)/2] + 1$, $n = 2[(p-1)/2]$, $s = 1$, $\alpha = 0$

$$\| w_i - w_{h,i} \|_{1-2i} \leqslant C \{ h^{2p-2} (\| f \|_{-1} + | \vec{\sigma} |_{-1/2}) + k^{2p-1/2} | \vec{\sigma} |_1 \} .$$

LEMMA 5.2 : *Suppose the hypotheses of Lemma* (5.1) *are satisfied. Then for $\vec{\sigma}$ sufficiently smooth, $h \leqslant k$ and all $\vec{\beta} \in \dot{V}_k$, we have for all $s \leqslant 1$ and $r = 2$ that*

$$\begin{aligned} &\| w_i(\vec{\sigma}) - w_{h,i}(\vec{\beta}) \|_s + | w_i(\vec{\sigma}) - w_{h,i}(\vec{\beta}) |_{s-1/2} \\ &\qquad \leqslant C h^{t+2} | \vec{\sigma} |_{t+1/2} + C h^{\bar{t}+2} | \vec{\sigma} - \vec{\beta} |_{\bar{t}+1/2} + C | \vec{\sigma} - \vec{\beta} |_{s+2i-2p+1/2} , \end{aligned}$$

where $-1 \leqslant t \leqslant \min(0, -s)$ *and* $-1 \leqslant \bar{t} \leqslant \min(0, -s, n - 1/2)$.

Proof : Following the proof of Lemma 5.1 we have that

$$\begin{aligned} ||| w_i(\vec{\sigma}) - w_{h,i}(\vec{\beta}) |||_s &\leqslant ||| w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma}) |||_s \\ &\quad + ||| w_{h,i}(\vec{\sigma} - \vec{\beta}) - w_i(\vec{\sigma} - \vec{\beta}) |||_s + ||| w_i(\vec{\sigma}) - w_i(\vec{\beta}) |||_s \\ &\leqslant ||| w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma}) |||_s \\ &\quad + ||| w_{h,i}(\vec{\sigma} - \vec{\beta}) - w_i(\vec{\sigma} - \vec{\beta}) |||_s + | \vec{\sigma} - \vec{\beta} |_{s+2i-2p+1/2} . \end{aligned}$$

Now from Lemma (4.1) we have that for $0 \leqslant \bar{s} \leqslant 1$

$$||| w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma}) |||_{\bar{s}} \leqslant C h^{t+2} | \vec{\sigma} |_{t+1/2}, \qquad -1 \leqslant t \leqslant -\bar{s}.$$

Hence choosing $\bar{s} = \max(s, 0)$, we get for $s \leqslant \min(1, n + 1/2)$ that

$$||| w_i(\vec{\sigma}) - w_{h,i}(\vec{\beta}) |||_s \leqslant C h^{t+2} | \vec{\sigma} |_{t+1/2} + C h^{\bar{t}+2} | \vec{\sigma} - \vec{\beta} |_{\bar{t}+1/2} + C | \vec{\sigma} - \vec{\beta} |_{s+2i-2p+1/2},$$

where $-1 \leqslant t \leqslant \min(0, -s)$ and $-1 \leqslant \bar{t} \leqslant \min(0, -s, n - 1/2)$.

THEOREM 5.3 : *Let $\vec{\sigma}$ and $\vec{\sigma}_k$ be the respective solutions of Problems Q and Q_h^k. Suppose that $r = 2, p \geqslant 2$, and $\dot{S}_k \subset H^n(\Gamma)$ with $n \geqslant \max(0, -1/2 + 2[(p-1)/2])$. Let $\alpha = -\min(0, n - 2[(p-1)/2] - 1/2)$. Then for ε sufficiently small, we have for $\vec{\sigma}$ and f sufficiently regular that*

$$| \vec{\sigma} - \vec{\sigma}_k |_{s+1/2-2p} \leqslant C \{ h^{2-s} (k/h)^{\max(0, -n-s+1/2)} (\| f \|_{-s} + | \vec{\sigma} |_{1/2-s}) + k^{t-s+2p-1/2} | \vec{\sigma} |_t \} \quad (5.5)$$

holds for $0 \leqslant s \leqslant \alpha$, $-1/2 \leqslant t \leqslant \dot{r} - 2[(p-1)/2]$ when $h \leqslant \varepsilon k^{(2p-\alpha-s)/(2-\alpha)}$ and

$$| \vec{\sigma} - \vec{\sigma}_k |_{s+1/2-2p} \leqslant C \{ h^{2-\alpha} k^{\alpha-s} (\| f \|_{-\alpha} + | \vec{\sigma} |_{1/2-\alpha}) + k^{t-s+2p-1/2} | \vec{\sigma} |_t \} \quad (5.6)$$

holds for $\alpha \leqslant s \leqslant 2p-1$ and $-1/2 \leqslant t \leqslant \dot{r} - 2[(p-1)/2]$ when $h \leqslant \varepsilon k^{2(p-\alpha)/(2-\alpha)}$.

Proof : Following exactly the proof of Theorem 5.1 we get by Theorem 4.2 for $0 \leqslant s \leqslant \alpha$ and $h \leqslant \varepsilon k^{(2p-\alpha-s)/(2-\alpha)}$ that

$$| \vec{\sigma}_k - \Pi_k \vec{\sigma} |_{s+1/2-2p} \leqslant C \sum_{i=0}^{[\frac{p-1}{2}]} \{ | (T^{p-i} - T_h^{p-i}) f |_{s-1/2-2i} + k^{\alpha_i - s + 2i + 1/2} | (T^{p-i} - T_h^{p-i}) f |_{\alpha_i} + | w_i(\vec{\sigma}) - w_{h,i}(\Pi_k \vec{\sigma}) |_{s-1/2-2i} + k^{\alpha_i - s + 2i + 1/2} | w_i(\vec{\sigma}) - w_{h,i}(\Pi_k \vec{\sigma}) |_{\alpha_i} \},$$

where $\alpha_i = \max(-n, s - 1/2 - 2i)$.

Using Theorem 3.1 and Lemma 5.2 and observing that $s - 1/2 - 2i \leqslant -1/2$ for $i \geqslant 1$ we get

$$| (T^{p-i} - T_h^{p-i}) f |_{s-1/2-2i} + | w_i(\vec{\sigma}) - w_{h,i}(\Pi_k \vec{\sigma}) |_{s-1/2-2i} \leqslant C h^{2-s} \| f \|_{-s} + C h^{2-s} | \vec{\sigma} |_{1/2-s} + C h^{2+\min(-s, n-1/2)} | \vec{\sigma} - \Pi_k \vec{\sigma} |_{\min(1/2-s, n)} + C | \vec{\sigma} - \Pi_k \vec{\sigma} |_{s-2p+1/2} \cdot$$

Now for $0 \leqslant s \leqslant \alpha$, $h \leqslant k$, and $n \geqslant 1$,

$$Ch^{2+\min(-s,n-1/2)} \,|\, \vec{\sigma} - \Pi_k \vec{\sigma} \,|_{\min(1/2-s,n)} \leqslant Ch^{2-s} \,|\, \vec{\sigma} \,|_{1/2-s} \qquad \text{(by (3.4))},$$

and for $n = 0$ (and hence $p = 2$), $0 \leqslant s \leqslant \alpha = 1/2$, and $h \leqslant k$

$$\begin{aligned} Ch^{2+\min(-s,n-1/2)} \,|\, \vec{\sigma} - \Pi_k \vec{\sigma} \,|_{\min(1/2-s,n)} &\leqslant Ch^{3/2} k^{1/2-s} \,|\, \vec{\sigma} \,|_{1/2-s} \\ &\leqslant Ch^{2-s} \,|\, \vec{\sigma} \,|_{1/2-s} . \end{aligned}$$

Hence we obtain

$$\begin{aligned} &|\,(T^{p-i} - T_h^{p-i}) f \,|_{s-1/2-2i} + |\, w_i(\vec{\sigma}) - w_{h,i}(\Pi_k \vec{\sigma}) \,|_{s-1/2-2i} \\ &\qquad \leqslant Ch^{2-s}(\| f \|_{-s} + |\, \vec{\sigma} \,|_{1/2-s}) + C \,|\, \vec{\sigma} - \Pi_k \vec{\sigma} \,|_{s-2p+1/2} . \end{aligned}$$

For $n \geqslant 1$ we note that $\alpha_i \leqslant -1/2$ for $i \geqslant 1$ and $\alpha_i = s - 1/2$ for $i = 0$. Hence we get

$$\begin{aligned} &|\,(T^{p-i} - T_h^{p-i}) f \,|_{\alpha_i} + |\, w_i(\vec{\sigma}) - w_{h,i}(\Pi_k \vec{\sigma}) \,|_{\alpha_i} \leqslant Ch^{2-s} \| f \|_{-s} \\ &\quad + Ch^{2-s} \,|\, \vec{\sigma} \,|_{1/2-s} + Ch^{2+\min(-s,n-1/2)} \,|\, \vec{\sigma} - \Pi_k \vec{\sigma} \,|_{\min(1/2-s,n)} \\ &\quad + C \,|\, \vec{\sigma} - \Pi_k \vec{\sigma} \,|_{\alpha_i+2i-2p+1} \\ &\quad \leqslant Ch^{2-s}(\| f \|_{-s} + |\, \vec{\sigma} \,|_{1/2-s}) + C \,|\, \vec{\sigma} - \Pi_k \vec{\sigma} \,|_{\alpha_i+2i-2p+1} . \end{aligned}$$

For $n = 0$ and $0 \leqslant s \leqslant 1/2 = \alpha$, $\alpha_i = 0$ and so we get

$$\begin{aligned} &|\,(T^{p-i} - T_h^{p-i}) f \,|_{\alpha_i} + |\, w_i(\vec{\sigma}) - w_{h,i}(\Pi_k \vec{\sigma}) \,|_{\alpha_i} \leqslant Ch^{3/2} \| f \|_{-1/2} \\ &\qquad + Ch^{3/2} \,|\, \vec{\sigma} \,|_0 + Ch^{3/2} \,|\, \vec{\sigma} - \Pi_k \vec{\sigma} \,|_0 + C \,|\, \vec{\sigma} - \Pi_k \vec{\sigma} \,|_{2i-2p+1} \\ &\qquad\qquad \leqslant Ch^{3/2}(\| f \|_{-1/2} + |\, \vec{\sigma} \,|_0) + C \,|\, \vec{\sigma} - \Pi_k \vec{\sigma} \,|_{2i-2p+1} . \end{aligned}$$

The above can also be simplified since $n = 0$ implies $p = 2$ which implies $i = 0$.

Combining results we get for $n \geqslant 1$, $h \leqslant \varepsilon k^{(2p-\alpha-s)/(2-\alpha)}$, and $0 \leqslant s \leqslant \alpha$ that

$$\begin{aligned} |\, \vec{\sigma}_k - \Pi_k \vec{\sigma} \,|_{s+1/2-2p} &\leqslant Ch^{2-s}(\| f \|_{-s} + |\, \vec{\sigma} \,|_{1/2-s}) + C \,|\, \vec{\sigma} - \Pi_k \vec{\sigma} \,|_{s-2p+1/2} \\ &\quad + \sum_{i=0}^{[\frac{p-1}{2}]} Ck^{\alpha_i - s + 2i + 1/2} \,|\, \vec{\sigma} - \Pi_k \vec{\sigma} \,|_{\alpha_i+2i-2p+1} \\ &\leqslant Ch^{2-s}(\| f \|_{-s} + |\, \vec{\sigma} \,|_{1/2-s}) + Ck^{t-s+2p-1/2} \,|\, \vec{\sigma} \,|_t , \\ &\quad -1/2 \leqslant t \leqslant \dot{r} + 2[(p-1)/2] \end{aligned} \tag{5.7}$$

and for $n = 0$ (and hence $p = 2$), $h \leqslant \varepsilon k^{(7-2s)/3}$ and $0 \leqslant s \leqslant 1/2 = \alpha$ that

$$|\vec{\sigma}_k - \Pi_k \vec{\sigma}|_{s-7/2} \leqslant Ch^{2-s}(1 + (k/h)^{1/2-s})(\|f\|_{-s} + |\vec{\sigma}|_{1/2-s}) + Ck^{t+7/2-s}|\vec{\sigma}|_t, \quad -1/2 \leqslant t \leqslant \dot{r}. \quad (5.8)$$

Now for $\alpha \leqslant s \leqslant 2p - 1$ we have by (3.3) that

$$|\vec{\sigma}_k - \Pi_k \vec{\sigma}|_{s+1/2-2p} \leqslant Ck^{\alpha-s}|\vec{\sigma}_k - \Pi_k \vec{\sigma}|_{\alpha+1/2-2p}.$$

Now using (5.7) and (5.8) with $s = \alpha$, we get for $n \geqslant 0$, $h \leqslant \varepsilon k^{2(p-\alpha)/(2-\alpha)}$, and $\alpha \leqslant s \leqslant 2p - 1$ that

$$|\vec{\sigma}_k - \Pi_k \vec{\sigma}|_{s+1/2-2p} \leqslant Ck^{\alpha-s}(h^{2-\alpha}\|f\|_{-\alpha} + |\vec{\sigma}|_{1/2-\alpha}) + Ck^{t+2p-s-1/2}|\vec{\sigma}|_t,$$
$$-1/2 \leqslant t \leqslant \dot{r} - 2[(p-1)/2].$$

The theorem now follows from (3.4) and the triangle inequality.

THEOREM 5.4 : *Suppose the hypotheses of Theorem* (5.3) *are satisfied, and that w_i and $w_{h,i}$ are defined by* (2.8) *and* (4.10) *respectively* (*in particular $u = w_0$ and $u_h = w_{h,0}$*). *Then for all $\alpha \leqslant s \leqslant 1$ and $h \leqslant \varepsilon k^{2(p-\alpha)/(2-\alpha)}$*

$$\|w_i - w_{h,i}\|_s + |w_i - w_{h,i}|_{s-1/2} \leqslant C\{h^{2-s}(|\vec{\sigma}|_{1/2-s} + \|f\|_{-s}) + (h^{2-s}k^{s-\alpha} + h^{2-\alpha}k^{\alpha-s-2i})(\|f\|_{-\alpha} + |\vec{\sigma}|_{1/2-\alpha}) + k^{\dot{r}-2[(p-1)/2]-s-2i+2p-1/2}|\vec{\sigma}|_{\dot{r}-2[(p-1)/2]}\},$$

$i = 0, 1, \ldots, p-2$.

For $i = p-1$, the estimate holds with $\|f\|_{-s}$ replaced by $\|f\|_0$.

Proof : From (2.8) and (4.10) we have that

$$|||w_i - w_{h,i}|||_s \leqslant |||w_i(\vec{\sigma}) - w_{h,i}(\vec{\sigma}_k)|||_s + |||T^{p-i}f - T_h^{p-i}f|||_s.$$

From Theorem 3.1 we get for $0 \leqslant s \leqslant 1$ that

$$|||T^{p-i}f - T_h^{p-i}f|||_s \leqslant \begin{cases} Ch^{2-s}\|f\|_{-s}, & i = 0, 1, \ldots, p-2 \\ Ch^{2-s}\|f\|_0, & i = p-1. \end{cases}$$

From Lemma 5.2 we obtain for $0 \leqslant s \leqslant 1$ that

$$|||w_i(\vec{\sigma}) - w_i(\vec{\sigma}_k)|||_s \leqslant Ch^{t+2}|\vec{\sigma}|_{t+1/2} + Ch^{\bar{t}+2}|\vec{\sigma} - \vec{\sigma}_k|_{\bar{t}+1/2} + C|\vec{\sigma} - \vec{\sigma}_k|_{s+2i-2p+1/2},$$

$-1 \leqslant t \leqslant s$, $-1 \leqslant \bar{t} \leqslant \min(-s, n-1/2)$.

Now by (3.3),

$$|\vec{\sigma} - \vec{\sigma}_k|_{\bar{t}+1/2} \leqslant |\vec{\sigma} - \Pi_k \vec{\sigma}|_{\bar{t}+1/2} + Ck^{-\bar{t}-p} \{ |\vec{\sigma} - \Pi_k \vec{\sigma}|_{1/2-p} + |\vec{\sigma} - \vec{\sigma}_k|_{1/2-p} \} .$$

Hence for $0 \leqslant s \leqslant 1$, we get using Theorem 5.3 and choosing $t = -s$ and $\bar{t} = \min(-s, n - 1/2)$ that

$$\begin{aligned} |\vec{\sigma} - \vec{\sigma}_k|_{t+1/2} &\leqslant Ck^{-\bar{t}-\alpha} |\vec{\sigma}|_{1/2-\alpha} + Ck^{-\bar{t}-p} |\vec{\sigma} - \vec{\sigma}_k|_{1/2-p} \\ &\leqslant Ck^{-\bar{t}-\alpha} |\vec{\sigma}|_{1/2-\alpha} + Ck^{-\bar{t}-p} \{h^{2-\alpha} k^{\alpha-p}(\| f \|_{-\alpha} + |\vec{\sigma}|_{1/2-\alpha}) \\ &\qquad + k^{p-\alpha} |\vec{\sigma}|_{1/2-\alpha} \} . \end{aligned}$$

Hence

$$\begin{aligned} Ch^{\bar{t}+2} |\vec{\sigma} - \vec{\sigma}_k|_{\bar{t}+1/2} &\leqslant Ch^{\bar{t}+2} \{ k^{-\bar{t}-\alpha} |\vec{\sigma}|_{1/2-\alpha} \\ &\qquad + h^{2-\alpha} k^{\alpha-2p-\bar{t}}(\| f \|_{-\alpha} + |\vec{\sigma}|_{1/2-\alpha}) \} \\ &\leqslant Ch^{\bar{t}+2} k^{-\bar{t}-\alpha} \{ |\vec{\sigma}|_{1/2-\alpha} + h^{2-\alpha} k^{2\alpha-2p}(\| f \|_{-\alpha} \\ &\qquad + |\vec{\sigma}|_{1/2-\alpha}) \} . \end{aligned}$$

Now for $\alpha \leqslant s \leqslant 1$, $-s \leqslant -\alpha \leqslant n - 2[(p-1)/2] - 1/2 \leqslant n - 1/2$ so that $\bar{t} = -s$. Hence for $\alpha \leqslant s \leqslant 1$ and $h \leqslant \varepsilon k^{2(p-\alpha)/(2-\alpha)}$ we get

$$\begin{aligned} ||| w_i(\vec{\sigma}) - w_i(\vec{\sigma}_k) |||_s &\leqslant Ch^{2-s} |\vec{\sigma}|_{1/2} \\ &\quad + Ch^{2-s} k^{s-\alpha}(\| f \|_{-\alpha} + |\vec{\sigma}|_{1/2-\alpha}) + C |\vec{\sigma} - \vec{\sigma}_k|_{s+2i-2p+1/2} . \end{aligned}$$

Noting that for $\alpha \leqslant s \leqslant 1$, $\alpha \leqslant s + 2i \leqslant 2p - 1$ for $i = 0, 1, \dots, p-1$ we get by Theorem 5.3 that

$$\begin{aligned} |\vec{\sigma}-\vec{\sigma}_k|_{s+2i-2p+1/2} &\leqslant C \{ h^{2-\alpha} k^{\alpha-s-2i}(\| f \|_{-\alpha} + |\vec{\sigma}|_{1/2-\alpha}) \\ &\qquad + k^{\dot{r}-2[(p-1)/2]-s-2i+2p+1/2} |\vec{\sigma}|_{\dot{r}-2[(p-1)/2]} \} . \end{aligned}$$

Combining our results we have for $\alpha \leqslant s \leqslant 1$ and $h \leqslant \varepsilon k^{2(p-\alpha)/(2-\alpha)}$ that

$$\begin{aligned} ||| w_i - w_{h,i} |||_s \leqslant C \{ & h^{2-s}(|\vec{\sigma}|_{1/2-s} + \| f \|_{-s}) \\ & + (h^{2-s} k^{s-\alpha} + h^{2-\alpha} k^{\alpha-s-2i}) (\| f \|_{-\alpha} + |\vec{\sigma}|_{1/2-\alpha}) \\ & + k^{\dot{r}-2[(p-1)/2]-s-2i+2p-1/2} |\vec{\sigma}|_{\dot{r}-2[(p-1)/2]} \} , \end{aligned}$$

$i = 0, 1, \dots, p-2$.

For $i = p - 1$, the estimate holds with $\| f \|_{-s}$ replaced by $\| f \|_0$.

Although we will not do so here, we remark that similar arguements can be used to derive estimates in the case $s < \alpha$.

Example 5.3 : $p=2$, $\dot{r}=2$, $n=1$, $h \leqslant \varepsilon k^2$, $s=0$, $\alpha=0$

$$\| u - u_h \|_0 \leqslant Ch^2(\| f \|_0 + | \vec{\sigma} |_{1/2}) + Ck^{11/2} | \vec{\sigma} |_2 .$$

Choosing $h = Ck^{11/4}$ we get that $h \leqslant \varepsilon k^2$ for k sufficiently small.

Example 5.4 : $p = 2$, $\dot{r} = 1$, $n = 0$, $h \leqslant \varepsilon k^2$, $s = 1$, $\alpha = 1/2$

$$\| u - u_h \|_1 \leqslant C \{ h(| \vec{\sigma} |_{-1/2} + \| f \|_{-1}) + (hk^{1/2} + h^{3/2} k^{-1/2}) (\| f \|_{-1/2} + | \vec{\sigma} |_0) + k^{7/2} | \vec{\sigma} |_1 \} = 0(h) + 0(k^{7/2}) .$$

6. EFFICIENT SOLUTION OF PROBLEM (Q_h^k)

In this section we show how some ideas developed in [4] can be extended to develop methods for the efficient solution of the linear systems of equations arising from Problem (Q_h^k). To describe these ideas we first define a discrete boundary Laplacian

$$l_k : \dot{S}_k \to \dot{S}_k \quad \text{by} \quad \langle l_k \phi, \theta \rangle = \langle \phi, \theta \rangle + \langle \phi_s, \theta_s \rangle$$

where ϕ_s is the tangential derivative of ϕ along Γ. We note that l_k is positive definite and symmetric and hence l_k^s may be defined in the usual way by taking powers of its eigenvalues.

The method we present depends on the following property of the operator l_k^s (*cf.* [5]).

LEMMA 6.1 : *Let* $\dot{S}_k \subset H^1(\Gamma)$. *Then for* $-\dot{r} \leqslant s \leqslant 1$, *there are constants* C_1 *and* C_2 *such that for* $\phi \in \dot{S}_k$

$$C_1 | \phi |_s^2 \leqslant | l_k^{s/2} \phi |_0^2 \leqslant C_2 | \phi |_s^2 . \tag{6.1}$$

Combining Lemma (6.1) with Lemma (4.3) we have for $r - 2p \geqslant -1$, $1 \geqslant 1/2 - p + 2[(p-1)/2] \geqslant -\dot{r}$, and $h \leqslant \varepsilon k$ that

$$C_0 \sum_{j=1}^{[\frac{p+1}{2}]} | l_k^{1/4 - p/2 + (j-1)} \sigma_j |_0^2 \leqslant \left| \sum_{i=1}^{[\frac{p+1}{2}]} \langle \sigma_i, w_{h,i-1}(\vec{\sigma}) \rangle \right| \leqslant C_1 \sum_{j=1}^{[\frac{p+1}{2}]} | l_k^{1/4 - p/2 + (j-1)} \sigma_j |_0^2$$

for all $\vec{\sigma} \in \dot{V}_k$.

Defining $\lambda_j = l_k^{1/4-p/2+(j-1)}\sigma_j$ and using the equivalence of vector norms and the symmetry of l_k^s we get

$$C_0 \sum_{j=1}^{[\frac{p+1}{2}]} |\lambda_j|_0^2 \leqslant \left| \sum_{i=1}^{[\frac{p+1}{2}]} \langle l_k^{-1/4-p/2+(i-1)} P_0\, w_{h,i-1}(\vec{\sigma}), \lambda_i \rangle \right| . \leqslant C_1 \sum_{j=1}^{[\frac{p+1}{2}]} |\lambda_j|^2 . \tag{6.2}$$

Further defining

$$Z_{h,i-1}^k(\vec{\lambda}) = l_k^{-1/4+p/2-(i-1)} P_0 \sum_{j=1}^{[\frac{p+1}{2}]} (-1)^{p-i-j}\, T_h^{p-j}\, G_h\, l_k^{-1/4+p/2-(j-1)} \lambda_j$$

we have that $Z_{h,i-1}^k(\vec{\lambda}) = l_k^{-1/4+p/2-(i-1)} P_0\, w_{h,i-1}(\vec{\sigma})$ so that (6.2) becomes

$$C_0 \sum_{j=1}^{[\frac{p+1}{2}]} |\lambda_j|_0^2 \leqslant \left| \sum_{i=1}^{[\frac{p+1}{2}]} \langle Z_{h,i-1}^k(\vec{\lambda}), \lambda_i \rangle \right| \leqslant C_1 \sum_{j=1}^{[\frac{p+1}{2}]} |\lambda_j|_0^2 . \tag{6.3}$$

Hence if instead of solving the system (4.14) we solve the equivalent system

$$Z_{h,i}^k(\vec{\lambda}) = l_k^{-1/4+p/2-i}(-1)^{p-i+1}\, T_h^{p-i} f, \qquad i=0, 1, \ldots, [(p-1)/2] , \tag{6.4}$$

then (6.3) implies that the matrix induced by this system will have a condition number bounded independent of k and h. Thus the conjugate gradient algorithm will provide an efficient means to solve (6.4) and to use it we need only be able to compute the action of the operators T_h, G_h and l_k^s for appropriate s (*cf.* [1]). The action of T_h and G_h are easily computed and so is l_k^s *via* discrete Fourier transform when $\dot{S}_k$ consists of smooth splines on a uniform mesh.

If we wish to avoid the computation of l_k^s for non-integer powers of s we can also obtain a well conditional system by using Theorem 4.1 with $s=3/2$ (provided $n \geqslant 1$) and Lemma (6.1).

We then get for

$$2p - 1 \leqslant \dot{r} \leqslant r - 2[(p-1)/2] - 1/2 + 2p$$

and $h \leqslant \varepsilon k$, that

$$C_0 \sum_{j=1}^{[\frac{p+1}{2}]} | l_k^{1-p+(j-1)} \sigma_j |_0^2 \leqslant \sum_{i=0}^{[\frac{p-1}{2}]} | l_k^{1/2-i} P_0\, w_{h,i}(\vec{\sigma}) |_0^2 \leqslant C_1 \sum_{j=1}^{[\frac{p+1}{2}]} | l_k^{1-p+(j-1)} \sigma_j |_0^2 . \tag{6.5}$$

Now set $\lambda_j = l_k^{1-p+(j-1)} \sigma_j$. Then

$$\sum_{i=0}^{\left[\frac{p-1}{2}\right]} | l_k^{1/2-i} P_0 w_{h,i}(\vec{\sigma}) |_0^2$$
$$= \sum_{i=0}^{\left[\frac{p-1}{2}\right]} \langle l_k^{1-2i} P_0 w_{h,i}(\vec{\sigma}), P_0 w_{h,i}(\vec{\sigma}) \rangle$$
$$= \sum_{i=0}^{\left[\frac{p-1}{2}\right]} \left\langle l_k^{1-2i} P_0 w_{h,i}(\vec{\sigma}), P_0 \sum_{j=1}^{\left[\frac{p+1}{2}\right]} (-1)^{p-i-j} T_h^{p-i-j} G_h \sigma_j \right\rangle$$
$$= \sum_{i=0}^{\left[\frac{p-1}{2}\right]} \sum_{j=1}^{\left[\frac{p+1}{2}\right]} (-1)^{p-i-j} \langle P_0 T_h^{p-i-j} G_h l_k^{1-2i} P_0 w_{h,i}(\vec{\sigma}), \sigma_j \rangle$$
$$= \sum_{j=1}^{\left[\frac{p+1}{2}\right]} \sum_{i=0}^{\left[\frac{p-1}{2}\right]} (-1)^{p-i-j} \langle l_k^{-1+p-(j-1)} P_0 T_h^{p-i-j} G_h l_k^{1-2i} P_0 w_{h,i}(\vec{\sigma}), \lambda_j \rangle ,$$

where we have used Lemma 3.3 and the symmetry of l_k^s.

If we now define $Z_{h,j}^k(\vec{\lambda})$

$$= \sum_{i=0}^{\left[\frac{p-1}{2}\right]} (-1)^{p-i-j} l_k^{-1+p-(j-1)} P_0 T_h^{p-i-j} G_h l_k^{1-2i} P_0 \times \sum_{m=1}^{\left[\frac{p+1}{2}\right]} T_h^{p-i-m} G_h l_k^{-1+p-(j-1)} \lambda_j ,$$

then

$$Z_{h,j}^k(\vec{\lambda}) = \sum_{i=0}^{\left[\frac{p-1}{2}\right]} (-1)^{p-i-j} l_k^{-1+p-(j-1)} P_0 T_h^{p-i-j} G_h l_k^{1-2i} P_0 w_{h,i}(\vec{\sigma})$$

so that (6.5) becomes

$$C_0 \sum_{j=1}^{\left[\frac{p+1}{2}\right]} | \lambda_j |_0^2 \leqslant \left| \sum_{j=1}^{\left[\frac{p+1}{2}\right]} \left\langle Z_{h,j}^k(\vec{\lambda}), \lambda_j \right\rangle \right| \leqslant C_1 \sum_{j=1}^{\left[\frac{p+1}{2}\right]} | \lambda_j |_0^2 . \quad (6.6)$$

Hence if instead of solving the system (4.14) we solve the equivalent system

$$Z_{h,j}^k(\vec{\lambda}) = \sum_{i=0}^{\left[\frac{p-1}{2}\right]} (-1)^{p-i-j} l_k^{-1+p-(j-1)} P_0 T_h^{p-i-j} G_h l_k^{1-2i} P_0 (-1)^{p-i+1} T_h^{p-i} f$$
$$j = 1, 2, \ldots, [(p-1)/2] , \quad (6.7)$$

then (6.7) implies that the matrix induced by this system is uniformly well-conditioned (independent of h and k). Once again the system can be efficiently solved using the conjugate gradient method. In this case only the action of T_h, G_h and integer powers of l_k need be computed.

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