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INTERIOR AND SUPERCONVERGENCE ESTIMATES FOR MIXED METHODS FOR SECOND ORDER ELLIPTIC PROBLEMS (*)

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Résumé. — *Nous établissons des estimations a priori à l'intérieur pour l'erreur dans des méthodes d'éléments finis mixtes pour des équations elliptiques linéaires ou semi-linéaires, l'erreur étant évaluée dans des espaces de Sobolev d'indice non positif. Les estimations a priori correspondantes sont prouvées pour des quotients aux différences finies de l'erreur quand l'espace des éléments finis est associé avec un réseau invariant par translation dans un sous-domaine intérieur. Dans ce cas la procédure de superconvergence de Bramble-Schatz est étendue aux méthodes d'éléments finis mixtes.*

Abstract. — *Interior estimates for the error in mixed finite element methods for linear and semilinear elliptic equations of second order are derived in Sobolev spaces of nonpositive index. Corresponding estimates are demonstrated for difference quotients of the error when the finite element space is associated with a translation-invariant grid on an interior subdomain, and the Bramble-Schatz superconvergence procedure is extended to the mixed method.*

0. INTRODUCTION

Let Ω be a bounded, planar domain with smooth boundary $\partial\Omega$. We shall assume that for each pair of functions (f, g) in $H^m(\Omega) \times H^{m+3/2}(\partial\Omega)$, $-1 \leqslant m \leqslant 2k+2$ there exists a unique solution $p \in H^{m+2}(\Omega)$ of the Dirichlet problem

$$\left.\begin{aligned} Np = -\operatorname{div}(a(p)\,\underset{\sim}{\nabla} p + \underset{\sim}{b}(p)) + c(p) = f \quad &\text{in } \Omega\,, \\ p = -g \text{ on } \partial\Omega\,, & \end{aligned}\right\} \quad (0.1)$$

where $\underset{\sim}{\nabla} w$ denotes the gradient of a scalar function w, $\operatorname{div} \underset{\sim}{v} = \nabla \cdot \underset{\sim}{v}$ denotes the divergence of a vector function $\underset{\sim}{v}$, and $a : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfies the requirement that $a(x, q) \geqslant a_0 > 0$.

If the coefficients a, $\underset{\sim}{b}$, and c have the properties that $\dfrac{\partial a}{\partial p} = 0$, $\underset{\sim}{b}(p) = \underset{\sim}{b}(x)\,p$, and $c(p) = c(x)\,p$, then the quasilinear problem (0.1) becomes the linear

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Dirichlet problem

$$\left.\begin{aligned} Lp = -\underset{\sim}{\nabla}.(a\underset{\sim}{\nabla}p + \underset{\sim}{b}p) + cp = f \quad \text{in } \Omega\,, \\ p = -g \text{ on } \partial\Omega\,, \end{aligned}\right\} \tag{0.2}$$

where we have omitted writing the variable x, as we shall do throughout the paper.

We shall denote by $(\,,)$ the natural inner product in $L^2(\Omega)$ or $L^2(\Omega)^2$, and by $\langle\,,\rangle$ the one in $L^2(\partial\Omega)$. We shall use the same notations to indicate the dualities between the $W^{r,s}(\Omega)$ and $W^{r,s}(\Omega)'$, and $H^s(\partial\Omega)$ and $H^{-s}(\partial\Omega)$.

Let

$$\underset{\sim}{V} = \underset{\sim}{H}(\mathrm{div}\,;\Omega) = \{\, \underset{\sim}{v} \in L^2(\Omega)^2 \mid \mathrm{div}\, \underset{\sim}{v} \in L^2(\Omega) \,\}\,,$$

normed by

$$\| \underset{\sim}{v} \|_{\underset{\sim}{V}}^2 = \| \underset{\sim}{v} \|_0^2 + \| \mathrm{div}\, \underset{\sim}{v} \|_0^2\,,$$

and

$$W = L^2(\Omega)\,.$$

If

$$\underset{\sim}{u} = -\,(a(p)\,\underset{\sim}{\nabla}p + \underset{\sim}{b}(p))\,, \quad \alpha = 1/a\,, \quad \underset{\sim}{\beta} = \alpha\underset{\sim}{b}\,, \tag{0.3}$$

then $(\underset{\sim}{u}, p) \in \underset{\sim}{V} \times W$ is a weak solution of (0.1) in the sense that it satisfies the equations

$$\left.\begin{aligned} (\alpha(p)\,\underset{\sim}{u}, \underset{\sim}{v}) - (\mathrm{div}\,\underset{\sim}{v}, p) + (\underset{\sim}{\beta}(p), \underset{\sim}{v}) = \langle g, \underset{\sim}{v}\cdot\underset{\sim}{\nu} \rangle,\ \underset{\sim}{v} \in \underset{\sim}{V}\,, \\ (\mathrm{div}\,\underset{\sim}{u}, w) + (c(p), w) = (f, w),\ w \in W\,, \end{aligned}\right\} \tag{0.4}$$

where $\underset{\sim}{\nu}$ denotes the unit outward normal vector to $\partial\Omega$.

Let $\mathcal{T}_h$ be a quasi-uniform polygonalization of Ω (by triangles or rectangles) of characteristic parameter $h \in (0, 1)$, where boundary polygons are allowed to have one curvilinear edge. That is, if we let r_T be the radius of the circle inscribed in T and R_T the radius of the circle circumscribing T, $T \in \mathcal{T}_h$, then there exist positive constants m and M, independent of h, such that

$$mh \leqslant r_T < R_T \leqslant Mh\,, \quad T \in \mathcal{T}_h\,.$$

Let

$$\underset{\sim}{V}_h \times W_h \subset \underset{\sim}{V} \times W$$

be the Raviart-Thomas-Nedelec space of index $k \geqslant 0$ associated with $\mathcal{C}_h$, [3, 6, 8, 11], which is defined as follows.

For $E \subset R^2$ let $P_k(E)$ denote the restrictions of polynomials of total degree k to the set E and let $Q_k(E)$ denote the restriction of $P_k(\mathbb{R}) \otimes P_k(\mathbb{R})$ to E. Then, let $R_k(E) = P_k(E)$ if E is a triangle (interior or boundary) and $R_k(E) = Q_k(E)$ if E is a rectangle (interior or boundary), and let $\underset{\sim}{R}_k(E) = R_k(E)^2$. For any $E \in \mathcal{C}_h$ let

$$\underset{\sim}{V}(E) = \underset{\sim}{R}_k(E) \otimes \underset{\sim}{x} R_k(E),$$

$$W(E) = R_k(E).$$

Set

$$\underset{\sim}{V}_h = \underset{\sim}{V}(k, \mathcal{C}_h) = \{ \underset{\sim}{v} \in \underset{\sim}{V} \mid \underset{\sim}{v}\ |_E \in \underset{\sim}{V}(E),\ E \in \mathcal{C}_h \}$$

$$= \left\{ \underset{\sim}{v} \in \prod_{E \in \mathcal{C}_h} \underset{\sim}{V}(E) \mid \underset{\sim}{v}\ |_{E_i} \cdot \underset{\sim}{\nu}_i + \underset{\sim}{v}\ |_{E_j} \cdot \underset{\sim}{\nu}_j = 0 \quad \text{on} \quad \bar{E}_i \cap \bar{E}_j \right\}$$

where $\underset{\sim}{\nu}_l$ is the outer normal to ∂E_l; also, let

$$W_h = W(k, \mathcal{C}_h) = \{ w \in W \mid w\ |_E \in W(E),\ E \in \mathcal{C}_h \}.$$

Let $P_h : W \to W_h$ be the orthogonal L^2-projection of W into W_h defined by

$$(P_h w - w, \chi) = 0, \quad w \in W, \quad \chi \in W_h, \tag{0.5}$$

which satisfies, for $1 \leqslant q \leqslant \infty$ and for either $G_0 \subset \subset G_1 \subset \subset \Omega$ and h sufficiently small or $G_0 = G_1 = \Omega$,

$$\| P_h w - w \|_{0,q;G_0} \leqslant Q \| w \|_{s,q;G_1} h^s, \quad 0 \leqslant s \leqslant k+1, \quad \text{if} \quad w \in W^{s,q}(G_1), \tag{0.6}$$

$$\| P_h w - w \|_{-r,G_0} \leqslant Q \| w \|_{s,G_1} h^{r+s}, \quad 0 \leqslant r, s \leqslant k+1, \quad \text{if} \quad w \in H^s(G_1), \tag{0.7}$$

where $\| \cdot \|_{s,q;G}$ denotes the usual Sobolev norm in $W^{s,q}(G)$, with $q = 2$ being omitted. Also, since div $\underset{\sim}{V}_h \subset W_h$,

$$(\text{div}\ \underset{\sim}{v}, w - P_h w) = 0,\ \underset{\sim}{v} \in \underset{\sim}{V}_h;\ w \in W. \tag{0.8}$$

Let $\pi_h : \underset{\sim}{V} \to \underset{\sim}{V}_h$ be the Raviart-Thomas projection, [8, 9], which satisfies

$$\operatorname{div} \pi_h = P_h \operatorname{div}, \tag{0.9}$$

$$\| \pi_h \underset{\sim}{v} - \underset{\sim}{v} \|_{0,q;G_0} \leqslant Q \| \underset{\sim}{v} \|_{s,q;G_1} h^s, \ 1/q < s \leqslant k+1, \quad \text{if} \quad v \in W^{s,q}(G_1)^2 . \tag{0.10}$$

The mixed finite element method for (0.1) is the discrete form of (0.4) : Find $(\underset{\sim}{u}_h, p_h) \in \underset{\sim}{V}_h \times W_h$ such that

$$\left.\begin{aligned} (\alpha(p_h)\, \underset{\sim}{u}_h, \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, p_h) + (\underset{\sim}{\beta}(p_h), \underset{\sim}{v}) &= \langle g, \underset{\sim}{v} \cdot \underset{\sim}{\nu} \rangle, & \underset{\sim}{v} &\in \underset{\sim}{V}_h , \\ (\operatorname{div} \underset{\sim}{u}_h, w) + (c(p_h), w) &= (f, w), & w &\in W_h . \end{aligned}\right\} \tag{0.11}$$

The existence and uniqueness of a solution of (0.11) and global convergence rates for the approximation $(\underset{\sim}{u}_h, p_h)$ to $(\underset{\sim}{u}, p)$ in various Sobolev spaces have been established in [3] for the linear problem (0.2). Existence, a form of uniqueness, and the convergence of the corresponding approximate solution have been demonstrated in [5] for the quasilinear problem.

Let

$$\underset{\sim}{\zeta} = \underset{\sim}{u} - \underset{\sim}{u}_h , \quad \xi = p - p_h , \quad \underset{\sim}{\sigma} = \pi_h \underset{\sim}{u} - \underset{\sim}{u}_h , \quad \tau = P_h p - p_h ,$$
$$\theta = \tau - \xi = P_h p - p .$$

Let

$$\underset{\sim}{\kappa} = \left[\int_0^1 (1-t)\, \alpha_{pp}(p - t\xi)\, dt \right] \underset{\sim}{u} + \int_0^1 (1-t)\, \underset{\sim}{\beta}_{pp}(p - t\xi)\, dt ,$$

$$\lambda = \int_0^1 \alpha_p(p - t\xi)\, dt , \qquad \rho = \int_0^1 (1-t)\, c_{pp}(p - t\xi)\, dt ,$$

for the nonlinear problem, and set $\underset{\sim}{\kappa} = \underset{\sim}{0}$ and $\lambda = \rho = 0$ for the linear problem. Also, let

$$\underset{\sim}{\Gamma} = \alpha_p(p)\, \underset{\sim}{u} + \underset{\sim}{\beta}_p(p), \quad \gamma = c_p(p)$$

for the nonlinear problem and

$$\underset{\sim}{\Gamma} = \underset{\sim}{\beta} , \quad \gamma = c$$

for the linear problem.

If then follows from (0.4) and (0.11) that $\underset{\sim}{\zeta}$ and ξ satisfy the following error equations (see [3, 5]) :

$$\left.\begin{aligned}(\alpha(p)\,\underset{\sim}{\zeta},\underset{\sim}{v}) - (\operatorname{div}\underset{\sim}{v},\xi) + (\underset{\sim}{\Gamma}\xi,\underset{\sim}{v}) &= ([\underset{\sim}{\kappa}\xi + \lambda\underset{\sim}{\zeta}]\,\xi,\underset{\sim}{v}), && \underset{\sim}{v}\in\underset{\sim}{V}_h\,,\\ (\operatorname{div}\underset{\sim}{\zeta},w) + (\gamma\xi,w) &= (\rho\xi^2,w), && w\in W_h\,.\end{aligned}\right\} \quad (0.12)$$

Let $M : H^2(\Omega) \to L^2(\Omega)$ be the operator given by

$$Mw = -\,\underset{\sim}{\nabla}\cdot(a(p)\,\underset{\sim}{\nabla}w + a(p)\,\underset{\sim}{\Gamma}w) + \gamma w\,,$$

and let M^* be its formal adjoint; that is,

$$M^*\chi = -\,\underset{\sim}{\nabla}\cdot(a(p)\,\underset{\sim}{\nabla}\chi) + a(p)\,\underset{\sim}{\Gamma}\cdot\underset{\sim}{\nabla}\chi + \gamma\chi\,.$$

Note that $M = L$ and $M^* = L^*$ in the linear case.

For any open set $E \subset \Omega$ set

$$\| z \|_{-s,E} = \sup_{\substack{y\in H_0^s(E)\\ y\neq 0}} \frac{(z,y)}{\| y \|_{s,E}}\,. \quad (0.13)$$

It follows from [10] that the operator M^* is coercive over $H_0^1(G)$ for any ball $G \subset \Omega$ of diameter not greater than some d_0. Furthermore, if the coefficients of (0.1) (or (0.2)) are sufficiently smooth, then, for any fixed $s \geqslant 0$, M^* has a bounded inverse from $H^s(G)$ onto $H^{s+2}(G) \cap H_0^1(G)$. *All balls appearing in the remainder of this paper will be assumed to be of diameters less than* d_0.

Let

$$\begin{aligned}\underset{\sim}{V}_h(G) &= \{\,\underset{\sim}{v}\in\underset{\sim}{V}_h : \operatorname{supp}\underset{\sim}{v}\subset G\,\}\,,\\ W_h(G) &= \{\,w\in W_h : \operatorname{supp} w\subset G\,\}\,.\end{aligned} \quad (0.14)$$

The pair $(\underset{\sim}{u}_h, p_h) \in \underset{\sim}{V}_h \times W_h$ is said to be an interior solution of the mixed method equations (0.11) on $G \subset\subset \Omega$ if it satisfies

$$\begin{aligned}(\alpha(p_h)\,\underset{\sim}{u}_h,\underset{\sim}{v}) - (\operatorname{div}\underset{\sim}{v},p_h) + (\underset{\sim}{\beta}(p_h),\underset{\sim}{v}) &= 0, && \underset{\sim}{v}\in\underset{\sim}{V}_h(G)\,,\\ (\operatorname{div}\underset{\sim}{u}_h,w) + (c(p_h),w) &= (f,w), && w\in W_h(G)\,.\end{aligned} \quad (0.15)$$

Note that the boundary integral from (0.4) is missing in (0.15) as $\underset{\sim}{v}\cdot\underset{\sim}{\nu} = 0$ on $\partial\Omega$ if $\underset{\sim}{v} \in \underset{\sim}{V}_h(G)$.

The plan of the paper is as follows. In section 1 we derive a local duality lemma which will be used for both the linear and nonlinear problems. We then derive local estimates in H^{-s}, $0 \leqslant s \leqslant k+1$, for the linear problem (section 2) and for the nonlinear (section 3). In section 4 we find rates of convergence for difference quotients for ξ, ζ, and div ζ in H^{-s}, $0 \leqslant s \leqslant k+1$. Finally, in section 5 we demonstrate local superconvergence in L^2 of some particular local averages of p_h, u_h, and div u_h to p, u, and div u, respectively.

1. THE INTERIOR DUALITY LEMMA

Global error estimates for the linear problem have been derived by Douglas and Roberts [3] using an analytical approach based strongly on a duality argument, and an analogous argument has been employed by Milner [5] in treating the quasilinear problem. Lemma 1.1, which will be fundamental in obtaining our interior estimates, is a local version of Lemmas 3.1 and 3.2 of Douglas-Roberts [3] and of Lemmas 2.1 and 5.1 of Milner [5]. Let δ_{lk} denote the Kronecker symbol below.

LEMMA 1.1 : *Let G' and G be concentric balls such that $G' \subset\subset G \subset\subset \Omega$. Assume that $\zeta \in V$, $f_0 \in L^2(G)^2$, $f_1 \in L^2(G)$, $g_0 \in L^2(G)$. Also, let $q \in L^{1+\delta}(G)^2$ and $\eta \in L^{1+\delta}(G)$ for some δ, $0 < \delta < 1$. Suppose that $z \in W_k$ satisfies the equations*

$$\left.\begin{aligned} (\alpha\zeta, v) - (\operatorname{div} v, z) + (\Gamma z, v) &= (f_0 + q, v) + (f_1, \operatorname{div} v), & v \in V_h(G)\,, \\ (\operatorname{div} \zeta, w) + (\gamma z, w) &= (g_0 + \eta, w)\,, & w \in W_h(G)\,, \end{aligned}\right\} \quad (1.1)$$

Let $s \geqslant 0$ and $l = \min(s, k)$. Then,

$$\begin{aligned} \| z \|_{-s,G'} \leqslant K\,[&h^{l+1} \| \zeta \|_{0,G} + h^{l+2-\delta_{lk}} \| \operatorname{div} \zeta \|_{0,G} \\ &+ h^{l+1} \| z \|_{0,G} + h^{l} \{ h \| f_0 \|_{0,G} + \| f_1 \|_{0,G} \\ &+ h^{2-\delta_{lk}} \| g \|_{0,G} \} + \| q \|_{0,1+\delta,G} + \| \eta \|_{0,1+\delta,G} \\ &+ \| \zeta \|_{-s-2,G} + \| z \|_{-s-1,G} + \| f_0 \|_{-s-1,G} \\ &+ \| f_1 \|_{-s,G} + \| g_0 \|_{-s-2,G}]\,. \end{aligned} \quad (1.2)$$

Proof: It will be convenient here and throughout the remainder of the paper to introduce a finite sequence $\{ G_i \}$ of balls concentric with G' and such that $G' \subset\subset G_i \subset\subset G_{i+1} \subset\subset G$. We shall use a corresponding sequence $\{ \omega_i \}$ of functions such that $\omega_i \in C_0^\infty(G_{i+1})$ and $\omega_i = 1$ on G_i; normally the index i will be omitted for ω_i.

Let $\psi \in H_0^s(G_1)$ and determine $\phi \in H^{s+2}(G_1) \cap H_0^1(G_1)$ as the solution of $M^* \phi = \psi$ in G_1. Then, $\| \phi \|_{s+2,G_1} \leqslant Q \| \psi \|_{s,G_1}$, and

$$v = \pi_h\, \omega a\, \nabla\phi \in V_h(G_2)\ (\subset V_h(G))$$

if h is sufficiently small. Hence, we see from (1.1) that

$$\begin{aligned}(\omega z, \psi) &= (\omega z, M^* \phi)\\
&= (z, -\underset{\sim}{\nabla}.(\omega a \underset{\sim}{\nabla}\phi) + \underset{\sim}{\Gamma}.\omega a \underset{\sim}{\nabla}\phi + \omega\gamma\phi) + (z, \underset{\sim}{\nabla}\omega . a \underset{\sim}{\nabla}\phi)\\
&= (z, -\operatorname{div}[\pi_h \omega a \underset{\sim}{\nabla}\phi]) + (\underset{\sim}{\Gamma} z, \pi_h \omega a \underset{\sim}{\nabla}\phi) + (\gamma z, P_h \omega\phi)\\
&\quad + (\underset{\sim}{\Gamma} z, \omega a \underset{\sim}{\nabla}\phi - \pi_h \omega a \underset{\sim}{\nabla}\phi) + (\gamma z, \omega\phi - P_h \omega\phi) + (z, \underset{\sim}{\nabla}\omega . a \underset{\sim}{\nabla}\phi)\\
&= (\underset{\sim}{q} + \underset{\sim}{f_0}, \pi_h \omega a \underset{\sim}{\nabla}\phi) + (f_1, \operatorname{div}[\pi_h \omega a \underset{\sim}{\nabla}\phi]) - (\alpha\underset{\sim}{\zeta}, \pi_h \omega a \underset{\sim}{\nabla}\phi)\\
&\quad + (g_0 + \eta, P_h \omega\phi) - (\operatorname{div} \underset{\sim}{\zeta}, P_h \omega\phi) + (\underset{\sim}{\Gamma} z, \omega a \underset{\sim}{\nabla}\phi - \pi_h \omega a \underset{\sim}{\nabla}\phi)\\
&\quad + (\gamma z, \omega\phi - P_h \omega\phi) + (z, \underset{\sim}{\nabla}\omega . a \underset{\sim}{\nabla}\phi)\\
&= (\underset{\sim}{q}, \omega a \underset{\sim}{\nabla}\phi) + (\underset{\sim}{q}, \pi_h \omega a \underset{\sim}{\nabla}\phi - \omega a \underset{\sim}{\nabla}\phi) + (\underset{\sim}{f_0}, \omega a \underset{\sim}{\nabla}\phi)\\
&\quad + (\underset{\sim}{f_0}, \pi_h \omega a \underset{\sim}{\nabla}\phi - \omega a \underset{\sim}{\nabla}\phi) + (f_1, \operatorname{div} \omega a \underset{\sim}{\nabla}\phi)\\
&\quad + (f_1, \operatorname{div}[\pi_h \omega a \underset{\sim}{\nabla}\phi - \omega a \underset{\sim}{\nabla}\phi])\\
&\quad + (g_0, \omega\phi) + (g_0, P_h \omega\phi - \omega\phi) + (\eta, \omega\phi) + (\eta, P_h \omega\phi - \omega\phi)\\
&\quad + (\alpha\underset{\sim}{\zeta} + \underset{\sim}{\Gamma} z, \omega a \underset{\sim}{\nabla}\phi - \pi_h \omega a \underset{\sim}{\nabla}\phi) + (\operatorname{div} \underset{\sim}{\zeta} + \gamma z, \omega\phi - P_h \omega\phi)\\
&\quad + (\underset{\sim}{\zeta}, (\underset{\sim}{\nabla}\omega)\phi) + (z, \underset{\sim}{\nabla}\omega . a \underset{\sim}{\nabla}\phi), \end{aligned} \tag{1.3}$$

since

$$-(\underset{\sim}{\zeta}, \omega \underset{\sim}{\nabla}\phi) - (\operatorname{div} \underset{\sim}{\zeta}, \omega\phi) = -(\omega, \operatorname{div}(\underset{\sim}{\zeta}\phi)) = (\zeta, (\underset{\sim}{\nabla}\omega)\phi).$$

First note that the embedding $H^{s+1}(G) \subset L^{1+(1/\delta)}(G)$ implies that

$$\begin{aligned}(\underset{\sim}{q}, \omega a \underset{\sim}{\nabla}\phi) &\leqslant K \| \underset{\sim}{q} \|_{0,1+\delta;G_2} \| \omega a \underset{\sim}{\nabla}\phi \|_{0,1+(1/\delta);G_2}\\
&\leqslant K \| \underset{\sim}{q} \|_{0,1+\delta;G_2} \| \underset{\sim}{\nabla}\phi \|_{s+1,G_2}. \end{aligned} \tag{1.4}$$

and that

$$\begin{aligned}(\eta, \omega\phi) &\leqslant K \| \eta \|_{0,1+\delta;G_2} \| \omega\phi \|_{0,1+(1/\delta);G_2}\\
&\leqslant K \| \eta \|_{0,1+\delta;G_2} \| \phi \|_{s+1,G_2}. \end{aligned} \tag{1.5}$$

Also, by (0.6), (0.10), and the embedding $H^{s+1}(G) \subset W^{\delta/(\delta+1),1+(1/\delta)}(G)$,

$$\begin{aligned}(\underset{\sim}{q}, \pi_h \omega a \underset{\sim}{\nabla}\phi - \omega a \underset{\sim}{\nabla}\phi) &\leqslant K \| \underset{\sim}{q} \|_{0,1+\delta;G_2} \| \pi_h \omega a \underset{\sim}{\nabla}\phi - \omega a \nabla\phi \|_{0,1+(1/\delta);G_2}\\
&\leqslant K \| \underset{\sim}{q} \|_{0,1+\delta;G_2} h^{\delta/(\delta+1)} \| \omega a \underset{\sim}{\nabla}\phi \|_{\delta/(\delta+1),1+(1/\delta);G_2}\\
&\leqslant K h^{\delta/(\delta+1)} \| \underset{\sim}{q} \|_{0,1+\delta;G_2} \| \underset{\sim}{\nabla}\phi \|_{s+1,G_2} \end{aligned} \tag{1.6}$$

and

$$\begin{aligned}(\eta, P_h\,\omega\phi - \omega\phi) &\leqslant K \,\|\eta\|_{0,1+\delta;G_2}\, \|P_h\,\omega\phi - \omega\phi\|_{0,1+(1/\delta);G_2}\\ &\leqslant K\,\|\eta\|_{0,1+\delta;G_2}\,\|\omega\phi\|_{\delta/(\delta+1),1+(1/\delta);G_2}\, h^{\delta/(\delta+1)}\\ &\leqslant K\,\|\eta\|_{0,1+\delta;G_2}\,\|\phi\|_{s+1,G_2}\, h^{\delta/(\delta+1)}. \end{aligned} \tag{1.7}$$

Next, note that

$$(\underset{\sim}{f}_0, \omega a\,\underset{\sim}{\nabla}\phi) \leqslant K\,\|\underset{\sim}{f}_0\|_{-s-1,G_2}\,\|\underset{\sim}{\nabla}\phi\|_{s+1,G_2}. \tag{1.8}$$

$$(f_1, \operatorname{div}\omega a\,\underset{\sim}{\nabla}\phi) \leqslant K\,\|f_1\|_{-s,G_2}\,\|\operatorname{div}\omega a\,\underset{\sim}{\nabla}\phi\|_{s,G_2}, \tag{1.9}$$

$$(g_0, \omega\phi) \leqslant K\,\|g_0\|_{-s-2,G_2}\,\|\phi\|_{s+2,G_2}. \tag{1.10}$$

Also, (0.6) and (0.10) imply that

$$\begin{aligned}&(\underset{\sim}{f}_0 - \alpha\underset{\sim}{\zeta} - \underset{\sim}{\Gamma}z, \pi_h\,\omega a\,\underset{\sim}{\nabla}\phi - \omega a\underset{\sim}{\nabla}\phi) \leqslant\\ &\quad\leqslant K(\|\underset{\sim}{f}_0\|_{0,G_2} + \|\underset{\sim}{\zeta}\|_{0,G_2} + \|z\|_{0,G_2})\,\|\pi_h\,\omega a\,\underset{\sim}{\nabla}\phi - \omega a\,\underset{\sim}{\nabla}\phi\|_{0,G_2} \qquad (1.11)\\ &\quad\leqslant Kh^{l+1}(\|\underset{\sim}{f}_0\|_{0,G_2} + \|\underset{\sim}{\zeta}\|_{0,G_2} + \|z\|_{0,G_2})\,\|\underset{\sim}{\nabla}\phi\|_{l+1,G_2},\end{aligned}$$

$$\begin{aligned}&(g_0 - \operatorname{div}\underset{\sim}{\zeta} - \gamma z, P_h\,\omega\phi - \omega\phi) \leqslant\\ &\quad\leqslant K(\|g_0\|_{0,G_2} + \|\operatorname{div}\underset{\sim}{\zeta}\|_{0,G_2} + \|z\|_{0,G_2})\,\|P_h\,\omega\phi - \omega\phi\|_{0,G_2} \qquad (1.12)\\ &\quad\leqslant Kh^{l+2-\delta_{lk}}(\|g_0\|_{0,G_2} + \|\operatorname{div}\underset{\sim}{\zeta}\|_{0,G_2} + \|z\|_{0,G_2})\,\|\phi\|_{l+2,G_2}\end{aligned}$$

$$\begin{aligned}&(f_1, \operatorname{div}(\pi_h\,\omega a\,\underset{\sim}{\nabla}\phi) - \operatorname{div}(\omega a\,\underset{\sim}{\nabla}\phi)) = (f_1, P_h\operatorname{div}(\omega a\,\underset{\sim}{\nabla}\phi) - \operatorname{div}(\omega a\,\underset{\sim}{\nabla}\phi))\\ &\qquad\leqslant K\,\|f_1\|_{0,G_2}\,\|P_h\operatorname{div}(\omega a\,\underset{\sim}{\nabla}\phi) - \operatorname{div}(\omega a\,\underset{\sim}{\nabla}\phi)\|_{0,G_2} \qquad (1.13)\\ &\qquad\leqslant Kh^{l}\,\|f_1\|_{0,G_2}\,\|\operatorname{div}(\omega a\,\underset{\sim}{\nabla}\phi)\|_{l,G_2}.\end{aligned}$$

Finally,

$$(\underset{\sim}{\zeta}, (\underset{\sim}{\nabla}\omega)\,\phi) \leqslant K\,\|\underset{\sim}{\zeta}\|_{-s-2,G_2}\,\|\phi\|_{s+2,G_2}, \tag{1.14}$$

$$(z, \underset{\sim}{\nabla}\omega . a\,\underset{\sim}{\nabla}\phi) \leqslant K\,\|z\|_{-s-1,G_2}\,\|a\,\underset{\sim}{\nabla}\phi\|_{s+1,G_2}. \tag{1.15}$$

The lemma now follows trivially from the substitution of (1.4)-(1.15) into (1.3).

We can now derive some preliminary local error estimates from this lemma.

1.2 : *If h is sufficiently small, $0 \leqslant r \leqslant k+1$, $0 \leqslant s \leqslant k$, and $p \in H^r(G)$, then*

(i) $$\begin{aligned}\|\xi\|_{-s,G'} \leqslant K[&h^{s+1}(\|\xi\|_{0,G} + \|\underset{\sim}{\zeta}\|_{0,G}) + h^{s+2-\delta_{sk}}\,\|\operatorname{div}\underset{\sim}{\zeta}\|_{0,G}\\ &+ h^{r+s}\,\|p\|_{r,G} + \|\xi^2\|_{0,1+\delta;G} + \|\xi\underset{\sim}{\zeta}\|_{0,1+\delta;G}\\ &+ \|\xi\|_{-k-1,G} + \|\underset{\sim}{\zeta}\|_{-k-1,G}],\end{aligned}$$

(ii) $\| \underset{\sim}{\zeta} \|_{-s,G'} \leqslant K[h^s(\| \xi \|_{0,G} + \| \underset{\sim}{\zeta} \|_{0,G}) + h^{s+1} \| \operatorname{div} \underset{\sim}{\zeta} \|_{0,G}$

$+ h^{r+s} \| p \|_{r,G} + \| \xi^2 \|_{0,1+\delta;G} + \| \xi \underset{\sim}{\zeta} \|_{0,1+\delta;G}$

$+ \| \xi \|_{-k-1,G} + \| \underset{\sim}{\zeta} \|_{-k-1,G}]$.

(iii) $\| \operatorname{div} \underset{\sim}{\zeta} \|_{-s,G'} \leqslant K[h^s(\| \xi \|_{0,G} + \| \operatorname{div} \underset{\sim}{\zeta} \|_{0,G}) + \| \rho \xi^2 \|_{0,1+\delta;G}$

$+ \| \xi \|_{-s,G}]$.

Proof: Rewrite (0.12) in the form

$$\left.\begin{aligned} (\alpha(p)\, \underset{\sim}{\zeta}, \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, \tau) + (\underset{\sim}{\Gamma}\tau, \underset{\sim}{v}) &= ([\underset{\sim}{\kappa}\xi + \lambda \underset{\sim}{\zeta}]\, \xi + \underset{\sim}{\Gamma}\theta, \underset{\sim}{v}), \quad \underset{\sim}{v} \in \underset{\sim}{V}_h(G), \\ (\operatorname{div} \underset{\sim}{\zeta}, w) + (\gamma\tau, w) &= (\rho\xi^2 + \gamma\theta, w), \quad w \in W_h(G). \end{aligned}\right\} \quad (1.16)$$

We can now apply Lemma 1.1 to (1.15), by setting $\underset{\sim}{q} = (\underset{\sim}{\kappa}\xi + \lambda\underset{\sim}{\zeta})\, \xi$, $\eta = \rho\xi^2$, $\underset{\sim}{f}_0 = \underset{\sim}{\Gamma}\theta$, $f_1 = 0$, and $g_0 = \gamma\theta$ in (1.2). Thus,

$$\begin{aligned} \| \tau \|_{-s,G'} \leqslant K[&h^{s+1} \| \underset{\sim}{\zeta} \|_{0,G_2} + h^{s+2-\delta_{sk}} \| \operatorname{div} \underset{\sim}{\zeta} \|_{0,G_2} \\ &+ \| (\underset{\sim}{\kappa}\xi + \lambda\underset{\sim}{\zeta})\, \xi \|_{0,1+\delta;G_2} + \| \rho\xi^2 \|_{0,1+\delta;G_2} \\ &+ h^{s+1} \| \underset{\sim}{\Gamma}\theta \|_{0,G_2} + h^{s+2-\delta_{sk}} \| \gamma\theta \|_{0,G_2} \\ &+ h^{s+1} \| \tau \|_{0,G_2} + \| \underset{\sim}{\Gamma}\theta \|_{-s-1,G_2} + \| \gamma\theta \|_{-s-2,G_2} \\ &+ \| \underset{\sim}{\zeta} \|_{-s-2,G_2} + \| \tau \|_{-s-1,G_2}]. \end{aligned} \quad (1.17)$$

For $s = 0$, (1.17) gives the estimate

$$\begin{aligned} \| \tau \|_{0,G'} &\leqslant K[h \| \underset{\sim}{\zeta} \|_{0,G_2} + h^{2-\delta_{0k}} \| \operatorname{div} \underset{\sim}{\zeta} \|_{0,G_2} + \| (\underset{\sim}{\kappa}\xi + \lambda\underset{\sim}{\zeta})\, \xi \|_{0,1+\delta;G_2} \\ &\quad + \| \rho\xi^2 \|_{0,1+\delta;G_2} + h \| \underset{\sim}{\Gamma}\theta \|_{0,G_2} + h^{2-\delta_{0k}} \| \gamma\theta \|_{0,G_2} \\ &\quad + h \| \tau \|_{0,G_2} + \| \underset{\sim}{\Gamma}\theta \|_{-1,G_2} + \| \gamma\theta \|_{-2,G_2} + \| \underset{\sim}{\zeta} \|_{-2,G_2} \\ &\quad + \| \tau \|_{-1,G_2}] \\ &\leqslant K[h \| \underset{\sim}{\zeta} \|_{0,G_2} + h^{2-\delta_{0k}} \| \operatorname{div} \underset{\sim}{\zeta} \|_{0,G_2} + \| (\underset{\sim}{\kappa}\xi + \lambda\underset{\sim}{\zeta})\, \xi \|_{0,1+\delta;G_2} \\ &\quad + \| \rho\xi^2 \|_{0,1+\delta;G_2} + h \| \underset{\sim}{\Gamma}\theta \|_{0,G_2} + h^{2-\delta_{0k}} \| \gamma\theta \|_{0,G_2} \\ &\quad + h \| \tau \|_{0,G_2} + h^{-s} \| \underset{\sim}{\Gamma}\theta \|_{-s-1,G_2} + h^{-s/2} \| \gamma\theta \|_{-s-2,G_2} \\ &\quad + h^{-s/2} \| \underset{\sim}{\zeta} \|_{-s-2,G_2} + h^{-s} \| \tau \|_{-s-1,G_2}], \end{aligned} \quad (1.18)$$

where we have used interpolation between Sobolev spaces, [4].

If we now substitute (1.18) recursively into itself we obtain the bound

$$\begin{aligned}\| \tau \|_{0,G'} \leqslant K[h \| \underset{\sim}{\zeta} \|_{0,G_{2k}} &+ h^{2-\delta_{0k}} \| \operatorname{div} \underset{\sim}{\zeta} \|_{0,G_{2k}} + \| (\underset{\sim}{\kappa}\xi + \lambda \underset{\sim}{\zeta}) \xi \|_{0,1+\delta;G_{2k}} \\ &+ \| \rho\xi^2 \|_{0,1+\delta;G_{2k}} + h \| \underset{\sim}{\Gamma}\theta \|_{0,G_{2k}} + h^{2-\delta_{0k}} \| \gamma\theta \|_{0,G_{2k}} \\ &+ h^k \| \tau \|_{0,G_{2k}} + h^{-s} \| \underset{\sim}{\Gamma}\theta \|_{-s-1,G_{2k}} + h^{-s/2} \| \gamma\theta \|_{-s-2,G_{2k}} \\ &+ h^{-s/2} \| \underset{\sim}{\zeta} \|_{-s-2,G_{2k}} + h^{-s} \| \tau \|_{-s-1,G_{2k}}],\end{aligned}$$

which when substituted into (1.17) gives the estimate

$$\begin{aligned}\| \tau \|_{-s,G'} \leqslant K[h^{s+1} \| \underset{\sim}{\zeta} \|_{0,G'_1} &+ h^{s+2-\delta_{sk}} \| \operatorname{div} \underset{\sim}{\zeta} \|_{0,G'_1} \\ &+ \| (\underset{\sim}{\kappa}\xi + \lambda \underset{\sim}{\zeta}) \xi \|_{0,1+\delta;G'_1} + \| \rho\xi^2 \|_{0,1+\delta;G'_1} \\ &+ h^{s+1} \| \underset{\sim}{\Gamma}\theta \|_{0,G'_1} + h^{s+2-\delta_{sk}} \| \gamma\theta \|_{0,G'_1} \\ &+ h^{s+k+1} \| \tau \|_{0,G'_1} + \| \underset{\sim}{\Gamma}\theta \|_{-s-1,G'_1} + \| \gamma\theta \|_{-s-2,G'_1} \\ &+ \| \underset{\sim}{\zeta} \|_{-s-2,G'_1} + \| \tau \|_{-s-1,G'_1}],\end{aligned} \tag{1.19}$$

(where $G'_1 = G_{2k}$) which in turn implies (using (0.6) and (0.7)) that

$$\begin{aligned}\| \xi \|_{-s,G'} \leqslant K[h^{s+1} \| \underset{\sim}{\zeta} \|_{0,G'_1} &+ h^{s+2-\delta_{sk}} \| \operatorname{div} \underset{\sim}{\zeta} \|_{0,G'_1} \\ &+ \| \underset{\sim}{\zeta}\xi \|_{0,1+\delta;G'_1} + \| \xi^2 \|_{0,1+\delta;G'_1} \\ &+ h^{r+s} \| p \|_{r,G_{2k+1}} + h^{k+s+1} \| \xi \|_{0,G'_1} \\ &+ \| \underset{\sim}{\zeta} \|_{-s-2,G'_1} + \| \xi \|_{-s-1,G'_1}]\end{aligned} \tag{1.20}$$

Let us consider the divergence of $\underset{\sim}{\zeta}$. Let $\phi \in H_0^s(G_1)$. Then, we see from (1.16) that

$$\begin{aligned}(\omega \operatorname{div} \underset{\sim}{\zeta}, \phi) &= (\operatorname{div} \underset{\sim}{\zeta}, P_h(\omega\phi)) + (\operatorname{div} \underset{\sim}{\zeta}, \omega\phi - P_h(\omega\phi)) \\ &= -(\gamma\xi, \omega\phi) + (\operatorname{div} \underset{\sim}{\zeta} + \gamma\xi, \omega\phi - P_h(\omega\phi)) + (\rho\xi^2, P_h(\omega\phi)),\end{aligned}$$

which implies the bound

$$\| \operatorname{div} \underset{\sim}{\zeta} \|_{-s,G'} \leqslant K[\| \xi \|_{-s,G_1} + h^s(\| \xi \|_{0,G_1} + \| \operatorname{div} \underset{\sim}{\zeta} \|_{0,G_1}) + \| \rho\xi^2 \|_{0,G_1}] \tag{1.21}$$

for $0 \leqslant s \leqslant k + 1$. This gives (iii) of the lemma.

Finally, let us obtain a bound for $\underset{\sim}{\zeta}$ in $H^{-s}(G')^2$. Let $\psi \in H_0^s(G_1)^2$ and take $\phi \in H^{s+1}(G_1) \cap H_0^1(G_1)$ as the solution of $-\underset{\sim}{\nabla}.(a(p)\underset{\sim}{\nabla}\phi) = \operatorname{div} \underset{\sim}{\psi}$ in G_1. Set $\underset{\sim}{\delta} = \underset{\sim}{\psi} + a(p)\underset{\sim}{\nabla}\phi$.. Then,

$$\operatorname{div} \underset{\sim}{\delta} = 0 \tag{1.22}$$

and

$$\| \underset{\sim}{\delta} \|_{s,G_1} + \| \phi \|_{s+1,G_1} \leqslant Q \| \underset{\sim}{\psi} \|_{s,G_1} , \quad s \geqslant 0 . \tag{1.23}$$

Also, integrating by parts, we obtain the relation

$$(\omega\alpha(p) \underset{\sim}{\zeta}, \underset{\sim}{\psi}) = (\text{div } \underset{\sim}{\zeta}, \omega\phi) + (\underset{\sim}{\zeta}, \phi\underset{\sim}{\nabla}\omega) + (\alpha(p) \underset{\sim}{\zeta}, \omega\underset{\sim}{\delta}) . \tag{1.24}$$

Using the relation that led to (1.21) with $(\rho\xi^2, P_h(\omega\phi))$ replaced by $(\rho\xi^2, \omega\phi) + (\rho\xi^2, P_h(\omega\phi) - \omega\phi)$, we obtain the estimate

$$\begin{aligned} | (\text{div } \underset{\sim}{\zeta}, \omega\phi) + (\underset{\sim}{\zeta}, \phi\underset{\sim}{\nabla}\omega) | &= | - (\gamma\xi, \omega\phi) + (\text{div } \underset{\sim}{\zeta} + \gamma\xi, \omega\phi - P_h(\omega\phi)) \\ &\quad + (\rho\xi^2, \omega\phi) + (\rho\xi^2, P_h(\omega\phi) - \omega\phi) + (\underset{\sim}{\zeta}, \phi\underset{\sim}{\nabla}\omega) | \\ &\leqslant K[\| \xi \|_{-s-1,G_1} + \| \underset{\sim}{\zeta} \|_{-s-1,G_1} + \| \rho\xi^2 \|_{0,1+\delta;G_1} \\ &\quad + h^{s+1}(\| \xi \|_{0,G_1} + \| \text{div } \underset{\sim}{\zeta} \|_{0,G_1})] \| \phi \|_{s+1,G_1} . \end{aligned} \tag{1.25}$$

Next, it follows from (0.12) that

$$\begin{aligned} (\alpha(p) \underset{\sim}{\zeta}, \omega\underset{\sim}{\delta}) &= (\alpha(p) \underset{\sim}{\zeta}, \pi_h \omega\underset{\sim}{\delta}) + (\alpha(p) \underset{\sim}{\zeta}, \omega\underset{\sim}{\delta} - \pi_h \omega\underset{\sim}{\delta}) \\ &= (\text{div } \pi_h \omega\underset{\sim}{\delta}, \xi) - (\underset{\sim}{\Gamma}\xi, \omega\underset{\sim}{\delta}) + (\alpha(p) \underset{\sim}{\zeta} + \underset{\sim}{\Gamma}\xi, \omega\underset{\sim}{\delta} - \pi_h \omega\underset{\sim}{\delta}) \\ &\quad + ([\underset{\sim}{\kappa}\xi + \lambda\underset{\sim}{\zeta}] \xi, \omega\underset{\sim}{\delta}) + ([\underset{\sim}{\kappa}\xi + \lambda\underset{\sim}{\zeta}] \xi, \pi_h \omega\underset{\sim}{\delta} - \omega\underset{\sim}{\delta}) . \end{aligned} \tag{1.26}$$

Now, (0.8), (0.9), and (1.22) imply (using integration by parts) that

$$\begin{aligned} (\text{div } \pi_h \omega\underset{\sim}{\delta}, \xi) &= (\text{div } [\pi_h \omega\underset{\sim}{\delta} - \omega\underset{\sim}{\delta}], \tau) + (\text{div } \omega\underset{\sim}{\delta}, \tau) \\ &= (\text{div } \omega\underset{\sim}{\delta}, \tau) \\ &= (\underset{\sim}{\delta} . \underset{\sim}{\nabla}\omega, \tau) , \end{aligned} \tag{1.27}$$

since $\tau = \xi + \theta \in W_h$; it now follows from (1.26) and (1.27) that

$$\begin{aligned} | (\alpha(p) \underset{\sim}{\zeta}, \omega\underset{\sim}{\delta}) | &\leqslant K[\| \tau \|_{-s,G_1} + \| \xi \|_{-s,G_1} + h^s(\| \xi \|_{0,G_1} + \| \underset{\sim}{\zeta} \|_{0,G_1}) \\ &\quad + \| (\underset{\sim}{\kappa}\xi + \lambda\underset{\sim}{\zeta} \|_{0,1+\delta;G_1}] \| \underset{\sim}{\delta} \|_{s,G_1} , \quad 1 \leqslant s \leqslant k + 1 . \end{aligned} \tag{1.28}$$

Using (0.6), (1.23), (1.25), and (1.28) in (1.24) we obtain the bound

$$\begin{aligned} \| \underset{\sim}{\zeta} \|_{-s,G'} &\leqslant K[\| \xi \|_{-s,G_1} + h^{r+s} \| p \|_{r,G_2} + h^s(\| \xi \|_{0,G_1} + \| \underset{\sim}{\zeta} \|_{0,G_1}) \\ &\quad + \| \xi^2 \|_{0,1+\delta;G_1} + \| \underset{\sim}{\zeta}\xi \|_{0,1+\delta;G_1} + h^{s+1} \| \text{div } \underset{\sim}{\zeta} \|_{0,G_1} + \| \underset{\sim}{\zeta} \|_{-s-1,G_1}] \end{aligned} \tag{1.29}$$

for $1 \leqslant s \leqslant k$, and since the relation holds trivially for $s = 0$, using interpolation, [4], we see that it holds for $0 \leqslant s \leqslant k$.

Recursive substitution of (1.20) into (1.29) gives the inequality

$$\begin{aligned}\| \underset{\sim}{\zeta} \|_{-s,G'} \leqslant K[h^s(\| \xi \|_{0,G_{k+1}} + \| \underset{\sim}{\zeta} \|_{0,G_{k+1}}) + h^{s+1} \| \operatorname{div} \underset{\sim}{\zeta} \|_{0,G_{k+1}} \\ + \| \xi^2 \|_{0,1+\delta;G_{k+1}} + \| \underset{\sim}{\zeta}\xi \|_{0,1+\delta;G_{k+1}} + h^{r+s} \| p \|_{r,G_{k+2}} \\ + \| \underset{\sim}{\zeta} \|_{-s-1,G_{k+1}} + \| \xi \|_{-k-1,G_{k+1}}] . \end{aligned} \quad (1.30)$$

Now substitute (1.30) recursively into itself to see that

$$\begin{aligned}\| \underset{\sim}{\zeta} \|_{-s,G'} \leqslant K[h^s(\| \xi \|_{0,G_{k(k+1)}} + \| \underset{\sim}{\zeta} \|_{0,G_{k(k+1)}}) + h^{s+1} \| \operatorname{div} \underset{\sim}{\zeta} \|_{0,G_{k(k+1)}} \\ + \| \xi^2 \|_{0,1+\delta;G_{k(k+1)}} + \| \underset{\sim}{\zeta}\xi \|_{0,1+\delta;G_{k(k+1)}} + h^{r+s} \| p \|_{r,G_{k(k+2)}} \\ + \| \xi \|_{-k-1,G_{k(k+1)}} + \| \underset{\sim}{\zeta} \|_{-k-1,G_{k(k+1)}}] ,\end{aligned}$$

which gives part (ii) of this lemma.

We now substitute (ii) into (1.20) (with G_1' shifted to G_1) to obtain the estimate

$$\begin{aligned}\| \xi \|_{-s,G'} \leqslant K[h^{s+1}(\| \xi \|_{0,G_2} + \| \underset{\sim}{\zeta} \|_{0,G_2}) + h^{s+2-\delta_{sk}} \| \operatorname{div} \underset{\sim}{\zeta} \|_{0,G_2} \\ + \| \xi^2 \|_{0,1+\delta;G_2} + \| \underset{\sim}{\zeta}\xi \|_{0,1+\delta;G_2} + h^{r+s} \| p \|_{r,G_2} \\ + \| \xi \|_{-s-1,G_2} + \| \underset{\sim}{\zeta} \|_{-k-1,G_2}] ,\end{aligned}$$

which recursively substituted into itself yields (i) of the lemma.

Observe that we have a quadratic error term with no powers of h in front of it (just in the nonlinear case). This will produce eventually bounds with a quite atypical term in them.

2. THE LOCAL ESTIMATES FOR THE LINEAR CASE

Observe that $s = 0$ in Lemma 1.2 (i) gives the bound

$$\begin{aligned}\| \xi \|_{0,G_0} \leqslant K[h(\| \xi \|_{0,G_1} + \| \underset{\sim}{\zeta} \|_{0,G_1}) + h^{2-\delta_{0k}} \| \operatorname{div} \underset{\sim}{\zeta} \|_{0,G_1} \\ + h^r \| p \|_{r,G_1} + \| \xi^2 \|_{0,1+\delta;G_1} + \| \underset{\sim}{\zeta}\xi \|_{0,1+\delta;G_1} \\ + \| \xi \|_{-k-1,G_1} + \| \underset{\sim}{\zeta} \|_{-k-1,G_1}] ,\end{aligned} \quad (2.1)$$

for $2 \leqslant r \leqslant k + 1$.

Also, $\kappa = 0, \lambda = 0$, and $\rho = 0$ in the linear problem, and the bound

$$\| \xi \|_{-k-1,\Omega} + \| \underset{\sim}{\zeta} \|_{-k-1,\Omega} + \| \operatorname{div} \underset{\sim}{\zeta} \|_{-k-1,\Omega} \leqslant K \| p \|_{j,\Omega} h^{j+k-1} \quad (2.2)$$

for $2 \leqslant j \leqslant k + 3$ if $p \in H^j(\Omega)$ is given in [3].

Thus, (2.1) can be rewritten in the form

$$\| \xi \|_{0,G_0} \leqslant K[h(\| \xi \|_{0,G_1} + \| \underset{\sim}{\zeta} \|_{0,G_1}) + h^{2-\delta_{0k}} \| \operatorname{div} \underset{\sim}{\zeta} \|_{0,G_1} + h^r \| p \|_{r,G_1} + h^{k+1} \| p \|_{2,\Omega}], \quad (2.3)$$

$2 \leqslant r \leqslant k+1$, if $p \in H^2(\Omega) \cap H^r(G_1)$. We do not have similar estimates for $\underset{\sim}{\zeta}$ and div $\underset{\sim}{\zeta}$.

We shall need to have error estimates for a local solution of the homogeneous equations.

LEMMA 2.1 : *Let* $(\underset{\sim}{v}_h, q_h) \in \underset{\sim}{V}_h \times W_h$ *be a solution of the system*

$$\left.\begin{aligned} (\alpha \underset{\sim}{v}_h, \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, q_h) + (\underset{\sim}{\beta} q_h, \underset{\sim}{v}) &= 0, \quad \underset{\sim}{v} \in \underset{\sim}{V}_h(G), \\ (\operatorname{div} \underset{\sim}{v}_h, w) + (c q_h, w) &= 0, \quad w \in W_h(G), \end{aligned}\right\} \quad (2.4)$$

where G is a ball of sufficiently small diameter. Then, for fixed $\varepsilon > 0$, *any* $m \geqslant 0$, *and h sufficiently small,*

$$h^{-\varepsilon}(\| q_h \|_{0,G'} + \| \underset{\sim}{v}_h \|_{0,G'}) + \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G'} \leqslant \leqslant Q[h^m(\| \underset{\sim}{v}_h \|_{0,G} + \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G} + \| q_h \|_{0,G}) + (\| \underset{\sim}{v}_h \|_{-k-1,G} + \| q_h \|_{-k-1,G}) h^{-\varepsilon}].$$

Proof : Lemma 1.1 implies directly the bound

$$\| q_h \|_{0,G_0} \leqslant K[h(\| q_h \|_{0,G_1} + \| \underset{\sim}{v}_h \|_{0,G_1}) + h^{2-\delta_{0k}} \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1} + \| q_h \|_{-1,G_1} + \| \underset{\sim}{v}_h \|_{-2,G_1}]. \quad (2.5)$$

We should now like to take $w = \operatorname{div} \underset{\sim}{v}_h$ in the second equation of (2.4), but there is no reason to believe that div $\underset{\sim}{v}_h \in W_h(G)$. However, there is $\chi \in W_h(G_1)$ such that

$$\| \chi - \omega \operatorname{div} \underset{\sim}{v}_h \|_0 \leqslant Qh \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}. \quad (2.6)$$

Thus, (2.4)-(2.6) imply that

$$\begin{aligned} \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G'}^2 &\leqslant (\omega \operatorname{div} \underset{\sim}{v}_h, \operatorname{div} \underset{\sim}{v}_h) \\ &= (\omega \operatorname{div} \underset{\sim}{v}_h - \chi, \operatorname{div} \underset{\sim}{v}_h + c q_h) - (\omega \operatorname{div} \underset{\sim}{v}_h, c q_h) \\ &\leqslant Q \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1} (h \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1} + \| q_h \|_{0,G_1}) \quad (2.7) \\ &\leqslant Q[h(\| q_h \|_{0,G_2} + \| \underset{\sim}{v}_h \|_{0,G_2} + \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}) + \| q_h \|_{-1,G_2} \\ &\quad + \| \underset{\sim}{v}_h \|_{-2,G_2}] \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}. \end{aligned}$$

Let us try to estimate $\| v_h \|_{0,G_0}$. Again, there are $\bar{\chi} \in V_h(G_1)$ and $\psi \in W_h(G_1)$ such that

$$\left.\begin{aligned} &\| \omega v_h - \bar{\chi} \|_0 \leqslant Qh \| v_h \|_{0,G_1}, \\ &\| \operatorname{div}(\omega v_h - \bar{\chi}) \|_0 \leqslant Qh \| \operatorname{div} v_h \|_{0,G_1}, \\ &\| \omega q_h - \psi \|_0 \leqslant Qh \| q_h \|_{0,G_1}. \end{aligned}\right\} \tag{2.8}$$

Then, it follows from (2.4) and (2.8) that

$$\begin{aligned} \| v_h \|_{0,G'}^2 &\leqslant Q(\omega\alpha v_h, v_h) \\ &= Q[(\alpha v_h, \omega v_h - \bar{\chi}) + (\operatorname{div}[\bar{\chi} - \omega v_h], q_h) - (\beta q_h, \bar{\chi} - \omega v_h) \\ &\quad + (\operatorname{div} \omega v_h, q_h) - (\omega\beta q_h, v_h)] \\ &\leqslant Q[h(\| v_h \|_{0,G_1}^2 + \| q_h \|_{0,G_1} \| \operatorname{div} v_h \|_{0,G_1}) \\ &\quad + \| q_h \|_{0,G_1} \| v_h \|_{0,G_1} + (\omega \operatorname{div} v_h, q_h) + (v_h \cdot \nabla\omega, q_h)] \\ &\leqslant Q[h(\| v_h \|_{0,G_1}^2 + \| q_h \|_{0,G_1} \| \operatorname{div} v_h \|_{0,G_1}) \\ &\quad + \| q_h \|_{0,G_1} \| v_h \|_{0,G_1} - (\omega c q_h, q_h) + 0(h \| q_h \|_{0,G_1} [\| q_h \|_{0,G_1} \\ &\quad + \| \operatorname{div} v_h \|_{0,G_1}])] \\ &\leqslant Q[h(\| v_h \|_{0,G_1}^2 + \| q_h \|_{0,G_1} \| \operatorname{div} v_h \|_{0,G_1}) \\ &\quad + \| q_h \|_{0,G_1} (\| q_h \|_{0,G_1} + \| v_h \|_{0,G_1})]. \end{aligned} \tag{2.9}$$

Lemma 1.2, (i) and (ii) imply that, for $0 \leqslant s \leqslant k$,

$$\begin{aligned} \| q_h \|_{-s,G'} &\leqslant Q[h^{s+1}(\| q_h \|_{0,G_1} + \| v_h \|_{0,G_1}) + h^{s+2-\delta_{sk}} \| \operatorname{div} v_h \|_{0,G_1} \\ &\quad + \| q_h \|_{-k-1,G_1} + \| v_h \|_{-k-1,G_1}], \\ \| v_h \|_{-s,G'} &\leqslant Q[h^s(\| q_h \|_{0,G_1} + \| v_h \|_{0,G_1}) + h^{s+1} \| \operatorname{div} v_h \|_{0,G_1} \\ &\quad + \| q_h \|_{-k-1,G_1} + \| v_h \|_{-k-1,G_1}], \end{aligned}$$

where we have omitted the term $h^{r+s} \| p \|_{r,G_1}$ as it came from the projection errors. Substitution of these relations into (2.5) and (2.7) will replace the terms $\| q_h \|_{-1,G_1}$ and $\| v_h \|_{-2,G_1}$ by $\| q_h \|_{-k-1,G_2}$ and $\| v_h \|_{-k-1,G_2}$, respectively, giving the bounds

$$\begin{aligned} \| q_h \|_{0,G'} &\leqslant K[h(\| q \|_{0,G_3} + \| v_h \|_{0,G_3}) + h^{2-\delta_{0k}} \| \operatorname{div} v_h \|_{0,G_3} \\ &\quad + \| q_h \|_{-k-1,G_3} + \| v_h \|_{-k-1,G_3}], \end{aligned} \tag{2.10}$$

$$\begin{aligned} \| \operatorname{div} v_h \|_{0,G'}^2 &\leqslant K[h(\| q_h \|_{0,G_3} + \| v_h \|_{0,G_3} + \| \operatorname{div} v_h \|_{0,G_3}) + \\ &\quad + \| q_h \|_{-k-1,G_3} + \| v_h \|_{-k-1,G_3}] \| \operatorname{div} v_h \|_{0,G_3}. \end{aligned} \tag{2.11}$$

Recursive substitution of (2.10) into itself (and relabeling the G_i's) will give the estimate

$$\| q_h \|_{0,G'} \leqslant K[h^t \| q_h \|_{0,G_1} + h \| \underset{\sim}{v}_h \|_{0,G_1} + h^{2-\delta_{0k}} \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1} + \| q_h \|_{-k-1,G_1} + \| \underset{\sim}{v}_h \|_{-k-1,G_1}], \tag{2.12}$$

for any fixed integer $t > 0$.

Substituting (2.12) into (2.9), we see that

$$\begin{aligned} \| \underset{\sim}{v}_h \|_{0,G'}^2 \leqslant\ & Q[h \| \underset{\sim}{v}_h \|_{0,G_1}^2 + h^{t+1} \| q_h \|_{0,G_1} \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1} \\ & + h^2 \| \underset{\sim}{v}_h \|_{0,G_1} \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1} + h^{3-\delta_{0k}} \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}^2 \\ & + h(\| q_h \|_{-k-1,G_1} + \| \underset{\sim}{v}_h \|_{-k-1,G_1}) \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1} \\ & + h^{2t} \| q_h \|_{0,G_1}^2 + \| q_h \|_{-k-1,G_1}^2 + \| \underset{\sim}{v}_h \|_{-k-1,G_1}^2] \\ \leqslant\ & Q[h^{2T} \| q_h \|_{0,G_1}^2 + h \| \underset{\sim}{v}_h \|_{0,G_1}^2 + h^2 \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}^2 \\ & + \| q_h \|_{-k-1,G_1}^2 + \| \underset{\sim}{v}_h \|_{-k-1,G_1}^2], \end{aligned}$$

where T is any fixed positive integer. By repeated substitution of this relation into itself (and relabeling of the G_i's) we obtain the bound

$$\| \underset{\sim}{v}_h \|_{0,G'}^2 \leqslant Q\, [h^{2T}(\| q_h \|_{0,G_1}^2 + \| \underset{\sim}{v}_h \|_{0,G_1}^2) + h^2 \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}^2 + \| q_h \|_{-k-1,G_1}^2 + \| v_h \|_{-k-1,G_1}^2]. \tag{2.13}$$

It follows from (2.11), (2.12), and (2.13) that

$$\begin{aligned} \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G'}^2 \leqslant\ & Q[h(\| q_h \|_{0,G_1}^2 + \| \underset{\sim}{v}_h \|_{0,G_1}^2 + \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}^2) \\ & + (\| q_h \|_{-k-1,G_1} + \| \underset{\sim}{v}_h \|_{-k-1,G_1}) \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}] \\ \leqslant\ & Q[h^{2T}(\| q_h \|_{0,G_1}^2 + \| \underset{\sim}{v}_h \|_{0,G_1}^2) + h \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}^2 \\ & + h(\| q_h \|_{-k-1,G_1}^2 + \| \underset{\sim}{v}_h \|_{-k-1,G_1}^2) + (\| q_h \|_{-k-1,G_1} \\ & + \| \underset{\sim}{v}_h \|_{-k-1,G_1}) \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}]. \end{aligned} \tag{2.14}$$

If we now substitute repeatedly (2.14) into (2.13) we see that

$$\| \underset{\sim}{v}_h \|_{0,G'}^2 \leqslant Q[h^{2T}(\| q_h \|_{0,G_1}^2 + \| \underset{\sim}{v}_h \|_{0,G_1}^2 + \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}^2) + \| q_h \|_{-k-1,G_1}^2 + \| \underset{\sim}{v}_h \|_{-k-1,G_1}^2]. \tag{2.15}$$

Also, (2.12) implies, using (2.14) repeatedly and (2.15), that

$$\| q_h \|_{0,G'}^2 \leqslant Q[h^{2T}(\| q_h \|_{0,G_1}^2 + \| \underset{\sim}{v}_h \|_{0,G_1}^2 + \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}^2) + \| q_h \|_{-k-1,G_1}^2 + \| \underset{\sim}{v}_h \|_{-k-1,G_1}^2], \tag{2.16}$$

for any fixed positive integer T.

Finally, (2.14) implies, using (2.15) and (2.16), that

$$\| \operatorname{div} \underset{\sim}{v}_h \|^2_{0,G'} \leqslant Q[h^{2T}(\| q_h \|^2_{0,G_1} + \| \underset{\sim}{v}_h \|^2_{0,G_1}) + h^{\varepsilon} \| \operatorname{div} \underset{\sim}{v}_h \|^2_{0,G_1} + h^{-\varepsilon}(\| q_h \|^2_{-k-1,G_1} + \| \underset{\sim}{v}_h \|^2_{-k-1,G_1})]$$

for any fixed $\varepsilon > 0$, and this in turn implies, by repeated substitution into itself, the bound

$$\| \operatorname{div} \underset{\sim}{v}_h \|^2_{0,G'} \leqslant Q[h^{2T}(\| q_h \|^2_{0,G_1} + \| \underset{\sim}{v}_h \|^2_{0,G_1} + \| \operatorname{div} \underset{\sim}{v}_h \|^2_{0,G_1}) + h^{-\varepsilon}(\| q_h \|^2_{-k-1,G_1} + \| \underset{\sim}{v}_h \|^2_{-k-1,G_1})],$$

which, with (2.15) and (2.16), proves the lemma.

We are now ready to derive the local L^2-error estimates.

THEOREM 2.1 : *If* $p \in H^2(\Omega) \cap H^r(\Omega_1)$ *and* h *is sufficiently small then*

(i) $\| \xi \|_{0,\Omega_0} \leqslant Q[\| p \|_{r,\Omega_1} h^{r-\delta_{0k}} + \| p \|_{2,\Omega} h^{k+1}], \quad 2 \leqslant r \leqslant k+1+\delta_{0k}$

(ii) $\| \underset{\sim}{\zeta} \|_{0,\Omega_0} \leqslant Q[\| p \|_{r,\Omega_1} h^{r-1} + \| p \|_{2,\Omega} h^{k+1}], \quad 2 \leqslant r \leqslant k+2,$

(iii) $\| \operatorname{div} \underset{\sim}{\zeta} \|_{0,\Omega_0} \leqslant Q[\| p \|_{r,\Omega_1} h^{r-2} + \| p \|_{2,\Omega} h^{k+1-\varepsilon}], \quad 2 \leqslant r \leqslant k+3,$

for any fixed $\varepsilon > 0$ *and for* $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$.

Proof : Let $(\overline{\underset{\sim}{u}}, \overline{p}) = (\omega \underset{\sim}{u}, \omega p)$ where $\omega \in C_0^\infty(G_2)$ and $\omega = 1$ on G_1. There is no reason to assume that $\{ E \cap G_1 \}_{E \in \mathcal{T}_h}$ is a quasi-regular polygonalization of G_1, but, if $h < \operatorname{dist}(G_1, G_2)$, we can take $\tilde{G}_1$, $G_1 \subset \tilde{G}_1 \subset G_2$, such that $\tilde{G}_1$ is a union of polygons in $\mathcal{T}_h$. Therefore, it follows from [3] that there exists a unique $(\overline{\underset{\sim}{v}}_h, \overline{q}_h) \in \underset{\sim}{V}_h(G_2) \times W_h(G_2)$ such that

$$\left.\begin{aligned} (\alpha[\overline{\underset{\sim}{u}} - \overline{\underset{\sim}{v}}_h], \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, \overline{p} - \overline{q}_h) + (\underset{\sim}{\beta}[\overline{p} - \overline{q}_h], \underset{\sim}{v}) &= 0, \quad \underset{\sim}{v} \in \underset{\sim}{V}_h(G_2), \\ (\operatorname{div} [\overline{\underset{\sim}{u}} - \overline{\underset{\sim}{v}}_h], w] + (c[\overline{p} - \overline{q}_h], w) &= 0, \quad w \in W_h(G_2), \end{aligned}\right\} \quad (2.17)$$

and, furthermore, if $p \in H^r(G_2)$, we have the bounds

$$\left.\begin{aligned} &\| \overline{p} - \overline{q}_h \|_{-s,G'} \leqslant Q \| \overline{p} \|_{r,G_1} h^{r+s-\delta_{sk}} \leqslant Q \| p \|_{r,G_2} h^{r+s-\delta_{sk}}, \\ &\qquad\qquad 2 \leqslant r \leqslant k+1+\delta_{sk'} \\ &\| \overline{\underset{\sim}{u}} - \overline{\underset{\sim}{v}}_h \|_{-s,G'} \leqslant Q \| p \|_{r,G_2} h^{r+s-1}, \quad 2 \leqslant r \leqslant k+2, \\ &\| \operatorname{div} (\overline{\underset{\sim}{u}} - \overline{\underset{\sim}{v}}_h) \|_{-s,G'} \leqslant Q \| p \|_{r,G_2} h^{r+s-2}, \quad 2 \leqslant r \leqslant k+3, \end{aligned}\right\} \quad (2.18)$$

for $0 \leqslant s \leqslant k$.

Now, if we restrict $(\underset{\sim}{v}, w)$ to the space $\underset{\sim}{V}_h(G_1) \times W_h(G_1)$ in (2.17), then $(\underset{\sim}{\mathrm{u}}, \overline{p})$

can be replaced by $(\underset{\sim}{u}, p)$; then, using (0.12) with $(\underset{\sim}{v}, w) \in \underset{\sim}{V}_h(G_1) \times W_h(G_1)$ we can replace $(\underset{\sim}{u}, p)$ by $(\underset{\sim}{u}_h, p_h)$ so that

$$\left.\begin{aligned}(\alpha[\underset{\sim}{u}_h - \bar{\underset{\sim}{v}}_h], \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, p_h - \bar{q}_h) + (\underset{\sim}{\beta}[p_h - \bar{q}_h], \underset{\sim}{v}) = 0, \quad & \underset{\sim}{v} \in \underset{\sim}{V}_h(G_1), \\ (\operatorname{div}[\underset{\sim}{u}_h - \bar{\underset{\sim}{v}}_h], w) + (c[p_h - \bar{q}_h], w) = 0, \quad & w \in W_h(G_1).\end{aligned}\right\}$$

Since $(\underset{\sim}{u}_h - \bar{\underset{\sim}{v}}_h, p_h - \bar{q}_h)$ is a local solution of the mixed method equations, Lemma 2.1 implies (taking $\underset{\sim}{v}_h = \underset{\sim}{u}_h - \bar{\underset{\sim}{v}}_h$, $q_h = p_h - \bar{q}_h$ in (2.4) and using (2.18)) that

$$\begin{aligned}(\| p_h - \bar{q}_h \|_{0,G'} + \| \underset{\sim}{u}_h - \bar{\underset{\sim}{v}}_h \|_{0,G'}) h^{-\varepsilon} + \| \operatorname{div}(\underset{\sim}{u}_h - \bar{\underset{\sim}{v}}_h) \|_{0,G'} \\ \leqslant Q[h^m(\| p_h - \bar{q}_h \|_{0,G_1} + \| \underset{\sim}{u}_h - \bar{\underset{\sim}{v}}_h \|_{0,G_1} + \| \operatorname{div}(\underset{\sim}{u}_h - \bar{\underset{\sim}{v}}_h) \|_{0,G_1}) \\ + h^{-\varepsilon}(\| p_h - \bar{q}_h \|_{-k-1,G_1} + \| \underset{\sim}{u}_h - \bar{\underset{\sim}{v}}_h \|_{-k-1,G_1})] \qquad (2.19) \\ \leqslant Q[h^m(\| \xi \|_{0,G_1} + \| \underset{\sim}{\zeta} \|_{0,G_1} + \| \operatorname{div} \underset{\sim}{\zeta} \|_{0,G_1}) \\ + h^{k+r-1-\varepsilon} \| p \|_{r,G_1} + h^{-\varepsilon}(\| \xi \|_{-k-1,G_1} + \| \underset{\sim}{\zeta} \|_{-k-1,G_1})],\end{aligned}$$

for $2 \leqslant r \leqslant k + 3$.

If we now take $s = 0$ in (2.18), $r = 2$, and $\varepsilon + m = k + 1$ in (2.19), we see that

$$\begin{aligned}\| \xi \|_{0,G'} \leqslant \| \bar{p} - \bar{q}_h \|_{0,G'} + \| p_h - \bar{q}_h \|_{0,G'} \\ \leqslant Q[\| p \|_{r,G}(h^{r-\delta_{0k}} + h^{k+1}) \\ + h^{k+1}(\| \xi \|_{0,G} + \| \underset{\sim}{\zeta} \|_{0,G} + \| \operatorname{div} \underset{\sim}{\zeta} \|_{0,G}) \\ + \| \xi \|_{-k-1,G} + \| \underset{\sim}{\zeta} \|_{-k-1,G}], \quad 2 \leqslant r \leqslant k + 1 + \delta_{0k}, \qquad (2.20)\end{aligned}$$

$$\begin{aligned}\| \underset{\sim}{\zeta} \|_{0,G'} \leqslant Q[\| p \|_{r,G}(h^{r-1} + h^{k+1}) + h^{k+1}(\| \xi \|_{0,G} + \| \underset{\sim}{\zeta} \|_{0,G} + \| \operatorname{div} \underset{\sim}{\zeta} \|_{0,G}) + \\ + \| \xi \|_{-k-1,G} + \| \underset{\sim}{\zeta} \|_{-k-1,G}], \quad 2 \leqslant r \leqslant k+2, \qquad (2.21)\end{aligned}$$

$$\begin{aligned}\| \operatorname{div} \underset{\sim}{\zeta} \|_{0,G'} \leqslant Q[\| p \|_{r,G}(h^{r-2} + h^{k+1-\varepsilon}) + h^{k+1}(\| \xi \|_{0,G} + \| \underset{\sim}{\zeta} \|_{0,G} + \| \operatorname{div} \underset{\sim}{\zeta} \|_{0,G}) \\ + h^{-\varepsilon}(\| \xi \|_{-k-1,G} + \| \underset{\sim}{\zeta} \|_{-k-1,G})], \quad 2 \leqslant r \leqslant k+3. \qquad (2.22)\end{aligned}$$

If we now use a finite cover of Ω_1 by a collection of G's such that the corresponding (G')'s will cover Ω_0, we obtain again the bounds (2.20)-(2.22) with G' replaced by Ω_0 and G by Ω_1.

Finally, [3] implies that for $p \in H^2(\Omega)$,

$$\left.\begin{aligned}\| \xi \|_{0,\Omega_1} + \| \underset{\sim}{\zeta} \|_{0,\Omega_1} + \| \operatorname{div} \underset{\sim}{\zeta} \|_{0,\Omega_1} \leqslant Q \| p \|_{2,\Omega} \\ \| \xi \|_{-k-1,\Omega_1} + \| \underset{\sim}{\zeta} \|_{-k-1,\Omega_1} \leqslant Q h^{k+1} \| p \|_{2,\Omega},\end{aligned}\right\}$$

which completes the proof of the theorem.

We have also obtained (by combining Lemma 1.2, (2.2), and Theorem 2.1) the following local negative norm error estimates.

THEOREM 2.2 : *If $p \in H^j(\Omega) \cap H^r(\Omega_1)$, $2 \leqslant j \leqslant k+3$, and h is sufficiently small, then, for $0 \leqslant s \leqslant k+1$,*

(i) $$\| \xi \|_{-s,\Omega_0} \leqslant K[\| p \|_{r,\Omega_1} h^{r+s-(s-k+1)^+} + \| p \|_{j,\Omega} h^{j+k-1}], \quad 2 \leqslant r \leqslant k+1+(s-k+1)^+,$$

(ii) $$\| \underset{\sim}{\zeta} \|_{-s,\Omega_0} \leqslant K[\| p \|_{r,\Omega_1} h^{r+s-1-(s-k)^+} + \| p \|_{j,\Omega} h^{j+k-1}], \quad 2 \leqslant r \leqslant k+2+(s-k)^+,$$

(iii) $$\| \operatorname{div} \underset{\sim}{\zeta} \|_{-s,\Omega_0} \leqslant K[\| p \|_{r,\Omega_1} h^{r+s-2} + \| p \|_{j,\Omega} h^{k-1+\min\{j,s+2-\varepsilon\}}], \quad 2 \leqslant r \leqslant k+3.$$

Theorem 2.2 for $s = 0$ restates the results of Theorem 2.1, and (i) is, of course, what was expected from (2.3).

3. THE LOCAL ESTIMATES FOR THE NONLINEAR CASE

Let $\omega_i \in C_0^\infty(G_{i+2})$, $\omega_i = 1$ on G_{i+1}, just as before.
Set

$$(\underset{\sim}{\bar{u}}_i, \bar{p}_i) = (\omega_i \underset{\sim}{u}, \omega_i p),$$

$$\underset{\sim}{\bar{\Gamma}} = \int_0^1 [\alpha_p(p_h + t\xi)\, dt]\, \underset{\sim}{u}_h + \int_0^1 \underset{\sim}{\beta}_p(p_h + t\xi)\, dt \in L^\infty(\bar{\Omega})^2, \tag{3.1}$$

$$\bar{\gamma} = \int_0^1 c_p(p_h + t)\, dt \in L^\infty(\bar{\Omega}).$$

We can now rewrite the error equations (0.12) in the form (see [5])

$$\left.\begin{aligned} (\alpha(p)\, \underset{\sim}{\zeta}, \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, \xi) + (\underset{\sim}{\bar{\Gamma}}\xi, \underset{\sim}{v}) &= 0, \quad \underset{\sim}{v} \in \underset{\sim}{V}_h, \\ (\operatorname{div} \underset{\sim}{\zeta}, w) + (\bar{\gamma}\xi, w) &= 0, \quad w \in W_h, \end{aligned}\right\} \tag{3.2}$$

which are the mixed method equations corresponding to the operator $\bar{M} : H^2(\Omega) \to L^2(\Omega)$ given by

$$\bar{M}w = -\underset{\sim}{\nabla}.(a(p)\, \underset{\sim}{\nabla} w + a(p)\underset{\sim}{\bar{\Gamma}} w) + \bar{\gamma} w.$$

It was shown in [5] that $\bar{M}^*$ has a bounded inverse $L^2(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega)$. However, it does not follow that $(\bar{M}^*)^{-1}$ maps $H^s(\Omega)$ into $H^{s+2}(\Omega) \cap H_0^1(\Omega)$ boundedly if $s > 0$. Therefore, the duality argument developed in Section 2 for the linear problem (more specifically, the use of Lemma 1.1 to prove Lemma

2.1, and the bounds (2.18)) does not apply to the nonlinear problem. We *can* obtain a result analogous to Lemma 2.1 for the equations (3.2) with (ζ, ξ) replaced by a local solution $(\underset{\sim}{v}_h, q_h)$ as in (2.4), which we shall do for the sake of completeness.

LEMMA 3.1 : *Let $(\underset{\sim}{v}_h, q_h) \in \underset{\sim}{V}_h \times W_h$ be a local solution of the homogeneous mixed method equations for the operator $\overline{M}$ on G; that is,*

$$\left.\begin{aligned}(\alpha(p)\, \underset{\sim}{v}_h, \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, q_h) + (\overline{\underset{\sim}{\Gamma}} q_h, \underset{\sim}{v}) &= 0\,, \quad \underset{\sim}{v} \in \underset{\sim}{V}_h(G)\,, \\ (\operatorname{div} \underset{\sim}{v}_h, w) + (\overline{\gamma} q_h, w) &= 0\,, \quad w \in W_h(G)\,.\end{aligned}\right\} \tag{3.3}$$

Let $\varepsilon > 0$ and $m \geqslant 0$. Then, if h is sufficiently small,

$$\| q_h \|_{0,G} + \| \underset{\sim}{v}_h \|_{0,G} + \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G} \leqslant K[h^{-\varepsilon}(\| q_h \|_{-k-1,G_1} + \| \underset{\sim}{v}_h \|_{-k-1,G_1}) + \\ + h^m(\| q_h \|_{0,G_1} + \| \underset{\sim}{v}_h \|_{0,G_1} + \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1})]\,.$$

Proof: The existence of a bounded inverse $(\overline{M}^*)^{-1} : L^2(G) \to H^2(G) \cap H_0^1(G)$ implies that Lemma 1.1 is valid on (3.3) with $s = 0$. Thus,

$$\begin{aligned}\| q_h \|_{0,G} \leqslant K[h \| \underset{\sim}{v}_h \|_{0,G_1} &+ h^{2-\delta_{0k}} \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1} + h \| q_h \|_{0,G_1} \\ &+ \| \underset{\sim}{v}_h \|_{-2,G_1} + \| q_h \|_{-1,G_1}]\,.\end{aligned} \tag{3.4}$$

Using again interpolation between Sobolev spaces (see [4]), we obtain from (3.4) the bound

$$\begin{aligned}\| q_h \|_{0,G} \leqslant K[h^{\varepsilon/2k}(\| q_h \|_{0,G_1} &+ \| \underset{\sim}{v}_h \|_{0,G_1}) + h \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1} + \\ &+ h^{-\varepsilon/2}(\| q_h \|_{-k-1,G_1} + \| \underset{\sim}{v}_h \|_{-k-1,G_1})]\,, \quad k > 0\,.\end{aligned} \tag{3.5}$$

Next, $\overline{\gamma} \in L^\infty(G)$ implies that (2.7) holds again, so that

$$\begin{aligned}\| \operatorname{div} \underset{\sim}{v}_h \|_{0,G}^2 &\leqslant Q \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}(h \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1} + \| q_h \|_{0,G_1}) \\ &\leqslant Q[h^{\varepsilon/2k} \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}^2 + h^{-\varepsilon/2k} \| q_h \|_{0,G_1}^2]\,.\end{aligned} \tag{3.6}$$

Also, $\overline{\underset{\sim}{\Gamma}} \in L^\infty(G)^2$ implies that (2.9) still holds, so that

$$\begin{aligned}\| \underset{\sim}{v}_h \|_{0,G}^2 &\leqslant Q[h(\| q_h \|_{0,G_1}^2 + \| \underset{\sim}{v}_h \|_{0,G_1}^2 + \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}^2) \\ &\quad + \| q_h \|_{0,G_1}(\| q_h \|_{0,G_1} + \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1})] \\ &\leqslant K[h^{\varepsilon/2k} \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}^2 + h \| \underset{\sim}{v}_h \|_{0,G_1}^2 + h^{-\varepsilon/2k} \| q_h \|_{0,G_1}^2]\,.\end{aligned} \tag{3.7}$$

Also, we can obtain from (3.5) the estimate

$$\| q_h \|_{0,G}^2 \leqslant K[h^{\varepsilon/k}(\| q_h \|_{0,G_1}^2 + \| \underset{\sim}{v}_h \|_{0,G_1}^2) + h \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}^2 + h^{-\varepsilon}(\| q_h \|_{-k-1,G_1}^2 + \| \underset{\sim}{v}_h \|_{-k-1,G_1}^2)],$$

which when recursively substituted into itself will give (after relabeling the G_i's) the bound

$$\| q_h \|_{0,G}^2 \leqslant K[h^{t} \| q_h \|_{0,G_1}^2 + h^{\varepsilon/k} \| \underset{\sim}{v}_h \|_{0,G_1}^2 + h \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}^2 + h^{-\varepsilon}(\| q_h \|_{-k-1,G_1}^2 + \| \underset{\sim}{v}_h \|_{-k-1,G_1}^2)], \tag{3.8}$$

where t is any fixed positive number.

Substituting (3.8) into (3.7), we see that

$$\| \underset{\sim}{v}_h \|_{0,G'}^2 \leqslant K[h^{2T} \| q_h \|_{0,G_1}^2 + h^{\varepsilon/2k} \| \underset{\sim}{v}_h \|_{0,G_1}^2 + h^{\varepsilon/2k} \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}^2 + h^{-3\varepsilon/2}(\| q_h \|_{-k-1,G_1}^2 + \| v_h \|_{-k-1,G_1}^2)],$$

for any fixed $T > 0$, and recursive substitution of this relation into itself (and relabeling of the G_i's) will show that

$$\| \underset{\sim}{v}_h \|_{0,G'}^2 \leqslant K[h^{2T}(\| q_h \|_{0,G_1}^2 + \| \underset{\sim}{v}_h \|_{0,G_1}^2) + h^{\varepsilon/2k} \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}^2 + h^{-3\varepsilon/2}(\| q_h \|_{-k-1,G_1}^2 + \| \underset{\sim}{v}_h \|_{-k-1,G_1}^2)]. \tag{3.9}$$

We now substitute (3.9) into (3.8) to obtain the estimate

$$\| q_h \|_{0,G'}^2 \leqslant K[h^{2T}(\| q_h \|_{0,G_2}^2 + \| \underset{\sim}{v}_h \|_{0,G_2}^2) + h^{3\varepsilon/2k} \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_2}^2 + h^{-3\varepsilon/2}(\| q_h \|_{-k-1,G_2}^2 + \| \underset{\sim}{v}_h \|_{-k-1,G_2}^2)]. \tag{3.10}$$

If we now substitute (3.10) into (3.6), and the resulting relation repeatedly into itself, we see (after relabeling the G_i's) that, for any fixed positive number T,

$$\| \operatorname{div} \underset{\sim}{v}_h \|_{0,G'}^2 \leqslant K[h^{2T}(\| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}^2 + \| \underset{\sim}{v}_h \|_{0,G_1}^2 + \| q_h \|_{0,G_1}^2) + h^{-2\varepsilon}(\| q_h \|_{-k-1,G_1}^2 + \| \underset{\sim}{v}_h \|_{-k-1,G_1}^2)]. \tag{3.11}$$

Substitution of (3.11) into (3.9) and (3.10) finishes the proof of the lemma when $k > 0$. The case $k = 0$ is covered by the proof of Lemma 2.1 starting from (3.4), (2.7), and (2.9) without making use of Lemma 1.2, which is obviously unnecessary anyway if $k = 0$.

This result, though, will not be sufficient to allow us to reproduce the proof of Theorem 2.1 for the nonlinear problem, since, although the boundedness of

$(\overline{M}^*)^{-1} : L^2(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega)$ still guarantees the existence and uniqueness of the solution $(\overline{v}_h, \overline{q}_h)$ of the analogue of (2.17) and also the error bounds (2.18) for $s = 0$ (only), we do not have any estimates to replace (2.18) for $s = k$.

Let now $\omega \in C_0^\infty(G_2)$, $\omega = 1$ on G_1, and set $(\overline{u}, \overline{p}) = (\omega u, \omega p)$. Let

$$t(j, k) = 2j - 2 - \varepsilon - 2(j - k - 2)^+$$

for $\varepsilon > 0$ and $2 + \varepsilon \leqslant j \leqslant k + 3$.

THEOREM 3.1 : *Let $\Omega_0 \subset \subset \Omega_1 \subset \subset \Omega$ be open subdomains of Ω. If $p \in H^j(\Omega) \cap H^r(\Omega_1)$ and $k \geqslant 1$, then*

(i) $$\| \xi \|_{0,\Omega_0} \leqslant K[h^r \| p \|_{r,\Omega_1} + h^{t(j,k)} \| p \|_{j,\Omega} (\| p \|_{j,\Omega} + 1)], \quad 2 \leqslant r \leqslant k+1,$$

(ii) $$\| \zeta \|_{0,\Omega_0} \leqslant K[h^{r-1} \| p \|_{r,\Omega_1} + h^{t(j,k)-\varepsilon} \| p \|_{j,\Omega} (\| p \|_{j,\Omega} + 1)], \quad 2 \leqslant r \leqslant k + 2 - \varepsilon,$$

(iii) $$\| \operatorname{div} \zeta \|_{0,\Omega_0} \leqslant K[h^{r-2} \| p \|_{r,\Omega_1} + h^{t(j,k)-\varepsilon} \| p \|_{j,\Omega} (\| p \|_{j,\Omega} + 1)], \quad 2 \leqslant r \leqslant k + 3 - \varepsilon.$$

Proof : Throughout this proof, the subindeces will be shifted so that the smaller ball on any norm bound will be called G, and the larger one G_1.

Let $(\overline{v}_h, \overline{q}_h) \in V_h(G_1) \times W_h(G_1)$ satisfy

$$\left.\begin{aligned} (\alpha(p)[\overline{u} - \overline{v}_h], v) - (\operatorname{div} v, \overline{p} - \overline{q}_h) + (\overline{\Gamma}[\overline{p} - \overline{q}_h], v) = 0, &\quad v \in V_h(G_1), \\ (\operatorname{div}[\overline{u} - \overline{v}_h], w) + (\overline{\gamma}[\overline{p} - \overline{q}_h], w) = 0, &\quad w \in W_h(G_1). \end{aligned}\right\} \quad (3.12)$$

As it was pointed out earlier, $(\overline{v}_h, \overline{q}_h)$ exists and is unique. Furthermore, if $p \in H^r(G_1)$,

$$\left.\begin{aligned} \| \overline{p} - \overline{q}_h \|_{0,G} &\leqslant Q \| p \|_{r,G_1} h^r, & 2 \leqslant r \leqslant k + 1, \\ \| \overline{u} - \overline{v}_h \|_{0,G} &\leqslant Q \| p \|_{r,G_1} h^{r-1}, & 2 \leqslant r \leqslant k + 2, \\ \| \operatorname{div}(\overline{u} - \overline{v}_h) \|_{0,G} &\leqslant Q \| p \|_{r,G_1} h^{r-2}, & 2 \leqslant r \leqslant k + 3, \end{aligned}\right\} \quad (3.13)$$

just as in (2.18) for $s = 0$, where the constant Q depends on $\| p \|_{2+\varepsilon}$.

It follows from [5] that, if $k \geqslant s$, then

$$\| \xi \|_{-k-1} + \| \zeta \|_{-k-1} + \| \operatorname{div} \zeta \|_{-k-1} \leqslant C h^{t(j,k)} \| p \|_{j,\Omega} (\| p \|_{j,\Omega} + 1), \quad (3.14)$$

and also that

$$\| (\underset{\sim}{\kappa}\xi + \lambda\underset{\sim}{\zeta}) \xi \|_0 \leqslant K(\| \underset{\sim}{\zeta} \|_0 + \| \xi \|_0) \| \xi \|_{0,\infty} \leqslant Ch^{t(j,k)} \| p \|_{j,\Omega} (\| p \|_{j,\Omega} + 1) . \tag{3.15}$$

It now follows from (2.1), (3.14), and (3.15) that

$$\| \xi \|_{0,G} \leqslant K[h(\| \xi \|_{0,G_1} + \| \underset{\sim}{\zeta} \|_{0,G_1}) + h^2 \| \operatorname{div} \underset{\sim}{\zeta} \|_{0,G_1} + h^r \| p \|_{r,G_1} + h^{t(j,k)} \| p \|_{j,\Omega} (\| p \|_{j,\Omega} + 1)] , \quad 2 \leqslant r \leqslant k + 1 . \tag{3.16}$$

As in the proof of Theorem 2.1, we can replace $(\bar{\underset{\sim}{u}}, \bar{p})$ in (3.12) by $(\underset{\sim}{u}_h, p_h)$ using (3.2). What we have to do next is to take $(\underset{\sim}{v}_h, q_h) = (\underset{\sim}{u}_h - \bar{\underset{\sim}{v}}_h, p_h - \bar{q}_h)$ in (3.3) and to change Lemma 3.1 slightly. It follows from (3.13) and (3.16) that

$$\| q_h \|_{0,G} \leqslant K[h(\| q_h \|_{0,G_1} + \| \underset{\sim}{v}_h \|_{0,G_1}) + h^2 \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1} + h^r \| p \|_{r,G_1} + h^{t(j,k)} \| p \|_{j,\Omega} (\| p \|_{j,\Omega} + 1)] , \quad 2 \leqslant r \leqslant k + 1 ,$$

so that, by nesting and relabeling the G_i's, we can see that

$$\| q_h \|_{0,G}^2 \leqslant K[h^{2T} \| q_h \|_{0,G_1}^2 + h^2 \| \underset{\sim}{v}_h \|_{0,G_1}^2 + h^4 \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}^2 + h^{2r} \| p \|_{r,G_1}^2 + h^{2t(j,k)}(\| p \|_{j,\Omega}^2 + \| p \|_{j,\Omega}^4)] , \quad 2 \leqslant r \leqslant k + 1 , \tag{3.17}$$

where T is any fixed positive number. Next, (3.6) and (3.17) imply that

$$\| \operatorname{div} \underset{\sim}{v}_h \|_{0,G}^2 \leqslant K[h^{2T} \| q_h \|_{0,G_1}^2 + h^{2-2\varepsilon} \| \underset{\sim}{v}_h \|_{0,G_1}^2 + h^{2\varepsilon} \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}^2 + h^{2r-2\varepsilon} \| p \|_{r,G_1}^2 + h^{2t(j,k)-2\varepsilon}(\| p \|_{j,\Omega}^2 + \| p \|_{j,\Omega}^4)] , \quad 2 \leqslant r \leqslant k + 1 ,$$

which repeatedly substituted into itself will give (after renaming the G_i's) the bound

$$\| \operatorname{div} \underset{\sim}{v}_h \|_{0,G}^2 \leqslant K[h^{2T}(\| q_h \|_{0,G_1}^2 + \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}^2) + h^{2-2\varepsilon} \| \underset{\sim}{v}_h \|_{0,G_1}^2 + h^{2r-2\varepsilon} \| p \|_{r,G_1} + h^{2t(j,k)-2\varepsilon}(\| p \|_{j,\Omega}^2 + \| p \|_{j,\Omega}^4)] , \quad 2 \leqslant r \leqslant k + 1 . \tag{3.18}$$

We see from (3.17) and (3.18) that

$$\| q_h \|_{0,G}^2 \leqslant K[h^{2T}(\| q_h \|_{0,G_1}^2 + \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}^2) + h^2 \| \underset{\sim}{v}_h \|_{0,G_1}^2 + h^{2r} \| p \|_{r,G_1} + h^{2t(j,k)}(\| p \|_{j,\Omega}^2 + \| p \|_{j,\Omega}^4)] , \quad 2 \leqslant r \leqslant k + 1 . \tag{3.19}$$

Also, (3.7), (3.18), and (3.19) imply that

$$\| \underset{\sim}{v}_h \|_{0,G}^2 \leqslant K[h^{2T}(\| q_h \|_{0,G_1}^2 + \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}^2) + h \| \underset{\sim}{v}_h \|_{0,G_1}^2 + h^{2r-2\varepsilon} \| p \|_{r,G_1}^2 + h^{2t(j,k)-2\varepsilon}(\| p \|_{j,\Omega}^2 + \| p \|_{j,\Omega}^4)], \quad 2 \leqslant r \leqslant k+1,$$

and substituting this bound recursively into itself leads to the bound

$$\| \underset{\sim}{v}_h \|_{0,G}^2 \leqslant K[h^{2T}(\| q_h \|_{0,G_1}^2 + \| \underset{\sim}{v}_h \|_{0,G_1}^2 + \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}^2) + h^{2r-2\varepsilon} \| p \|_{r,G_1}^2 + h^{2t(j,k)-2\varepsilon}(\| p \|_{j,\Omega}^2 + \| p \|_{j,\Omega}^4)], \quad 2 \leqslant r \leqslant k+1. \tag{3.20}$$

Substituting (3.20) into (3.18) and (3.19) finally gives

$$h^{-\varepsilon} \| q_h \|_{0,G} + \| \underset{\sim}{v}_h \|_{0,G} + \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G} \leqslant K[h^T(\| q_h \|_{0,G_1} + \| \underset{\sim}{v}_h \|_{0,G_1} + \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G_1}) + h^{r-\varepsilon} \| p \|_{r,G_1} + h^{t(j,k)-\varepsilon} \| p \|_{j,\Omega} (\| p \|_{j,\Omega} + 1)] \tag{3.21}$$

for $2 \leqslant r \leqslant k+1$ and any fixed $T > 0$.

It follows from [5] that

$$\| \xi \|_0 + \| \underset{\sim}{\zeta} \|_0 + \| \operatorname{div} \underset{\sim}{\zeta} \|_0 \leqslant K \| p \|_{2+\varepsilon,\Omega} \,. \tag{3.22}$$

Now, (3.13), (3.21), and (3.22) imply that

$$\begin{aligned} \| q_h \|_{0,G} &\leqslant K[h^T(\| \bar{p} - \bar{q}_h \|_{0,G_1} + \| \xi \|_{0,G_1} + \| \bar{\underset{\sim}{u}} - \bar{\underset{\sim}{v}}_h \|_{0,G_1} + \| \underset{\sim}{\zeta} \|_{0,G_1} \\ &\quad + \| \operatorname{div}(\bar{\underset{\sim}{u}} - \bar{\underset{\sim}{v}}_h) \|_{0,G_1} + \| \operatorname{div} \underset{\sim}{\zeta} \|_{0,G_1}) + h^r \| p \|_{r,G_1} \\ &\quad + h^{t(j,k)} \| p \|_{j,\Omega} (\| p \|_{j,\Omega} + 1)] \\ &\leqslant K[h^r \| p \|_{r,G_1} + h^{t(j,k)} \| p \|_{j,\Omega} (\| p \|_{j,\Omega} + 1)], \quad 2 \leqslant r \leqslant k+1, \end{aligned}$$

and

$$\| \underset{\sim}{v}_h \|_{0,G} + \| \operatorname{div} \underset{\sim}{v}_h \|_{0,G} \leqslant K[h^{r-\varepsilon} \| p \|_{r,G_1} + h^{t(j,k)-\varepsilon} \| p \|_{j,\Omega} (\| p \|_{j,\Omega} + 1)], \quad 2 \leqslant r \leqslant k+1.$$

These estimates together with (3.13) and the finite covering process used in Theorem 2.1 complete the proof of the theorem.

We have also obtained (by combining Lemma 1.2, (3.14), (3.15), and Theorem 3.1) the following local negative norm error estimates.

THEOREM 3.2 : *If* $p \in H^j(\Omega) \cap H^r(\Omega_1)$, $2 + \varepsilon \leqslant j \leqslant k + 3$, $k > 0$, *and* h *is sufficiently small, then, for* $0 \leqslant s \leqslant k + 1$,

(i) $$\| \xi \|_{-s,\Omega_0} \leqslant K[\| p \|_{r,\Omega_1} h^{r+s-(s-k+1)^+} + \| p \|_{j,\Omega} (\| p \|_{j,\Omega} + 1) h^{t(j,k)}], \quad 2 \leqslant r \leqslant k+1+(s-k+1)^+,$$

(ii) $$\| \underset{\sim}{\zeta} \|_{-s,\Omega_0} \leqslant K[\| p \|_{r,\Omega_1} h^{r+s-1-(s-k)^+} + \| p \|_{j,\Omega}(\| p \|_{j,\Omega}+1) h^{t(j,k)-(\varepsilon-s)^+}], \quad 2 \leqslant r \leqslant k+2-\varepsilon+(s-k)^+,$$

(iii) $$\| \operatorname{div} \underset{\sim}{\zeta} \|_{-s,\Omega_0} \leqslant K[\| p \|_{r,\Omega_1} h^{r+s-2} + \| p \|_{j,\Omega}(\| p \|_{j,\Omega}+1) h^{t(j,k)-(\varepsilon-s)^+}], \quad 2 \leqslant r \leqslant k+3-\varepsilon.$$

4. CONVERGENCE OF DIFFERENCE QUOTIENTS

Assume a *uniform* polygonalization in the interior of Ω for the remainder of this paper.

Let $\mu = (\mu_1, \mu_2) \in Z^2$, and let

$$T_h^{\mu} v(x) = v(x + \mu h).$$

Set

$$\begin{aligned} \partial_{h,j} v(x) &= h^{-1}(T_h^{e_j} - I)\, v(x), \\ &= \frac{v(x + he_j) - v(x)}{h}, \quad j = 1, 2; \\ \partial^{\alpha} u &= \partial_h^{\alpha} u = \partial_{h,1}^{\alpha_1}\, \partial_{h,2}^{\alpha_1} u. \end{aligned}$$

DÉFINITION 4.1 : *An operator Q_h of the form*

$$Q_h\, w = \sum_{\mu,\nu} c_{\mu,\nu}\, T^{\mu}\, \partial^{\nu} w,$$

where only finitely many $c_{\mu,\nu}$'s are nontrivial, is called a difference operator. *Observe that for some $M \geqslant 0$,*

$$Q_h\, w = \sum_{|\delta| \leqslant M} \tilde{c}_{\delta}\, T_h^{\delta}\, w.$$

DÉFINITION 4.2 : *We say that the difference operator Q_h* is of order *m if*

$$\inf_{\delta} \{ l_{\delta} / \tilde{c}_{\delta} = 0(h^{l_{\delta}}) \} = -m.$$

We shall make the assumption throughout the rest of the paper that, for any fixed μ and $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$, there exists h_0 such that, for $h \in (0, h_0]$,

$$(T_h^{\mu}\, \underset{\sim}{v}, T_h^{\mu}\, w) \in \underset{\sim}{V}_h(\Omega_1) \times W_h(\Omega_1) \quad \text{if} \quad (\underset{\sim}{v}, w) \in \underset{\sim}{V}_h(\Omega_0) \times W_h(\Omega_0),$$

where $T_h^{\mu}\, \underset{\sim}{v} = T_h^{\mu}(v_1, v_2) = (T_h^{\mu}\, v_1, T_h^{\mu}\, v_2)$.

We shall also make use of the discrete Leibnitz formula :

$$\partial_h^{\phi}(uv) = \sum_{\psi \leqslant \phi} \binom{\phi}{\psi} T_h^{\psi}\, \partial_h^{\phi}\, u\, \partial_h^{\psi} v, \tag{4.1}$$

where $\begin{pmatrix}\phi\\\psi\end{pmatrix} = \begin{pmatrix}\phi_1\\\psi_1\end{pmatrix}\begin{pmatrix}\phi_2\\\psi_2\end{pmatrix}$. Note that Q_h commutes with any differential operator with constant coefficients, as well as with P_h and π_h.

We wish to obtain, following the ideas developed in [7], estimates for $Q_h\,\xi$ and Q_h div $\underset{\sim}{\zeta}$ in $H^s(\Omega_0)$, and for $Q_h\,\underset{\sim}{\zeta}$ in $H^s(\Omega_0)^2$, with $s = -k-1$ or $s = 0$. It is clearly sufficient to derive estimates for the case $Q_h = T_h^{\mu}\,\partial^{\nu}$. Moreover, since

$$\| T_h^{\mu}\, v \|_{0,G} \leqslant \| v \|_{0,G_1}$$

for sufficiently small h, it suffices to consider $Q_h = \partial_h^{\nu}$.

We need to find equations satisfied by $\partial^{\nu}\xi$, $\partial^{\nu}\underset{\sim}{\zeta}$ and ∂^{ν} div $\underset{\sim}{\zeta}$. We see that (4.1) and (0.12) imply that, for $|\,\nu\,| > 0$,

$$\begin{aligned}(\alpha(p)\,\partial^{\nu}\underset{\sim}{\zeta}, v) &- (\operatorname{div} \underset{\sim}{v}, \partial^{\nu}\xi) + (\underset{\sim}{\Gamma}\,\partial^{\nu}\xi, \underset{\sim}{v})\\ &= (\alpha(p)\,\underset{\sim}{\zeta}, \partial^{-\nu}\underset{\sim}{v}) - (\partial^{-\nu} \operatorname{div} \underset{\sim}{v}, \xi) + (\underset{\sim}{\Gamma}\xi, \partial^{-\nu}\underset{\sim}{v})\\ &\quad - \sum_{\mu<\nu}\binom{\nu}{\mu}\{([T_h^{\mu}\,\partial^{\nu-\mu}\alpha(p)]\,\partial^{\mu}\underset{\sim}{\zeta}, \underset{\sim}{v}) + ([T_h^{\mu}\,\partial^{\nu-\mu}\underset{\sim}{\Gamma}]\,\partial^{\mu}\xi, v)\}\\ &= (\partial^{\nu}[\underset{\sim}{\kappa}\xi^2 + \lambda\underset{\sim}{\zeta}\xi], \underset{\sim}{v}) - \sum_{\mu<\nu}\binom{\nu}{\mu}([T_h^{\mu}\,\partial^{\nu-\mu}\alpha(p)]\,\partial^{\mu}\underset{\sim}{\zeta}\\ &\quad + [T_h^{\mu}\,\partial^{\nu-\mu}\underset{\sim}{\Gamma}]\,\partial^{\mu}\xi, \underset{\sim}{v}), \quad \underset{\sim}{v} \in \underset{\sim}{V}_h(G)\,,\end{aligned} \tag{4.2}$$

since $\partial^{-\nu}\underset{\sim}{v} \in \underset{\sim}{V}_h(G)$ if $\underset{\sim}{v} \in \underset{\sim}{V}_h(G_1)$. Also,

$$(\operatorname{div} \partial^{\nu}\underset{\sim}{\zeta}, w) + (\gamma\,\partial^{\nu}\xi, w) = (\partial^{\nu}[\rho\xi^2], w) - \sum_{\mu<\nu}\binom{\nu}{\mu}([T_h^{\mu}\,\partial^{\nu-\mu}\gamma]\,\partial^{\mu}\xi, w), \quad w \in W_h(G)\,. \tag{4.3}$$

We are now ready to derive estimates for the linear problem.

THEOREM 4.1 : *If* $p \in H^j(\Omega) \cap H^r(\Omega_1)$, $2 \leqslant j \leqslant k+3$, *the operator* M *of* (1.1) *is linear, and* h *is sufficiently small, then for* $0 \leqslant s \leqslant k+1$,

(i) $$\| Q_h\,\xi \|_{-s,\Omega_0} \leqslant K[\| p \|_{r,\Omega_1}\, h^{r+s-(s-k+1)^+-m} + \| p \|_{j,\Omega}\, h^{j+k-1}]\,,$$
$$m+2 \leqslant r \leqslant m+k+1+(s-k+1)^+\,,$$

(ii) $$\| Q_h\,\underset{\sim}{\zeta} \|_{-s,\Omega_0} \leqslant K[\| p \|_{r,\Omega_1}\, h^{r+s-1-(s-k)^+-m} + \| p \|_{j,\Omega}\, h^{j+k-1}]\,,$$
$$m+2 \leqslant r \leqslant m+k+2+(s-k)^+\,,$$

(iii) $\| Q_h \operatorname{div} \underset{\sim}{\zeta} \|_{-s,\Omega_0} \leqslant K[\| p \|_{r,\Omega_1} h^{r+s-2-m} + \| p \|_{j,\Omega} h^{k-1+\min\{j;s+2-\varepsilon\}}]$,

$$m + 2 \leqslant r \leqslant m + k + 3,$$

where $m \geqslant 0$ is the order of Q_h.

Proof : The proof will proceed by induction. As we pointed out earlier, it suffices to consider $Q_h = \partial^\nu$, with $m = |\nu|$. The case $|\nu| = 0$ is just Theorem 2.2, so that we shall assume that $|\nu| \geqslant 1$ and that (i), (ii), and (iii) hold for ∂^μ, with $|\mu| \leqslant |\nu| - 1$.

Let us recall that in the linear problem $\alpha(p) = \alpha$, $\underset{\sim}{\Gamma} = \underset{\sim}{\beta}$, $\gamma = c$, $\underset{\sim}{\kappa} = \underset{\sim}{0}$, $\lambda = \rho = 0$. Therefore, since $\xi = \tau - \theta$, we can rewrite (4.2) and (4.3) as follows :

$$\left.\begin{aligned} &(\alpha\partial^\nu \underset{\sim}{\zeta}, \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, \partial^\nu \tau) + (\underset{\sim}{\beta}\, \partial^\nu \tau, \underset{\sim}{v}) = (\beta\, \partial^\nu \theta, \underset{\sim}{v}) \\ &\quad - \sum_{\mu<\nu} \binom{\nu}{\mu} ([T_h^\mu\, \partial^{\nu-\mu}\alpha]\, \partial^\mu \underset{\sim}{\zeta} + [T_h^\mu\, \partial^{\nu-\mu}\underset{\sim}{\beta}]\, \partial^\mu \xi, \underset{\sim}{v}), \quad \underset{\sim}{v} \in \underset{\sim}{V}_h(G), \\ &(\operatorname{div} \partial^\nu \underset{\sim}{\zeta}, w) + (c\, \partial^\nu \xi, w) = (c\, \partial^\nu \theta, w) - \sum_{\mu<\nu} \binom{\nu}{\mu} ([T_h^\mu\, \partial^{\nu-\mu} c]\, \partial^\mu \xi, w), \\ &\qquad\qquad w \in W_h(G). \end{aligned}\right\} \quad (4.4)$$

Since equations (4.4) are of the form (1.1), it is clear that the same argument that led from Lemma 1.1 to Theorems 2.1 and 2.2 shows that, for $0 \leqslant s \leqslant k+1$ and $0 \leqslant \bar{r} \leqslant k+1+(s-k+1)^+$,

$$\begin{aligned} \| \partial^\nu \xi \|_{-s,G} &\leqslant K\Big[\| \partial^\nu p \|_{\bar{r},G_1} h^{\bar{r}+s-(s-k+1)^+} + h^{s+1}\Big(\sum_{\mu<\nu} \| \partial^\mu \xi \|_{0,G_1} \\ &\quad + \sum_{\mu<\nu} \| \partial^\mu \underset{\sim}{\zeta} \|_{0,G_1} \Big) + \sum_{\mu\leqslant\nu} (\| \partial^\mu \xi \|_{-k-1-|\mu|} + \| \partial^\mu \underset{\sim}{\zeta} \|_{-k-1-|\mu|}) \Big] \\ &\leqslant K[\| p \|_{r,G_1} h^{r+s-(s-k+1)^+-|\nu|} + \| p \|_{j,\Omega} h^{j+k} \qquad (4.5) \\ &\quad + \| \xi \|_{-k-1} + \| \underset{\sim}{\zeta} \|_{-k-1}] \\ &\leqslant K[\| p \|_{r,G_1} h^{r+s-(s-k+1)^+-|\nu|} + \| p \|_{j,\Omega} h^{j+k-1}], \\ &\qquad |\nu| + 2 \leqslant r \leqslant |\nu| + k + 1 + (s-k+1)^+, \end{aligned}$$

$$\| \partial^\nu \underset{\sim}{\zeta} \|_{-s,G} \leqslant K[\| p \|_{r,G_1} h^{r+s-1-(s-k)^+-|\nu|} + \| p \|_{j,\Omega} h^{j+k-1}], \qquad (4.6)$$

$$|\nu| + 2 \leqslant r \leqslant |\nu| + k + 2 + (s-k)^+ ,$$

$$\| \partial^\nu \operatorname{div} \underset{\sim}{\zeta} \|_{-s,G} \leqslant K[\| p \|_{r,G_1} h^{r+s-2-|\nu|} + \| p \|_{j,\Omega} h^{j+k-1}], \tag{4.7}$$

$$|\nu| + 2 \leqslant r \leqslant |\nu| + k + 3 ,$$

since

$$\| \partial^\nu \theta \|_{0,G} = \| P_h(\partial^\nu p) - \partial^\nu p \|_{0,G} \leqslant K \| p \|_{r+|\nu|,G_1} h^r, \quad 0 \leqslant r \leqslant k+1 ,$$

and

$$\| \partial^\nu \theta \|_{-k-1-|\nu|,G} \leqslant K \| \theta \|_{-k-1,G_1} \leqslant K \| p \|_{r,G_1} h^{r+k+1}, \quad 0 \leqslant r \leqslant k+1 .$$

Finally, the finite covering process used in the previous sections, together with (4.5), (4.6), and (4.7), finishes the proof of the theorem.

Note that, in particular, we have shown that

$$\| Q_h \xi \|_{-k-1,\Omega_0} + \| Q_h \underset{\sim}{\zeta} \|_{-k-1,\Omega_0} + \| Q_h \operatorname{div} \underset{\sim}{\zeta} \|_{-k-1,\Omega_0} \leqslant K h^{2k+2} (\| p \|_{k+3,\Omega} + \| p \|_{k+3+m,\Omega_1}) . \tag{4.8}$$

We can also derive the estimates for the nonlinear problem. Let $m \leqslant k+1$ be the order of Q_h, and let

$$l(j,k) = 2(j-1) + s - m - \varepsilon - 2(j-k-2)^+$$

so that $l(j,k) \geqslant t(j,k)$ if $s \geqslant m$.

THEOREM 4.2 : *Assume* $k \geqslant 1$, $p \in H^j(\Omega) \cap H^r(\Omega_1)$, $2 + \varepsilon \leqslant j \leqslant k+3$, *the operator M of* (1.1) *is nonlinear, and h is sufficiently small ; then, for* $0 \leqslant s \leqslant k+1$,

(i) $$\| Q_h \xi \|_{-s,\Omega_0} \leqslant K[\| p \|_{r,\Omega_1} h^{r+s-(s-k+1)^+ - m} + \| p \|_{j,\Omega} (\| p \|_{j,\Omega} + 1) h^{l(j,k)}] ,$$

$$m + 2 \leqslant r \leqslant m + k + 1 + (s-k+1)^+ ,$$

(ii) $$\| Q_h \underset{\sim}{\zeta} \|_{-s,\Omega_0} \leqslant K[\| p \|_{r,\Omega_1} h^{r+s-1-(s-k)^+ - m} + \| p \|_{j,\Omega} (\| p \|_{j,\Omega} + 1) h^{l(j,k)-(\varepsilon-s)^+}] ,$$

$$m + 2 \leqslant r \leqslant m + k + 2 + (s-k)^+ - \varepsilon ,$$

(iii) $$\| Q_h \operatorname{div} \underset{\sim}{\zeta} \|_{-s,\Omega_0} \leqslant K[\| p \|_{r,\Omega_1} h^{r+s-2-m} + \| p \|_{j,\Omega} (\| p \|_{j,\Omega} + 1) h^{l(j,k)-(\varepsilon-s)^+}] ,$$

$$m + 2 \leqslant r \leqslant m + k + 3 - \varepsilon .$$

Proof: Let us rewrite (4.2) and (4.3) in the form

$$\left.\begin{aligned}
&(\alpha(p)\,\partial^{\nu}\underset{\sim}{\zeta},\underset{\sim}{v})-(\operatorname{div}\underset{\sim}{v},\partial^{\nu}\tau)+(\underset{\sim}{\Gamma}\,\partial^{\nu}\tau,\underset{\sim}{v})=(\underset{\sim}{\Gamma}\,\partial^{\nu}\theta,\underset{\sim}{v})\\
&+(\partial^{\nu}[\underset{\sim}{\kappa}\xi^{2}+\lambda\xi\underset{\sim}{\zeta}],\underset{\sim}{v})-\sum_{\mu<\nu}\binom{\nu}{\mu}([T_h^{\mu}\,\partial^{\nu-\mu}\alpha(p)]\,\partial^{\mu}\underset{\sim}{\zeta}\\
&+[T_h^{\mu}\,\partial^{\nu-\mu}\underset{\sim}{\Gamma}]\,\partial^{\mu}\xi,\underset{\sim}{v}),\quad \underset{\sim}{v}\in\underset{\sim}{V}_h(G),\\
&(\operatorname{div}\partial^{\nu}\underset{\sim}{\zeta},w)+(\gamma\,\partial^{\nu}\xi,w)=(\gamma\,\partial^{\nu}\theta,w)+(\partial^{\nu}[\rho\xi^{2}],w)\\
&\qquad\qquad-\sum_{\mu<\nu}\binom{\nu}{\mu}([T_h^{\mu}\,\partial^{\nu-\mu}\gamma]\,\partial^{\mu}\xi,w),\\
&\qquad\qquad\qquad\qquad w\in W_h(G).
\end{aligned}\right\}\tag{4.9}$$

Just as in the linear case (Theorem 4.1) it suffices to consider $Q_h=\partial^{\nu}$ and to proceed by induction. Again, $|\nu|=0$ is just Theorem 3.2, and so, we assume that $|\nu|>0$ and that (i), (ii), and (iii) are valid for $m\leqslant|\nu|-1$. The proof will be the same as in Theorem 4.1, but before we can apply Lemma 1.1 to (4.9) we need estimates for $\|\partial^{\nu}(\underset{\sim}{\kappa}\xi^{2}+\lambda\xi\underset{\sim}{\zeta})\|_{0,G_1}$ and $\|\partial^{\nu}[\rho\xi^{2}]\|_{0,G_1}$. Note that, by definition,

$$\partial^{\nu}w=h^{-|\nu|}(T_h^{e_2}-I)^{\nu_2}(T_h^{e_1}-I)^{\nu_1}w,$$

which implies that

$$\|\partial^{\nu}w\|_{0,\Omega_0}\leqslant Qh^{-|\nu|}\|w\|_{0,\Omega_1}.$$

Therefore, this estimate and (3.15) imply that

$$\begin{aligned}\|\partial^{\nu}(\underset{\sim}{\kappa}\xi^{2}+\lambda\xi\underset{\sim}{\zeta})\|_{0,\Omega_0}&\leqslant Qh^{-|\nu|}\|\kappa\xi^{2}+\lambda\xi\underset{\sim}{\zeta}\|_{0,\Omega_1}\\
&\leqslant Qh^{t(j,k)-|\nu|}\|p\|_{j,\Omega}^{2}(\|p\|_{j,\Omega}+1),\quad|\nu|\leqslant k+1.\end{aligned}\tag{4.10}$$

Similarly,

$$\|\partial^{\nu}(\rho\xi^{2})\|_{0,\Omega_0}\leqslant Qh^{t(j,k)-|\nu|}\|p\|_{j,\Omega}^{2}(\|p\|_{j,\Omega}+1),\quad|\nu|\leqslant k+1.\tag{4.11}$$

We can now apply Lemma 1.1, then the same argument that led to Theorems 4.1 and 4.2 will show (in view of (3.14), (4.10), and (4.11)) that

$$\| \partial^\nu \xi \|_{-s,G} \leqslant K\Big[\| \partial^\nu p \|_{\bar{r},G_1} h^{\bar{r}+s-(s-k+1)^+} + h^{s+1}\Big(\sum_{\mu<\nu} \| \partial^\nu \xi \|_{0,G_1} + \sum_{\mu<\nu} \| \partial^\mu \underset{\sim}{\zeta} \|_{0,G_1} + \| \partial^\nu (\underset{\sim}{\kappa}\xi^2 + \lambda \xi \underset{\sim}{\zeta}) \|_{0,G_1} + \| \partial^\nu (\rho \xi^2) \|_{0,G_1} \Big) + \sum_{\mu \leqslant \nu} (\| \partial^\mu \xi \|_{-k-1-|\mu|} + \| \partial^\mu \underset{\sim}{\zeta} \|_{-k-1-|\mu|}) + \| \partial^\mu (\underset{\sim}{\kappa}\xi^2) + \lambda \xi \underset{\sim}{\zeta} \|_{-k-1-|\nu|} + \| \partial^\nu (\rho \xi^2) \|_{-k-1-|\nu|} \Big] \tag{4.12}$$

$$\leqslant K[\| p \|_{r,G_1} h^{r+s-(s-k+1)^+ - |\nu|} + \| p \|_{j,\Omega} (\| p \|_{j,\Omega} + 1) h^{l(j,k)} + \| \xi \|_{-k-1} + \| \underset{\sim}{\zeta} \|_{-k-1}]$$

$$\leqslant K[\| p \|_{r,G_1} h^{r+s-(s-k+1)^+ - |\nu|} + \| p \|_{j,\Omega} (\| p \|_{j,\Omega} + 1) h^{l(j,k)}],$$

$$|\nu| + 2 \leqslant r \leqslant |\nu| + k + 1 + (s-k+1)^+,$$

$$\| \partial^\nu \underset{\sim}{\zeta} \|_{-s,G} \leqslant K[\| p \|_{r,G_1} h^{r+s-1-(s-k)^+ - |\nu|} + \| p \|_{j,\Omega} (\| p \|_{j,\Omega} + 1) \times h^{l(j,k)-(\varepsilon-s)^+}], \quad |\nu| + 2 \leqslant r \leqslant |\nu| + k + 2 + (s-k)^+ - \varepsilon \tag{4.13}$$

$$\| \partial^\nu \operatorname{div} \underset{\sim}{\zeta} \|_{0,G} \leqslant K[\| p \|_{r,G_1} h^{r+s-2-|\nu|} + \| p \|_{j,\Omega} (\| p \|_{j,\Omega} + 1) \times h^{l(j,k)-(\varepsilon-s)^+}], \quad |\nu| + 2 \leqslant r \leqslant |\nu| + k + 3 - \varepsilon . \tag{4.14}$$

We now finish the proof using (4.12), (4.13), and (4.14) with the finite covering process described before.

Note that, in particular, Theorem 4.2 shows that (4.8) holds also for the nonlinear problem, with the exponent $2k + 2$ replaced by $2k + 2 - \varepsilon$.

5. THE SUPERCONVERGENCE ESTIMATES

We shall introduce in this section a kernel K_h as defined in [1, 2], which will produce by convolution with $\underset{\sim}{u}_h$ and p_h a new approximation to $(\underset{\sim}{u}, p)$ in $\underset{\sim}{V} \times W$, the convergence of which will be (for sufficiently smooth p) twice as fast as that of $(\underset{\sim}{u}_h, p_h)$ to $(\underset{\sim}{u}, p)$. Recall that [3] and [5] show that if $p \in H^{k+3}(\Omega)$, then

$$\| \xi \|_{0,\Omega_0} + \| \underset{\sim}{\zeta} \|_{0,\Omega_0} + \| \operatorname{div} \underset{\sim}{\zeta} \|_{0,\Omega_0} \leqslant \| \xi \|_0 + \| \underset{\sim}{\zeta} \|_0 + \| \operatorname{div} \underset{\sim}{\zeta} \|_0 \leqslant C \| p \|_{k+3} h^{k+1} \tag{5.1}$$

We shall obtain in a moment the following estimates :

$$\| p - K_h * p_h \|_{0,\Omega_0} + \| \underset{\sim}{u} - K_h * \underset{\sim}{u}_h \|_{0,\Omega} + \| \operatorname{div} \underset{\sim}{u} - K_h * \operatorname{div} \underset{\sim}{u}_h \|_{0,\Omega_0}$$
$$\leqslant C(\| p \|_{2k+4-\varepsilon,\Omega_1} + \| p \|_{k+3,\Omega})\, h^{2k+2-\varepsilon}, \tag{5.2}$$
$$\text{if } p \in H^{k+3}(\Omega) \cap H^{2k+4-\varepsilon}(\Omega_1),$$

where the exponent $-\varepsilon$ does not appear in the linear case. Comparing (5.1) and (5.2) we see that we have indeed doubled the rate of convergence.

Let

$$\chi(t) = \begin{cases} 1, & |t| \leqslant \frac{1}{2}, \\ 0, & |t| > \frac{1}{2}, \end{cases}$$

and for l a positive integer set

$$\psi_1^{(l)}(t) = \chi * \chi * \cdots * \chi, \quad \text{convolution } l - 1 \text{ times}.$$

Let c_i, $0 \leqslant i \leqslant k$, be determined as the unique solution of the linear system of algebraic equations (see [2])

$$\sum_{i=0}^{k} c_i \int_{\mathbb{R}} \psi_1^{(k+2)}(y)\,(y + i)^{2m}\, dy = \delta_{0m}, \quad \text{for} \quad 0 \leqslant m \leqslant k,$$

and define K_h by

$$K_h(x) = K_{h,k+2}^{2k+2}(x) = \prod_{m=1}^{2} \left(\sum_{i=-k}^{k} h^{-1} c_i' \psi_1^{(k+2)}(h^{-1} x_m - i) \right), \quad x \in \mathbb{R}^2, \tag{5.3}$$

where the constants c_j' are given by $c_{-i}' = c_i' = c_i/2$, $1 \leqslant i \leqslant k$, and $c_0' = c_0$.

It follows from [2] that

$$\| K_h * w - w \|_{0,\Omega_0} \leqslant C \| w \|_{r,\Omega_1} h^r, \quad \text{if} \quad w \in H^r(\Omega_1), \quad 0 \leqslant r \leqslant 2k+2, \tag{5.4}$$

$$\| D^\nu(K_h * w) \|_{s,\Omega_0} \leqslant C \| \partial^\nu w \|_{s,\Omega_1}, \quad \text{if} \quad w \in H^s(\Omega_1). \tag{5.5}$$

Also, it is easily shown by induction (see [2] for a detailed proof) that

$$\| w \|_{0,\Omega_0} \leqslant C \sum_{|\nu| \leqslant s} \| D^\nu w \|_{-s,\Omega_1}, \quad s \in \mathbb{Z}, \quad s \geqslant 0, \quad \text{if} \quad w \in L^2(\Omega). \tag{5.6}$$

Next, note that

$$\underset{\sim}{u} - K_h * \underset{\sim}{u}_h = (\underset{\sim}{u} - K_h * \underset{\sim}{u}) + K_h * \underset{\sim}{\zeta}\,, \tag{5.7}$$

$$p - K_h * p_h = (p - K_h * p) + K_h * \xi\,, \tag{5.8}$$

$$\operatorname{div} \underset{\sim}{u} - K_h * \operatorname{div} \underset{\sim}{u}_h = (\operatorname{div} \underset{\sim}{u} - K_h * \operatorname{div} \underset{\sim}{u}) + K_h * \operatorname{div} \underset{\sim}{\zeta}\,. \tag{5.9}$$

We are now ready to demonstrate the superconvergence estimates announced in (5.2).

THEOREM 5.1 : *Assume the operator N is linear (i.e., $N = L$), $p \in H^j(\Omega) \cap H^r(\Omega_1)$, $2 \leqslant j \leqslant k + 3$, and $k + 1 \leqslant r \leqslant 2k + 4$; then, for sufficiently small h,*

(i) $\| p - K_h * p_h \|_{0,\Omega_0} \leqslant C[\| p \|_{r,\Omega_1} h^{r-(r/2-k)^+} + \| p \|_{j,\Omega} h^{j+k-1}]$,

(ii) $\| \underset{\sim}{u} - K_h * \underset{\sim}{u}_h \|_{0,\Omega_0} \leqslant C[\| p \|_{r,\Omega_1} h^{r-1-(r/2-k-1)^+} + \| p \|_{j,\Omega} h^{j+k-1}]$;

(iii) $\| \operatorname{div} \underset{\sim}{u} - K_h * \operatorname{div} \underset{\sim}{u}_h \|_{0,\Omega_0}$
$\leqslant C[\| p \|_{r,\Omega_1} h^{r-2} + \| p \|_{j,\Omega} h^{j+k-1-(j-k-3+\varepsilon)^+}]$.

THEOREM 5.2 : *Assume the operator N is nonlinear, $k \geqslant 1$, $p \in H^j(\Omega) \cap H^r(\Omega_1)$, $2 + \varepsilon \leqslant j \leqslant k + 3$, and $k + 1 \leqslant r \leqslant 2k + 4 - \varepsilon$; then, for sufficiently small h,*

(i) $\| p - K_h * p_h \|_{0,\Omega_0}$
$\leqslant C[\| p \|_{r,\Omega_1} h^{r-(r/2-k)^+} + \| p \|_{j,\Omega} (\| p \|_{j,\Omega} + 1)\, h^{t(j,k)}]$,

(ii) $\| \underset{\sim}{u} - K_h * \underset{\sim}{u}_h \|_{0,\Omega_0}$
$\leqslant C[\| p \|_{r,\Omega_1} h^{r-1-(r/2-k-1)^+} + \| p \|_{j,\Omega} (\| p \|_{j,\Omega} + 1)\, h^{t(j,k)}]$,

(iii) $\| \operatorname{div} \underset{\sim}{u} - K_h * \operatorname{div} \underset{\sim}{u}_h \|_{0,\Omega_0}$
$\leqslant C[\| p \|_{r,\Omega_1} h^{r-2} + \| p \|_{j,\Omega} (\| p \|_{j,\Omega} + 1)\, h^{t(j,k)}]$,

where $t(j, k) = 2(j - 1) - \varepsilon - 2(j - k - 2)^+$.

Proof of Theorems 5.1 *and* 5.2. Use the triangle inequality on (5.7), (5.8), and (5.9), and apply then (5.4), (5.6), (5.5), and Theorems 4.1 and 4.2, respectively.

REFERENCES

[1] J. H. BRAMBLE and A. H. SCHATZ, *Estimates for spline projections,* R.A.I.R.O., Anal. numér., 10 (1976), pp. 5-37.

[2] *Higher order local accuracy by averaging in the finite element method,* Math. of Comp., 31 (1977), pp. 94-111.

[3] J. DOUGLAS, Jr. and J. E. ROBERTS, *Global estimates for mixed methods for second order elliptic problems,* Math. of Comp., 44 (1985), pp. 39-52.

[4] J. L. LIONS and E. MAGENES, *Non homogeneous boundary value problems and applications,* I, Springer-Verlag, Berlin, 1970.

[5] F. A. MILNER, *Mixed finite element methods for quasilinear second order elliptic* problems, Math. of Comp., 44 (1985), pp. 303-320.

[6] J. C. NEDELEC, *Mixed finite elements in* $\mathbb{R}^3$, Numer. Math., 35 (1980), pp. 315-341.

[7] J. A. NITSCHE and A. H. SCHATZ, *Interior estimates for Ritz-Galerkin methods,* Math. of Comp., 28 (1974), pp. 937-958.

[8] P. A. RAVIART and J. M. THOMAS, *A mixed finite element method for 2nd order elliptic problems,* in Proceedings of a Conference on Mathematical Aspects of Finite Element Methods, Lecture Notes in Mathematics 606, Springer-Verlag, Berlin, 1977, pp. 292-315.

[9] R. SCHOLZ, L_∞*-convergence of saddle-point approximations for second order problems,* R.A.I.R.O., Anal. numér., 11 (1977), pp. 209-216.

[10] G. STAMPACCHIA, *Equations elliptiques du second ordre à coefficients discontinus,* Les Presses de l'Université de Montréal, Montréal, 1966.

[11] J. M. THOMAS, *Sur l'analyse numérique des méthodes d'éléments finis hybrides et mixtes,* Thèse, Université P.-et-M. Curie, Paris, 1977.