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ON THE ORDER OF POINTWISE CONVERGENCE OF SOME BOUNDARY ELEMENT METHODS. PART I. OPERATORS OF NEGATIVE AND ZERO ORDER (*)

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Summary. — *The paper deals with the approximate solution of boundary integral equations and strongly elliptic pseudodifferential equations by the finite element Galerkin method. For operators of order* $2\alpha \leq 0$ *it is shown that the discrete solutions and for the case of some first kind integral equations also the traces of the corresponding potentials converge uniformly with almost the same optimal order as is known for their convergence in the mean-square sense. The proof is based on error estimates for discrete Green functions which are derived by using weighted Sobolev norms and Gårding's inequality.*

Résumé. — *Cet article porte sur la résolution approchée par la méthode d'éléments finis de Galerkin d'équations intégrales sur la frontière, qui sont pseudo-différentielles fortement elliptiques. Pour des opérateurs d'ordre* $2\alpha \leq 0$, *on montre que les solutions discrètes et aussi la trace des potentiels correspondants, pour certaines équations intégrales de première espèce, convergent uniformément. En outre l'ordre de convergence optimal est presque identique à celui déjà connu pour la convergence en moyenne quadratique. La démonstration est basée sur des estimations d'erreur pour les fonctions de Green discrètes qui sont obtenues par utilisation des espaces de Sobolev avec poids et de l'inégalité de Garding.*

1. BOUNDARY INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

Let Γ be a smooth simple closed $(n-1)$-dimensional surface in $\mathbb{R}^n$, $n = 2$ or $n = 3$. On Γ, we consider a strongly elliptic boundary integro-differential equation and corresponding boundary element methods. As a special model problem we first consider the integral equations of the first kind,

$$Vu(x) = \int_\Gamma \gamma_n(x-y)\, u(y)\, do_y + \int_\Gamma K(x,y)\, u(y)\, do_y = f(x)\,, \quad (1.1)$$

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where $\gamma_n(z) = -\log|z|$, if $n = 2$, and $\gamma_n(z) = |z|^{-1}$, if $n = 3$, and $K(x, y)$ is a sufficiently smooth kernel These special integral equations (1 1) arise in many applications as in conformal mapping, viscous flow problems, electrostatics, acoustics and elasticity in case $n = 2$ (for references see e g in [10]) and for $n = 3$ in electrostatics (see [17], [18]), acoustics (see [1], [31]) and electromagnetic waves (see [25])

In many of the applications (1 1) needs to be modified by additional unknown constant quantities and corresponding complementing conditions, or by letting u become an unknown vector valued charge and $K(x, y)$ a matrix valued kernel Our following error analysis also applies to these cases with only very minor technical modifications which we omit here for brevity

Equation (1 1) is a rather special case belonging to the large class of so-called strongly elliptic boundary integro differential equations (or pseudodifferential equations) which are solved numerically by boundary element methods (for a brief survey we refer to [30]) Here V can be written as the sum of a strongly elliptic pseudodifferential operator of order 2α on Γ and an additional sufficiently smoothing operator Since for these operators the Galerkin method converges of optimal order in appropriate Sobolev spaces (see [11], [12], [26]) we shall derive the pointwise error estimates also for the general case containing equation (1 1) as a special case

If the general pseudodifferential equation is required to be solved on a *bounded* sub-domain of Γ then in general boundary conditions have to be associated (see [6], [26]) Equations of this type include the boundary value problems of strongly elliptic partial differential equations, and the above mentioned boundary element methods then become the well known domain finite element methods where pointwise convergence of Galerkin's method is well established for the most popular boundary conditions (see [7], [16], [19], [21], [23], [24]) For the more general pseudodifferential operators, however, the boundary conditions introduce new complications Therefore, we consider only the case of compact surfaces Γ without boundary here

We denote by $L^p(\Gamma)$ and $W^{r\,p}(\Gamma)$, $1 \leqslant p \leqslant \infty$, $r \in \mathbb{R}$, the Lebesgue and Sobolev spaces on Γ provided with the usual norms, $(.,.)$ and $\|\,.\,\|$ are the inner product and norm of $L^2(\Omega)$, respectively, and $\|\,.\,\|_r$ is the norm of the Hilbert space $H^r(\Gamma) = W^{r\,2}(\Gamma)$, $r \in \mathbb{R}$ For convenience, we shall denote by c a generic positive constant which may vary within the context (but will usually be independent of the mesh-width and of the solution)

As will be indicated in the Appendix, for sufficiently smooth Γ and $K(x, y)$, the operator V given by (1 1) admits the decomposition

$$V = V_0 + V_1, \tag{1 2}$$

where V_0 is strongly $H^{-1/2}(\Gamma)$-coercive, i.e., V_0 satisfies the inequality

$$(V_0 v, v) \geqslant \gamma \| v \|^2_{-1/2} \quad \text{for all} \quad v \in H^{-1/2}(\Gamma), \tag{1.3}$$

with a positive constant γ, and V_1 maps $H^s(\Gamma)$ continuously into $H^{s+2}(\Gamma)$ for some appropriate s, in particular, for $s = -\frac{1}{2}$ and $s = 0$, i.e.,

$$\| V_1 v \|_{s+2} \leqslant c \| v \|_s . \tag{1.4}$$

Then, with the compact embedding $H^{s+2}(\Gamma) \to H^{s+1}(\Gamma)$ and (1.4), V is a Fredholm operator of index zero from $H^{-1/2}(\Gamma)$ into $H^{1/2}(\Gamma)$. We shall further require that the solution of (1.1) is unique which implies the unique solvability of (1.1) with $u \in H^s(\Gamma)$ for any $f \in H^{s+1}(\Gamma)$ and the a priori estimate

$$\| u \|_s \leqslant c \| f \|_{s+1} . \tag{1.5}$$

Equation (1.1) is a special case of a strongly elliptic equation,

$$\mathrm{Au} = f \quad \text{on} \quad \Gamma, \tag{1.6}$$

where A is a strongly elliptic pseudodifferential operator of order $2\alpha \in \mathbb{R}$ on Γ. The latter means that A admits a decomposition

$$A = A_0 + A_1, \tag{1.7}$$

where A_0 is strongly $H^\alpha(\Gamma)$-coercive, i.e., A_0 satisfies

$$\mathrm{Re}\,(A_0 v, v) = \mathrm{Re}\,(v, A_0^* v) \geqslant \gamma_0 \| v \|^2_\alpha, \quad \gamma_0 > 0, \tag{1.8}$$

(see [8]) and A_1 maps $H^{s+\alpha}(\Gamma)$ continuously into $H^{s-\alpha+1}(\Gamma)$ for appropriate s, i.e.,

$$\| A_1 v \|_{s-\alpha+1} \leqslant c \| v \|_{s+\alpha} . \tag{1.9}$$

Again, uniqueness of (1.6) implies unique solvability of (1.6) for any right hand side $f \in H^{s-2\alpha}(\Gamma)$, and the unique solution u satisfies the a priori estimate

$$\| u \|_s \leqslant c \| f \|_{s-2\alpha} . \tag{1.10}$$

Many of the boundary element methods are defined by boundary integro-differential operators of the above type. For special applications and examples we refer to [1], [2], [10], [17], [25], [30], [31]. For all these applications we show almost optimal order pointwise error estimates for Galerkin's method provided A has principal symbol of *integer* order $2\alpha \leqslant 0$.

For *arbitrary real* orders $2\alpha \leqslant -1$ all our proofs and results remain valid without modifications. In the case $-1 < 2\alpha < 0$ we find the same results for higher degree elements excluding piecewise constant approximations.

For positive orders $2\alpha > 0$ the proofs need several technical modifications. This case $2\alpha > 0$ will be presented in the forthcoming Part II. Since in [2] collocation methods with odd degree splines have been treated as modified Galerkin methods our pointwise error estimates in Part II extend to these collocations. The extension to collocation with even degree splines as in [22] and to collocation in higher dimensions, however, has yet to be carried out.

2. THE FINITE ELEMENT GALERKIN METHOD

Let $\Pi_h = \{ K \}$ be finite decompositions of the surface Γ into closed subsets K with mutually disjoint interiors $\overset{\circ}{K}$; $h \in \left(0, \frac{1}{2}\right]$ denotes a discretization parameter corresponding to the maximum diameter of K. We further set $\Gamma_h \doteq \cup \{ \overset{\circ}{K}, K \in \Pi_h \}$, and use corresponding Sobolev norms in $H^r(\Gamma_h)$. Depending on h, we shall consider the family $\{ \Pi_h \}$ of decompositions. For properties involving $\{ \Pi_h \}$ the generic constant c will always be independent of h. For $\{ \Pi_h \}$ we assume quasiregularity in the following sense :

(A1) *Associated with* $\{ \Pi_h \}$ *there exist two positive constants,* c_1 *and* c_2, *such that each element* $K \in \Pi_h$ *is contained in the intersection of* Γ *with some ball* $B_1 \subset \mathbb{R}^n$ *of radius* $c_1 h$, *and contains the intersection of* Γ *with some ball* $B_2 \subset \mathbb{R}^n$ *of radius* $c_2 h$.

For fixed integers $k \geqslant 1$ and $m \geqslant 0$, $m \leqslant k - 1$, let $S_h^{k,m}$ be a so-called (k, m)-system on Γ corresponding to the family of decompositions Π_h (see [4] and for splines [3], [20]: where the notation $S_{k-1}(\Pi_h, m - 1)$ is used). The first parameter, k, refers to the local approximation order of $S_h^{k,m}$ which usually consists of piecewise polynomial functions (or isoparametric splines) of degree $k - 1$; the second parameter, m, indicates the global smoothness of these functions,

$$S_h^{k,m} \subset H^m(\Gamma) \cap H^k(\Gamma_h) .$$

For our purpose, we need to require the following approximation and inverse properties :

(A2) *There exists a continuous operator* $p_h : H^m(\Gamma) \cup H^{\min(k,m-2\alpha)}(\Gamma_h) \to S_h^{k,m}$, *such that for all* $v \in H^m(\Gamma) \cap H^r(\Gamma_h)$, $\min \{ k, m - 2\alpha \} \leqslant r \leqslant k$, *there holds* *the global estimate*

$$\| v - p_h v \|_j \leqslant ch^{r-j} \| v \|_{H^r(\Gamma_h)}, \quad 0 \leqslant j \leqslant m, \tag{2.1}$$

and, in addition the local estimate

$$\| v - p_h v \|_{H^j(K)} \leqslant ch^{r-j} \| v \|_{H^r(K_h)}, \quad 0 \leqslant j \leqslant r, \tag{2.2}$$

for each $K \in \Pi_h$, *where* K_h *may be* K *or, if necessary, the union of the open interiors of all neighboring elements of* K *intersecting a ball* $B_K^h \subset \mathbb{R}^n$ *of radius* ch *having its center in* K *and with* c *independent of* h.

(A3) *For all* $\phi_h \in S_h^{k,m}$ *there holds, on each* $K \in \Pi_h$,

$$\| \phi_h \|_{H^k(K)} \leqslant c \| \phi_h \|_{H^{k-1}(K)}, \tag{2.3}$$

$$\| \phi_h \|_{H^j(K)} \leqslant ch^{l-j} \| \phi_h \|_{H^l(K)} \textit{ for integers } 0 \leqslant l \leqslant j \leqslant k-1, \tag{2.4}$$

and, globally,

$$\| \phi_h \|_\beta \leqslant ch^{\gamma-\beta} \| \phi_h \|_\gamma, \quad \textit{for real } \gamma, \beta \textit{ with } -k \leqslant \gamma \leqslant \beta \leqslant m. \tag{2.5}$$

These are typical properties of isoparametric finite element spaces $S_h^{k,m}$ of order k.

In case $n = 2$, for one-dimensional splines on a uniform grid, these properties can easily be obtained from [3], Chap. 4 (with $p_h = p_h^m r_h$ and $k = m + 1$). For higher dimensional finite elements similar projection operators can be constructed for piecewise polynomials of degree $k - 1$ e.g. from the results in [5] and [27]. (For tensor product splines, however, (A2) and (A3) must be modified. This is omitted here.)

Note that (2.3) becomes trivial if $\phi_h|_K$ is a polynomial of degree $k - 1$. Usually, the systems $S_h^{k,m}$ also satisfy the pointwise estimate

$$\inf_{\phi_h \in S^{k,m}} \| v - \phi_h \|_{L^\infty} \leqslant ch^r \| v \|_{W^{r,\infty}}, \quad 0 \leqslant r \leqslant k. \tag{2.6}$$

The Galerkin approximations $u_h \in S_h^{k,m}$ to the solution u of problem (1.1) are determined by the finite dimensional analogues of (1.1),

$$(Au_h, \phi_h) = (f, \phi_h) = (Au, \phi_h) \quad \text{for all} \quad \phi_h \in S_h^{k,m}. \tag{2.7}$$

Since we assume that the family $S_h^{k,m}$ is dense in H^α (which is a consequence of (2.1)), since (1.1) is uniquely solvable and since A admits the properties (1.3)-(1.5) (with $s = \alpha$), the problems (2.7) are uniquely solvable for sufficently small h (see [26]). Furthermore, the approximation property (2.6) of the spaces

$S_h^{k,m}$ implies that u_h converges with optimal order and provides even superconvergence in « negative » Sobolev norms (see [11], [12], [17])

$$\| u - u_h \|_p + \| A(u - u_h) \|_{p-2\alpha} \leqslant ch^{q-p} \| u \|_q \tag{2.8}$$

for $2\alpha - k \leqslant p \leqslant m$ and $\max\{\alpha, p\} \leqslant q \leqslant k$. Using in the usual manner the inverse inequality for $S_h^{k,m}$, the mean-square result (2.8) also gives a pointwise error estimate of the form (see also [11])

$$\| u - u_h \|_{L^\infty} \leqslant ch^{k - \frac{n-1}{2}} (\| u \|_k + \| u \|_{W^{r,\infty}}), \tag{2.9}$$

where $r = \max\left\{ m, k - \dfrac{n-1}{2} \right\}$.

In view of (2.6), this estimate is not of optimal order if $u \in W^{k,\infty}(\Gamma)$. Under the foregoing assumptions (A1)-(A3), we can improve (2.9) as follows :

THEOREM 1 : *Suppose that 2α is an integer, $-k \leqslant \alpha \leqslant 0$, and that $u \in L^\infty(\Gamma)$. Then there holds for the Galerkin solution $u_h \in S_h^{k,m}$ the pointwise estimate*

$$\| u - u_h \|_{L^\infty} \leqslant c \left(\log \frac{1}{h} \right)^{n/2 - 1} \inf_{\phi_h \in S_h^{k,m}} \| u - \phi_h \|_{L^\infty} . \tag{2.10}$$

Remarks : Theorem 1 and its proof remain valid without any modifications for non integer real orders $2\alpha \leqslant -1$. For $-1 < 2\alpha < 0$ Theorem 1 also holds provided $m \geqslant 1$. For piecewise constant approximations in the latter case one needs modifications of the approximation property (2.1) and of (2.8) as well as of the proof of Lemma 3.4 which are omitted here.

We did not attempt to avoid the logarithm in (2.10) in order to keep the proof as short and simple as possible.

In view of (2.6), we obtain from (2.10)

$$\| u - u_h \|_{L^\infty} \leqslant ch^k \left(\log \frac{1}{h} \right)^{n/2 - 1} \| u \|_{W^{k,\infty}}, \tag{2.11}$$

provided that $u \in W^{k,\infty}(\Gamma)$.

The global L^∞-result (2.10) can be refined to a local estimate of the form

$$| (u - u_h)(z) | \leqslant c_\rho \left(\log \frac{1}{h} \right)^{n/2 - 1} \inf_{\phi_h \in S_h^{k,m}} \{ \| u - \phi_h \|_{L^\infty(B_z^\rho \cap \Gamma)} + \| u - \phi_h \| \}, \tag{2.12}$$

where B_z^ρ denotes some ball in $\mathbb{R}^n$ of radius $\rho = 0(1)$ as $h \to 0$ with center in z.

Note that for $\alpha \leqslant 0$ Theorem 1 yields the L^∞-stability of the Galerkin scheme (2.7) in the form

$$\| u_h \|_{L^\infty} \leqslant c\left(\log \frac{1}{h}\right)^{(n-2)/2} \| u \|_{L^\infty}. \tag{2.13}$$

However, this estimate does not imply stability of the discrete equations (2.7) with right hand sides (f, ϕ_h) (see also [13]).

Optimal order L^∞-error estimates for the standard finite element method applied to properly elliptic partial differential equations are known e.g. by the work of Natterer [16], Nitsche [19], Scott [24], Frehse and Rannacher [7] and Schatz and Wahlbin [23] for second order equations ($\alpha = 1$) and from Rannacher [21] for the bi-Laplacian ($\alpha = 2$). In proving (2.10) for the case $\alpha \leqslant 0$, we shall adapt techniques from [7] and [21]. Some technical complications will arise from the fact that in the present situation the governing operator A is non-local and perhaps of negative order. However, localization techniques still work since the principle part A_0 of A is assumed to be a pseudo-differential operator on Γ of the class $OPS^{2\alpha}_{1,0}$ (see [28]). In particular, for any C^∞-multiplier ϕ, the commutator $[\phi A_0 - A_0 \phi]$ becomes a pseudo-differential operator of order $2\alpha - 1$ (see [28]).

Having shown optimal order pointwise estimates for the error $u - u_h$ one is led to the question whether the error in the corresponding potentials, $Au - Au_h$, admits a corresponding improved bound. We shall analyze this problem here only for the simple layer model operator in (1.1) with $2\alpha = -1$,

$$Au(x) = \int_\Gamma \gamma_n(x - y)\, u(y)\, do_y \tag{2.14}$$

where $\gamma_n(z) = -\log |z|$, if $n = 2$, and $\gamma_n(z) = |z|^{-1}$, if $n = 3$, More general cases can be treated analogously.

THEOREM 2 : *Suppose that the operator A is of the form* (2.14) *with order* $2\alpha = -1$, *and that $u \in L^\infty(\Gamma)$. Then there holds for the traces of the potentials generated by the Galerkin solutions $u_h \in S_h^{k,m}$ the pointwise estimate*

$$\| Au - Au_h \|_{L^\infty} \leqslant ch \log \frac{1}{h} \| u - u_h \|_{L^\infty}. \tag{2.15}$$

Using (2.11) in (2.15), we obtain the error estimate

$$\| Au - Au_h \|_{L^\infty} \leqslant ch^{k+1}\left(\log \frac{1}{h}\right)^{n/2} \| u \|_{W^{k,\infty}}, \tag{2.16}$$

provided that $u \in W^{k,\infty}(\Gamma)$.

3 PROOF OF THEOREM 1

For any fixed $z \in \Gamma$ we introduce the weight function

$$\sigma(x) = (|x - z|^2 + \kappa^2 h^2)^{1/2},$$

where the parameter $\kappa \geqslant 1$ is chosen to be sufficiently large (depending on c_i in condition (A1)) such that for any real β there holds

$$\max_{K \in \Pi_h} \left\{ \max_{x \in K} \sigma^\beta(x) / \min_{x \in K} \sigma^\beta(x) \right\} \leqslant c,$$

where c might depend on β (see [19])

Here and in the following, the generic constant c is always independent of h and z For integer r and real β, we introduce the weighted norms

$$\| v \|_{r\,\beta} = \left(\sum_{|j| \leqslant r} \sum_{K \in \Pi_h} \int_K \sigma^\beta(x) \, | D^j v(x) |^2 \, do_x \right)^{1/2}$$

Here D^j denote the covariant derivatives of order j on Γ and can be replaced by the j-powers of the gradient in the $(n-1)$-dimensional parameter domains of regular local parameter representations of Γ

The local approximation and inverse properties of $S_h^{k\,m}$ imply for $v \in H^m(\Gamma) \cap H^r(\Gamma_h)$, $\min \{ k, m - 2\alpha \} \leqslant r \leqslant k$, the corresponding properties with weighted norms,

$$\| v - p_h v \|_{j\,\beta} \leqslant c h^{r-j} \| v \|_{r\,\beta}, \quad 0 \leqslant j \leqslant r, \tag{3 1}$$

and, for $\phi_h \in S_h^{k\,m}$,

$$\| \phi_h \|_{r\,\beta} \leqslant c h^{s-r} \| \phi_h \|_{s\,\beta}, \quad 0 \leqslant s \leqslant r \leqslant k-1, \tag{3 2}$$

$$\| \phi_h \|_{k\,\beta} \leqslant c \| \phi_h \|_{k-1\,\beta} \tag{3 3}$$

Next, let δ be any smooth function on Γ, and let g be the solution of

$$A_0^* g = \delta \quad \text{on} \quad \Gamma \tag{3 4}$$

Correspondingly, let $g_h \in S_h^{k\,m}$ be defined by

$$(\phi_h, A_0^* g_h) = (\phi_h, A_0^* g) \quad \text{for all} \quad \phi_h \in S_h^{k\,m} \tag{3 5}$$

Below, we shall take δ to be a smooth approximation of the Dirac functional in order to represent the pointwise error $(u - u_h)(z)$ in terms of a local integral

expression. Then, g_h can be considered as a discrete Green function corresponding to the operator A_0^* on $S_h^{k,m}$ (discrete pseudo-inverse to A_0^*).
For abbreviation, we set $e = u - u_h$ and $\eta = g - g_h$. From the orthogonality properties of e and η, we obtain

$$\begin{aligned}(e, \delta) = (e, A_0^* g) &= (e, A_0^* \eta) + (e, A_0^* g_h) \\ &= (u - \phi_h, A_0^* \eta) - (e, A_1^* g_h)\end{aligned} \tag{3.6}$$

where $\phi_h \in S_h^{k,m}$ is arbitrary. For the first term on the right hand side there holds

$$|(u - \phi_h, A_0^* \eta)| \leqslant \| u - \phi_h \|_{L^\infty} \| A_0^* \eta \|_{L^1} . \tag{3.7}$$

The second term can be estimated using $k \geqslant 1$, the a priori estimate (1.5) with $s = 0$, and the error estimate (2.8) for η :

$$\begin{aligned}|(e, A_1^* g_h)| &\leqslant \| e \|_{2\alpha-1} \| A_1^* g_h \|_{1-2\alpha} \\ &\leqslant c \| e \|_{2\alpha-1} \{ \| \eta \|_0 + \| g \|_0 \} \\ &\leqslant c \| e \|_{2\alpha-1} \{ h^k \| g \|_k + \| g \|_0 \} .\end{aligned} \tag{3.8}$$

Next, we estimate the crucial term $\| A_0^* \eta \|_{L^1}$. By the definition of the weight function σ, we find with elementary computations

$$\| A_0^* \eta \|_{L^1} \leqslant c\left(\log \frac{1}{h}\right)^{n/2-1} h^{n/2-3/2} \| A_0^* \eta \|_{0;2} . \tag{3.9}$$

In order to estimate the term $\| A_0^* \eta \|_{0;2}$, we first provide the following three lemmas. For $1 \leqslant i \leqslant n$, let $\xi_i = x_i - z_i$.

LEMMA 3.1 : *The commutator* $A_0^* \xi_i - \xi_i A_0^*$ *satisfies*

$$\| [A_0^* \xi_i - \xi_i A_0^*] \phi \|_{r-2\alpha} \leqslant c \| \phi \|_{r-1} , \quad 1 \leqslant i \leqslant n , \tag{3.10}$$

for. $\phi \in H^{r-1}(\Gamma)$ *and* $\alpha \leqslant r \leqslant k$.

Proof : A_0^* is a pseudodifferential operator of order 2α with a symbol in the class $S_{1,0}^{2\alpha}$ (see [9], [28] and also the Appendix) and the principal symbol $|\theta|^{2\alpha}$. Multiplication by the smooth function ξ_i defines another pseudodifferential operator of order zero with the symbol $x_i(p) - z_i$, $p \in \Gamma$. Hence, the commutator defines a pseudodifferential operator of order $2\alpha+0-1=2\alpha-1$ (see [29], Corollary 4.2, p. 39). This implies (3.9) (see [29], Corollary 1.3, p. 50). q.e.d.

LEMMA 3.2 : *There holds the estimate*

$$\| \xi_i \eta \|_{k,0} + \| \xi_i^2 \eta \|_{k,-2} \leqslant c \| g \|_{k,2}, \quad 1 \leqslant i \leqslant n. \tag{3.10}$$

Proof : The definition of weighted norms implies with $k \geqslant 1$

$$\| \xi_i \eta \|_{k,0} \leqslant c \{ \| \eta \|_{k,2} + \| \eta \|_{k-1,0} \}$$
$$\| \xi_i^2 \eta \|_{k,-2} \leqslant c \{ \| \eta \|_{k,2} + \| \eta \|_{k-1,0} + \| \eta \|_{k-2,-2} \}.$$

Now we use (3.1), for $j = k, k-1$, and (3.3) to find

$$\begin{aligned} \| \eta \|_{k,2} &\leqslant \| \eta - p_h \eta \|_{k,2} + \| p_h \eta \|_{k,2}, \\ &\leqslant \| g - p_h g \|_{k,2} + c \| p_h \eta \|_{k,2}, \\ &\leqslant c \| g \|_{k,2} + c \| \eta \|_{k-1,0}. \end{aligned}$$

It remains to estimate $\| \eta \|_{k-j,0}$ for $j = 1, 2$. For $m > k - j$ we use the inverse assumption (3.2) and the error estimate (2.8), to obtain

$$\begin{aligned} \| \eta \|_{k-j,0} &\leqslant \| g - p_h g \|_{k-j,0} + \| p_h \eta \|_{k-j,0}, \\ &\leqslant ch^j \| g \|_k + ch^{j-k+m} \| p_h \eta \|_m, \\ &\leqslant ch^j \| g \|_k + ch^{j-k+m} \| \eta \|_m, \end{aligned}$$

and hence

$$\| \eta \|_{k-j,0} \leqslant ch^j \| g \|_k.$$

For $k - j \leqslant m$ we use the error estimate (2.8) directly to obtain the above estimate. This implies with the definition of $\sigma(x)$ and the weighted norms the estimates

$$\| \eta \|_{k-1,0} \leqslant ch \| g \|_k \leqslant c \| g \|_{k,2}$$
$$\| \eta \|_{k-2,-2} \leqslant ch^{-1} \| \eta \|_{k-2,0} \leqslant c \| g \|_{k,2}.$$

Collecting the above estimates gives (3.10). q.e.d.

LEMMA 3.3 : *For real $\alpha \leqslant \beta \leqslant m$ there holds*

$$\| \xi_i \eta \|_\beta \leqslant ch^{\alpha-\beta} \| \xi_i \eta \|_\alpha + ch^{k-\beta} \| g \|_{k,2}, \quad 1 \leqslant i \leqslant n. \tag{3.11}$$

Proof : In view of assumption (A.2), the L^2-projektion $L_h : L^2(\Gamma) \to S_h^{k,m}$ satisfies

$$\| v - L_h v \|_\gamma \leqslant ch^{k-\gamma} \| v \|_{k,0},$$

for any $\gamma \in \mathbb{R}$ with $-k \leqslant \gamma \leqslant m$ and any $v \in H^m(\Gamma) \cap H^k(\Gamma_h)$. Using this and the inverse assumption (A.3), we conclude that

$$\begin{aligned}\| \xi_i \eta \|_\beta &\leqslant \| \xi_i \eta - L_h[\xi_i \eta] \|_\beta + \| L_h[\xi_i \eta] \|_\beta \\ &\leqslant ch^{k-\beta} \| \xi_i \eta \|_{k;0} + ch^{\alpha-\beta} \| L_h[\xi_i \eta] \|_\alpha \\ &\leqslant ch^{k-\beta} \| \xi_i \eta \|_{k;0} + ch^{\alpha-\beta} \| \xi_i \eta \|_\alpha .\end{aligned}$$

Hence, by Lemma 3.2,

$$\| \xi_i \eta \|_\beta \leqslant ch^{k-\beta} \| g \|_{k;2} + ch^{\alpha-\beta} \| \xi_i \eta \|_\alpha . \quad \text{q.e.d.}$$

We are now prepared to estimate $\| A_0^* \eta \|_{0;2}$.

LEMMA 3.4 : *There holds the estimate*

$$\| A_0^* \eta \|_{0;2} \leqslant ch^{-\alpha} \sum_{i=1}^{n} \| \xi_i \eta \|_\alpha + ch^{k-2\alpha} \| g \|_{k;2} . \tag{3.12}$$

Proof : First, we consider the case $\alpha = 0$. Using Lemma 3.1 and the continuity of A_0^*, it follows that

$$\begin{aligned}\| A_0^* \eta \|_{0;2}^2 &= (\kappa h)^2 \| A_0^* \eta \|^2 + \sum_{i=1}^{n} \| \xi_i A_0^* \eta \|^2 \\ &\leqslant ch^2 \| \eta \|^2 + c \| \eta \|_{-1}^2 + c \sum_{i=1}^{n} \| \xi_i \eta \|^2 .\end{aligned}$$

Now, the error estimate (2.8), with $p = 0, -1$, and Lemma 3.3 yield

$$\| A_0^* \eta \|_{0;2} \leqslant ch^{k+1} \| g \|_k + ch^k \| g \|_{k;2} + c \sum_{i=1}^{n} \| \xi_i \eta \| .$$

From this the assertion follows by observing that

$$\| g \|_k \leqslant ch^{-1} \| g \|_{k;2} .$$

Next, we consider the case $\alpha < 0$. Let the integer $r = r(\alpha)$ be defined by

$$r = s \in \mathbb{N}_0 \quad \text{for} \quad -s \leqslant \alpha < -s + 1 .$$

Using (3.5), we obtain the estimate

$$\begin{aligned}\| A_0^* \eta \|_{0;2}^2 &= (\sigma^2 A_0^* \eta, A_0^* \eta) = (\sigma^2 A_0^* \eta - p_h[\sigma^2 A_0^* \eta], A_0^* \eta) \\ &\leqslant \| A_0^* \eta \|_{0;2} \| \sigma^2 A_0^* \eta - p_h[\sigma^2 A_0^* \eta] \|_{0;-2} .\end{aligned}$$

Observing that $k \geqslant r$ and $\sigma^2 A_0^* \eta \in H^r(\Gamma)$, we apply the weighted norm estimate (3.1) to obtain

$$\| A_0^* \eta \|_{0,2} \leqslant ch^r \| \sigma^2 A_0^* \eta \|_{r,-2}$$
$$\leqslant ch^r \{ \| A_0^* \eta \|_{r,2} + \| A_0^* \eta \|_{r-1} + a \}$$

where $a = \| A_0^* \eta \|_{r-2,-2}$, for $r \geqslant 2$, and $a = 0$, for $r = 1$.

For the first term on the right hand side we find in an analogous way as above :

$$\| A_0^* \eta \|_{r,2}^2 \leqslant ch^2 \| A_0^* \eta \|_r^2 + c \sum_{i=1}^{n} \| [A_0^* \xi_i - \xi_i A_0^*] \eta \|_r^2 +$$
$$+ c \sum_{i=1}^{n} \| A_0^*[\xi_i \eta] \|_r^2 + c \| A_0^* \eta \|_{r-1}^2$$
$$\leqslant ch^2 \| \eta \|_{r+2\alpha}^2 + c \| \eta \|_{r+2\alpha-1}^2 + c \sum_{i=1}^{n} \| \xi_i \eta \|_{r+2\alpha}^2 .$$

Hence,

$$\| A_0^* \eta \|_{0,2} \leqslant ch^r \left\{ h \| \eta \|_{r+2\alpha} + \| \eta \|_{r+2\alpha-1} + \sum_{i=1}^{n} \| \xi_i \eta \|_{r+2\alpha} + b \right\},$$

where $b = h^{-1} \| \eta \|_{r+2\alpha-2}$, for $r \geqslant 2$, and $b = 0$, for $r = 1$ The error estimate (2.8) with $p = r + 2\alpha, r + 2\alpha - 1, r + 2\alpha - 2$ (if $r \geqslant 2$), and Lemma 3.3 again lead to the estimate

$$\| A_0^* \eta \|_{0,2} \leqslant ch^{k+1-2\alpha} \| g \|_k + ch^{k-2\alpha} \| g \|_{k,2} + ch^{-\alpha} \sum_{i=1}^{n} \| \xi_i \eta \|_{\alpha} .$$

q.e.d.

Remark : Note that in order to apply (2.8) we need the assumption $p = r + 2\alpha \leqslant m$ which is always satisfied for integer $2\alpha < 0$ and also for real $2\alpha \leqslant -1$ for any $m \in \mathbb{N}_0$. In case $-1 < 2\alpha < 0$, however, we have $r = 1$ and $0 < p = 1 + 2\alpha = r + 2\alpha < 1$ which requires $m \geqslant 1$.

LEMMA 3.5 : *For any $\varepsilon > 0$ there holds*

$$\| \xi_i \eta \|_{\alpha} \leqslant \varepsilon h^{\alpha} \| A_0^* \eta \|_{0,2} + c\left(1 + \frac{1}{\varepsilon}\right) h^{k-\alpha} \| g \|_{k,2}, \quad 1 \leqslant i \leqslant n, \qquad (3.13)$$

where c is independent of ε.

Proof : From the coerciveness of A_0^* on $H^\alpha(\Gamma)$ we obtain

$$\| \xi_i \eta \|_\alpha^2 \leqslant \frac{1}{\gamma_0} \operatorname{Re}(\xi_i \eta, A_0^*[\xi_i \eta]) .$$

Further, by using (3.5) and Lemma 3.1, we find

$$(\xi_i \eta, A_0^*[\xi_i \eta]) = (\xi_i \eta, [A_0^* \xi_i - \xi_i A_0^*] \eta) + (\xi_i^2 \eta - p_h[\xi_i^2 \eta], A_0^* \eta)$$
$$\leqslant c \| \xi_i \eta \|_\alpha \| \eta \|_{\alpha-1} + c \| \xi_i^2 \eta - p_h[\xi_i^2 \eta] \|_{0;-2} \| A_0^* \eta \|_{0;2} .$$

From (3.1) and Lemma 3.2, it follows that

$$\| \xi_i^2 \eta - p_h[\xi_i^2 \eta] \|_{0;-2} \leqslant ch^k \| \xi_i^2 \eta \|_{k;-2} \leqslant ch^k \| g \|_{k;2} .$$

Also, by the error estimate (2.8),

$$\| \eta \|_{\alpha-1} \leqslant ch^{k+1-\alpha} \| g \|_k \leqslant ch^{k-\alpha} \| g \|_{k;2} .$$

Combining the foregoing estimates, we arrive at

$$\| \xi_i \eta \|_\alpha^2 \leqslant c \| \xi_i \eta \|_\alpha h^{k-\alpha} \| g \|_{k;2} + ch^k \| A_0^* \eta \|_{0;2} \| g \|_{k;2} ,$$

which implies (3.13) q.e.d.

From Lemma 3.4 and 3.5 we obtain by choosing ε sufficiently small that

$$\| A_0^* \eta \|_{0;2} \leqslant ch^{k-2\alpha} \| g \|_{k;2} . \tag{3.14}$$

Combining this with (3.6)-(3.9) leads us to the preliminary result

$$| (e, \delta) | \leqslant c\left(\log \frac{1}{h}\right)^{\frac{n}{2}-1} h^{\frac{n}{2}-\frac{3}{2}} h^{k-2\alpha} \| g \|_{k;2} \inf_{\phi_h \in S_h^{k,m}} \| u - \phi_h \|_{L\infty} +$$
$$+ c \{ h^k \| g \|_k + \| g \| \} \| e \|_{2\alpha-1} . \tag{3.15}$$

To bound the norms for g in (3.15), we provide the following.

LEMMA 3.6 : *There hold the estimates*

$$\| g \|_r \leqslant c \| \delta \|_{r-2\alpha} , \quad 0 \leqslant r \leqslant k , \tag{3.16}$$

$$\| g \|_{k;2} \leqslant c \left\{ \| \delta \|_{k-2\alpha-1} + h \| \delta \|_{k-2\alpha} + \sum_{i=1}^{n} \| \xi_i \delta \|_{k-2\alpha} \right\} . \tag{3.17}$$

Proof : (3.16) is an immediate consequence of the a priori estimate (1.10) for the operator A_0^*. To prove (3.17), we write

$$\| g \|_{k;2}^2 \leqslant c \left\{ h^2 \| g \|_k^2 + \sum_{i=1}^{n} \| \xi_i g \|_k^2 + \| g \|_{k-1}^2 \right\}.$$

Then by the a priori estimate (1.10), it follows that

$$\| g \|_{k,2} \leqslant c \left\{ h \| \delta \|_{k-2\alpha} + \| \delta \|_{k-2\alpha-1} + \sum_{i=1}^{n} \| A_0^*[\xi_i g] \|_{k-2\alpha} \right\}.$$

Using Lemma 3.1 we have

$$\| A_0^*[\xi g] \|_{k-2\alpha} \leqslant c \| g \|_{k-1} + c \| \xi_i \delta \|_{k-2\alpha}.$$

Hence, in view of (3.16), we obtain

$$\| g \|_{k,2} \leqslant c \left\{ h \| \delta \|_{k-2\alpha} + \| \delta \|_{k-2\alpha-1} + \sum_{i=1}^{n} \| \xi_i \delta \|_{k-2\alpha} \right\}, \quad \text{q.e.d.}$$

Now we are prepared to prove the pointwise error estimate (2.10). To this end, we take the function δ corresponding to the point $z \in K \in \Pi_h$ as a regularized Dirac function.

LEMMA 3.7 : *There exists a function $\delta \in C_0^\infty(K)$ with the following properties* :

$$\phi_h(z) = (\phi_h, \delta) \quad \text{for all} \quad \phi_h \in S_h^{k,m}, \tag{3.18}$$

$$\| \delta \|_{L^1} \leqslant c, \tag{3.19}$$

$$h^r \| \delta \|_{r-2\alpha} \leqslant c h^{2\alpha+1/2-n/2}, \quad 0 \leqslant r \leqslant k, \tag{3.20}$$

$$h^k \| \xi_i \delta \|_{k-2\alpha} \leqslant c h^{2\alpha+3/2-n/2}, \quad 1 \leqslant i \leqslant n. \tag{3.21}$$

The constants c are independent of z.

Proof of Lemma 3.7 : For constructing the function δ we use a Sobolev representation formula as in the original proof of the embedding theorem; for technical details see [5]. Let ω be a smooth mollifier on $\mathbb{R}^n$ satisfying $\omega \geqslant 0$, $\omega(x) = 0$ for $|x| \geqslant 1$ and $\int \omega \, dx = 1$. On Γ we define the smoothing kernel

$$\omega_h(x) = \left(\int_\Gamma \omega\left(\frac{y}{h}\right) do_y \right)^{-1} \omega\left(\frac{x}{h}\right).$$

According to our assumption (A.1) there are balls B in $\mathbb{R}^n$ with radius $c_1 h$

such that $B \cap \Gamma \subset K$. Then, for any $\phi \in H^k(K)$ there holds the representation

$$\phi(x) = P_k \phi(x) + R_k \phi(x), \quad x \in K,$$

where

$$P_k \phi(x) = \int_K \phi(y) \chi(x, y) \, do_x,$$

$$\chi(x, y) = \sum_{|\alpha| \leqslant k} \frac{(-1)^\alpha}{\alpha !} D_y^\alpha [\omega_{c_1 h}(y) (x - y)^\alpha],$$

$$R_k \phi(x) = k \sum_{|\alpha| = k} \int_K D^\alpha \phi(y) k_\alpha(x, y) \, do_y,$$

$$k_\alpha(x, y) = \frac{1}{\alpha !} (x - y)^\alpha \int_0^1 s^{-n-1} \omega_{c_1 h}\left(x + \frac{1}{s}(y - x)\right) ds$$

with α a multiindex.

Clearly, if ϕ is a polynomial of degree less than k, we have $\phi \equiv P_k \phi$ on K. In the case that $S_h^{k,m}$ consists of functions being piecewise polynomials of degree less than $k - 1$, we may take (for fixed $x \in K$)

$$\delta(y) = \chi(x, y), \quad y \in K,$$

to obtain (3.18). The bounds (3.19) and (3.20) then follow readily from the pointwise bound

$$| D^\alpha \omega_{c_1 h} | \leqslant c h^{1-n-|\alpha|}.$$

In the case of isoparametric elements the functions $\phi_h \in S_h^{k,m}$ may be piecewise polynomials of degree less than k only modulo local coordinate transformations, i.e., $\phi_h|_K(\lambda_K^{-1}[\cdot])$ is a polynomial with some regular transformation onto a reference element $\hat{K}$, $\lambda_K : K \to \hat{K}$. We now construct first a function $\chi \in C_0^\infty(\hat{K})$ as above and then obtain the desired function $\chi \in C_0^\infty(K)$ by using the local transformation λ_K. Since λ_K is assumed to be sufficiently smooth, the bounds (3.18)-(3.20) remain valid. q.e.d.

In view of Lemma 3.7 we find, with any $\phi_h \in S_h^{k,m}$,

$$\begin{aligned} | e(z) | &\leqslant | (u - \phi_h)(z) | + | (\phi_h - u_h)(z) | \\ &\leqslant | (u - \phi_h)(z) | + | (\phi_h - u_h, \delta) | \\ &\leqslant | (u - \phi_h)(z) | + | (e, \delta) | + \| u - \phi_h \|_{L^\infty(K)} \| \delta \|_{L^1}. \end{aligned}$$

Consequently,

$$| e(z) | \leqslant c \inf_{\phi_h \in S_r^{k,m}} \| u - \phi_h \|_{L^\infty} + | (e, \delta) | \tag{3 22}$$

Combining the estimates in Lemma 3 6 and 3 7 with (3 15), we conclude in a straightforward way

$$| (e, \delta) | \leqslant c\left(\log \frac{1}{h}\right)^{\frac{n}{2} - 1} \inf_{\phi_h \in S_r^{k,m}} \| u - \phi_h \|_{L^\infty} + ch^{2\alpha - \frac{n}{2} + \frac{1}{2}} \| e \|_{2\alpha - 1} \tag{3 23}$$

Inserting this estimate into (3 22), we eventually obtain the desired pointwise error estimate (2 10) by applying the following lemma with $\beta = 2\alpha - 1$

LEMMA 3 8 *For any real* β, *with* $2\alpha - k \leqslant \beta \leqslant \alpha$, *there holds*

$$\| e \|_\beta \leqslant ch^{-\beta} \inf_{\phi_h \in S_r^{k,m}} \| u - \phi_h \| \tag{3 24}$$

Proof We employ a standard duality argument For any $\psi \in H^{-\beta}(\Gamma)$ let $v \in H^{2\alpha - \beta}(\Gamma)$ be the solution of $A^* v = \psi$ With the Galerkin approximation $v_h \in S_h^{k,m}$ of v and an arbitrary $\phi_h \in S_h^{k,m}$, there holds

$$\begin{aligned}(e, \psi) &= (Ae, v) = (Ae, v - v_h) = (A[u - \phi_h], v - v_h) \\ &\leqslant c \| u - \phi_h \| \| v - v_h \|_{2\alpha}\end{aligned}$$

Hence, the error estimate (2 8), with $p = 2\alpha$ and $q = 2\alpha - \beta$, and the a priori estimate (1 10), with $s = 2\alpha - \beta$, imply

$$(e, \psi) \leqslant ch^{-\beta} \| \psi \|_{-\beta} \| u - \phi_h \| ,$$

which clearly proves (3 24) q e d

4 PROOF OF THEOREM 2

Let us introduce the operator

$$A_h \phi(z) = \int_\Gamma \gamma_n^h(z - y) \phi(y) \, do_y ,$$

where

$$\gamma_n^h(x) = \int_\Gamma \omega_h(x - y) \gamma_n(y) \, do_y$$

LEMMA 4.1 : *There holds the estimate*

$$\| A\phi - A_h \phi \|_{L^\infty} \leqslant ch \log \frac{1}{h} \| \phi \|_{L^\infty}. \tag{4.1}$$

Proof : We give the argument only for $n = 3$, the minor modifications for $n = 2$ being left to the reader. There holds

$$\begin{aligned} | A\phi(z) - A_h \phi(z) | &= \left| \int_\Gamma \{ \gamma_n(z - y) - \gamma_n^h(z - y) \} \phi(y) \, do_y \right| \\ &\leqslant \| \phi \|_{L^\infty} \int_\Gamma | \gamma_n(z - y) - \gamma_n^h(z - y) | \, do_y . \end{aligned}$$

The integral on the right hand side equals

$$\begin{aligned} &\int_\Gamma \left| \gamma_n(z - y) - \int_\Gamma \omega_h(z - y - x) \gamma_n(x) \, do_x \right| do_y \\ &= \int_\Gamma \int_\Gamma \omega_h(z - y - x) \, | \gamma_n(z - y) - \gamma_n(x) | \, do_x \, do_y \\ &= \int_\Gamma \int_\Gamma \omega_h(z - x) \, | \gamma_n(z - y) - \gamma_n(x - y) | \, do_x \, do_y , \end{aligned}$$

where the coordinate transformation $x' = x + y$ is used. The last integral can be estimated by (see [15], p. 83)

$$\begin{aligned} &ch^{1-n} \int_{|x-z| \leqslant h} \left[\int_\Gamma \frac{| x - z |}{| z - y | \, | x - y |} do_y \right] do_x \\ &\leqslant ch^{2-n} \int_{|x-z| \leqslant h} \left[\int_\Gamma \frac{do_y}{| z - y | \, | x - y |} \right] do_x \\ &\leqslant ch^{2-n} \int_{|x-z| \leqslant h} (1 + | \log | x - y | |) \, do_x \\ &\leqslant ch^{2-n} \int_0^h r^{n-2} (1 + | \log r |) \, \mathrm{d}r \\ &\leqslant ch \log \frac{1}{h} . \end{aligned}$$

q.e.d.

Hence, with $e = u - u_h$ there holds for any $z \in \Gamma$,

$$A_h\, e(z) = (e, \gamma_n^h(z - .)) . \tag{4.2}$$

Now take $\delta = \gamma_n^h(z - .)$ as the right hand side in (3.4). To complete the argument, we need the following lemma (observe here $\alpha = -1/2$) :

LEMMA 4.2 : *There hold the estimates*

$$h^r \| \delta \|_{r+1} \leqslant c h^{\frac{1}{2} - \frac{n}{2}}, \quad 0 \leqslant r \leqslant k , \tag{4.3}$$

$$h^r \| \xi_i \delta \|_{r+1} \leqslant c h^{\frac{3}{2} - \frac{n}{2}}, \quad 1 \leqslant i \leqslant n . \tag{4.4}$$

Proof : By construction, the regularized kernels γ_n^h satisfy

$$| D_y^\alpha \gamma_n^h(z - y) | \leqslant c \sigma(y)^{1-n-|\alpha|}, \quad | \alpha | \geqslant 1 ,$$

where again $\sigma(y) = (| z - y |^2 + (\kappa h^2))^{1/2}$. Observing that

$$\int_\Gamma \sigma^{-r}\, do_x \leqslant c h^{n-1-r}, \quad r \geqslant n ,$$

the bounds (4.3) and (4.4) follow by a straightforward calculation. q.e.d.

Next, we combine the estimates of Lemma 4.2 and 3.7 with (3.15) to obtain

$$| (e, \delta) | \leqslant c \left(\log \frac{1}{h} \right)^{\frac{n}{2} - 1} h \inf_{\phi_h \in S_h^{k,m}} \| u - \phi_h \|_{L^\infty} + c h^{\frac{1}{2} - \frac{n}{2}} \| e \|_{-2} . \tag{4.5}$$

This together with (4.2) and Lemma 4.1 implies that

$$\begin{aligned} | Ae(z) | &\leqslant | Ae(z) - A_h\, e(z) | + | (e, \delta) | \\ &\leqslant c h \log \frac{1}{h} \| e \|_{L^\infty} + c h^{1/2 - n/2} \| e \|_{-2} \\ &\qquad + c \left(\log \frac{1}{h} \right)^{n/2 - 1} h \inf_{\phi_h \in S_h^{k,m}} \| u - \phi_h \|_{L^\infty} . \end{aligned}$$

Combining this with the error estimates (2.10) and (3.24) for e, we obtain the desired result (2.15) for Ae.

APPENDIX

LEMMA A.1 (see also [1]) : *Let* Γ *be sufficiently smooth* (for simplicity C^∞). *Then the single layer operator in* (1.1),

$$Vu(x) = \int_\Gamma \gamma_n(x-y)\, u(y)\, do_y$$

is a pseudodifferential operator of order -1 *on* Γ *having the principal symbol* $\frac{(n-1)}{|\xi|}$ ($n = 2$ or 3) *with respect to the natural representation of* Γ *for* $n = 2$ *and surface polar coordinates for* $n = 3$.

Proof : (i) For $n = 2$, Γ is a curve which can be given by a regular parameter representation $x = x(s)$ being a C^∞ L-periodic $\mathbb{R}^2$-valued function of the arc length. Let $\chi(t)$ be a C_0^∞-function which is identically one in a fixed neighbourhood of zero. We then write

$$\begin{aligned}(Vu)(x(t)) &= -\int \chi(|s-t|) \log|t-s|\, u(x(s))\, ds - \\ &\quad - \int\left\{ \chi(|s-t|) \log\left|\frac{x(t)-x(s)}{t-s}\right| + (1-\chi(|s-t|)) \right. \\ &\quad \left. \times \log|x(t)-x(s)| \right\} u(x(s))\, ds \\ &= \frac{1}{2\pi}\int e^{-i\xi t}\, \hat{u}(\xi)\, a(t,\xi)\, d\xi + Su \qquad \text{(A.1)}\end{aligned}$$

where $Su = \int \{\dots\}\, u(x(s))\, ds$ is an operator with C^∞-kernel and, hence, of order $-\infty$ and where

$$\begin{aligned}a(t,\xi) &= -\int_{\mathbb{R}} e^{i\xi(t-s)}\, \chi(|t-s|) \log|t-s|\, ds \\ &= -\int_{-\infty}^{+\infty} e^{i\xi\sigma}\, \chi(|\sigma|) \log|\sigma|\, d\sigma = a(\xi)\,. \qquad \text{(A.2)}\end{aligned}$$

As is well known, $a(\xi)$ is an analytic function of ξ and admits an asymptotic expansion for large ξ, i.e.,

$$a(\xi) = -\sum_{j=0}^{N-1}\left\{\left(\frac{d}{d\sigma}\right)^j [\chi(|\sigma|)]\right\}\Big|_{\sigma=0}\int_{-\infty}^{+\infty} e^{i\xi\sigma}\frac{1}{j!}\sigma^j \log|\sigma|\,d\sigma + \int_{-\infty}^{+\infty} R_N(\chi;\sigma)\, e^{i\xi\sigma}\log|\sigma|\,d\sigma$$

$$= \frac{\pi}{|\xi|} + O_N(|\xi|^{-N-1}) \quad \text{for} \quad |\xi| \geqslant 1 \tag{A.3}$$

and any natural N. By differentiation of (A.3) with respect to ξ we find

$$\left(\frac{d}{d\xi}\right)^l a(\xi) = (-1)^l \pi l!\, \xi^{-l}|\xi|^{-1} + \int_{-\infty}^{+\infty} (i\sigma)^l R_N(\chi;\sigma)\, e^{i\xi\sigma}\log|\sigma|\,d\sigma \tag{A.4}$$

where the last term is of order $|\xi|^{-N-1-l}$ for $|\xi| \geqslant 1$. Hence, $a(\xi)$ is an amplitude in $S^{-1}_{1,0}$ and the first term in (A.1) defines a standard pseudo-differential operator (see [29] Definition 2.3, p. 16). Since $(Vu)(x(t))$ is C^∞ whenever $t \notin \operatorname{supp} u(x(.))$, our Lemma A.1 is proved (see [29] Proposition 5.1, p. 49).

(ii) For $n = 3$ we represent Γ locally about x by surface polar coordinates $\rho = |x - y|$ and ϕ as in [14], Chap. 2.1.5. We may then write Vu in the form

$$(Vu)(x) = \int_{\rho=0}^{\infty}\int_{\phi=0}^{2\pi} \chi(\rho)\frac{u(y(\rho,\phi))}{\rho}\,do(\rho,\phi) + \int_\Gamma (1 - \chi(|x-y|)) \times \frac{u(y)}{|x-y|}\,do_y . \tag{A.5}$$

The second term in (A.5) has a C^∞-kernel and, hence, defines a pseudodifferential operator of order $-\infty$. The first term can be written as

$$\int_{\rho=0}^{\infty}\int_{\phi=0}^{2\pi} \chi(\rho)\frac{u(y)}{\rho}\,d\sigma_y = \frac{1}{(2\pi)^2}\int e^{-i\xi . t}\,\hat{u}(\xi)\, a(t,\xi)\,d\xi$$

where $x = x(t_1, t_2)$, $t_1 = r\cos\psi$, $t_2 = r\sin\psi$ and

$$a(t,\xi) = \int_{\rho=0}^{\infty}\int_{\phi=0}^{2\sigma} e^{-i(\xi_1\rho\cos\phi+\xi_2\rho\sin\phi)}.\chi(\rho) \times \left\{1 + \frac{1}{4}(2\, b_{lj}(x(t))\, e^l e^j - b_l^l(x(t)))\, b_{km}\, e^k e^m \rho^2 + \text{higher order terms of } \rho\right\} d\rho d\phi \tag{A.6}$$

with $e^1 = \cos \phi$, $e^2 = \sin \phi$, $b_{lj}(x(t))$, respectively $b_l^j(x(t)) = g^{lj}(x(t)) \times b_{jl}(x(t))$ the components of the second fundamental tensor of Γ with respect to the coordinates t_1, t_2 associated to the observation point $x(t)$.

Here the amplitude $a(t, \xi)$ is again analytic with respect to ξ and C^∞ with respect to t. Taylor's expansion with respect to ρ about 0 shows an asymptotic expansion of the form

$$a(t, \xi) = \frac{2\pi}{|\xi|} + A_3(t, \xi) + \cdots, \qquad (A.7)$$

where $A_k(t, \xi)$ is C^∞ with respect to t and is a positive homogeneous function of degree $-k$ of ξ. It follows explicitly that $a(.)$ is an amplitude in $S_{1,0}^{-1}$, defining a standard pseudodifferential operator. By definition, Vu is C^∞ whenever $x \notin \operatorname{supp} u$. Thus, our lemma is valid also in case $n = 3$. q.e.d.

LEMMA A.2 : *The operator V satisfies Gårding's inequality*

$$(Vv, v) \geqslant \gamma_0 \| v \|_{-1/2}^2 - (V_1 v, v) \qquad (A.8)$$

where $\gamma_0 > 0$ and V_1 is a pseudodifferential operator of order -2. (For $n = 2$ see also [11] and for $n = 3$ [18].)

Proof : To the principal symbol of V there exists a symbol σ_0 in $S_{1,0}^{+1}$ such that $\sigma_0 \in C^\infty$ and

$$\sigma_0(\xi) = (\pi(n-1))^{-1} |\xi| \quad \text{for} \quad |\xi| \geqslant 1$$

and $\sigma_0(\xi) \geqslant \kappa > 0$ for *all* ξ. To σ_0 there exists a pseudodifferential operator W_0 having this symbol and satisfying Gårding's inequality (see [8])

$$(W_0 w, w) \geqslant c_0 \| w \|_{1/2}^2 - c_1 \| w \|_0^2, \quad c_0 > 0 .$$

Then $V_0^{-1} := W_0 + c_1 I$ is strongly coercive and, hence invertible. V_0 has the symbol

$$(\sigma_0(\xi) + c_1)^{-1}$$

with the asymptotic expansion $\dfrac{\pi(n-1)}{|\xi|} \displaystyle\sum_{j=0}^{\infty} \left(- \frac{\pi(n-1)\, c_1}{|\xi|} \right)^j$ for large enough $|\xi|$. With $w = V_0 v$ we then find

$$(V_0 v, v) = (w, V_0^{-1} w) \geqslant c_0 \| w \|_{1/2}^2 \geqslant \gamma_0 \| V_0^{-1} w \|_{-1/2}^2 = \gamma_0 \| v \|_{-1/2}^2 .$$

The difference $V_1 = V - V_0$ then has a symbol in the class $S_{1,0}^{-2}$ and is a pseudodifferential operator of order -2. q.e.d.

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