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A TECHNIQUE OF UPSTREAM TYPE APPLIED TO A LINEAR NONCONFORMING FINITE ELEMENT APPROXIMATION OF CONVECTIVE DIFFUSION EQUATIONS (*)

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Summary — *We present a technıque of upstream type ın a lınear nonconformıng fınıte element approxımatıon of convectıve dıffusıon equatıons It ıs then shown that thıs scheme satısfies the dıscrete maxımum prıncıple and leads to an $O(h)$ error estımate ın H^1-norm Some numerıcal examples are gıven for the model problem*

Résume — *On presente une technıque de décentrage dans l'approxımatıon par un élément finı de degre un non conforme des equatıons de dıffusıon-convectıon Ensuıte, ıl est montré que ce schéma satısfaıt au prıncıpe du maxımum dıscret et conduıte à l'estımatıon $O(h)$ d'erreur dans $H^1(\Omega)$ Quelques exemples numérıques sont donnés pour le problème modèle*

INTRODUCTION

In this note a technıque of upstream type is introduced in a linear nonconforming finite element approximation of convective diffusion equations. The nonconforming element under consideration here ıs so-called a piecewise linear element using *Loof connections*, which were thoroughly investigated by Crouzeıx and Raviart [7] and Temam [16] from the theoretical interest

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occured in studying the approximations of incompressible flow problems. For practical computations, see also the recent book by Thomasset [17].

On the other hand, several techniques of upstream type to the usual piecewise linear element were developed in recent years in Japan (Baba and Tabata [1], Ikeda [10], Kanayama [11], Kikuchi and Ushijima [13]). Our present method is an extension of one of such techniques to the considered nonconforming element, which is obtained along the way of the modification of the bilinear form corresponding to the convective term, mentioned in Kikuchi and Ushijima [13]. Then we introduce barycentric domains corresponding to mid-points of sides of all triangles belonging to the triangulation T_h in order to define the lumped regions. Recently, Dervieux and Thomasset [8] also proposed the barycentric domain associated with the linear nonconforming finite element in order to derive an upwind scheme to the convective term. However, their scheme is different from our scheme.

An outline of the paper is as follows. In Section 1, notation and the model problem are presented. Section 2 is devoted to the construction of a lumping method based on the considered nonconforming element. In Section 3, our technique of upstream type is proposed. Then we show the discrete maximum principle for our scheme in Section 4, and an $O(h)$ H^1 error estimate in Section 5. In Section 6, we give some numerical examples.

The authors would like to express their sincere thanks to the referee of this paper for his valuable comments and constructive recommendations which are most helpful to improve an earlier version of the paper. Following his idea Lemma 3 is obtained, which clarifies a feature of our technique of upstream type.

1. NOTATION AND PRELIMINARIES

Let Ω be a polygonal bounded connected domain of $\mathbb{R}^2$ with the boundary Γ. For a non-negative integer m, let $H^m(\Omega)$ be the usual mth order Sobolev space equipped with the norm and the semi-norm

$$\| v \|_{m,\Omega} = \left(\sum_{|\alpha| \leqslant m} \| D^\alpha v \|_{0,\Omega}^2 \right)^{1/2}, \tag{1.1}$$

$$| v |_{m,\Omega} = \left(\sum_{|\alpha| = m} \| D^\alpha v \|_{0,\Omega}^2 \right)^{1/2}, \tag{1.2}$$

where $\| \cdot \|_{0,\Omega}$ is the norm of $L^2(\Omega)$. The scalar product in $L^2(\Omega)$ is given by $(\cdot, \cdot)$. We set as usual

$$H_0^1(\Omega) = \{ v \in H^1(\Omega) ; v |_\Gamma = 0 \} . \tag{1.3}$$

We consider the following stationary convective diffusion equation

$$(E)\quad \begin{cases} -\nu\,\Delta u + (\mathbf{b}.\nabla)\,u = f & \text{in } \Omega\,, \quad (1.4)\\ \qquad\qquad\qquad u = u_0 & \text{on } \Gamma\,, \quad (1.5)\end{cases}$$

where ν is a positive constant, $\mathbf{b} = \mathbf{b}(x) \in C^1(\bar{\Omega})^2$, $f \in L^2(\Omega)$ and $u_0 \in H^1(\Omega)$.

Let $a(u, v)$ and $b(u, v)$ be two bilinear forms on $H^1(\Omega) \times H^1(\Omega)$ defined by

$$a(u, v) = \int_\Omega \nabla u . \nabla v \, dx\,, \tag{1.6}$$

$$b(u, v) = \int_\Omega (\mathbf{b}.\nabla u)\, v \, dx\,. \tag{1.7}$$

Furthermore, we set

$$t(u, v) = \nu a(u, v) + b(u, v)\,. \tag{1.8}$$

We consider the variational formulation (Π) for (E) :

$$(\Pi)\quad \begin{cases} \text{Find } u \in H^1(\Omega) \text{ such that} \\ t(u, v) = (f, v) \quad \text{for all} \quad v \in H_0^1(\Omega)\,, & (1.9)\\ u - u_0 \in H_0^1(\Omega)\,. & (1.10)\end{cases}$$

This problem has a unique solution under the condition that there exists a positive constant α_1 such that

$$\nu\alpha_0 - 1/2.\operatorname{div}\mathbf{b} \geqslant \alpha_1 > 0 \quad \text{in } \Omega\,, \tag{1.11}$$

where $\alpha_0 > 0$ is less than or equal to the minimum eigenvalue of $-\Delta$ with Dirichlet boundary condition.

It is well known that the *maximum principle* holds for the solution of (Π) in the following form (*cf.* Courant and Hilbert [6]) :

Assume that the solution u of (Π) *is continuous on* $\bar{\Omega}$ *and twice continuously differentiable in* Ω. *Then it holds that*

$$\max_{x \in \bar{\Omega}} u(x) \leqslant \max_{x \in \Gamma} u_0(x) \quad \textit{when} \quad f \leqslant 0 \quad \textit{in } \Omega\,. \tag{1.12}$$

2. NONCONFORMING FINITE ELEMENT AND LUMPING OPERATOR

In this section we shall consider an approximation of upstream type for the convective term using the linear nonconforming finite element.

Let $\{ T_h \}$ be a family of triangulation of $\overline{\Omega}$ made of open triangles K, that is

$$\overline{\Omega} = \bigcup_{K \in T_h} \overline{K}, \tag{2.1}$$

where any two triangles are either disjoint or share at most one side or one vertex. Let h_K be the maximum side length of $K \in T_h$ and ρ_K be the diameter of the inscribed circle in K. Moreover, we set $h = \max_{K \in T_h} h_K$.

In what follows, we assume that $\{ T_h \}$ is *regular*. That is, when h tends to 0, there exists a constant $\sigma > 0$, independent of h and K, such that

$$\sigma_K = h_K/\rho_K \leqslant \sigma \quad \text{for all} \quad K \in T_h . \tag{2.2}$$

Let us recall the linear nonconforming finite element studied by Crouzeix and Raviart [7]. Let B_i, $1 \leqslant i \leqslant N$, be the mid-points of sides lying in the interior of Ω and B_i, $N + 1 \leqslant i \leqslant N + M$, be the mid-points of sides lying on Γ. Let V_h be the linear nonconforming finite element approximate space of $H^1(\Omega)$ defined by

$$V_h = \{ v_h \in L^2(\Omega) : v_h \text{ is linear on } K \in T_h \text{ and is continuous at } B_i, \\ 1 \leqslant i \leqslant N + M \} . \tag{2.3}$$

Furthermore, we define

$$V_{0h} = \{ v_h \in V_h ; v_h = 0 \quad \text{at } B_i, N + 1 \leqslant i \leqslant N + M \} . \tag{2.4}$$

Observe that $V_h \not\subset H^1(\Omega)$ and $V_{0h} \not\subset H_0^1(\Omega)$.

Let w_{ih}, $1 \leqslant i \leqslant N + M$, be the elements of V_h such that

$$w_{ih}(B_j) = \delta_{ij} \quad \text{for} \quad 1 \leqslant i, j \leqslant N + M , \tag{2.5}$$

where δ_{ij} is the Kronecker delta. Then the sets of functions $\{ w_{ih} ; 1 \leqslant i \leqslant N + M \}$, and $\{ w_{ih} ; 1 \leqslant i \leqslant N \}$, form bases of V_h, and V_{0h}, respectively. This element, however, satisfies the following *compatibility conditions*:

$(N1)$ For any K_1, $K_2 \in T_h$, it holds that

$$\int_{\partial\Gamma_{12}} (v_h |_{K_1} - v_h |_{K_2}) \, d\gamma = 0 \quad \text{for all} \quad v_h \in V_h , \tag{2.6}$$

where $\partial\Gamma_{12} = \partial K_1 \cap \partial K_2$.

$(N2)$ For any $K \in T_h$, it holds that

$$\int_{\partial K \cap \Gamma} v_h |_K \, d\gamma = 0 \quad \text{for all} \quad v_h \in V_{0h} . \tag{2.7}$$

We provide the space V_h with the following norm and semi-norm :

$$\| v_h \|_{1,h} = \left(\sum_{K \in T_h} \| v_h \|_{1,K}^2 \right)^{1/2}, \tag{2.8}$$

$$\| v_h \|_h = \left(\sum_{K \in T_h} | v_h |_{1,K}^2 \right)^{1/2}. \tag{2.9}$$

The above conditions $(N1)$ and $(N2)$ imply that $\| \cdot \|_h$ is a norm over the space V_{0h}.

Next, we define the *barycentric domain* associated with the linear nonconforming finite element. For any $K \in T_h$ with vertices $P_{i,K}$, $1 \leqslant i \leqslant 3$, let $B_{i,K}$ be the mid-point of the side K_i' opposite to $P_{i,K}$, $1 \leqslant i \leqslant 3$, and G_K be the barycenter of K. Consider the triangle S_{ij}, $1 \leqslant i, j \leqslant 3$, $i \neq j$, with vertices G_K, $B_{i,K}$ and $P_{k,K}$, where $k \neq i, j$. We say that S_{ij} is a *barycentric fragment* of K. Then, with each $B_{i,K}$, $1 \leqslant i \leqslant 3$, we associate a *barycentric subdomain* S_i^K as follows :

$$S_i^K = \bigcup_{j \neq i} S_{ij}. \tag{2.10}$$

If K_1 and K_2 are adjacent elements having B_i as its common mid-point, we say that $\Omega_i = S_i^{K_1} \cup S_i^{K_2}$ is the *barycentric domain* with respect to B_i. If $B_i \in \Gamma$, we set $\Omega_i = S_i^K$. Furthermore, with each B_i, $1 \leqslant i \leqslant N + M$, we associate the index set

$$\Lambda_i = \{ j \neq i ;\ B_j \text{ is the mid-point of the side of a triangle having } B_i \text{ as another one} \}. \tag{2.11}$$

For any $j \in \Lambda_i$, $1 \leqslant i \leqslant N$, we set as follows :

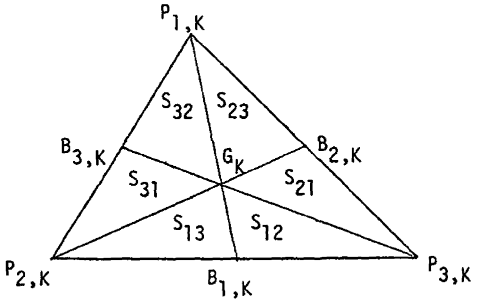
$$\Gamma_{ij}^S = \partial S_i^K \cap \partial S_j^K. \tag{2.12}$$

If B_i is the mid-point of the side lying in the interior of Ω, we have

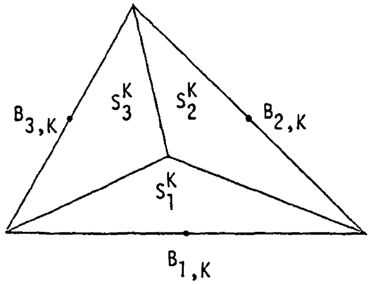
$$\partial \Omega_i = \bigcup_{j \in \Lambda_i} \Gamma_{ij}^S. \tag{2.13}$$

In our linear nonconforming finite element approximation, this barycentric domain plays the role of the *lumping region* in the usual conforming finite element approximation (see Kikuchi and Ushijima [13]).

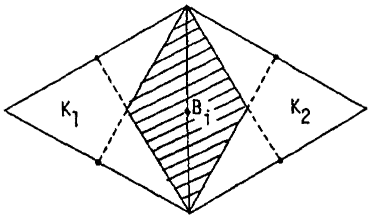
Let $\overline{w}_{ih}$ be the characteristic function of Ω_i and $\overline{V}_h$ be the linear space spanned by the functions $\overline{w}_{jh}$, $1 \leqslant j \leqslant N + M$. Let L_h be the *lumping* operator from



Barycentric fragments S_{ij} .



Barycentric subdomains S_i^K .



Barycentric domain Ω_i

Fig. 1. — Lumping region.

V_h onto $\bar{V}_h$ defined by

$$V_h \ni v_h = \sum_{j=1}^{N+M} V_j\, w_{jh} \mapsto L_h\, v_h = \bar{v}_h = \sum_{j=1}^{N+M} V_j\, \bar{w}_{jh} \in \bar{V}_h . \tag{2.14}$$

It is easily seen that the lumping operator L_h satisfies the following properties :

$$\| v_h \|_{0,\Omega} = \| L_h\, v_h \|_{0,\Omega} \quad \text{for all} \quad v_h \in V_h , \tag{2.15}$$

$$\| v_h - L_h\, v_h \|_{0\,\Omega} \leqslant h \,\| v_h \|_h \quad \text{for all} \quad v_h \in V_h . \tag{2.16}$$

3. UPSTREAM-LIKE SCHEME OF THE NONCONFORMING TYPE

We define the following approximate bilinear forms on $V_h \times V_h$:

$$a_h(u_h, v_h) = \sum_{K \in T_h} \int_K \nabla u_h . \nabla v_h \, dx , \tag{3.1}$$

$$b_h(u_h, v_h) = \sum_{K \in T_h} \int_K (\mathbf{b} . \nabla u_h) \, v_h \, dx . \tag{3.2}$$

Then we set for any $u_h, v_h \in V_h$

$$t_h(u_h, v_h) = \nu a_h(u_h, v_h) + b_h(u_h, v_h) . \tag{3.3}$$

In [14] we have considered the following approximate problem of Galerkin type :

$$(\Pi_h) \quad \begin{cases} \text{Find } u_h \in V_h \text{ such that} & \\ t_h(u_h, v_h) = (f, v_h) \quad \text{for all} \quad v_h \in V_{0h} , & (3.4) \\ u_h - u_{0h} \in V_{0h} , & (3.5) \end{cases}$$

where $u_{0h} \in V_h$ is chosen so that $u_{0h}(B_i) = u_0(B_i)$, $N + 1 \leqslant i \leqslant N + M$.

Now, we shall consider the modification of $b_h(u_h, v_h)$ by using the lumping process with the barycentric domain, following the procedure stated in Kikuchi and Ushijima [13] for the case of conforming piecewise linear approximation. In the first time, we rewrite $b_h(u_h, v_h)$ as follows :

$$b_h(u_h, v_h) = b_h^1(u_h, v_h) + b_h^2(u_h, v_h) , \tag{3.6}$$

where

$$b_h^1(u_h, v_h) = \sum_{K \in T_h} \int_K (\operatorname{div} u_h \, \mathbf{b}) \, v_h \, dx , \tag{3.7}$$

and

$$b_h^2(u_h, v_h) = - \sum_{K \in T_h} \int_K (\operatorname{div} \mathbf{b}) \, u_h \, v_h \, dx . \tag{3.8}$$

Then we modify $b_h^1(u_h, v_h)$ by $b_h^1(u_h, L_h v_h)$. According to the patch-wise application of the Gauss divergence formula, it can be easily verified that

$$b_h^1(u_h, L_h v_h) = \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}^S} (\mathbf{b} . \mathbf{n}_j) \, u_h \, d\gamma \, v_h(B_j) + \sum_{j=N+1}^{N+M} \int_{\partial \Omega_j \cap \Gamma} (\mathbf{b} . \mathbf{n}_j) \, u_h \, d\gamma \, v_h(B_j) , \tag{3.9}$$

where $\mathbf{n}_j$ is the unit outer normal vector along $\partial\Omega_j$. Taking account of (3.9) we define the modified form $\tilde{b}_h^1(u_h, v_h)$ as follows :

$$\tilde{b}_h^1(u_h, v_h) = \sum_{j=1}^{N+M} \sum_{k\in\Lambda_j} \int_{\Gamma_{jk}^S} (\mathbf{b}.\mathbf{n}_j)\, u_h^{jk}\, d\gamma\, v_h(B_j) + \sum_{j=N+1}^{N+M} \int_{\partial\Omega_j\cap\Gamma} (\mathbf{b}.\mathbf{n}_j)\, d\gamma\, u_h(B_j)\, v_h(B_j)\,, \quad (3.10)$$

where

$$\left.\begin{aligned} u_h^{jk} &= \lambda_{jk}\, u_h(B_j) + (1-\lambda_{jk})\, u_h(B_k)\,,\\ \lambda_{jk} &= 1 - \lambda_{kj}\,,\\ |\lambda_{jk}| &\leqslant \Lambda \quad (\Lambda \text{ is a constant independent of } j,\ k \text{ and } h)\,. \end{aligned}\right\} \quad (3.11)$$

Furthermore, $b_h^2(u_h, v_h)$ is modified by $b_h^2(L_h u_h, L_h v_h)$ which is denoted by $\tilde{b}_h^2(u_h, v_h)$. Then we have

$$\tilde{b}_h^2(u_h, v_h) = -\sum_{j=1}^{N+M} \sum_{k\in\Lambda_j} \int_{\Gamma_{jk}^S} (\mathbf{b}.\mathbf{n}_j)\, d\gamma\, u_h(B_j)\, v_h(B_j) - \sum_{j=N+1}^{N+M} \int_{\partial\Omega_j\cap\Gamma} (\mathbf{b}.\mathbf{n}_j)\, d\gamma\, u_h(B_j)\, v_h(B_j)\,. \quad (3.12)$$

Thus we can define the *modified* form $\tilde{b}_h(u_h, v_h)$ as follows :

$$\begin{aligned} \tilde{b}_h(u_h, v_h) &= \tilde{b}_h^1(u_h, v_h) + \tilde{b}_h^2(u_h, v_h)\\ &= \sum_{j=1}^{N+M} \sum_{k\in\Lambda_j} \int_{\Gamma_{jk}^S} (\mathbf{b}.\mathbf{n}_j)\,(u_h^{jk} - u_h(B_j))\, d\gamma\, v_h(B_j)\,. \end{aligned} \quad (3.13)$$

Remark 1 : If we take λ_{jk} as follows, then (3.13) yields the upstream scheme for the convective term :

$$\lambda_{jk} = \begin{cases} 1 & \left(\text{if } \displaystyle\int_{\Gamma_{jk}^S} (\mathbf{b}.\mathbf{n}_j)\, d\gamma \geqslant 0\right),\\ 0 & \text{(otherwise)}\,. \end{cases} \quad (3.14)$$

Finally, we define our modified form of $t_h(u_h, v_h)$:

$$\tilde{t}_h(u_h, v_h) = \nu a_h(u_h, v_h) + \tilde{b}_h(u_h, v_h)\,. \quad (3.15)$$

Hence, our approximate problem is written as follows :

$$(\tilde{\Pi}_h) \begin{cases} \text{Find } \tilde{u}_h \in V_h \text{ such that} & \\ \tilde{l}_h(\tilde{u}_h, v_h) = (f, v_h) \quad \text{for all} \quad v_h \in V_{0h}, & (3.16) \\ \tilde{u}_h - u_{0h} \in V_{0h}. & (3.17) \end{cases}$$

4. DISCRETE MAXIMUM PRINCIPLE

This section is devoted to the study of the discrete maximum principle for the upstream-like scheme $(\tilde{\Pi}_h)$. From now on, we assume that the triangulation T_h is of *acute type*, that is, it holds that

$$\tau_K \leqslant 0 \quad \text{for all} \quad K \in T_h, \tag{4.1}$$

where

$$\tau_K = \max_{i \neq j} \{ \cos (\nabla \mu_i, \nabla \mu_j)_{\mathbf{R}^2} \} \tag{4.2}$$

and μ_i, $1 \leqslant i \leqslant 3$, are barycentric coordinates of $x \in K$ with respect to the mid-points $B_{i,K}$, $1 \leqslant i \leqslant 3$, of the sides of K.

Remark 2 : We note that the above definition of the *acuteness* is equivalent to the usual one (*cf.* Fujii [9]). Hence (4.1) implies that all the angles of the triangles of T_h are less than or equal to $\pi/2$.

Let

$$\tilde{b}_{jk} = \tilde{b}_h(w_{kh}, w_{jh}) \quad \text{for} \quad 1 \leqslant j, k \leqslant N + M, \tag{4.3}$$

then we have

LEMMA 1 : *It holds that*

$$\tilde{b}_{jj} = - \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}^S} (1 - \lambda_{jk}) (\mathbf{b}.\mathbf{n}_j) \, d\gamma, \tag{4.4}$$

$$\tilde{b}_{jk} = \begin{cases} \int_{\Gamma_{jk}^S} (1 - \lambda_{jk}) (\mathbf{b}.\mathbf{n}_j) \, d\gamma & (\text{if} \quad k \in \Lambda_j), \\ 0 & (\text{if} \quad k \notin \Lambda_j), \end{cases} \tag{4.5}$$

and

$$\sum_{k=1}^{N+M} \tilde{b}_{jk} = 0 \quad \text{for} \quad 1 \leqslant j \leqslant N + M. \tag{4.6}$$

Proof : By the definition of $\tilde{b}_h(.,.)$ we have

$$\tilde{b}_{jk} = \sum_{l=1}^{N+M} \delta_{jl} \sum_{m \in \Lambda_l} \int_{\Gamma_{lm}^S} (\mathbf{b}.\mathbf{n}_l) (1 - \lambda_{lm}) (\delta_{km} - \delta_{kl}) \, d\gamma,$$

where δ_{ij} is the Kronecker delta. Hence, (4.4) and (4.5) hold. On the other hand, it holds that

$$\sum_{k=1}^{N+M} \tilde{b}_{jk} = \tilde{b}_{jj} + \sum_{k \in \Lambda_j} \tilde{b}_{jk} .$$

Then (4.6) follows from (4.4) and (4.5).

If we take λ_{jk} as in (3.14), then we have from Lemma 1,

$$\tilde{b}_{jk} \leqslant 0 \quad \text{for} \quad 1 \leqslant j, k \leqslant N + M , \quad j \neq k . \tag{4.7}$$

We set

$$a_{jk} = a_h(w_{kh}, w_{jh}) \quad \text{for} \quad 1 \leqslant j \leqslant N , \quad 1 \leqslant k \leqslant N + M . \tag{4.8}$$

By an analogous discussion to Kikuchi [12] we have

$$\left.\begin{aligned} \sum_{k=1}^{N+M} a_{jk} = 0 \quad & \text{for} \quad 1 \leqslant j \leqslant N , \\ a_{jk} \leqslant 0 \quad & \text{for} \quad 1 \leqslant j \leqslant N , \quad 1 \leqslant k \leqslant N + M , \quad j \neq k . \end{aligned}\right\} \tag{4.9}$$

The proof of (4.9) can be found in [14].

Now let us return to the scheme $(\tilde{\Pi}_h)$. Observe that $(\tilde{\Pi}_h)$ is equivalent to the following linear system :

$$\left.\begin{aligned} TU + T_1 V &= F , \qquad (4.10) \\ V &= G , \qquad (4.11) \end{aligned}\right\}$$

where

$$\left.\begin{aligned} T &= (\tilde{t}_{ij}) = (\nu a_{ij} + \tilde{b}_{ij}) \quad \text{for} \quad 1 \leqslant i, j \leqslant N , \\ T_1 &= (\tilde{t}_{ij}) \quad \text{for} \quad 1 \leqslant i \leqslant N , \quad N + 1 \leqslant j \leqslant N + M , \\ U &= (U_j) = (\tilde{u}_h(B_j)) \quad \text{for} \quad 1 \leqslant j \leqslant N , \\ V &= (U_j) \quad \text{for} \quad N + 1 \leqslant j \leqslant N + M , \\ F &= (F_j) = (f(B_j).\text{mes}\,(\text{supp}\,(w_{jh}))/3) \quad \text{for} \quad 1 \leqslant j \leqslant N , \\ G &= (G_j) = (u_0(B_j)) \quad \text{for} \quad N + 1 \leqslant j \leqslant N + M . \end{aligned}\right\} \tag{4.12}$$

We define the matrix

$$T_0 = (\tilde{t}_{ij}) \quad \text{for} \quad 1 \leqslant i \leqslant N , \quad 1 \leqslant j \leqslant N + M . \tag{4.13}$$

Then we have the following Lemma from (4.6), (4.7) and (4.9).

LEMMA 2 : *If we take λ_{jk} as in* (3.14), *then the matrix T_0 is of non-negative type, that is*

$$\left.\begin{array}{l} \tilde{t}_{ij} \leqslant 0 \quad \textit{for} \quad i \neq j, \quad 1 \leqslant i \leqslant N, \quad 1 \leqslant j \leqslant N+M, \\ \sum_{j=1}^{N+M} \tilde{t}_{ij} \geqslant 0 \quad \textit{for} \quad 1 \leqslant i \leqslant N. \end{array}\right\} \tag{4.14}$$

THEOREM 1 : *Assume that the triangulation T_h is of acute type and that the matrix T is invertible. If we take λ_{jk} as in* (3.14), *then we have*

$$\max_{1 \leqslant j \leqslant N} U_j \leqslant \max\left(0, \max_{1 \leqslant j \leqslant M} G_{N+j}\right) \quad \text{if} \quad \max_{1 \leqslant j \leqslant N} F_j \leqslant 0. \tag{4.15}$$

Proof : This fact comes from general considerations due to Ciarlet [3]. For the sake of completeness we, however, give a direct proof as follows.

First we prove (4.15) in the case of

$$\max_{1 \leqslant j \leqslant N} F_j < 0. \tag{4.16}$$

Let $U_i = \max_{1 \leqslant j \leqslant N} U_j$. When $U_i \leqslant 0$, then the assertion is trivial. Then we let $U_i > 0$. Assume that $U_i > \max_{1 \leqslant j \leqslant M} U_{N+j}$. Since

$$\sum_{j=1}^{N} \tilde{t}_{ij} U_j + \sum_{j=1}^{M} \tilde{t}_{i,N+j} U_{N+j} = F_i \quad \text{for} \quad 1 \leqslant i \leqslant N,$$

we have

$$\begin{aligned} \tilde{t}_{ii} U_i &= \sum_{\substack{j=1 \\ j \neq i}}^{N} (-\tilde{t}_{ij}) U_j + \sum_{j=1}^{M} (-\tilde{t}_{i,N+j}) U_{N+j} + F_i \\ &\leqslant -U_i \sum_{\substack{j=1 \\ j \neq i}}^{N+M} \tilde{t}_{ij} + F_i, \end{aligned} \tag{4.17}$$

where we use in the last inequality the fact that the matrix T_0 is non-negative type by Lemma 2. Therefore we have

$$0 > F_i \geqslant U_i \sum_{j=1}^{N+M} \tilde{t}_{ij} \geqslant 0, \tag{4.18}$$

which is a contradiction. Thus we obtain (4.15).

Next, we prove (4.15) for the case

$$\max_{1 \leqslant j \leqslant N} F_j \leqslant 0. \tag{4.19}$$

We set a column vector F_ε for any $\varepsilon > 0$ as follows :

$$F_\varepsilon = (F_{\varepsilon j}) = (F_j - \varepsilon) \quad \text{for} \quad 1 \leqslant j \leqslant N . \tag{4.20}$$

Then we have $\max\limits_{1 \leqslant j \leqslant N} F_{\varepsilon j} < 0$. Let U_ε be the solution of the following equation :

$$\left.\begin{aligned} TU + T_1 V &= F_\varepsilon , \\ V &= G . \end{aligned}\right\} \begin{aligned} (4.21) \\ (4.22) \end{aligned}$$

For the solution U_ε of (4.21)-(4.22), (4.15) holds by the first half of this proof. Furthermore, since the matrix T is invertible, we have

$$U_\varepsilon = T^{-1}(F_\varepsilon - T_1 V) \tag{4.23}$$

and

$$U_\varepsilon \to U \quad \text{as} \quad \varepsilon \to 0 . \tag{4.24}$$

Thus our assertion is completely proved.

Remark 3 : In Theorem 1, if the triangulation T_h is of *strictly acute type*, namely if all the angles of triangles of T_h are less than $\pi/2$, then the matrix T is invertible. Another condition to assert the invertibility of T is that div $\mathbf{b}$ is non-positive in Ω, which will be shown in Theorem 3.

5. ERROR ESTIMATE FOR UPSTREAM-LIKE APPROXIMATION

In the first time, we show that the modified form $\tilde{b}_h(.,.)$ is *admissible* in the sence defined in Kikuchi and Ushijima [13].

THEOREM 2 : *Assume that there exists a constant $D > 0$ independent of j, k and h such that*

$$h.\text{mes}\,(\Gamma_{jk}^S) \leqslant D.\text{mes}\,(S_{jk}) , \tag{5.1}$$

then there exists a constant $C > 0$ independent of h $(0 < h \leqslant \bar{h})$ such that

$$| b_h(u_h, v_h) - \tilde{b}_h(u_h, v_h) | \leqslant Ch \, \| u_h \|_{1,h} \, \| v_h \|_{1,h} \tag{5.2}$$

for all $u_h, v_h \in V_h$.

Proof : We follow the proof of Proposition 2 of [13] with suitable modifications. We may write for all $u_h, v_h \in V_h$

$$\begin{aligned} b_h(u_h, v_h) - \tilde{b}_h(u_h, v_h) = b_h^1(u_h, v_h) - b_h^1(u_h, L_h v_h) + b_h^1(u_h, L_h v_h) \\ - \tilde{b}_h^1(u_h, v_h) + b_h^2(u_h, v_h) - \tilde{b}_h^2(u_h, v_h) . \end{aligned}$$

Using (2.13) and (2.14), one can easily check that

$$| b_h^1(u_h, v_h) - b_h^1(u_h, L_h v_h) | \leqslant C_1 h \| u_h \|_{1,h} \| v_h \|_{1,h} \tag{5.3}$$

and

$$| b_h^2(u_h, v_h) - \tilde{b}_h^2(u_h, v_h) | \leqslant C_2 h \| u_h \|_{1,h} \| v_h \|_{1,h} \tag{5.4}$$

with the appropriate constants C_i, $1 \leqslant i \leqslant 2$.

Thus it suffices to estimate $b_h^1(u_h, L_h v_h) - \tilde{b}_h^1(u_h, v_h)$. From (3.7) and (3.8) we find that

$$\begin{aligned} b_h^1(u_h, L_h v_h) - \tilde{b}_h^1(u_h, v_h) &= \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}^S} (\mathbf{b}.\mathbf{n}_j)(u_h - u_h^{jk})\, d\gamma\, v_h(B_j) \\ &+ \sum_{j=N+1}^{N+M} \int_{\Omega_j \cap \Gamma} (\mathbf{b}.\mathbf{n}_j)(u_h - u_h(B_j))\, d\gamma\, v_h(B_j) = I_1 + I_2 . \end{aligned} \tag{5.5}$$

Taking into account that $\Gamma_{jk}^S = \Gamma_{kj}^S$ and $\mathbf{n}_j = -\mathbf{n}_k$, we have

$$\begin{aligned} I_1 = 1/2 \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}^S} (\mathbf{b}.\mathbf{n}_j) \{ \lambda_{jk}(u_h - u_h(B_j)) \\ + (1 - \lambda_{jk})(u_h - u_h(B_k)) \}\, d\gamma\, (v_h(B_j) - v_h(B_k)) . \end{aligned} \tag{5.6}$$

On the other hand, since u_h is linear on $\bar{S}_{jk}$ it is easy to check that for any $x \in \Gamma_{jk}^S$

$$| u_h(x) - u_h(B_j) | \leqslant h \operatorname{mes}(S_{jk})^{-1/2} \| \nabla v_h \|_{L^2(S_{jk})} \tag{5.7}$$

and

$$| v_h(B_j) - v_h(B_k) | \leqslant h \operatorname{mes}(S_{jk})^{-1/2} \| \nabla v_h \|_{L^2(S_{jk})} . \tag{5.8}$$

Then from the properties of λ_{jk}, (5.7), (5.8) and the assumption (5.1), we find that

$$| I_1 | \leqslant C_3 h \| u_h \|_{1,h} \| v_h \|_{1,h} . \tag{5.9}$$

Next, let us estimate for I_2. Here we assume that the mid-points of sides lying on Γ, B_i, $N + 1 \leqslant i \leqslant N + M$, are located consecutively on the boundary in anti-clockwise orientation such that

$$B_j = \text{mid-point of } \overline{P_j P_{j+1}} \quad \text{for} \quad N + 1 \leqslant j \leqslant N + M , \tag{5.10}$$

where P_j, $N + 1 \leqslant j \leqslant N + M$, are boundary vertices with $P_{N+M+1} = P_{N+1}$.

Then we may write

$$I_2 = \sum_{j=N+1}^{N+M} \int_{P_j}^{P_{j+1}} (\mathbf{b}(B_j).\mathbf{n}_j)\,(u_h - u_h(B_j))\,d\gamma\; v_h(B_j)$$

$$+ \sum_{j=N+1}^{N+M} \int_{P_j}^{P_{j+1}} ((\mathbf{b} - \mathbf{b}(B_j)).\mathbf{n}_j)\,(u_h - u_h(B_j))\,d\gamma\; v_h(B_j)\,. \quad (5.11)$$

Since u_h is linear on $\overline{P_j P}_{j+1}$, $N + 1 \leqslant j \leqslant N + M$, the first term of I_2 vanishes. Since the second term of I_2 is equal to

$$\sum_{j=N+1}^{N+M} \left\{ \int_{P_j}^{B_j} ((\mathbf{b} - \mathbf{b}(B_j)).\mathbf{n}_j)\,(u_h - u_h(B_j))\,d\gamma \right.$$

$$\left. + \int_{B_j}^{P_{j+1}} ((\mathbf{b} - \mathbf{b}(B_j)).\mathbf{n}_j)\,(u_h - u_h(B_j))\,d\gamma \right\} v_h(B_j)\,,$$

using (5.1), (5.7) and the following fact

$$|\,(\mathbf{b} - \mathbf{b}(B_j)).\mathbf{n}_j\,| \leqslant C_4\, h \quad \text{for} \quad x \in \overline{P_j B_j} \quad (\text{resp. } \overline{B_j P}_{j+1})\,,$$

we conclude that there exists a constant C_5 such that

$$|\, I_2 \,| \leqslant C_5\, h \,\|\, u_h \,\|_{1,h} \,\|\, v_h \,\|_{1,h}\,. \quad (5.12)$$

Combining (5.3), (5.4), (5.9) and (5.12), we obtain (5.2).

Remark 4 : Since from Proposition 4.13 of Temam [16] the following discrete Poincaré inequality holds

$$\|\, v_h \,\|_{0,\Omega} \leqslant C(\Omega) \,\|\, v_h \,\|_h \quad \text{for all} \quad v_h \in V_{0h} \quad (5.13)$$

with a constant $C(\Omega) > 0$ independent of $h \in (0, \bar{h}]$, $\|\,.\,\|_{1,h}$ and $\|\,.\,\|_h$ are equivalent norms on V_{0h}.

Hence we find a constant $C > 0$ independent of $h \in (0, \bar{h}]$ such that

$$|\, b_h(u_h, v_h) - \tilde{b}_h(u_h, v_h) \,| \leqslant Ch \,\|\, u_h \,\|_h \,\|\, v_h \,\|_h \quad \text{for all} \quad u_h, v_h \in V_{0h}\,. \quad (5.14)$$

In what follows, we shall restrict our attention to the homogeneous Dirichlet problem for simplicity, which is denoted by (E^0). Thus we consider the following problem :

$$(\tilde{\Pi}_h^0) \quad \begin{cases} \text{Find } \tilde{u}_h \in V_{0h} \text{ such that} \\ \tilde{t}_h(\tilde{u}_h, v_h) = (f, v_h) \quad \text{for all} \quad v_h \in V_{0h}\,. \end{cases}$$

LEMMA 3 : *It holds that for any $v_h \in V_{0h}$*

$$\tilde{b}_h^1(v_h, v_h) + \frac{1}{2}\tilde{b}_n^2(v_h, v_h)$$
$$= \sum_{j=1}^{N+M} \sum_{K \in \Lambda_j} \int_{\Gamma_{jk}^S} \frac{1}{2}(\mathbf{b}.\mathbf{n}_j)(v_h(B_j) - v_h(B_k))^2 \left(\lambda_{jk} - \frac{1}{2}\right) d\gamma . \quad (5.15)$$

Proof : Let v_h be an arbitrary element in V_{0h}. From (3.10) and (3.12) we have

$$\tilde{b}_h^1(v_h, v_h) + \frac{1}{2}\tilde{b}_h^2(v_h, v_h) = \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}^S} (\mathbf{b}.\mathbf{n}_j)\left(v_h^{jk} - \frac{1}{2}v_h(B_j)\right) v_h(B_j)\, d\gamma$$

Since $\Gamma_{jk}^S = \Gamma_{kj}^S$ and $\mathbf{n}_j = -\mathbf{n}_k$, we obtain by using (3.11)

$$\tilde{b}_h^1(v_h, v_h) + \frac{1}{2}\tilde{b}_h^2(v_h, v_h)$$
$$= \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}^S} \frac{1}{2}(\mathbf{b}.\mathbf{n}_j)\left\{\left(v_h^{jk} - \frac{1}{2}v_h(B_j)\right) v_h(B_j) - \left(v_h^{kj} - \frac{1}{2}v_h(B_k)\right) v_h(B_k)\right\} d\gamma$$
$$= \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}^S} \frac{1}{2}(\mathbf{b}.\mathbf{n}_j)\left(\lambda_{jk} - \frac{1}{2}\right)\{ v_h(B_j)^2 - 2\, v_h(B_j)\, v_h(B_k) + v_h(B_k)^2 \}\, d\gamma$$
$$= \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}^S} \frac{1}{2}(\mathbf{b}.\mathbf{n}_j)(v_h(B_j) - v_h(B_k))^2 \left(\lambda_{jk} - \frac{1}{2}\right) d\gamma .$$

THEOREM 3 : *Assume that there exists a constant $\alpha' > 0$ such that*

$$\nu C(\Omega)^{-2} - \frac{1}{2}\cdot \operatorname{div} \mathbf{b} \geqslant \alpha' > 0 \quad in \quad \Omega , \quad (5.17)$$

where $C(\Omega)$ is the constant in the discrete Poincaré inequality (5.13). *If we take λ_{jk} as in* (3.14), *then the problem $(\tilde{\Pi}_h^0)$ has a unique solution $\tilde{u}_h \in V_{0h}$.*

Proof : It is sufficient to show the V_{0h}-coercivity of $\tilde{t}_h(.,.)$. From the definition of $\tilde{t}_h(.,.)$ and Lemma 3, we have for all $v_h \in V_{0h}$

$$\tilde{t}_h(v_h, v_h) = \nu a_h(v_h, v_h) + \tilde{b}_h^1(v_h, v_h) + \frac{1}{2}\tilde{b}_h^2(v_h, v_h) + \frac{1}{2}\tilde{b}_h^2(v_h, v_h)$$
$$= \nu \| v_h \|_h^2 + \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}^S} \frac{1}{2}(\mathbf{b}.\mathbf{n}_j)(v_h(B_j) - v_h(B_k))^2 \left(\lambda_{jk} - \frac{1}{2}\right) d\gamma$$
$$- \frac{1}{2}\sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}^S} (\mathbf{b}.\mathbf{n}_j)\, d\gamma\, v_h(B_j)^2 . \quad (5.18)$$

By virtue of the choice of λ_{jk}, we find that

$$\sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}^S} \frac{1}{2} (\mathbf{b}.\mathbf{n}_j) (v_h(B_j) - v_h(B_k))^2 \left(\lambda_{jk} - \frac{1}{2}\right) d\gamma \geqslant 0 . \quad (5.19)$$

Then we have

$$\begin{aligned} \tilde{t}_h(v_h, v_h) &\geqslant \nu \| v_h \|_h^2 - \frac{1}{2} \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}^S} (\mathbf{b}.\mathbf{n}_j)\, d\gamma\, v_h(B_j)^2 \\ &= \nu \| v_h \|_h^2 - \frac{1}{2} \left\{ \sum_{j=1}^{N+M} \sum_{k \in \Lambda_j} \int_{\Gamma_{jk}^S} (\mathbf{b}.\mathbf{n}_j)\, d\gamma\, v_h(B_j)^2 \right. \\ &\qquad \left. + \sum_{j=N+1}^{N+M} \int_{\partial\Omega_j \cap \Gamma} (\mathbf{b}.\mathbf{n}_j)\, d\gamma\, v_h(B_j)^2 \right\}, \end{aligned}$$

where in the last equality we used the fact that $v_h(B_j)=0$ for $N+1 \leqslant j \leqslant N+M$.

According to the patch-wise application of the Gauss divergence formula, it holds that

$$\tilde{t}_h(v_h, v_h) \geqslant \nu \| v_h \|_h^2 - \frac{1}{2} \sum_{j=1}^{N+M} \int_{\Omega_j} (\operatorname{div} \mathbf{b}) \,| L_h v_h |^2\, dx . \quad (5.20)$$

Then for any constant ε satisfying $0 < \varepsilon < \min \{ 1, \alpha' C(\Omega)^2/\nu \}$ we have

$$\tilde{t}_h(v_h, v_h) \geqslant \nu\varepsilon \| v_h \|_h^2 \quad \text{for all} \quad v_h \in V_{0h} . \quad (5.21)$$

THEOREM 4 : *Assume the hypotheses of Theorem 3. Then $t_h(.,.)$ is coercive on V_{0h} for any sufficiently small h.*

Proof . For all $v_h \in V_{0h}$ we have

$$t_h(v_h, v_h) = \tilde{t}_h(v_h, v_h) + b_h(v_h, v_h) - \tilde{b}_h(v_h, v_h) . \quad (5.22)$$

Therefore, from Theorem 3 and Remark 4 it holds that

$$t_h(v_h, v_h) \geqslant (\nu\varepsilon - Ch) \| v_h \|_h^2 \quad \text{for all} \quad v_h \in V_{0h} , \quad (5.23)$$

where C is the positive constant independent of h in (5.14).

Thus, we find that there is a constant $h_0 > 0$ such that it holds for some constant $\alpha^* > 0$

$$t_h(v_h, v_h) \geqslant \alpha^* \| v_h \|_h^2 \quad \text{for all} \quad v_h \in V_{0h} , \quad (5.24)$$

provided that $h \leqslant h_0$.

Now, let us derive a bound for the error $\| \tilde{u}_h - u_h \|_h$, where $\tilde{u}_h$ is the solution of $(\tilde{\Pi}_h^0)$ with λ_{jk} as in (3.14) and u_h is the solution of (Π_h^0) which is the Galerkin approximation of (E^0).

THEOREM 5 : *Assume the hypotheses of Theorem* 3. *Then we have*

$$\| \tilde{u}_h - u_h \|_h \leqslant Ch \| f \|_{0,\Omega} \tag{5.25}$$

for some constant $C > 0$.

Proof : Since

$$\tilde{t}_h(\tilde{u}_h, v_h) = (f, v_h) = t_h(u_h, v_h) \quad \text{for all} \quad v_h \in V_{0h}, \tag{5.26}$$

we may write

$$t_h(u_h, v_h) - \tilde{t}_h(u_h, v_h) + \tilde{t}_h(u_h, v_h) = \tilde{t}_h(\tilde{u}_h, v_h) \quad \text{for all} \quad v_h \in V_{0h}. \tag{5.27}$$

Hence we have

$$\begin{aligned} \tilde{t}_h(\tilde{u}_h - u_h, v_h) &= t_h(u_h, v_h) - \tilde{t}_h(u_h, v_h) \\ &= b_h(u_h, v_h) - \tilde{b}_h(u_h, v_h). \end{aligned} \tag{5.28}$$

According to Remark 4, it holds that

$$| \tilde{t}_h(\tilde{u}_h - u_h, v_h) | \leqslant Ch \| u_h \|_h \| v_h \|_h. \tag{5.29}$$

Taking $v_h = \tilde{u}_h - u_h$ in (5.29) we obtain from the coercivity of $\tilde{t}_h(.,.)$

$$\nu\varepsilon \| \tilde{u}_h - u_h \|_h^2 \leqslant Ch \| u_h \|_h \| \tilde{u}_h - u_h \|_h.$$

Hence we have

$$\| \tilde{u}_h - u_h \|_h \leqslant C' h \| u_h \|_h. \tag{5.30}$$

On the other hand, from the coercivity of $t_h(.,.)$ we can show that

$$\| u_h \|_h \leqslant 1/\alpha^* . \| f \|_{0,\Omega}. \tag{5.31}$$

Then, from (5.30) and (5.31) the assertion follows.

Remark 5 : In order to obtain the error estimate for $u - u_h$ in the norm $\| . \|_h$, we can apply the *primal hybrid* finite element method introduced by Raviart-Thomas [15], since the linear nonconforming finite element is one of the *hybrid* elements. Then we have the following result :

THEOREM 6 : *In addition to the hypotheses of Theorem 3, suppose that* $\{ T_h \}$ *is regular. If* $u \in H^2(\Omega) \cap H_0^1(\Omega)$, *then we have*

$$\| u - u_h \|_h \leqslant Ch \,| u |_{2,\Omega} . \tag{5.32}$$

Using Theorems 5 and 6, we can derive an error estimate for the *upstream-like* approximation with the linear nonconforming finite element.

THEOREM 7 : *Assume the hypotheses of Theorem 6. Then we have*

$$\| u - \tilde{u}_h \|_h \leqslant Ch(| u |_{2,\Omega} + \| f \|_{0,\Omega}) . \tag{5.33}$$

6. NUMERICAL EXAMPLES

As an illustration, here we adopt one of the problems treated in Kikuchi and Ushijima [13]. Namely, our model problem is :

$$\left.\begin{aligned} - \nu \,\Delta u + (\mathbf{b}.\nabla)\, u &= 1 \quad \text{in} \quad \Omega = (0, 1) \times (0, 1) , \\ u &= 0 \quad \text{on}\, \Gamma , \end{aligned}\right\} \tag{6.1}$$

where $\mathbf{b} = (1, 0)$. In [13], the following initial boundary value problem (6.2) is taken as an approximation of (6.1) for sufficiently small ν in the region far from $x_1 = 1$.

$$\left.\begin{aligned} \frac{\partial u}{\partial x_1} - \nu \frac{\partial^2 u}{\partial x_2^2} = 1 \quad &\text{in}\, \Omega , \\ u(0, x_2) = 0 \quad &\text{for} \quad 0 < x_2 < 1 , \\ u(x_1, 0) = u(x_1, 1) = 0 \quad &\text{for} \quad 0 < x_1 < 1 . \end{aligned}\right\} \tag{6.2}$$

Examples of employed meshes are pictured in figure 2, where N denotes a number of elements along the side $x_2 = 0$ (or $x_1 = 0$) of the domain Ω. Figures 3, 4, 5 and 6 show the distributions of the numerical solutions u_h and $\tilde{u}_h$ along the line $x_1 = 1/2$ of the square domain Ω, where u_h is the linear nonconforming finite element approximation and $\tilde{u}_h$ is the linear nonconforming finite element approximation of upstream type. In these figures, continuous curves are the profiles of numerical solutions of the problem (6.2), which are denoted by PEA. Among these results, the Galerkin method gives a strongly oscilating solution for the coarse meshes and the small values of ν, but gives an improved one for sufficiently fine meshes. On the other hand, our method gives a non-oscilating and reasonable solution.

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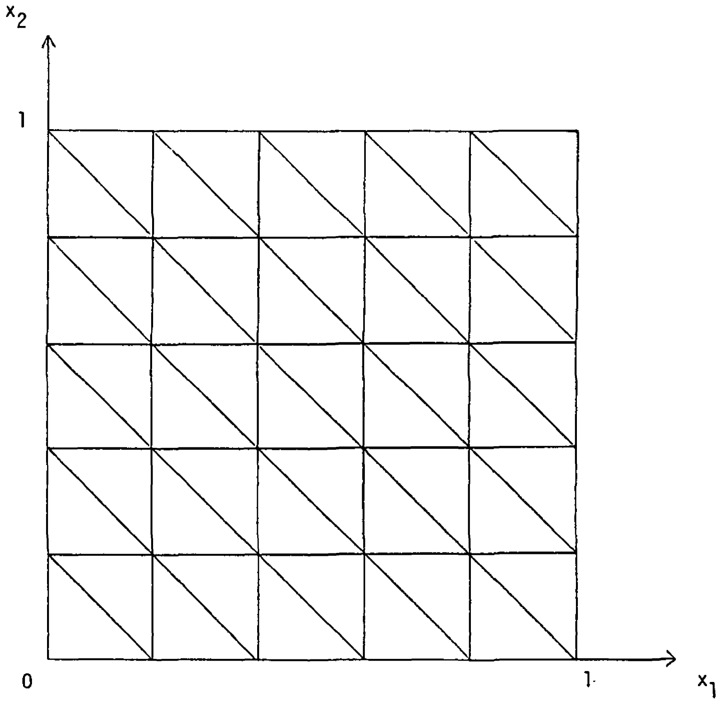

Fig. 2. — Finite element meshes for $N = 5$.

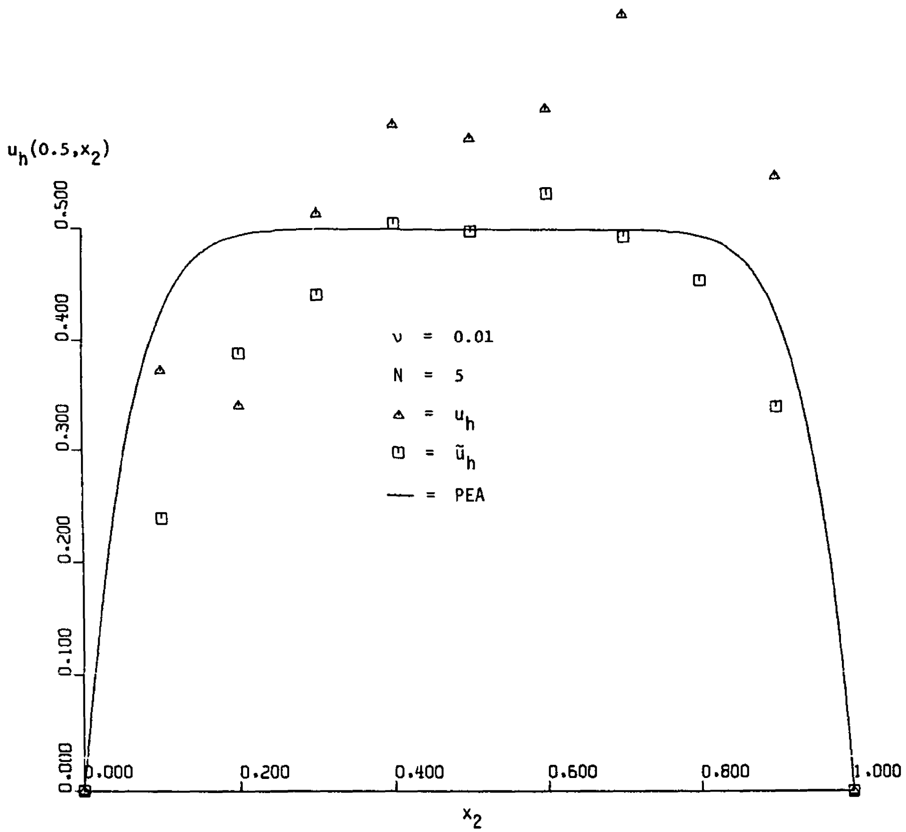

Fig. 3. — Distributions of $u_h(0.5, x_2)$ and $\tilde{u}_h(0.5, x_2)$ with $\nu = 0.01$ and $N = 5$.

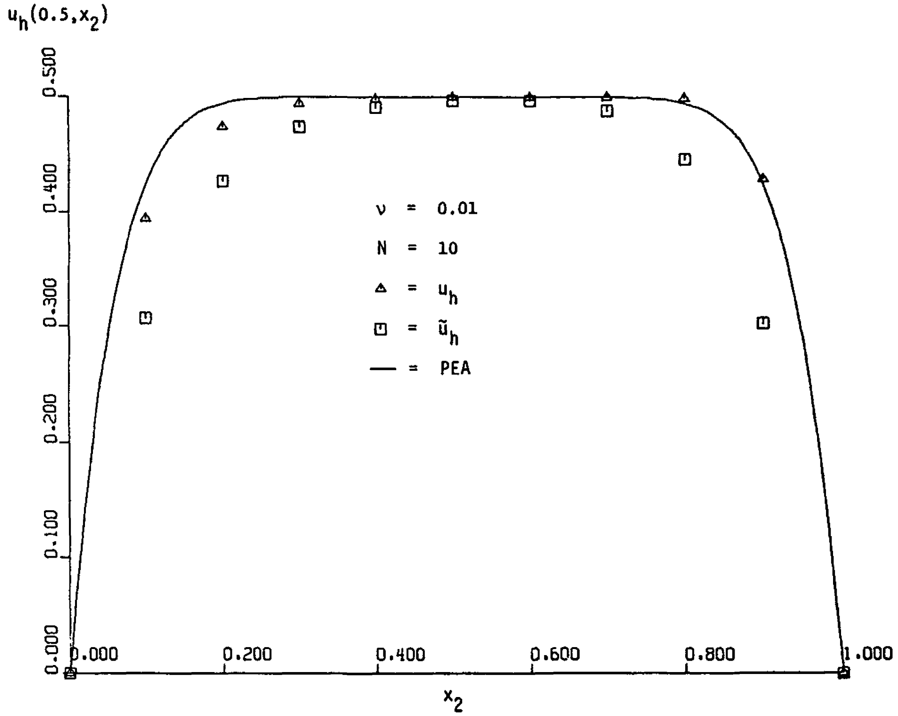

Fig. 4. — Distributions of $u_h(0.5, x_2)$ and $\tilde{u}_h(0.5, x_2)$ with $\nu = 0.01$ and $N = 10$.

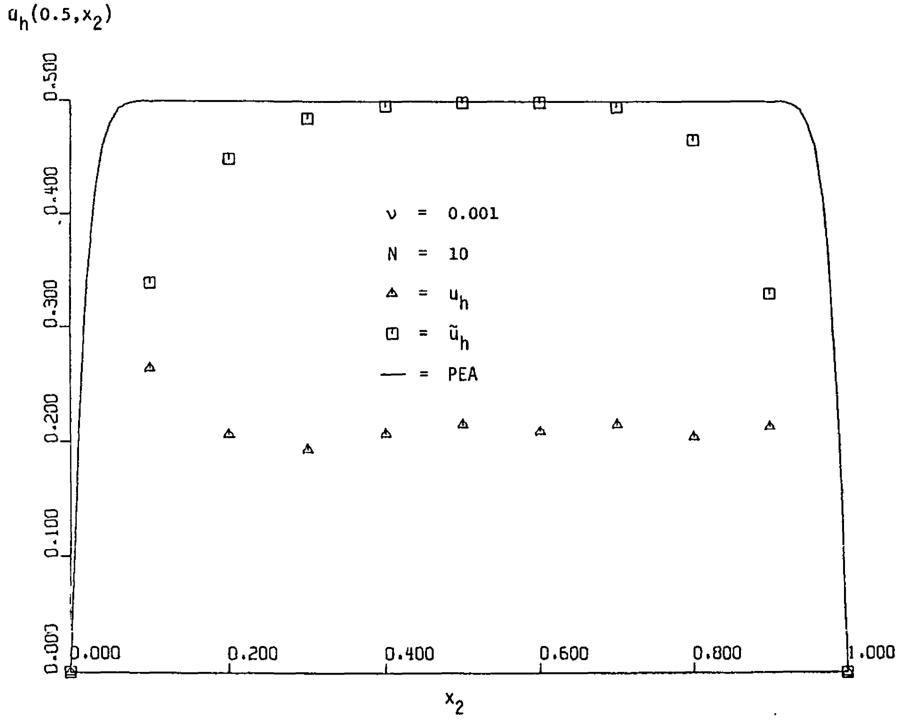

Fig. 5. — Distributions of $u_h(0.5, x_2)$ and $\tilde{u}_h(0.5, x_2)$ with $\nu = 0.001$ and $N = 10$.

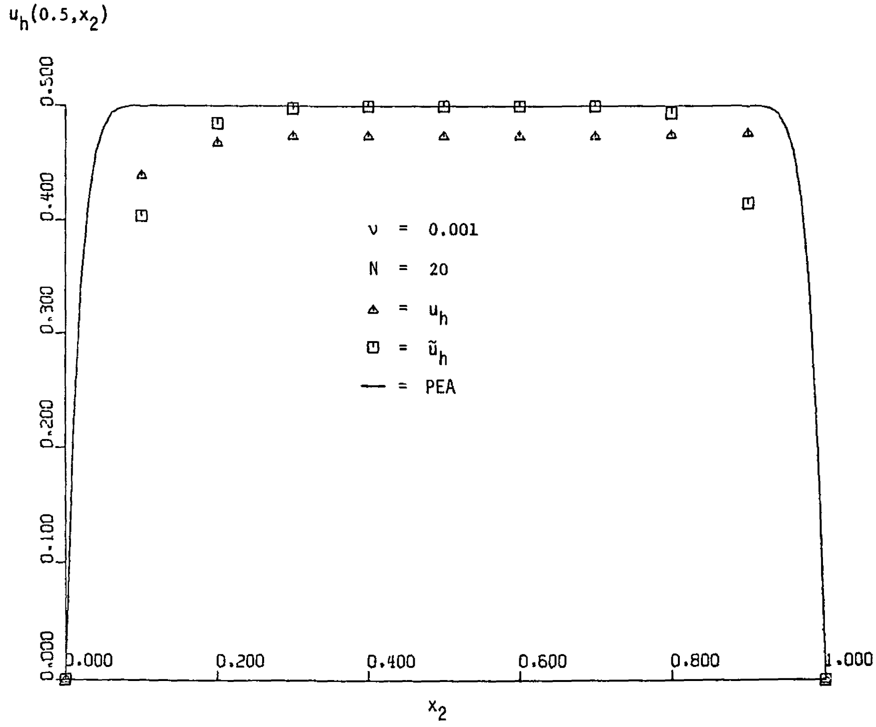

Fig. 6. — Distributions of $u_h(0.5, x_2)$ and $\tilde{u}_h(0.5, x_2)$ with $\nu = 0.001$ and $N = 20$.

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