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ADDENDUM TO THE PAPER « FINITE ELEMENT SOLUTION OF QUASISTATIONARY NONLINEAR MAGNETIC FIELD » (*)

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Résumé. — Une version corrigée et étendue du Théorème 2 du papier ci-dessus (ce journal, vol. 16 (1982), pp. 161-191) est établie.

Abstract. — A corrected and extended version of Theorem 2 of the above mentioned paper (this journal, vol. 16 (1982), pp. 161-191) is given.

In the above mentioned paper (this journal, vol. 16 (1982), pp. 161-191 ; in what follows it is denoted by [Z]) we proved Theorem 2 on a weak as well as on a strong convergence of a fully discrete approximate solution U^δ to the exact solution u of the problem P' . The assertion $\|u_R - U_R^\delta\|_{C([0, T]; H_R)} \rightarrow 0$ is not correct because the approximate solution U^δ is not continuously extended on the interval $[0, T]$. Here, we correct it in two ways and prove it even without any regularity requirement on u . Before doing it we remark that (1.1), (1.2), ..., (2.1), (2.2), etc. mean equations from [Z]. The equations in this addendum are denoted by (1), (2), etc. Further, we extend the discrete values $U^i \in V^h$ defined by (3.24) continuously on $[0, T]$. We define a new approximate solution $\mathcal{U}^\delta \in C([0, T]; V)$ by

$$\mathcal{U}^\delta = U^1 \quad \text{in } [0, t_1], \quad \mathcal{U}^\delta = U^{i-1} + \frac{t - t_{i-1}}{\Delta t} \Delta U^i \quad \text{in } (t_{i-1}, t_i], \quad (1)$$

$$2 \leq i \leq r, \quad r = T \Delta t^{-1}$$

$$(\Delta U^i = U^i - U^{i-1}).$$

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THEOREM : *Let the assumptions (1)-(5) of [Z] be fulfilled, let*

$$f^M \in L^{p'}(0, T; \overline{V}'_M), \quad M=R, S, \quad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1 \quad \text{and} \quad u_0 \in H_R.$$

Then there exists a unique function

$$u \in W_R = \{ u \mid u \in L^p(0, T; V); u'_R \in L^{p'}(0, T; \overline{V}'_R) \}$$

satisfying (3.17) and (3.18). Further, the approximate solutions U^δ and \mathcal{U}^δ defined by (3.24) and by (3.25) and (1), respectively, exist, are unique and for $\delta \equiv (h, \Delta t) \rightarrow 0$

$$U^\delta \rightarrow u \quad \text{in } L^p(0, T; V) \quad \text{weakly,} \tag{2}$$

$$\sup_{0 < t \leq T} |u_R(t) - U_R^\delta(t)|_R \rightarrow 0, \quad \|u_R - \mathcal{U}_R^\delta\|_{C((0,T);H_R)} \rightarrow 0. \tag{3}$$

If, in addition,

$$a(u, u-v) - a(v, u-v) \geq \gamma[u-v]^p \quad \forall u, v \in V, \quad \gamma = \text{const} > 0 \tag{4}$$

where $[\cdot]$ is a seminorm with property

$$[v] + \lambda |v_R|_R \geq \beta \|v\| \quad \forall v \in V, \quad \lambda, \beta = \text{const} > 0 \tag{5}$$

then

$$\|u - U^\delta\|_{L^p(0,T;V)} \rightarrow 0, \quad \|u - \mathcal{U}^\delta\|_{L^p(0,T;V)} \rightarrow 0. \tag{6}$$

Remark 1 : Let H be a Hilbert space which is dense and continuously imbedded in a separable reflexive Banach space V , let $u_0 \in H$, $f \in L^{p'}(0, T; V')$ and let $A(u)$ be a nonlinear operator from V in V' . We consider the problem

$$u' + A(u) = f, \quad u(0) = u_0 \tag{7}$$

and we assume that

i) $A(u)$ is hemicontinuous and monotone, $\|A(u)\|_{V'} \leq C \|u\|^{p-1}$ and $\langle A(u), u \rangle \geq \alpha[u]^p \quad \forall u \in V, \alpha = \text{const} > 0$ ($[\cdot]$ satisfies

$$[v] + \lambda |v| \geq \beta \|v\| \quad \forall v \in V, \quad \lambda, \beta = \text{const} > 0).$$

Then

$$U^\delta \rightarrow u \quad \text{in } L^p(0, T; V) \quad \text{weakly,}$$

$$\sup_{0 < t \leq T} |u(t) - U^\delta(t)| \rightarrow 0, \quad \|u - \mathcal{U}^\delta\|_{C((0,T);H)} \rightarrow 0.$$

If, in addition,

ii) $\langle A(u), u-v \rangle - \langle A(v), u-v \rangle \geq \gamma[u-v]^p \quad \forall u, v \in V, \quad \gamma = \text{const} > 0$
 then

$$\|u - U^\delta\|_{L^p(0,T;V)} \rightarrow 0, \quad \|u - \mathcal{U}^\delta\|_{L^p(0,T;V)} \rightarrow 0.$$

These assertions follow from the Theorem ($H_R = H, H_S = \emptyset$).

Remark 2 : We consider the problem (7) with a linear operator $A(u)$ from V in V' . We assume that

i) $\langle A(u), u \rangle \geq \alpha[u]^2 \quad \forall u \in V, \quad \alpha = \text{const} > 0.$

Then

$$\|u - \mathcal{U}^\delta\|_{C([0,T];H)} \rightarrow 0, \quad \|u - \mathcal{U}^\delta\|_{L^2(0,T;V)} \rightarrow 0.$$

From i) it follows

$$\langle A(u), u-v \rangle - \langle A(v), u-v \rangle = \langle A(u-v), u-v \rangle \geq \alpha[u-v]^2 \quad \forall u, v \in V.$$

Hence $A(u)$ is monotone and satisfies ii) of Remark 1 with $p = 2$. From monotonicity and linearity it follows that $A(u)$ is continuous. Evidently, the assumptions i), ii) of Remark 1 are fulfilled.

Remark 3 : Theorem 3 of [Z] is true without assuming $u \in C([0, T]; H_0^1(\Omega))$ if we correct (4.8) as follows :

$$\sup_{0 < t \leq T} \|u(t) - U^\delta(t)\|_{L^2(R)} \rightarrow 0, \quad \|u - \mathcal{U}^\delta\|_{C([0,T];L^2(R))} \rightarrow 0,$$

$$\|u - U^\delta\|_{L^2(0,T;H_0^1(\Omega))} \rightarrow 0, \quad \|u - \mathcal{U}^\delta\|_{L^2(0,T;H_0^1(\Omega))} \rightarrow 0.$$

Remark 4 : We consider a set of, in general, nonuniform partitions of the interval $[0, T] : 0 = t_0 < t_1 < \dots < t_r = T$. We denote $\Delta t = \max_{1 \leq i \leq r} \Delta t_i$, $\Delta t_i = t_i - t_{i-1}$, and we assume that the set of partitions has the following properties : 1) $\Delta t \rightarrow 0$, 2) $\min_{1 \leq i \leq r} \Delta t_i \geq \sigma_0 \Delta t$ for all partitions where σ_0 is a positive constant which does not depend on the chosen partition (nor on h). Let the approximate solution be defined by means of the Euler backward scheme, i.e. by

$$(U_R^i - U_R^{i-1}, z_R)_R + \Delta t_i a(U^i, z) = \Delta t_i \langle f^i, z \rangle \quad \forall z \in V^h, \quad U_R^0 = u_0$$

where now $f^i = \frac{1}{\Delta t_i} \int_{t_{i-1}}^{t_i} f(\tau) d\tau$. Then all results of the paper [Z] as well as of this addendum are true for the extended approximate solutions U^δ and \mathcal{U}^δ . The changes in the proofs are very small.

Proof of the Theorem : We have to prove (3) and (6). The other assertions are proved in [Z].

As $u_R \in \{ \omega \mid \omega \in L^p(0, T; \bar{V}_R); \omega' \in L^p(0, T; \bar{V}'_R) \}$ u_R belongs to $C([0, T]; H_R)$ (see lemma 1 in [Z]). To prove the first part of (3) we use a discrete version of an idea which is used in [4] (see references in [Z]) in the proof of Theorem 1.2, pp. 209-210. We derive an estimate for $\max_{1 \leq i \leq j} |U_R^i - v_R^i|_R$ where v is an arbitrary function from $C^1([0, T]; V^h)$. Then we take for v an approximation of u .

Let $v^0 = v(0)$ and let $v_{\Delta t}$ denote the function $v_{\Delta t} = v^i \equiv v(t_i)$ in (t_{i-1}, t_i) , $i = 1, \dots, r$. First, we consider $\{U^i\}_{i=1}^r$ defined by (3.29). Using (3.23), (3.9) and (3.10) we get

$$\begin{aligned} & \frac{1}{2} |U_R^j - v_R^j|_R^2 - \frac{1}{2} |U_R^0 - v_R^0|_R^2 \leq \sum_{i=1}^j (\Delta(U_R^i - v_R^i), U_R^i - v_R^i)_R = \\ & = \sum_{i=1}^j (\Delta U_R^i, U_R^i - v_R^i)_R - \sum_{i=1}^j (\Delta v_R^i, U_R^i - v_R^i)_R = \\ & = -\Delta t \sum_{i=1}^j a(U^i, U^i - v^i) + \Delta t \sum_{i=1}^j \langle f^i, U^i - v^i \rangle \\ & \quad - \sum_{i=1}^j \left(\int_{t_{i-1}}^{t_i} v' dt, U_R^\delta - (v_{\Delta t})_R \right)_R = - \int_0^{t_j} a(U^\delta, U^\delta - v_{\Delta t}) dt \\ & + \int_0^{t_j} \langle f, U^\delta - v_{\Delta t} \rangle dt - \int_0^{t_j} (v'_R, U_R^\delta - (v_{\Delta t})_R) dt = \\ & = - \int_0^{t_j} [a(U^\delta, U^\delta - u) - a(u, U^\delta - u)] dt \\ & + \int_0^{t_j} [a(u, u - v_{\Delta t}) - a(U^\delta, u - v_{\Delta t})] dt + \int_0^{t_j} \langle u'_R - v'_R, U_R^\delta - (v_{\Delta t})_R \rangle_R dt. \end{aligned}$$

From (3.37) and (3.13) it follows that the second integral is bounded from above by

$$\begin{aligned} C \|u - v_{\Delta t}\|_{L^p(0, T; V)} & \leq C \{ \|u - v\|_{L^p(0, T; V)} + \|v - v_{\Delta t}\|_{L^p(0, T; V)} \} \\ & \leq C \{ \|u - v\|_{W_R} + \Delta t \|v\|_{C^1([0, T], V)} \}. \end{aligned}$$

Here $\|u\|_{W_R} = \|u\|_{L^p(0, T; V)} + \|u'_R\|_{L^p(0, T; \bar{V}'_R)}$ and C is a generic constant not necessarily the same at any two places. C as well as C_i ($i = 0, \dots, 3$) introduced later do not depend on $\delta = (h, \Delta t)$ and on the functions v, z (C depends

on u). The third integral is bounded from above by

$$\begin{aligned} \| u'_R - v'_R \|_{L^{p'}(0,T;\bar{V}_R)} \| U^\delta - (v_{\Delta t})_R \|_{L^p(0,T;\bar{V}_R)} &\leq \\ &\leq C \| u - v \|_{W_R} \| U^\delta - v + v - v_{\Delta t} \|_{L^p(0,T;V)} \\ &\leq C \| u - v \|_{W_R} \{ C + \| v \|_{L^p(0,T;V)} + \| v - v_{\Delta t} \|_{L^p(0,T;V)} \} \\ &\leq C \{ \| u - v \|_{W_R} + \Delta t \| v \|_{C^1([0,T];V)} \} \end{aligned}$$

if we assume that

$$\| u - v \|_{W_R} \leq 1. \tag{8}$$

The result of all these estimates can be written in this way :

$$\begin{aligned} | U^j_R - v^j_R |^2_R + \int_0^{t_j} [a(U^\delta, U^\delta - u) - a(u, U^\delta - u)] dt &\leq \\ &\leq | U^0_R - v^0_R |^2_R + C \{ \| u - v \|_{W_R} + \Delta t \| v \|_{C^1([0,T];V)} \}. \end{aligned}$$

As (see lemma 1 in [Z])

$$| U^0_R - v^0_R |_R = | u(0)_R - v(0)_R |_R \leq \| u_R - v_R \|_{C([0,T];H_R)} \leq C \| u - v \|_{W_R}$$

and

$$| u^j_R - U^j_R |_R \leq \| u_R - v_R \|_{C([0,T];H_R)} + | U^j_R - v^j_R |_R$$

it easily follows that

$$Y^\delta \leq C_0 \{ \| u - v \|_{W_R} + \Delta t \| v \|_{C^1([0,T];V)} \} \tag{9}$$

where

$$Y^\delta = \max_{1 \leq i \leq r} | u^i_R - U^i_R |^2_R + \int_0^T [a(U^\delta, U^\delta - u) - a(u, U^\delta - u)] dt.$$

By means of (9) we prove that

$$Y^\delta \rightarrow 0. \tag{10}$$

To this end we remark that $C^1([0, T]; V)$ is dense in W_R (the proof is the same as in case $H_R = H, H_S = \emptyset$; see, e.g., [4], p. 144) and that $\bigcup_{j=1}^\infty C^1([0, T]; V^{hj})$ is dense in $\dot{C}^1([0, T]; V)$ if $h_j \rightarrow 0$ (the proof is the same as the proof of lemma 1.5 in [4], p. 209). We also remark that

$$\| z \|_{W_R} \leq C_1 \| z \|_{C^1([0,T];V)} \quad \forall z \in C^1([0, T]; V).$$

If (10) is not true there exists an $\varepsilon_0 > 0$ and a sequence

$$\{ \delta_n \}_{n=1}^\infty, \delta_n = (h_n \Delta t_n) \rightarrow 0,$$

with the property $Y^{\delta_n} \geq \varepsilon_0$ for $n \geq 1$. Let $w \in C^1([0, T]; V)$ be such that

$$\| u - w \|_{W_R} < \frac{1}{4 C_0} \varepsilon_0$$

and n_0 be so large that

$$\Delta t_n < \frac{\varepsilon_0}{2 C_0} (\| w \|_{C^1([0, T]; V)} + 1)^{-1}, \quad n \geq n_0.$$

As $\bigcup_{n=n_0}^\infty C^1([0, T]; V^{h_n})$ is dense in $C^1([0, T]; V)$ there exists $v \in C^1([0, T]; V^{h_{n_1}})$ with $n_1 \geq n_0$ such that

$$\| w - v \|_{C^1([0, T]; V)} < \frac{1}{4 C_0 C_1} \varepsilon_0.$$

We may assume that $\varepsilon_0 < \min(2 C_0, 4 C_0 C_1)$. Then

$$\begin{aligned} \| u - v \|_{W_R} &\leq \| u - w \|_{W_R} + \| w - v \|_{W_R} < \frac{1}{4 C_0} \varepsilon_0 + \\ &+ C_1 \| w - v \|_{C^1([0, T]; V)} < \frac{1}{2 C_0} \varepsilon_0 < 1. \end{aligned}$$

Therefore we may use (9) with this v and we get

$$Y^{\delta_{n_1}} < C_0 \left[\frac{1}{2 C_0} \varepsilon_0 + \Delta t_{n_1} \| v \|_{C^1([0, T]; V)} \right] < \varepsilon_0$$

which is in contradiction with $Y^{\delta_n} \geq \varepsilon_0$ for $n \geq 1$.

Now we show that (10) is true in case that $\{ U^i \}_{i=1}^r$ is defined by (3.59). To this end we remark that (3.61) is true if we replace U_R^i by any $\omega_R^i \in H_R$ such that $\omega_R^0 = \omega_R^{-1}$. Estimating the term $-(\omega_R^j, \omega_R^{j-1})_R$ from below by

$$- | \omega_R^j |_R^2 - \frac{1}{4} | \omega_R^{j-1} |_R^2 \text{ we get}$$

$$\begin{aligned} \frac{1}{4} | \omega_R^j |_R^2 - \frac{1}{2} | \omega_R^0 |_R^2 &\leq \sum_{i=1}^j \left(\frac{3}{2} \omega_R^i - 2 \omega_R^{i-1} + \frac{1}{2} \omega_R^{i-2}, \omega_R^i \right)_R = \\ &= \sum_{i=1}^j \left(\frac{3}{2} \Delta \omega_R^i - \frac{1}{2} \Delta \omega_R^{i-1}, \omega_R^i \right)_R. \quad (11) \end{aligned}$$

We choose $\omega_R^i = U_R^i - v_R^i$ (as before, $v^i = v(t_i)$, $i = 0, \dots, r$; in addition, we define $v^{-1} = v^0$ so that $\Delta v^0 = 0$) and we extend u'_R by zero outside $(0, T)$ keeping the same notation u'_R for this extension. Similarly as before we obtain

$$\begin{aligned} \frac{1}{4} |U_R^j - v_R^j|_R^2 - \frac{1}{2} |U_R^0 - v_R^0|_R^2 &\leq \sum_{i=1}^j \left(\frac{3}{2} U_R^i - 2 U_R^{i-1} + \frac{1}{2} U_R^{i-2}, U_R^i - v_R^i \right)_R - \\ &- \sum_{i=1}^j \left(\frac{3}{2} \Delta v_R^i - \frac{1}{2} \Delta v_R^{i-1}, U_R^i - v_R^i \right)_R = - \Delta t \sum_{i=1}^j a(U^i, U^i - v^i) \\ &+ \Delta t \sum_{i=1}^j \langle f^i, U^i - v^i \rangle - \frac{3}{2} \sum_{i=1}^j \left(\int_{t_{i-1}}^{t_i} v'_R(t), U_R^\delta - (v_{\Delta t})_R \right)_R \\ &+ \frac{1}{2} \sum_{i=2}^j \left(\int_{t_{i-1}}^{t_i} v'_R(t - \Delta t), U_R^\delta - (v_{\Delta t})_R \right)_R = - \int_0^{t_j} [a(U^\delta, U^\delta - u) \\ &- a(u, U^\delta - u)] dt + \int_0^{t_j} [a(u, u - v_{\Delta t}) - a(U^\delta, u - v_{\Delta t})] dt \\ &+ \int_0^{t_j} \langle u'_R(t), U_R^\delta - (v_{\Delta t})_R \rangle_R dt - \frac{3}{2} \int_0^{t_j} \langle v'_R(t), U_R^\delta - (v_{\Delta t})_R \rangle_R dt \\ &+ \frac{1}{2} \int_{t_1}^{t_j} \langle v'_R(t - \Delta t), U_R^\delta - (v_{\Delta t})_R \rangle_R dt . \end{aligned}$$

The first two integrals on the right-hand side are the same as before. The last three are equal to

$$\begin{aligned} \frac{3}{2} \int_0^{t_j} \langle u'_R(t) - v'_R(t), U_R^\delta - (v_{\Delta t})_R \rangle_R dt - \\ - \frac{1}{2} \int_{t_1}^{t_j} \langle u'_R(t - \Delta t) - v'_R(t - \Delta t), U_R^\delta - (v_{\Delta t})_R \rangle_R dt \\ - \frac{1}{2} \int_0^{t_j} \langle u'_R(t) - u'_R(t - \Delta t), U_R^\delta - (v_{\Delta t})_R \rangle_R dt . \end{aligned}$$

The first two terms can be estimated as before. The last is bounded by

$$C\alpha(\Delta t)(1 + \Delta t \|v\|_{C^1([0, T]; V)}), \quad \alpha(\Delta t) = \|u'_R(t) - u'_R(t - \Delta t)\|_{L^p([0, T]; \bar{V}_R)}.$$

Y^δ is bounded from above by

$$Y^\delta \leq C_2 \{ \|u - v\|_{W_R} + \Delta t \|v\|_{C^1([0, T]; V)} \} + C_3 \alpha(\Delta t). \tag{12}$$

As $u'_R \in L^p(0, T; \bar{V}_R)$ it holds $\alpha(\Delta t) \rightarrow 0$, hence $Y^\delta \rightarrow 0$.

Let $M(\delta) = \max_{1 \leq i \leq r} |u_R^i - U_R^i|_R$, $m(\Delta t) = \max |u_R(\tau_1) - u_R(\tau_2)|_R$, $\tau_1, \tau_2 \in [0, T]$, $|\tau_1 - \tau_2| \leq \Delta t$. Then

$$M(\delta) \rightarrow 0, \quad m(\Delta t) \rightarrow 0 \quad \text{if } \delta \rightarrow 0.$$

We have $|u_R(t) - U_R^\delta(t)|_R \leq m(\Delta t) + M(\delta) \rightarrow 0$ which proves the first part of (3). Now, let $\|u_R - \mathcal{U}_R^\delta\|_{C([0, T]; H_R)} = |u_R(\tau_0) - \mathcal{U}_R^\delta(\tau_0)|_R$. If $\tau_0 \in [0, t_1]$ then

$$|u_R(\tau_0) - \mathcal{U}_R^\delta(\tau_0)|_R \leq |u_R(\tau_0) - u_R^1|_R + |u_R^1 - U_R^1|_R \leq m(\Delta t) + M(\delta).$$

If $\tau_0 \in (t_{i-1}, t_i)$, $i \geq 2$ then

$$\begin{aligned} |u_R(\tau_0) - \mathcal{U}_R^\delta(\tau_0)|_R &\leq |u_R(\tau_0) - u_R^i|_R + |u_R^i - U_R^i|_R + |U_R^\delta(\tau_0) - \mathcal{U}_R^\delta(\tau_0)|_R \leq \\ &\leq m(\Delta t) + M(\delta) + \left| \frac{t_i - \tau_0}{\Delta t} \Delta U_R^i \right|_R \leq m(\Delta t) + M(\delta) + |\Delta U_R^i|_R. \end{aligned}$$

As $|\Delta U_R^i|_R \leq |U_R^i - u_R^i|_R + |\Delta u_R^i|_R + |u_R^{i-1} - U_R^{i-1}|_R \leq m(\Delta t) + 2M(\delta)$ we see that $\|u_R - \mathcal{U}_R^\delta\|_{C([0, T]; H_R)} \leq 2m(\Delta t) + 3M(\delta) \rightarrow 0$.

If $a(u, v)$ satisfies (4) it follows from (9) and (10) that $\int_0^T [u - U^\delta]^p dt \rightarrow 0$.

Further,

$$\begin{aligned} \int_0^T |u_R - U_R^\delta|_R^p dt &= \sum_{i=1}^r \int_{t_{i-1}}^{t_i} |u_R - U_R^i|_R^p dt \leq \\ &\leq \sum_{i=1}^r \int_{t_{i-1}}^{t_i} [|u_R(t) - u_R^i|_R + |u_R^i - U_R^i|_R]^p dt \\ &\leq T[m(\Delta t) + M(\delta)]^p \rightarrow 0. \end{aligned}$$

Consequently, $\|u - U^\delta\|_{L^p(0, T; V)} \rightarrow 0$ owing to (5).

Finally, from $\int_0^T \|u(t) - U^\delta(t)\|^p dt \rightarrow 0$ it follows

$$\int_0^T \|u(t - \Delta t) - U^\delta(t - \Delta t)\|^p dt \rightarrow 0$$

(here u and U are extended by zero outside $(0, T)$). As

$$\|u(t) - u(t - \Delta t)\|_{L^p(0, T; V)} \rightarrow 0$$

we get $\int_0^T \| U^\delta(t) - U^\delta(t - \Delta t) \|^p dt \rightarrow 0$. On the other hand

$$\int_0^T \| U^\delta(t) - U^\delta(t - \Delta t) \|^p dt = \Delta t \left\{ \| U^1 \|^p + \sum_{i=2}^r \| \Delta U^i \|^p \right\},$$

thus $\Delta t \sum_{i=2}^r \| \Delta U^i \|^p \rightarrow 0$. We have

$$\int_0^T \| u - \mathcal{U}^\delta \|^p dt \leq 2^{p-1} \left\{ \int_0^T \| u - U^\delta \|^p dt + \int_0^T \| U^\delta - \mathcal{U}^\delta \|^p dt \right\}$$

and

$$\int_0^T \| U^\delta - \mathcal{U}^\delta \|^p dt = \sum_{i=2}^r \int_{t_{i-1}}^{t_i} \left\| \frac{t_i - t}{\Delta t} \Delta U^i \right\|^p dt \leq \Delta t \sum_{i=2}^r \| \Delta U^i \|^p \rightarrow 0$$

which proves the last assertion of the Theorem.