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TWO MIXED FINITE ELEMENT METHODS FOR THE SIMPLY SUPPORTED PLATE PROBLEM (*)

by James H. BRAMBLE [1] and Richard S. FALK [2]

Résumé. — *On analyse deux méthodes d'éléments finis mixtes pour l'approximation du modèle biharmonique du problème d'une plaque simplement appuyée. On présente ensuite un procédé numérique efficace pour résoudre le système d'équations linéaires correspondant.*

Abstract. — *Two mixed finite element methods are analyzed for the approximation of the biharmonic model of the simply supported plate problem. An efficient numerical procedure for solving the resulting linear system of equations is then presented.*

1. INTRODUCTION

In this paper we wish to study two mixed finite element methods for the approximation of a boundary value problem modeling a simply supported plate, i.e. we consider the biharmonic equation

$$\Delta^2 \tilde{u} = f \quad \text{in} \quad \Omega \tag{1.1}$$

subject to the boundary conditions

$$\Delta \tilde{u} - \tau(\tilde{u}_{ss} + K\tilde{u}_n) = 0 \tag{1.2}$$

and

$$\tilde{u} = 0 \quad \text{on} \quad \Gamma, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary Γ, f is a given function, K is the curvature of Ω, $1 - \tau$ is Poisson's ratio, and $\tilde{u}_s$ and $\tilde{u}_n$ denote the tangential and exterior normal derivatives of $\tilde{u}$ respectively along Γ.

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In the standard variational formulation of (1.1)-(1.3), (1.2) is a natural boundary condition and so the solution $\tilde{u}$ may be characterized by

$$\text{Find} \quad \tilde{u} \in H^2(\Omega) \cap H_0^1(\Omega) \tag{1.4}$$

such that

$$(\Delta\tilde{u}, \Delta v) - \tau\{(\tilde{u}_{xx}, v_{yy}) + (\tilde{u}_{yy}, v_{xx}) - 2(\tilde{u}_{xy}, v_{xy})\} = (f, v)$$
$$\text{for all} \quad v \in H^2(\Omega) \cap H_0^1(\Omega)$$

(where $(.,.)$ denotes the $L_2(\Omega)$ inner product).

If one bases a finite element method on this variational principle, one is faced with the difficulty of constructing subspaces of $H^2(\Omega) \cap H_0^1(\Omega)$. This requires the use of C^1 finite elements which must vanish on $\partial\Omega$.

By using the mixed method technique of introducing new independent variables (e.g. $\tilde{w} = -\Delta\tilde{u}$), we are able to reformulate this problem as a lower order system of equations. This will allow us to define a conforming finite element method using only C^0 finite elements.

The first approximation scheme we will discuss will be based on the following variational formulation of (1.1)-(1.3). Let $\langle .,. \rangle$ denote the $L^2(\Gamma)$ inner product and also the pairing between $H^s(\Gamma)$ and $H^{-s}(\Gamma)$ and let

$$A_\alpha(u, v) = (\text{grad } u, \text{grad } v) + \alpha \langle u, v \rangle,$$

where α is chosen sufficiently large so that $2\alpha + K > 0$. We will consider :

Problem $(\tilde{P})$: Find $(\tilde{u}, \tilde{w}, \lambda, \sigma) \in H^1(\Omega) \times H^1(\Omega) \times H^{3/2}(\Gamma) \times H^{-1/2}(\Gamma)$ such that

$$A_\alpha(\tilde{w}, v) = (f, v) + \langle \sigma, v \rangle - \tau \langle \lambda_s, v_s \rangle \quad \text{for all} \quad v \in H^1(\Omega), \tag{1.5}$$

$$A_\alpha(\tilde{u}, z) = (\tilde{w}, z) + \langle \lambda, z \rangle \quad \text{for all} \quad z \in H^1(\Omega), \tag{1.6}$$

$$\tau \langle K[\lambda - \alpha\tilde{u}], \mu \rangle - \tau \langle \tilde{u}_s, \mu_s \rangle + \langle \tilde{w}, \mu \rangle = 0 \quad \text{for all} \quad \mu \in H^{3/2}(\Gamma), \tag{1.7}$$

and

$$\langle \tilde{u}, \beta \rangle = 0 \quad \text{for all} \quad \beta \in H^{-1/2}(\Gamma). \tag{1.8}$$

To understand the relation between Problem $(\tilde{P})$ and the biharmonic problem (1.1)-(1.3), observe first that equation (1.5) is the weak form of the boundary value problem

$$-\Delta\tilde{w} = f \quad \text{in} \quad \Omega$$

$$\frac{\partial\tilde{w}}{\partial n} + \alpha\tilde{w} = \sigma + \tau\lambda_{ss} \quad \text{on} \quad \Gamma,$$

and equation (1.6) is the weak form of the boundary value problem

$$-\Delta\tilde{u} = \tilde{w} \quad \text{in} \quad \Omega$$
$$\frac{\partial\tilde{u}}{\partial n} + \alpha\tilde{u} = \lambda \quad \text{on} \quad \Gamma .$$

Equations (1.7) and (1.8) give the boundary conditions

$$\tau[K(\lambda - \alpha\tilde{u}) + \tilde{u}_{ss}] + \tilde{w} = 0 \quad \text{on} \quad \Gamma$$

and

$$\tilde{u} = 0 \quad \text{on} \quad \Gamma .$$

Suppose now that for $\tilde{u}$ a smooth solution of (1.1)-(1.3) we set

$$\tilde{w} = -\Delta\tilde{u}, \tag{1.9}$$
$$\lambda = \frac{\partial\tilde{u}}{\partial n} + \alpha\tilde{u} \tag{1.10}$$

and

$$\sigma = -\left[\frac{\partial}{\partial n}\Delta\tilde{u} + \alpha\,\Delta\tilde{u} + \tau(\tilde{u}_{nss} + \alpha\tilde{u}_{ss})\right]. \tag{1.11}$$

Then from (1.1), $-\Delta\tilde{w} = f$ and by (1.9)-(1.11)

$$\sigma = \frac{\partial}{\partial n}\tilde{w} + \alpha\tilde{w} - \tau\lambda_{ss}$$

which implies that $(\tilde{u}, \tilde{w}, \lambda, \sigma)$ satisfies (1.5). Now from (1.9) and (1.10), it easily follows that $(\tilde{u}, \tilde{w}, \lambda, \sigma)$ satisfies (1.6). Using (1.2), (1.9), and (1.10) we get that $\tilde{w} + \tau[\tilde{u}_{ss} + K(\lambda - \alpha\tilde{u})] = 0$ on Γ and so (1.7) is satisfied. Finally, (1.3) implies (1.8) so that $(\tilde{u}, \tilde{w}, \lambda, \sigma)$, with $\tilde{w}, \lambda, \sigma$ defined by (1.9)-(1.11) is a solution of Problem $(\tilde{P})$.

When the curvature K of Ω is strictly positive we are able to give a much simpler variational formulation of (1.1)-(1.3).

Problem $(\tilde{P}^*)$: Find $(\tilde{u}, \tilde{w}, \sigma) \in H^1(\Omega) \times H^1(\Omega) \times H^{-1/2}(\Gamma)$ such that

$$A_\alpha(\tilde{w}, v) = (f, v) + \langle \sigma, v \rangle \quad \text{for all} \quad v \in H^1(\Omega), \tag{1.12}$$
$$A_\alpha(\tilde{u}, z) = (\tilde{w}, z) - \left\langle \frac{\tilde{w}}{\tau K}, z \right\rangle \quad \text{for all} \quad z \in H^1(\Omega), \tag{1.13}$$

and

$$\langle \tilde{u}, \beta \rangle = 0 \quad \forall\beta \in H^{-1/2}(\Gamma). \tag{1.14}$$

To understand the relation between Problem ($\tilde{P}^*$) and the biharmonic problem (1.1)-(1.3) observe first that equation (1.12) is the weak form of the boundary value problem

$$- \Delta \tilde{w} = f \quad \text{in} \quad \Omega$$
$$\frac{\partial \tilde{w}}{\partial n} + \alpha \tilde{w} = \sigma \quad \text{on} \quad \Gamma ,$$

and equation (1.13) is the weak form of the boundary value problem

$$- \Delta \tilde{u} = \tilde{w} \quad \text{in} \quad \Omega$$
$$\frac{\partial \tilde{u}}{\partial n} + \alpha \tilde{u} = - \frac{\tilde{w}}{\tau K} \quad \text{on} \quad \Gamma .$$

Equation (1.14) gives the boundary condition $\tilde{u} = 0$ on Γ.

Suppose now that for $\tilde{u}$ a smooth solution of (1.1)-(1.3) we set

$$\tilde{w} = - \Delta \tilde{u} \tag{1.15}$$

and

$$\sigma = - \left[\frac{\partial}{\partial n} \Delta \tilde{u} + \alpha \, \Delta \tilde{u} \right]. \tag{1.16}$$

Then by (1.1), $- \Delta \tilde{w} = f$ and by (1.15)-(1.16), $\sigma = \frac{\partial}{\partial n} \tilde{w} + \alpha \tilde{w}$ which implies that $(\tilde{u}, \tilde{w}, \sigma)$ satisfies (1.12). Now from (1.3), $\tilde{u}_{ss} = 0$ on Γ so that by (1.2) and (1.15) $\frac{\partial \tilde{u}}{\partial n} + \alpha \tilde{u} = - \frac{\tilde{w}}{\tau K}$. Hence (1.13) is satisfied. Finally (1.3) implies (1.14) so that $(\tilde{u}, \tilde{w}, \sigma)$ with $\tilde{w}$ and σ defined by (1.15)-(1.16) is a solution of Problem ($\tilde{P}^*$).

It is the purpose of this paper to analyze finite element methods for the approximation of the biharmonic problem (1.1)-(1.3) based on the two variational formulations ($\tilde{P}$) and ($\tilde{P}^*$). Once these approximation schemes are developed and error estimates derived, we shall then show how the resulting approximations can be obtained as the limit of a rapidly converging sequence of functions requiring only the numerical solution of second order problems with natural boundary conditions. The techniques used to obtain these results are based on those in Bramble [7], where analogous results are obtained for the second order Dirichlet problem.

We also note that many of the ideas used in this paper appear previously in several sources. The use of Lagrange multipliers was first analyzed by Babuška [3] for the second order Dirichlet problem. An analysis of a mixed finite element method for the biharmonic model of the clamped plate problem is given by

Ciarlet and Raviart in [8] and the idea of the solution of this problem using an iteration scheme requiring the solution of second order Dirichlet problems at each iteration is discussed in Ciarlet and Glowinski [9]. Further ideas in this direction can be found in Glowinski and Pironneau [11]. In Falk [10] a mixed finite element method is presented for the biharmonic problem with Dirichlet type boundary conditions whose solution involves an iteration scheme requiring the solution of two Neumann problems at each iteration.

An outline of the paper is as follows. In Section 2 we introduce the notation to be used and then state and prove some a priori estimates that will be needed in the subsequent analysis. Section 3 defines the approximating subspaces to be used in the finite element method and collects some results on the approximation properties of these subspaces. In Section 4, we define the approximation scheme for Problem $(\tilde{P})$ and prove some additional a priori estimates analogous to those in Section 2. Section 5 then contains the derivation of error estimates for Problem $(\tilde{P})$.

In Sections 6, 7, and 8 we analyze a finite element method based on the variational formulation $(\tilde{P}^*)$ for the case where Ω has strictly positive curvature. Section 6 has some preliminary results for this case, Section 7 discusses the finite element approximation scheme, and Section 8 contains the error estimates. Finally in Section 9 we discuss efficient computational procedures to solve our approximate problems.

2. NOTATION AND PRELIMINARY RESULTS

For $s \geqslant 0$ let $H^s(\Omega)$ and $H^s(\Gamma)$ denote the Sobolev spaces of order s of functions on Ω and Γ respectively, with associated norms $\|.\|_s$ and $|.|_s$ respectively (*cf.* [12]). For $s < 0$ let $H^s(\Omega)$ and $H^s(\Gamma)$ be the respective duals of $H^{-s}(\Omega)$ and $H^{-s}(\Gamma)$ with the usual dual norm.

To simplify the exposition of this paper we shall also use the norm $|||.|||_s$ defined on $H^{s+1/2}(\Omega) \cap H^s(\Gamma)$ by

$$|||\phi|||_s = |\phi|_s + \|\phi\|_{s+1/2}\,.$$

In order to analyze the mixed formulation denoted by Problem $(\tilde{P})$ and its finite element approximation it will be convenient to introduce the following notation.

Define operators

$$T : H^s(\Omega) \rightarrow H^{s+2}(\Omega)$$

and

$$G : H^s(\Gamma) \rightarrow H^{s+3/2}(\Omega)$$

by $$A_\alpha(Tf, v) = (f, v) \quad \text{for all} \quad v \in C^\infty(\overline{\Omega})$$

and $$A_\alpha(G\sigma, v) = \langle \sigma, v \rangle \quad \text{for all} \quad v \in C^\infty(\overline{\Omega}),$$

i.e. Tf is the weak solution of the boundary value problem

$$-\Delta(Tf) = f \quad \text{in} \quad \Omega$$

$$\frac{\partial}{\partial n}(Tf) + \alpha(Tf) = 0 \quad \text{on} \quad \Gamma$$

and $G\sigma$ is the weak solution of the boundary value problem

$$-\Delta(G\sigma) = 0 \quad \text{in} \quad \Omega$$

$$\frac{\partial}{\partial n}(G\sigma) + \alpha(G\sigma) = \sigma \quad \text{on} \quad \Gamma.$$

We remark that it is well known (*cf.* [13]) that T and G satisfy the estimates

$$\| Tf \|_{s+2} \leqslant C \| f \|_s \tag{2.1a}$$

and

$$\| G\sigma \|_{s+3/2} \leqslant C | \sigma |_s \tag{2.1b}$$

for all real s, where C is a constant independent of σ and f.

Using (2.1) we can also prove :

LEMMA 2.1 : *There exists a constant C independent of σ and f such that for all real s*

$$| Tf |_{1/2-s} \leqslant C \| f \|_{-1-s} \tag{2.2a}$$

and

$$| G\sigma |_{1/2-s} \leqslant C | \sigma |_{-1/2-s}. \tag{2.2b}$$

Proof : For $s < 1/2$ (2.2) follows from (2.1) by standard trace theorems. For $s \geqslant 1/2$ we use the definitions of T and G to write :

$$\langle Tf, \beta \rangle = A_\alpha(G\beta, Tf) = (f, G\beta) \leqslant \| f \|_{-1-s} \| G\beta \|_{1+s}$$

and

$$\langle G\sigma, \beta \rangle = A_\alpha(G\sigma, G\beta) = \langle \sigma, G\beta \rangle \leqslant | \sigma |_{-1/2-s} | G\beta |_{1/2+s}.$$

Hence from (2.1) we get

$$| Tf |_{1/2-s} = \sup_{\beta \in H^{s-1/2}} \frac{\langle Tf, \beta \rangle}{| \beta |_{s-1/2}}$$

$$\leqslant \sup_{\beta \in H^{s-1/2}} \frac{\| f \|_{-1-s} \| G\beta \|_{1+s}}{| \beta |_{s-1/2}} \leqslant C \| f \|_{-1-s}$$

and since $1/2 + s > 0$, we can use (2.2*b*) to get

$$| G\sigma |_{1/2-s} = \sup_{\beta \in H^{s-1/2}} \frac{\langle G\alpha, \beta \rangle}{| \beta |_{s-1/2}}$$

$$\leqslant \sup_{\beta \in H^{s-1/2}} \frac{| \sigma |_{-1/2-s} \, | G\beta |_{1/2+s}}{| \beta |_{s-1/2}} \leqslant C \, | \sigma |_{-1/2-s} .$$

Using these definitions we see from (1.5) that

$$\tilde{w} = Tf + G[\sigma + \tau\lambda_{ss}] \tag{2.3}$$

and from (1.6) that

$$\tilde{u} = T\tilde{w} + G\lambda = T^2 f + TG[\sigma + \tau\lambda_{ss}] + G\lambda . \tag{2.4}$$

Let us now define

$$w(\lambda, \sigma) = G[\sigma + \tau\lambda_{ss}] \tag{2.5}$$

and

$$u(\lambda, \sigma) = TG[\sigma + \tau\lambda_{ss}] + G\lambda . \tag{2.6}$$

Then
$$\tilde{w} = Tf + w(\lambda, \sigma)$$
and
$$\tilde{u} = T^2 f + u(\lambda, \sigma)$$

so that Problem $(\tilde{P})$ can be restated in the form :

Problem (*P*) : Find $(\lambda, \sigma) \in H^{3/2}(\Gamma) \times H^{-1/2}(\Gamma)$ such that

$$\tau K[\lambda - \alpha u(\lambda, \sigma)] + \tau u_{ss}(\lambda, \sigma) + w(\lambda, \sigma) = \tau K\alpha T^2 f - \tau(T^2 f)_{ss} - Tf \tag{2.7}$$

and

$$u(\lambda, \sigma) = - T^2 f . \tag{2.8}$$

It will be from this point of view that we will approximate $\tilde{u}$, i.e. we will approximate G, T, σ, and λ to obtain an approximation to $\tilde{u}$.

The analysis of the finite element method for the approximation of the mixed formulation of the biharmonic problem given in Problem (*P*) will depend heavily on the study of the function

$$u(\lambda, \sigma) = TG[\sigma + \tau\lambda_{ss}] + G\lambda .$$

From the definitions of T and G it easily follows that $u(\lambda, \sigma)$ is the solution of the biharmonic problem :

Problem (Q) : Given $(\lambda, \sigma) \in H^{3/2}(\Gamma) \times H^{-1/2}(\Gamma)$, find $u \in H^2(\Omega)$ satisfying

$$\Delta^2 u = 0 \quad \text{in} \quad \Omega \tag{2.9}$$

$$\frac{\partial u}{\partial n} + \alpha u = \lambda \quad \text{on} \quad \Gamma \tag{2.10}$$

$$-\frac{\partial}{\partial n}\Delta u - \alpha\,\Delta u - \tau[u_{nss} + \alpha u_{ss}] = \sigma \quad \text{on} \quad \Gamma\,. \tag{2.11}$$

In this section we wish to prove several a priori estimates for the solution of Problem (Q). To do so we first state some Green's formulas and standard a priori estimates for solutions of the biharmonic equation $\Delta^2 u = 0$.
For $u, v \in H^2(\Omega)$ define a bilinear form $E(u, v)$ by

$$E(u, v) = (\Delta u, \Delta v) - \tau\{(u_{xx}, v_{yy}) + (u_{yy}, v_{xx}) - 2(u_{xy}, v_{xy})\}\,. \tag{2.12}$$

We then have the following Green's formula (see e.g. [14]).

$$E(u, v) = (\Delta^2 u, v) - \left\langle \frac{\partial}{\partial n}\Delta u + \tau\frac{\partial}{\partial s}(u_{sn} - K u_s), v \right\rangle + \\ + \left\langle \Delta u - \tau(u_{ss} + K u_n), \frac{\partial v}{\partial n} \right\rangle. \tag{2.13}$$

Defining

$$M(u) = \Delta u - \tau(u_{ss} + K u_n) \tag{2.14}$$

and

$$V(u) = \frac{\partial}{\partial n}\Delta u + \tau\frac{\partial}{\partial s}(u_{sn} - K u_s) \tag{2.15}$$

and observing the symmetry of $E(u, v)$ we obtain for all u, v satisfying $\Delta^2 u = \Delta^2 v = 0$ that

$$-\langle V(u), v \rangle + \left\langle M(u), \frac{\partial v}{\partial n} \right\rangle = -\langle V(v), u \rangle + \left\langle M(v), \frac{\partial u}{\partial n} \right\rangle. \tag{2.16}$$

Setting $v = u$ in (2.12) we also easily obtain that for all $u \in H^2(\Omega)$

$$\begin{aligned} E(u, u) &= (\Delta u, \Delta u) - 2\,\tau(u_{xx}, u_{yy}) + 2\,\tau(u_{xy}, u_{xy}) \\ &= (1 - \tau)(\Delta u, \Delta u) + \tau\{(u_{xx}, u_{xx}) + (u_{yy}, u_{yy}) + 2(u_{xy}, u_{xy})\}\,. \end{aligned}$$

Hence for $0 < \tau < 1$,

$$\sum_{|\alpha|=2} \| D^{\alpha} u \|_0^2 \leqslant \frac{1}{\tau} E(u, u) . \tag{2.17}$$

A second standard Green's formula for solutions of the biharmonic equation $\Delta^2 u = 0$ is given by

$$(\Delta u, \Delta v) = (\Delta^2 u, v) - \left\langle \frac{\partial}{\partial n} \Delta u, v \right\rangle + \left\langle \Delta u, \frac{\partial v}{\partial n} \right\rangle . \tag{2.18}$$

It easily follows from (2.18) that

$$(\Delta u, \Delta v) = (\Delta^2 u, v) - \left\langle \frac{\partial}{\partial n} \Delta u + \alpha \Delta u, v \right\rangle + \left\langle \Delta u, \frac{\partial v}{\partial n} + \alpha v \right\rangle .$$

Using the symmetry of the form $(\Delta u, \Delta v)$ we then get for all u, v satisfying $\Delta^2 u = \Delta^2 v = 0$ that

$$- \left\langle \frac{\partial}{\partial n} \Delta u + \alpha \Delta u, v \right\rangle + \left\langle \Delta u, \frac{\partial v}{\partial n} + \alpha v \right\rangle =$$

$$= - \left\langle \frac{\partial}{\partial n} \Delta v + \alpha \Delta v, u \right\rangle + \left\langle \Delta v, \frac{\partial u}{\partial n} + \alpha u \right\rangle . \tag{2.19}$$

In the course of our analysis we shall also need to make use of an additional Green's identity, which we now derive.

LEMMA 2.2 : *If $u \in H^2(\Omega)$ is the solution of Problem (Q), then*

$$E(u, u) = \langle \sigma, u \rangle + \langle M(u), \lambda \rangle + \alpha\tau \langle \lambda, Ku \rangle - \alpha^2 \tau \langle Ku, u \rangle - \\ - \tau \langle [2\alpha + K] u_s, u_s \rangle .$$

Proof : Since $\Delta^2 u = 0$ we have from (2.13) that

$$\begin{aligned} E(u, u) &= - \left\langle \frac{\partial}{\partial n} \Delta u + \tau \frac{\partial}{\partial s} (u_{sn} - Ku_s), u \right\rangle \\ &\quad + \left\langle \Delta u - \tau(u_{ss} + Ku_n), \frac{\partial u}{\partial n} \right\rangle \\ &= - \left\langle \frac{\partial}{\partial n} \Delta u + \alpha \Delta u + \tau[u_{nss} + \alpha u_{ss}], u \right\rangle \\ &\quad + \left\langle \alpha \Delta u + \tau\left[\alpha u_{ss} + \frac{\partial}{\partial s}(Ku_s) \right], u \right\rangle \end{aligned}$$

$$
\begin{aligned}
&+ \left\langle \Delta u - \tau(u_{ss} + K u_n), \frac{\partial u}{\partial n} + \alpha u \right\rangle \\
&- \langle \alpha \Delta u - \alpha\tau(u_{ss} + K u_n), u \rangle \\
= &\langle \sigma, u \rangle + \langle M(u), \lambda \rangle \\
&+ 2\,\alpha\tau \langle u_{ss}, u \rangle - \tau \left\langle \frac{\partial}{\partial s}(K u_s), u \right\rangle + \alpha\tau \langle K u_n, u \rangle \\
= &\langle \sigma, u \rangle + \langle M(u), \lambda \rangle \\
&- \tau \langle [2\,\alpha + K]\, u_s, u_s \rangle + \alpha\tau \langle \lambda, K u \rangle - \alpha^2 \tau \langle K u, u \rangle .
\end{aligned}
$$

We shall also require the following a priori estimates satisfied by solutions of the biharmonic equation (*cf.* [13]).

LEMMA 2.3 : *Let v be a solution of the biharmonic equation $\Delta^2 v = 0$ in Ω. Then there exists a constant C (independent of v) such that v satisfies the following a priori estimates for all real s :*

$$\| v \|_{3+s} \leqslant C[| M(v) |_{1/2+s} + | v |_{5/2+s}] \tag{2.20}$$

and

$$\| v \|_{3+s} \leqslant C\left[\left| \frac{\partial}{\partial n} \Delta v + \alpha\, \Delta v \right|_{-1/2+s} + \left| \frac{\partial v}{\partial n} + \alpha v \right|_{3/2+s} \right]. \tag{2.21}$$

Using these results we now establish a series of a priori estimates for solutions of the biharmonic equation $\Delta^2 u = 0$.

LEMMA 2.4 : *If u is a solution of $\Delta^2 u = 0$, then there exists a constant C independent of u such that for all $s \geqslant 0$*

$$\left| \frac{\partial u}{\partial n} \right|_{-1/2-s} + | V(u) |_{-5/2-s} \leqslant C[| u |_{1/2-s} + | M(u) |_{-3/2-s}] .$$

Proof : To estimate $\left| \frac{\partial u}{\partial n} \right|_{-1/2-s}$, we use the Green's formula (2.16) and define v to be the solution of the biharmonic problem

$$
\begin{aligned}
\Delta^2 v &= 0 \quad \text{in} \quad \Omega \\
v &= 0 \quad \text{on} \quad \Gamma \\
M(v) &= \psi \quad \text{on} \quad \Gamma .
\end{aligned}
$$

From (2.16) we have that

$$\left\langle \frac{\partial u}{\partial n}, \psi \right\rangle = \left\langle M(u), \frac{\partial v}{\partial n} \right\rangle + \langle V(v), u \rangle$$
$$\leqslant | M(u) |_{-3/2-s} \left| \frac{\partial v}{\partial n} \right|_{3/2+s} + | V(v) |_{-1/2+s} | u |_{1/2-s} .$$

From (2.15),

$$| V(v) |_{-1/2+s} = \left| \frac{\partial}{\partial n} \Delta v + \tau \frac{\partial}{\partial s} (v_{sn} - K v_s) \right|_{-1/2+s}$$
$$\leqslant \left| \frac{\partial}{\partial n} \Delta v \right|_{-1/2+s} + \tau \left| \frac{\partial}{\partial s} (v_{sn} - K v_s) \right|_{-1/2+s} .$$

Since Δv is a harmonic function, we have the estimate

$$\left| \frac{\partial}{\partial n} \Delta v \right|_{-1/2+s} \leqslant C \| \Delta v \|_{1+s} \leqslant C \| v \|_{3+s} .$$

By standard trace theorems,

$$\left| \frac{\partial v}{\partial n} \right|_{3/2+s} \leqslant C \| v \|_{3+s}$$

and

$$\left| \frac{\partial}{\partial s} (v_{sn} - K v_s) \right|_{-1/2+s} \leqslant C[| v_n |_{3/2+s} + | v |_{3/2+s}] \leqslant C \| v \|_{3+s} .$$

Combining terms and applying (2.20) we get

$$\left\langle \frac{\partial u}{\partial n}, \psi \right\rangle \leqslant C[| M(u) |_{-3/2-s} + | u |_{1/2-s}] \| v \|_{3+s}$$
$$\leqslant C[| M(u) |_{-3/2-s} + | u |_{1/2-s}] | \psi |_{1/2+s} .$$

Finally,

$$\left| \frac{\partial u}{\partial n} \right|_{-1/2-s} = \sup_{\psi \in H^{1/2+s}(\Gamma)} \frac{\left\langle \frac{\partial u}{\partial n}, \psi \right\rangle}{| \psi |_{1/2+s}} \leqslant C[| M(u) |_{-3/2-s} + | u |_{1/2-s}] .$$

To estimate $| V(u) |_{-5/2-s}$ we again use the Green's formula (2.16) and define v to be the solution of the biharmonic problem

$$\begin{aligned} \Delta^2 v &= 0 \quad \text{in} \quad \Omega \\ M(v) &= 0 \quad \text{on} \quad \Gamma \\ v &= \psi \quad \text{on} \quad \Gamma . \end{aligned}$$

From (2.16) we have that

$$\begin{aligned} \langle V(u), \psi \rangle &= \left\langle M(u), \frac{\partial v}{\partial n} \right\rangle + \langle V(v), u \rangle \\ &\leqslant | M(u) |_{-3/2-s} \left| \frac{\partial v}{\partial n} \right|_{3/2+s} + | V(v) |_{-1/2+s} | u |_{1/2-s} . \end{aligned}$$

Estimating terms as before and again applying (2.20) we get

$$\begin{aligned} \langle V(u), \psi \rangle &\leqslant C[| M(u) |_{-3/2-s} + | u |_{1/2-s}] \| v \|_{3+s} \\ &\leqslant C[| M(u) |_{-3/2-s} + | u |_{1/2-s}] | \psi |_{5/2+s} . \end{aligned}$$

Finally,

$$| V(u) |_{-5/2-s} = \sup_{\psi \in H^{5/2-s}(\Gamma)} \frac{\langle V(u), \psi \rangle}{| \psi |_{5/2+s}} \leqslant C[| M(u) |_{-3/2-s} + | u |_{1/2-s}] .$$

LEMMA 2.5 : *If u is a solution of $\Delta^2 u = 0$, then there exists a constant C independent of u such that for all $s \geqslant 0$*

$$| u |_{1/2-s} + | \Delta u |_{-3/2-s} \leqslant C \left[\left| \frac{\partial}{\partial n} \Delta u + \alpha \Delta u \right|_{-5/2-s} + \left| \frac{\partial u}{\partial n} + \alpha u \right|_{-1/2-s} \right] .$$

Proof : To estimate $| u |_{1/2-s}$ we use the Green's formula (2.19) and define v to be the solution of the biharmonic problem

$$\begin{aligned} \Delta^2 v &= 0 \quad \text{in} \quad \Omega \\ \frac{\partial}{\partial n} \Delta v + \alpha \Delta v &= \psi \quad \text{on} \quad \Gamma \\ \frac{\partial v}{\partial n} + \alpha v &= 0 \quad \text{on} \quad \Gamma . \end{aligned}$$

From (2.19) we have that

$$\langle u, \psi \rangle = \left\langle \Delta v, \frac{\partial u}{\partial n} + \alpha u \right\rangle + \left\langle \frac{\partial}{\partial n} \Delta u + \alpha \Delta u, v \right\rangle$$

$$\leqslant \left| \frac{\partial u}{\partial n} + \alpha u \right|_{-1/2-s} | \Delta v |_{1/2+s} + \left| \frac{\partial}{\partial n} \Delta u + \alpha \Delta u \right|_{-5/2-s} | v |_{5/2+s} .$$

Applying standard trace theorems and the a priori estimate (2.21) we get

$$\langle u, \psi \rangle \leqslant C \left[\left| \frac{\partial u}{\partial n} + \alpha u \right|_{-1/2-s} + \left| \frac{\partial}{\partial n} \Delta u + \alpha \Delta u \right|_{-5/2-s} \right] \| v \|_{3+s}$$

$$\leqslant C \left[\left| \frac{\partial u}{\partial n} + \alpha u \right|_{-1/2-s} + \left| \frac{\partial}{\partial n} \Delta u + \alpha \Delta u \right|_{-5/2-s} \right] | \psi |_{-1/2+s} .$$

Finally

$$| u |_{1/2-s} = \sup_{\psi \in H^{-1/2+s}(\Gamma)} \frac{\langle u, \psi \rangle}{| \psi |_{-1/2+s}}$$

$$\leqslant C \left[\left| \frac{\partial u}{\partial n} + \alpha u \right|_{-1/2-s} + \left| \frac{\partial}{\partial n} \Delta u + \alpha \Delta u \right|_{-5/2-s} \right].$$

To estimate $| \Delta u |_{-3/2-s}$ we again use the Green's formula (2.19) and define v to be the solution of the biharmonic problem

$$\Delta^2 v = 0 \quad \text{in} \quad \Omega$$

$$\frac{\partial}{\partial n} \Delta v + \alpha \Delta v = 0 \quad \text{on} \quad \Gamma$$

$$\frac{\partial v}{\partial n} + \alpha v = \psi \quad \text{on} \quad \Gamma .$$

From (2.19) we have that

$$\langle \Delta u, \psi \rangle = \left\langle \Delta v, \frac{\partial u}{\partial n} + \alpha u \right\rangle + \left\langle \frac{\partial}{\partial n} \Delta u + \alpha \Delta u, v \right\rangle$$

$$\leqslant | \Delta v |_{1/2+s} \left| \frac{\partial u}{\partial n} + \alpha u \right|_{-1/2-s} + | v |_{5/2+s} \left| \frac{\partial}{\partial n} \Delta u + \alpha \Delta u \right|_{-5/2-s} .$$

Estimating terms as before and again applying (2.21) we get

$$\langle \Delta u, \psi \rangle \leqslant C\left[\left|\frac{\partial u}{\partial n} + \alpha u\right|_{-1/2-s} + \left|\frac{\partial}{\partial n}\Delta u + \alpha\,\Delta u\right|_{-5/2-s}\right] \| v \|_{3+s}$$
$$\leqslant C\left[\left|\frac{\partial u}{\partial n} + \alpha u\right|_{-1/2-s} + \left|\frac{\partial}{\partial n}\Delta u + \alpha\,\Delta u\right|_{-5/2-s}\right] | \psi |_{3/2+s}.$$

Finally

$$| \Delta u |_{-3/2-s} = \sup_{\psi \in H^{3/2+s}(\Gamma)} \frac{\langle \Delta u, \psi \rangle}{| \psi |_{3/2+s}}$$
$$\leqslant C\left[\left|\frac{\partial u}{\partial n} + \alpha u\right|_{-1/2-s} + \left|\frac{\partial}{\partial n}\Delta u + \alpha\,\Delta u\right|_{-5/2-s}\right].$$

Using these results we can now prove the following.

THEOREM 2.1 : *There exist positive constants C_1 and C_2 independent of σ and λ such that for all $s \geqslant 0$*

$$C_1[| \lambda |_{-1/2-s} + | \sigma |_{-5/2-s}] \leqslant | u(\lambda, \sigma) |_{1/2-s} + | M(\lambda, \sigma) |_{-3/2-s}$$
$$\leqslant C_2[| \lambda |_{-1/2-s} + | \sigma |_{-5/2-s}]$$

where $u(\lambda, \sigma)$ $(= TG[\sigma + \tau\lambda_{ss}] + G\lambda)$ is the solution of Problem (Q) and $M(\lambda, \sigma) = Mu(\lambda, \sigma)$.

Proof : To simplify notation we will simply denote $u(\lambda, \sigma)$ by u for the remainder of this proof.

To derive the lower inequality, we use (2.10) and Lemma 2.4 to obtain

$$| \lambda |_{-1/2-s} = \left|\frac{\partial u}{\partial n} + \alpha u\right|_{-1/2-s} \leqslant C\left[\left|\frac{\partial u}{\partial n}\right|_{-1/2-s} + | u |_{1/2-s}\right]$$
$$\leqslant C[| u |_{1/2-s} + | M(u) |_{-3/2-s}].$$

Now from (2.11), (2.14), and (2.15)

$$\sigma = -\frac{\partial}{\partial n}\Delta u - \alpha\,\Delta u - \tau[u_{nss} + \alpha u_{ss}]$$
$$= -\left[\frac{\partial}{\partial n}\Delta u + \tau u_{nss} - \tau\frac{\partial}{\partial s}Ku_s\right] - \alpha\,\Delta u - \tau\frac{\partial}{\partial s}Ku_s - \tau\alpha u_{ss}$$
$$= -V(u) - \alpha[\Delta u - \tau(u_{ss} + Ku_n)] - 2\,\alpha\tau u_{ss} - \tau\frac{\partial}{\partial s}Ku_s - \alpha\tau Ku_n$$
$$= -V(u) - \alpha M(u) - 2\,\alpha\tau u_{ss} - \tau\frac{\partial}{\partial s}Ku_s - \alpha\tau Ku_n\,.$$

Hence

$$|\sigma|_{-5/2-s} \leqslant C[|V(u)|_{-5/2-s} + |M(u)|_{-3/2-s} + |u|_{1/2-s} + |u_n|_{-1/2-s}].$$

Applying Lemma 2.4 we get

$$|\sigma|_{-5/2-s} \leqslant C[|u|_{1/2-s} + |M(u)|_{-3/2-s}].$$

To derive the upper inequality we use (2.10) and Lemma 2.5 to obtain

$$|u|_{1/2-s} + |\Delta u|_{-3/2-s} \leqslant C\left[\left|\frac{\partial}{\partial n}\Delta u + \alpha\,\Delta u\right|_{-5/2-s} + |\lambda|_{-1/2-s}\right].$$

But by (2.10) and (2.11)

$$\frac{\partial}{\partial n}\Delta u + \alpha\,\Delta u = -\sigma - \tau[u_{nss} + \alpha u_{ss}] = -\sigma - \tau\lambda_{ss}.$$

Hence

$$|u|_{1/2-s} + |\Delta u|_{-3/2-s} \leqslant C[|\sigma|_{-5/2-s} + |\lambda|_{-1/2-s}]. \qquad (2.22)$$

To complete the proof of the Lemma we observe from (2.10) and (2.14) that

$$M(u) = \Delta u - \tau(u_{ss} + Ku_n) = \Delta u - \tau(u_{ss} + K\lambda - \alpha K u).$$

Hence

$$|M(u)|_{-3/2-s} \leqslant C[|\Delta u|_{-3/2-s} + |u|_{1/2-s} + |\lambda|_{-1/2-s}].$$

The upper inequality now follows directly from (2.22).

3. APPROXIMATING SPACES ON Ω AND Γ

For $0 < h < 1$, let $\{S_h\}$ be a family of finite dimensional subspaces of $H^1(\Omega)$. Let $r \geqslant 2$ be an integer. We shall assume that for $\phi \in H^l(\Omega)$ with $1 \leqslant l \leqslant r$ there is a constant C such that

$$\inf_{\chi \in S_h} \|\phi - \chi\|_j \leqslant C h^{l-j} \|\phi\|_l, \quad j \leqslant 1. \qquad (3.1)$$

We now define the operators $G_h : H^{-1/2}(\Gamma) \to S_h$ and $T_h : H^{-1}(\Omega) \to S_h$ by

$$A_\alpha(G_h\theta, \chi) = \langle \theta, \chi \rangle, \quad \forall \chi \in S_h$$

and

$$A_\alpha(T_h f, \chi) = (f, \chi), \quad \forall \chi \in S_h.$$

These are just the standard Ritz-Galerkin approximations to G and T.

It follows from the approximations assumptions and standard duality arguments that we have the following well known results (*cf.* [2], [4]).

LEMMA 3.1 : *There exists a constant C independent of σ, f and h such that*

$$|(G - G_h)\sigma|_{j-1/2} + \|(G - G_h)\sigma\|_j \leqslant Ch^{l-j}\|G\sigma\|_l$$

and

$$|(T - T_h)f|_{j-1/2} + \|(T - T_h)f\|_j \leqslant Ch^{l-j}\|Tf\|_l$$

for $2 - r \leqslant j \leqslant 1 \leqslant l \leqslant r$, $\sigma \in H^{l-3/2}(\Gamma)$ *and* $f \in H^{l-2}(\Omega)$.

Note that the restriction to Ω of continuous piecewise polynomials of degree $r - 1$ on a quasi-uniform triangulation of R^2 or a rectangular mesh of " width " h are examples of spaces S_h satisfying Lemma 3.1.

In the analysis in the subsequent sections we shall require additional estimates for the approximation of the operators T and G by T_h and G_h. These are contained in the following two theorems.

THEOREM 3.1 : *Suppose* $0 \leqslant s \leqslant r - 1$ *and that* $f \in H^m(\Omega)$ *with*

$$-1 \leqslant m \leqslant \max(r - 4, s - 1)$$

and $\beta \in H^l(\Gamma)$ *with* $-1/2 \leqslant l \leqslant \max(r - 7/2, s - 1/2)$. *Then for* $r \geqslant 2$

$$|[T^2 - T_h^2]f|_{1/2-s} + \|[T^2 - T_h^2]f\|_{1-s} \leqslant Ch^{m+4+\min(s-1,r-4)}\|f\|_m$$

and

$$|[TG - T_h G_h]\beta|_{1/2-s} + \|[TG - T_h G_h]\beta\|_{1-s} \leqslant$$
$$\leqslant Ch^{l+3+\min(s-1/2,r-7/2)}|\beta|_l$$

where C is a constant independent of h and f.

Proof : To simplify the exposition, let us use the notation

$$|||\phi|||_s = |\phi|_s + \|\phi\|_{s+1/2}\,.$$

By the triangle inequality we have

$$|||[T^2 - T_h^2]f|||_{1/2-s} \leqslant |||T[T - T_h]f|||_{1/2-s} +$$
$$+ |||[T_h - T][T - T_h]f|||_{1/2-s} + |||[T - T_h]Tf|||_{1/2-s}$$

and

$$|||[TG - T_h G_h]\beta|||_{1/2-s} \leqslant |||T[G - G_h]\beta|||_{1/2-s} +$$
$$+ |||[T_h - T][G - G_h]\beta|||_{1/2-s} + |||T[G - G_h]\beta|||_{1/2-s}\,.$$

Using (2.1), (2.2), and Lemma 3.1 we get for $0 \leqslant s \leqslant r-1$, $-1 \leqslant m \leqslant r-2$, and $-1/2 \leqslant l \leqslant r-3/2$ that

$$\begin{aligned} |\!|\!| T[T-T_h]f |\!|\!|_{1/2-s} &\leqslant C \| [T-T_h]f \|_{-1-s} \\ &\leqslant Ch^{m+2+\min(s+1,r-2)} \| Tf \|_{m+2} \\ &\leqslant Ch^{m+4+\min(s-1,r-4)} \| f \|_m , \end{aligned}$$

$$\begin{aligned} |\!|\!| T[G-G_h]\beta |\!|\!|_{1/2-s} &\leqslant C \| [G-G_h]\beta \|_{-1-s} \\ &\leqslant Ch^{l+3/2+\min(s+1,r-2)} \| G\beta \|_{l+3/2} \\ &\leqslant Ch^{l+3+\min(s-1/2,r-7/2)} | \beta |_l , \end{aligned}$$

$$\begin{aligned} |\!|\!| [T_h-T][T-T_h]f |\!|\!|_{1/2-s} &\leqslant Ch^{s+1} \| T[T-T_h]f \|_2 \\ &\leqslant Ch^{s+1} \| [T-T_h]f \|_0 \leqslant Ch^{s+m+3} \| f \|_m \\ &\leqslant Ch^{m+4+\min(s-1,r-4)} \| f \|_m , \end{aligned}$$

and

$$\begin{aligned} |\!|\!| [T_h-T][G-G_h]\beta |\!|\!|_{1/2-s} &\leqslant Ch^{s+1} \| T[G-G_h]\beta \|_2 \\ &\leqslant Ch^{s+1} \| [G-G_h]\beta \|_0 \leqslant Ch^{s+l+5/2} | \beta |_l \\ &\leqslant Ch^{l+3+\min(s-1/2,r-7/2)} | \beta |_l . \end{aligned}$$

Now for $0 \leqslant s \leqslant 1$, $1 \leqslant t+4 \leqslant r$, and $1 \leqslant \bar{t}+7/2 \leqslant r$, we also have that

$$|\!|\!| [T-T_h]Tf |\!|\!|_{1/2-s} \leqslant Ch^{s-1+t+4} \| T^2 f \|_{t+4} \leqslant Ch^{s+t+3} \| f \|_t$$

and

$$|\!|\!| [T-T_h]G\beta |\!|\!|_{1/2-s} \leqslant Ch^{s-1+\bar{t}+7/2} \| TG\beta \|_{\bar{t}+7/2} \leqslant Ch^{s+\bar{t}+5/2} | \beta |_{\bar{t}} .$$

We now apply these results in two cases. When $0 \leqslant s \leqslant r-3$, we get choosing $t = m$ and $\bar{t} = l$ that for $-1 \leqslant m \leqslant r-4 = \max(s-1, r-4)$

$$|\!|\!| [T-T_h]Tf |\!|\!|_{1/2-s} \leqslant Ch^{s+m+3} \| f \|_m \leqslant Ch^{m+4+\min(s-1,r-4)} \| f \|_m$$

and for $-1/2 \leqslant l \leqslant r-7/2 = \max(s-1/2, r-7/2)$

$$|\!|\!| [T-T_h]G\beta |\!|\!|_{1/2-s} \leqslant Ch^{s+l+5/2} | \beta |_l \leqslant Ch^{l+3+\min(s-1/2,r-7/2)} | \beta |_l .$$

When $s > r-3$ we choose $t = m+r-s-3$ and $\bar{t} = l+r-s-3$ and get for $-1 \leqslant m \leqslant s-1 = \max(s-1, r-4)$ that

$$\begin{aligned} |\!|\!| [T-T_h]Tf |\!|\!|_{1/2-s} &\leqslant Ch^{m+r} \| f \|_{m+r-s-3} \\ &\leqslant Ch^{m+r} \| f \|_m \leqslant Ch^{m+4+\min(s-1,r-4)} \| f \|_m \end{aligned}$$

and for $-1/2 \leqslant l \leqslant s - 1/2 = \max(s - 1/2, r - 7/2)$ that

$$|||[T - T_h] G\beta|||_{1/2-s} \leqslant Ch^{l+r-1/2} |\beta|_{l+r-s-3}$$
$$\leqslant Ch^{l+r-1/2} |\beta|_l \leqslant Ch^{l+3+\min(s-1/2, r-7/2)} |\beta|_l .$$

The theorem follows by combining these results.

There are two special cases in which we shall use this theorem.

COROLLARY 3.1 : *Suppose* $0 \leqslant s \leqslant r - 3, f \in H^m(\Omega)$ *with* $-1 \leqslant m \leqslant r-4$ *and* $\beta \in H^l(\Gamma)$ *with* $-1/2 \leqslant l \leqslant r - 7/2$. *Then*

$$|[T^2 - T_h^2] f|_{1/2-s} + \|[T^2 - T_h^2] f\|_{1-s} \leqslant Ch^{m+s+3} \|f\|_m$$

and

$$|[TG - T_h G_h] \beta|_{1/2-s} + \|[TG - T_h G_h] \beta\|_{1-s} \leqslant Ch^{l+s+5/2} |\beta|_l .$$

Proof : Observe that $s - 1 \leqslant r - 4$ and $s - 1/2 \leqslant r - 7/2$.

COROLLARY 3.2 : *Suppose* $0 \leqslant s \leqslant r - 2, f \in H^m(\Omega)$ *with* $-1 \leqslant m \leqslant r - 3$ *and* $\beta \in H^l(\Gamma)$ *with* $-1/2 \leqslant l \leqslant r - 5/2$. *Then*

$$|[T^2 - T_h^2] f|_{1/2-s} + \|[T^2 - T_h^2] f\|_{1-s} \leqslant Ch^{m+s+2} \|f\|_m$$

and

$$|[TG - T_h G_h] \beta|_{1/2-s} + \|[TG - T_h G_h] \beta\|_{1-s} \leqslant Ch^{l+s+3/2} |\beta|_l .$$

Proof : Returning to the proof of Theorem 3.1 we note that for $0 \leqslant s \leqslant r-2$, $\min(s - 1, r - 4) \geqslant s - 2$ and $\min(s - 1/2, r - 7/2) \geqslant s - 3/2$. Hence for $0 \leqslant s \leqslant r - 2$, $-1 \leqslant m \leqslant r - 2$, and $-1/2 \leqslant l \leqslant r - 3/2$ we get :

$$|||T[T - T_h] f|||_{1/2-s} + |||[T_h - T][T - T_h] f|||_{1/2-s} \leqslant Ch^{m+s+2} \|f\|_m$$

and

$$|||T[G - G_h] \beta|||_{1/2-s} + |||[T_h - T][G - G_h] \beta|||_{1/2-s} \leqslant Ch^{l+s+3/2} |\beta|_l .$$

We also note that for $0 \leqslant s \leqslant r - 1$

$$|||[T - T_h] Tf|||_{1/2-s} \leqslant Ch^{s+t+3} \|f\|_t$$

holds for $1 \leqslant t + 4 \leqslant r$ and that

$$|||[T - T_h] G\beta|||_{1/2-s} \leqslant Ch^{s+\bar{t}+5/2} |\beta|_{\bar{t}}$$

holds for $1 \leqslant \bar{t} + 7/2 \leqslant r$.

Choosing $m = t + 1$ and $l = \bar{t} + 1$ we get for $-2 \leqslant m \leqslant r - 3$ and $-3/2 \leqslant l \leqslant r - 5/2$ that

$$||| [T - T_h] Tf |||_{1/2-s} \leqslant Ch^{s+2+m} \| f \|_{m-1} \leqslant Ch^{s+2+m} \| f \|_m$$

and

$$||| [T - T_h] G\beta |||_{1/2-s} \leqslant Ch^{s+l+3/2} | \beta |_{l-1} \leqslant Ch^{s+l+3/2} | \beta |_l .$$

The result follows by the triangle inequality.

THEOREM 3.2 : *Suppose* $0 \leqslant s \leqslant r - 2$ *and that* $f \in H^m(\Omega)$ *with* $-1 \leqslant m \leqslant r - 3$ *and* $\beta \in H^l(\Gamma)$ *with* $-1/2 \leqslant l \leqslant r - 5/2$.
Then for $r \geqslant 2$

$$\left| \left[G\left(\frac{1}{\tau K}\right) T - G_h\left(\frac{1}{\tau K}\right) T_h \right] f \right|_{1/2-s} + \left\| \left[G\left(\frac{1}{\tau K}\right) T - G_h\left(\frac{1}{\tau K}\right) T_h \right] f \right\|_{1-s} \leqslant Ch^{m+s+2} \| f \|_m$$

and

$$\left| \left[G\left(\frac{1}{\tau K}\right) G - G_h\left(\frac{1}{\tau K}\right) G_h \right] \beta \right|_{1/2-s} + \left\| \left[G\left(\frac{1}{\tau K}\right) G - G_h\left(\frac{1}{\tau K}\right) G_h \right] \beta \right\|_{1-s} \leqslant Ch^{l+s+3/2} | \beta |_l .$$

Proof : By the triangle inequality we have

$$\left||| \left[G\left(\frac{1}{\tau K}\right) T - G_h\left(\frac{1}{\tau K}\right) T_h \right] f \right|||_{1/2-s} \leqslant \leqslant \left||| G\left(\frac{1}{\tau K}\right) [T - T_h] f \right|||_{1/2-s} + \left||| [G_h - G]\left(\frac{1}{\tau K}\right) [T - T_h] f \right|||_{1/2-s} + \left||| [G - G_h]\left(\frac{1}{\tau K}\right) Tf \right|||_{1/2-s}$$

and

$$\left||| \left[G\left(\frac{1}{\tau K}\right) G - G_h\left(\frac{1}{\tau K}\right) G_h \right] \beta \right|||_{1/2-s} \leqslant \leqslant \left||| G\left(\frac{1}{\tau K}\right) [G - G_h] \beta \right|||_{1/2-s} + \left||| [G_h - G]\left(\frac{1}{\tau K}\right) [G - G_h] \beta \right|||_{1/2-s} + \left||| [G - G_h]\left(\frac{1}{\tau K}\right) G\beta \right|||_{1/2-s} .$$

Using (2.1), (2.2), and Lemma 3.1 we get for $0 \leqslant s \leqslant r-2$, $-1 \leqslant m \leqslant r-3$, and $-1/2 \leqslant l \leqslant r-5/2$ that

$$\left|\left|\left| G\left(\frac{1}{\tau K}\right)[T - T_h] f \right|\right|\right|_{1/2-s} \leqslant C \,|\, [T - T_h] f \,|_{-1/2-s}$$
$$\leqslant C h^{m+2+s} \,\|\, Tf \,\|_{m+2} \leqslant C h^{m+2+s} \,\|\, f \,\|_m ,$$

$$\left|\left|\left| G\left(\frac{1}{\tau K}\right)[G - G_h] \beta \right|\right|\right|_{1/2-s} \leqslant C \,|\, [G - G_h] \beta \,|_{-1/2-s}$$
$$\leqslant C h^{s+l+3/2} \,\|\, G\beta \,\|_{l+3/2} \leqslant C h^{s+l+3/2} \,|\, \beta \,|_l ,$$

$$\left|\left|\left| [G_h - G]\left(\frac{1}{\tau K}\right)[T - T_h] f \right|\right|\right|_{1/2-s} \leqslant C h^s \left|\left| G\left(\frac{1}{\tau K}\right)[T - T_h] f \right|\right|_1$$
$$\leqslant C h^s \,|\, [T - T_h] f \,|_{-1/2} \leqslant C h^{m+s+2} \,\|\, Tf \,\|_{m+2} \leqslant C h^{m+s+2} \,\|\, f \,\|_m ,$$

$$\left|\left|\left| [G_h - G]\left(\frac{1}{\tau K}\right)[G - G_h] \beta \right|\right|\right|_{1/2-s} \leqslant C h^s \left|\left| G\left(\frac{1}{\tau K}\right)[G - G_h] \beta \right|\right|_1$$
$$\leqslant C h^s \,|\, [G - G_h] \beta \,|_{-1/2} \leqslant C h^{s+l+3/2} \,\|\, G\beta \,\|_{l+3/2} \leqslant C h^{s+l+3/2} \,|\, \beta \,|_l ,$$

$$\left|\left|\left| [G - G_h]\left(\frac{1}{\tau K}\right) Tf \right|\right|\right|_{1/2-s} \leqslant C h^{m+s+2} \left|\left| G\left(\frac{1}{\tau K}\right) Tf \right|\right|_{m+3}$$
$$\leqslant C h^{m+s+2} \,|\, Tf \,|_{m+3/2} \leqslant C h^{m+s+2} \,\|\, f \,\|_m ,$$

and

$$\left|\left|\left| [G - G_h]\left(\frac{1}{\tau K}\right) G\beta \right|\right|\right|_{1/2-s} \leqslant C h^{s+l+3/2} \left|\left| G\left(\frac{1}{\tau K}\right) G\beta \right|\right|_{l+5/2}$$
$$\leqslant C h^{s+l+3/2} \,|\, G\beta \,|_{l+1} \leqslant C h^{s+l+3/2} \,|\, \beta \,|_l .$$

The theorem follows by combining these results.

For $0 < k < 1$, let $\{ \dot{S}_k \}$ be a family of finite dimensional subspaces of $H^n(\Gamma)$, $n \geqslant 0$. Let $\dot{r} \geqslant 1$ be an integer. We shall suppose that for $\phi \in H^l(\Gamma)$ with $j \leqslant n$ and $j \leqslant l \leqslant \dot{r}$, there is a constant C such that

$$\inf_{\chi \in S_k} |\, \phi - \chi \,|_j \leqslant C k^{l-j} \,|\, \phi \,|_l . \tag{3.2}$$

We further assume that for $j \leqslant i \leqslant n$ there is a constant C such that

$$|\, \phi \,|_i \leqslant C k^{j-i} \,|\, \phi \,|_j , \tag{3.3}$$

for all $\phi \in \dot{S}_k$.

The condition (3.2) and (3.3) together imply that for any given $j_0 \leqslant n$ there is an operator $\pi_k : H^{j_0}(\Gamma) \to \dot{S}_k$ with

$$|\, \phi - \pi_k \phi \,|_j \leqslant C_{j_0} k^{l-j} \,|\, \phi \,|_l \tag{3.4}$$

uniformly in j and l for $j \leqslant n$ and $j_0 \leqslant j \leqslant l \leqslant \dot{r}$. This result can be found in [6]. Finally we denote by P_0 the $L_2(\Gamma)$ orthogonal projection onto $\dot{S}_k$, i.e. for $\phi \in L_2(\Gamma) = H^0(\Gamma)$,

$$\langle P_0 \phi, \theta \rangle = \langle \phi, \theta \rangle \quad \text{for all} \quad \theta \in \dot{S}_k .$$

We now note for future reference the following property satisfied by the projection operator P_0. Using (3.3) for $j \geqslant 0$ and a standard duality argument for $j < 0$ we obtain :

LEMMA 3.2 : *For $-\dot{r} \leqslant j \leqslant n$ and $\max(-n, j) \leqslant l \leqslant \dot{r}$ there is a constant C such that*

$$| (I - P_0) \phi |_j \leqslant C k^{l-j} | \phi |_l \quad \text{for} \quad \phi \in H^l(\Gamma) .$$

Writing $P_0 \phi = \phi - (I - P_0) \phi$ and using Lemma 3.2 we get :

LEMMA 3.3 : *Under the hypotheses of Lemma 3.2, we have for all $\phi \in H^l(\Gamma)$ that*

$$| P_0 \phi |_j \leqslant C[| \phi |_j + k^{l-j} | \phi |_l] .$$

4. THE FINITE ELEMENT APPROXIMATION SCHEME FOR PROBLEM ($\tilde{P}$)

We now turn out attention to the study of a finite element method for the approximation of the simply supported plate problem, based on the mixed formulation given in Problem ($\tilde{P}$). We shall consider the following scheme under the assumptions that $\dot{S}_k \subset H^n(\Gamma)$, $n \geqslant 3/2$ and that S_h and $\dot{S}_k$ satisfy (3.1) and (3.2)-(3.3) respectively for some $r \geqslant 3$ and $\dot{r} \geqslant 3$.

Problem ($\tilde{P}_h^k$) : Find $(\tilde{u}_h, \tilde{w}_h, \lambda_k, \sigma_k) \in S_h \times S_h \times \dot{S}_k \times \dot{S}_k$ such that

$$A_\alpha(\tilde{w}_h, v_h) = (f, v_h) + \langle \sigma_k, v_h \rangle - \tau \langle (\lambda_k)_s, (v_h)_s \rangle \quad \text{for all} \quad v_h \in S_h \tag{4.1}$$

$$A_\alpha(\tilde{u}_h, z_h) = (\tilde{w}_h, z_h) + \langle \lambda_k, z_h \rangle \quad \text{for all} \quad z_h \in S_h \tag{4.2}$$

$$\tau \langle K[\lambda_k - \alpha \tilde{u}_h], \mu_k \rangle - \tau \langle (\tilde{u}_h)_s, (\mu_k)_s \rangle + \langle \tilde{w}_h, \mu_k \rangle = 0 \quad \text{for all} \quad \mu_k \in \dot{S}_k \tag{4.3}$$

and

$$\langle \tilde{u}_h, \beta_k \rangle = 0 \quad \text{for all} \quad \beta_k \in \dot{S}_k . \tag{4.4}$$

Using the operators T_h and G_h we can also rewrite Problem ($\tilde{P}_h^k$) in a form analogous to Problem (P). From (4.1) we have that

$$\tilde{w}_h = T_h f + G_h[\sigma_k + \tau(\lambda_k)_{ss}] \tag{4.5}$$

and from (4.2) that

$$\tilde{u}_h = T_h \tilde{w}_h + G_h \lambda_k = T_h^2 f + T_h G_h[\sigma_k + \tau(\lambda_k)_{ss}] + G_h \lambda_k . \quad (4.6)$$

We now define for $\lambda, \sigma \in H^{3/2}(\Gamma) \times H^{-1/2}(\Gamma)$

$$w_h(\lambda, \sigma) = G_h[\sigma + \tau\lambda_{ss}] \quad (4.7)$$

and

$$u_h(\lambda, \sigma) = T_h G_h[\sigma + \tau\lambda_{ss}] + G_h \lambda . \quad (4.8)$$

Then $\tilde{w}_h = T_h f + w_h(\lambda_k, \sigma_k)$

and $\tilde{u}_h = T_h^2 f + u_h(\lambda_k, \sigma_k)$

so that Problem $(\tilde{P}_h^k)$ can be restated in the form :

Problem (P_h^k) : Find $(\lambda_k, \sigma_k) \in \dot{S}_k \times \dot{S}_k$ such that

$$P_0 \{ \tau K[\lambda_k - \alpha u_h(\lambda_k, \sigma_k)] + \tau(u_h)_{ss}(\lambda_k, \sigma_k) + w_h(\lambda_k, \sigma_k) \} = P_0 \{ \tau K \alpha T_h^2 f - \tau(T_h^2 f)_{ss} - T_h f \} \quad (4.9)$$

and

$$P_0 u_h(\lambda_k, \sigma_k) = - P_0 T_h^2 f . \quad (4.10)$$

Our aim now is to study the function $u_h(\lambda_k, \sigma_k)$ and prove a result analogous to that of Theorem 2.1. We first note that from the definitions of T_h and G_h it easily follows that $\{ u_h(\lambda_k, \sigma_k), w_h(\lambda_k, \sigma_k) \}$ is the solution of :

Problem (Q_h^k) : Given $(\sigma_k, \lambda_k) \in \dot{S}_k \times \dot{S}_k$ find $(u_h, w_h) \in S_h \times S_h$ satisfying

$$A_\alpha(w_h, v_h) = \langle \sigma_k, v_h \rangle - \tau \langle (\lambda_k)_s, (v_h)_s \rangle \quad \text{for all} \quad v_h \in S_h$$

and

$$A_\alpha(u_h, z_h) = (w_h, z_h) + \langle \lambda_k, z_h \rangle \quad \text{for all} \quad z_h \in S_h .$$

To simplify the proof of the main result of this section and also the derivation of the error estimates in Section 5, it will be convenient to have the following result.

LEMMA 4.1 : *Let* $w(\lambda, \sigma)$, $u(\lambda, \sigma)$, $w_h(\lambda, \sigma)$, *and* $u_h(\lambda, \sigma)$ *be defined by* (2.5), (2.6), (4.7), *and* (4.8) *respectively. Then if* $(\lambda, \sigma) \in H^{l+2}(\Gamma) \times H^l(\Gamma)$ *we have for* $-1/2 \leqslant l \leqslant r - 3/2$ *and* $-2 \leqslant s \leqslant r - 3$ *that*

$$| w(\lambda, \sigma) - w_h(\lambda, \sigma) |_{-3/2-s} + \| w(\lambda, \sigma) - w_h(\lambda, \sigma) \|_{-1-s} \leqslant C h^{l+5/2+s} [| \sigma |_l + | \lambda |_{l+2}] \quad (4.11)$$

and for all $-1/2 \leqslant l \leqslant r - 7/2$ *and* $0 \leqslant s \leqslant r - 3$ *that*

$$| u(\lambda, \sigma) - u_h(\lambda, \sigma) |_{1/2-s} + \| u(\lambda, \sigma) - u_h(\lambda, \sigma) \|_{1-s} \leqslant C h^{l+5/2+s} [| \sigma |_l + | \lambda |_{l+2}] . \quad (4.12)$$

Proof : Using the notation $||| \phi |||_s = | \phi |_s + \| \phi \|_{s+1/2}$ and applying (2.5), (4.7), Lemma 3.1 and (2.1*b*) we have

$$||| w(\lambda, \sigma) - w_h(\lambda, \sigma) |||_{-3/2-s} = ||| [G - G_h] (\sigma + \tau\lambda_{ss}) |||_{-3/2-s} \leqslant C h^{l+5/2+s} \| G[\sigma + \tau\lambda_{ss}] \|_{l+3/2} \leqslant C h^{l+5/2+s} [| \sigma |_l + | \lambda |_{l+2}] .$$

From (2.6) and (4.1) we get

$$u(\lambda, \sigma) - u_h(\lambda, \sigma) = [TG - T_h G_h] (\sigma + \tau\lambda_{ss}) + [G - G_h] \lambda .$$

Using (2.1), Lemma 3.1, and Corollary 3.1 it follows that

$$\begin{aligned} ||| u(\lambda, \sigma) - u_h(\lambda, \sigma) |||_{1/2-s} & \\ \leqslant C h^{l+s+5/2} [| \sigma |_l + | \lambda |_{l+2}] &+ C h^{l+s+5/2} \| G\lambda \|_{l+7/2} \\ \leqslant C h^{l+s+5/2} [| \sigma |_l + | \lambda |_{l+2}] &. \end{aligned}$$

In order to state the analogue of Theorem 2.1 that we wish to prove we first define for each $(\lambda, \sigma) \in \dot{S}_k \times \dot{S}_k$ a finite dimensional version of the operator $M(\lambda, \sigma) = Mu(\lambda, \sigma)$ defined by (2.14). We first note that by (2.9), (2.10), and (2.11) the function $-\Delta u(\lambda, \sigma)$ satisfies

$$-\Delta[-\Delta u(\lambda, \sigma)] = 0 \quad \text{in} \quad \Omega$$

$$\frac{\partial}{\partial n} [-\Delta u(\lambda, \sigma)] + \alpha[-\Delta u(\lambda, \sigma)] = \sigma + \tau\lambda_{ss} \quad \text{on} \quad \Gamma .$$

Hence $-\Delta u(\lambda, \sigma) = G[\sigma + \tau\lambda_{ss}] = w(\lambda, \sigma)$. Thus it follows from (2.10) and (2.14) that $M(\lambda, \sigma)$ can be written in the form

$$\begin{aligned} M(\lambda, \sigma) = Mu(\lambda, \sigma) &= \Delta u(\lambda, \sigma) - \tau[u_{ss}(\lambda, \sigma) + K u_n(\lambda, \sigma)] \\ &= \Delta u(\lambda, \sigma) - \tau[u_{ss}(\lambda, \sigma) + K\lambda - \alpha K u(\lambda, \sigma)] \qquad (4.13) \\ &= -w(\lambda, \sigma) - \tau[u_{ss}(\lambda, \sigma)] - \tau K[\lambda - \alpha u(\lambda, \sigma)] . \end{aligned}$$

We then define an operator $M_h(\lambda, \sigma)$ by

$$M_h(\lambda, \sigma) = -w_h(\lambda, \sigma) - \tau[u_h(\lambda, \sigma)]_{ss} - \tau K[\lambda - \alpha u_h(\lambda, \sigma)] . \quad (4.14)$$

With this notation we now prove :

THEOREM 4.1 : *For $h \leqslant \varepsilon k$, with ε sufficiently small, there exist positive constants C_1 and C_2 independent of σ, λ, h, and k such that for all*

$$0 \leqslant s \leqslant \min(r - 3, \dot{r} - 3/2)$$

$$C_1[|\lambda|_{-1/2-s} + |\sigma|_{-5/2-s}] \leqslant |P_0 u_h(\lambda, \sigma)|_{1/2-s} + |P_0 M_h(\lambda, \sigma)|_{-3/2-s} \leqslant C_2[|\lambda|_{-1/2-s} + |\sigma|_{-5/2-s}]$$

for all $(\lambda, \sigma) \in \dot{S}_k \times \dot{S}_k$.

To simplify the proof of this theorem, we first prove the following preliminary result.

LEMMA 4.2 : *There exist positive constants C_1 and C_2 independent of λ, σ, and k such that for all $0 \leqslant s \leqslant \dot{r} - 3/2$*

$$C_1[|\lambda|_{-1/2-s} + |\sigma|_{-5/2-s}] \leqslant |P_0 u(\lambda, \sigma)|_{1/2-s} + |P_0 M(\lambda, \sigma)|_{-3/2-s} \leqslant C_2[|\lambda|_{-1/2-s} + |\sigma|_{-5/2-s}]$$

for all $(\lambda, \sigma) \in \dot{S}_k \times \dot{S}_k$ where $u(\lambda, \sigma)$ is the solution of Problem (Q) and $M(\lambda, \sigma) = Mu(\lambda, \sigma)$.

Proof : To simplify notation we will again simply denote $u(\lambda, \sigma)$ by u for the remainder of this proof.

Using Theorem 2.1 and the triangle inequality we have

$$\begin{aligned} C_1[|\lambda|_{-1/2-s} + |\sigma|_{-5/2-s}] - |(I - P_0)u|_{1/2-s} - |(I - P_0)M(u)|_{-3/2-s} \\ \leqslant |P_0 u|_{1/2-s} + |P_0 M(u)|_{-3/2-s} \\ \leqslant C_2[|\lambda|_{-1/2-s} + |\sigma|_{-5/2-s}] + |(I - P_0)u|_{1/2-s} + |(I - P_0)M(u)|_{-3/2-s}. \end{aligned} \tag{4.15}$$

Applying Lemma 3.2 and a standard trace theorem we get for $0 \leqslant s \leqslant \dot{r} + 1/2$ that

$$|(I - P_0)u|^2_{1/2-s} \leqslant Ck^{2+2s}|u|^2_{3/2} \leqslant Ck^{2+2s}\|u\|^2_2 \leqslant Ck^{2+2s}\left[\sum_{|\alpha|=2}\|D^\alpha u\|^2_0 + \|u\|^2_1\right]$$

and for $0 \leqslant s \leqslant \dot{r} - 3/2$ that

$$|(I - P_0)M(u)|^2_{-3/2-s} \leqslant Ck^{2+2s}|M(u)|^2_{-1/2}.$$

To estimate $| M(u) |_{-1/2}$ we define v to be the solution of the biharmonic problem $\Delta^2 v = 0$ in Ω, $v = 0$ and $\partial v/\partial n = \psi$ on Γ. Since $\Delta^2 u = 0$ in Ω it follows from (2.12), (2.13), and (2.14) that

$$\langle M(u), \psi \rangle = E(u, v) \leqslant C \| u \|_2 \| v \|_2 .$$

Since $\| v \|_2 \leqslant C | \psi |_{1/2}$ by a standard a priori estimate, we get

$$| M(u) |_{-1/2} = \sup_{\psi \in H^{1/2}(\Gamma)} \frac{\langle M(u), \psi \rangle}{| \psi |_{1/2}} \leqslant C \| u \|_2 .$$

Hence

$$| (I - P_0) M(u) |^2_{-3/2-s} \leqslant C k^{2+2s} \left[\sum_{|\alpha|=2} \| D^\alpha u \|_0^2 + \| u \|_1^2 \right].$$

Now from (2.17) and Lemma 2.2 we have for α sufficiently large (so that $2\alpha + K \geqslant 0$) that

$$\tau \sum_{|\alpha|=2} \| D^\alpha u \|_0^2 \leqslant E(u, u) + \tau \langle [2\alpha + K] u_s, u_s \rangle$$
$$\leqslant \langle \sigma, u \rangle + \langle M(u), \lambda \rangle + \alpha\tau \langle \lambda, Ku \rangle - \alpha^2 \tau \langle Ku, u \rangle .$$

Since $(\sigma, \lambda) \in \dot{S}_k \times \dot{S}_k$ we may write

$$\tau \sum_{|\alpha|=2} \| D^\alpha u \|_0^2 \leqslant \langle \sigma, P_0 u \rangle + \langle P_0 M(u), \lambda \rangle + \alpha\tau \langle \lambda, Ku \rangle - \alpha^2 \tau \langle Ku, u \rangle .$$

Applying standard trace theorems and (3.3) we get that

$$\tau \sum_{|\alpha|=2} \| D^\alpha u \|_0^2 \leqslant | \sigma |_{-1/2+s} | P_0 u |_{1/2-s} + | P_0 M(u) |_{-3/2-s} | \lambda |_{3/2+s}$$
$$+ C[| \lambda |_{-1/2} \| u \|_1 + \| u \|_1^2]$$
$$\leqslant C k^{-2-2s} [| \sigma |_{-5/2-s} | P_0 u |_{1/2-s} + | P_0 M(u) |_{-3/2-s} | \lambda |_{-1/2-s}]$$
$$+ C[k^{-s} | \lambda |_{-1/2-s} \| u \|_1 + \| u \|_1^2] .$$

Recalling that $u = u(\lambda, \sigma) = TG[\sigma + \tau\lambda_{ss}] + G\lambda$ and applying the a priori estimates (2.1) and (3.3) we get that

$$\| u \|_1 \leqslant \| TG[\sigma + \tau\lambda_{ss}] \|_1 + \| G\lambda \|_1$$
$$\leqslant C[\| G[\sigma + \tau\lambda_{ss}] \|_{-1} + | \lambda |_{-1/2}]$$
$$\leqslant C[| \sigma + \tau\lambda_{ss} |_{-5/2} + | \lambda |_{-1/2}]$$
$$\leqslant C[| \sigma |_{-5/2} + | \lambda |_{-1/2}]$$
$$\leqslant C k^{-s} [| \sigma |_{-5/2-s} + | \lambda |_{-1/2-s}] .$$

Applying the arithmetic-geometric mean inequality and combining results we get for arbitrary $\delta > 0$ that

$$\max\{\,|(I-P_0)u|^2_{1/2-s},\,|(I-P_0)M(u)|^2_{-3/2-s}\} \leqslant C[\delta+k^2]\,[|\sigma|^2_{-5/2-s}+|\lambda|^2_{-1/2-s}] + \frac{C}{\delta}[|P_0\,u|^2_{1/2-s}+|P_0\,M(u)|^2_{-3/2-s}]. \quad (4.16)$$

Hence for k and δ chosen sufficiently small we get using (4.15) that

$$C_1[|\lambda|_{-1/2-s}+|\sigma|_{-5/2-s}] \leqslant |P_0\,u(\lambda,\sigma)|_{1/2-s}+|P_0\,M(\lambda,\sigma)|_{-3/2-s},$$

and choosing δ sufficiently large in (4.16) we have from (4.15) that

$$|P_0\,u(\lambda,\sigma)|_{1/2-s}+|P_0\,M(\lambda,\sigma)|_{-3/2-s} \leqslant C_2[|\lambda|_{-1/2-s}+|\sigma|_{-5/2-s}].$$

To prove Theorem 4.1, we must now show that $u(\lambda, \sigma)$ and $M(\lambda, \sigma)$ can be replaced by $u_h(\lambda, \sigma)$ and $M_h(\lambda, \sigma)$ respectively.

Proof of Theorem 4.1 : Using Lemma 4.2 and the triangle inequality we have that for $0 \leqslant s \leqslant \dot{r} - 3/2$

$$\begin{aligned} C_1[|\lambda|_{-1/2-s}+|\sigma|_{-5/2-s}] &- |P_0[u(\lambda,\sigma)-u_h(\lambda,\sigma)]|_{1/2-s} \\ &- |P_0[M(\lambda,\sigma)-M_h(\lambda,\sigma)]|_{-3/2-s} \\ &\leqslant |P_0\,u_h(\lambda,\sigma)|_{1/2-s}+|P_0\,M_h(\lambda,\sigma)|_{-3/2-s} \\ &\leqslant C_2[|\lambda|_{-1/2-s}+|\sigma|_{-5/2-s}]+|P_0[u(\lambda,\sigma)-u_h(\lambda,\sigma)]|_{1/2-s} \\ &\quad + |P_0[M(\lambda,\sigma)-M_h(\lambda,\sigma)|_{-3/2-s}. \end{aligned}$$

Hence to prove Theorem 4.1, we need only show that for $0 \leqslant s \leqslant \min(r-3, \dot{r}-3/2)$

$$|P_0[u(\lambda,\sigma)-u_h(\lambda,\sigma)]|_{1/2-s}+|P_0[M(\lambda,\sigma)-M_h(\lambda,\sigma)]|_{-3/2-s} \leqslant \delta[|\lambda|_{-1/2-s}+|\sigma|_{-5/2-s}]$$

where δ is a constant which is small with $\varepsilon = h/k$. Applying the triangle inequality, Lemma 3.2, (4.13), and (4.14) we have for $0 \leqslant s \leqslant \dot{r} - 3/2$ that

$$\begin{aligned} |P_0[u(\lambda,\sigma)-u_h(\lambda,\sigma)]|_{1/2-s} &+ |P_0[M(\lambda,\sigma)-M_h(\lambda,\sigma)]|_{-3/2-s} \\ &\leqslant |u(\lambda,\sigma)-u_h(\lambda,\sigma)|_{1/2-s}+|M(\lambda,\sigma)-M_h(\lambda,\sigma)|_{-3/2-s} \\ &\quad + Ck^s\{\,|u(\lambda,\sigma)-u_h(\lambda,\sigma)|_{1/2}+|M(\lambda,\sigma)-M_h(\lambda,\sigma)|_{-3/2}\} \\ &\leqslant C\{\,|u(\lambda,\sigma)-u_h(\lambda,\sigma)|_{1/2-s}+|w(\lambda,\sigma)-w_h(\lambda,\sigma)|_{-3/2-s} \\ &\quad + k^s|u(\lambda,\sigma)-u_h(\lambda,\sigma)|_{1/2}+k^s|w(\lambda,\sigma)-w_h(\lambda,\sigma)|_{-3/2}\}. \end{aligned}$$

Now applying Lemma 4.1 with $l = -1/2$ and (3.3) we get

$$| P_0[u(\lambda, \sigma) - u_h(\lambda, \sigma)] |_{1/2-s} + | P_0[M(\lambda, \sigma) - M_h(\lambda, \sigma)] |_{-3/2-s}$$
$$\leqslant C[h^{2+s} + h^2 k^s] [| \sigma |_{-1/2} + | \lambda |_{3/2}]$$
$$\leqslant C\left[\left(\frac{h}{k}\right)^{2+s} + \left(\frac{h}{k}\right)^2\right][| \sigma |_{-5/2-s} + | \lambda |_{-1/2-s}].$$

The result now follows provided $h \leqslant \varepsilon k$ and ε is sufficiently small.

5. ERROR ESTIMATES FOR THE APPROXIMATION OF PROBLEM $\tilde{P}$

We begin this section by proving a preliminary lemma.

LEMMA 5.1 : *Suppose the hypotheses of Lemma* 4.1 *are satisfied. Then for all* $(\mu, \beta) \in \dot{S}_k \times \dot{S}_k$ *we have*

$$| w(\lambda, \sigma) - w_h(\mu, \beta) |_{-3/2-s} + \| w(\lambda, \sigma) - w_h(\mu, \beta) \|_{-1-s}$$
$$\leqslant C \{ h^{l+5/2+s}[| \sigma |_l + | \lambda |_{l+2}]$$
$$+ h^{2+s}[| \sigma - \beta |_{-1/2} + | \lambda - \mu |_{3/2}] + | \sigma - \beta |_{-5/2-s} + | \lambda - \mu |_{-1/2-s} \}$$

for $-1/2 \leqslant l \leqslant r - 3/2$ *and* $-2 \leqslant s \leqslant r - 3$ *and*

$$| u(\lambda, \sigma) - u_h(\mu, \beta) |_{1/2-s} + \| u(\lambda, \sigma) - u_h(\mu, \beta) \|_{1-s}$$
$$\leqslant C \{ h^{l+5/2+s}[| \sigma |_l + | \lambda |_{l+2}]$$
$$+ h^{2+s}[| \sigma - \beta |_{-1/2} + | \lambda - \mu |_{3/2}] + | \sigma - \beta |_{-5/2-s} + | \lambda - \mu |_{-1/2-s} \}$$

for $-1/2 \leqslant l \leqslant r - 7/2$ *and* $0 \leqslant s \leqslant r - 3$.

Proof : To simplify the exposition, let us use the notation

$$||| \phi |||_s = | \phi |_s + \| \phi \|_{s+1/2} .$$

Then by the triangle inequality

$$||| w(\lambda, \sigma) - w_h(\mu, \beta) |||_{-3/2-s} \leqslant ||| w(\lambda, \sigma) - w_h(\lambda, \sigma) |||_{-3/2-s}$$
$$+ ||| w_h(\lambda - \mu, \sigma - \beta) - w(\lambda - \mu, \sigma - \beta) |||_{-3/2-s} + ||| w(\lambda - \mu, \sigma - \beta) |||_{-3/2-s}$$

and

$$||| u(\lambda, \sigma) - u_h(\mu, \beta) |||_{1/2-s} \leqslant ||| u(\lambda, \sigma) - u_h(\lambda, \sigma) |||_{1/2-s}$$
$$+ ||| u_h(\lambda - \mu, \sigma - \beta) - u(\lambda - \mu, \sigma - \beta) |||_{1/2-s} + ||| u(\lambda - \mu, \sigma - \beta) |||_{1/2-s} .$$

Applying Lemma 4.1 we obtain for $-1/2 \leqslant l \leqslant r-3/2$ and $-2 \leqslant s \leqslant r-3$ that

$$|||\, w(\lambda, \sigma) - w_h(\lambda, \sigma)\, |||_{-3/2-s} \leqslant Ch^{l+5/2+s}[|\sigma|_l + |\lambda|_{l+2}]$$

and

$$|||\, w_h(\lambda-\mu, \sigma-\beta) - w(\lambda-\mu, \sigma-\beta)\, |||_{-3/2-s} \leqslant Ch^{2+s}[|\sigma-\beta|_{-1/2} + |\lambda-\mu|_{3/2}]$$

and for $-1/2 \leqslant l \leqslant r-7/2$ and $0 \leqslant s \leqslant r-3$ that

$$|||\, u(\lambda, \sigma) - u_h(\lambda, \sigma)\, |||_{1/2-s} \leqslant Ch^{l+5/2+s}[|\sigma|_l + |\lambda|_{l+2}]$$

and

$$|||\, u_h(\lambda-\mu, \sigma-\beta) - u(\lambda-\mu, \sigma-\beta)\, |||_{1/2-s} \leqslant Ch^{2+s}[|\sigma-\beta|_{-1/2} + |\lambda-\mu|_{3/2}].$$

Using (2.1), (2.2), (2.5) and (2.6) we obtain

$$\begin{aligned}|||\, w(\lambda-\mu, \sigma-\beta)\, |||_{-3/2-s} &= |||\, G[(\sigma-\beta+\tau(\lambda-\mu)_{ss}]\, |||_{-3/2-s} \\ &\leqslant C[|\sigma-\beta|_{-5/2-s} + |\lambda-\mu|_{-1/2-s}]\end{aligned}$$

and

$$\begin{aligned}|||\, u(\lambda-\mu, \sigma-\beta)\, |||_{1/2-s} &\leqslant |||\, TG[(\sigma-\beta)+\tau(\lambda-\mu)_{ss}]\, |||_{1/2-s} + |||\, G[\lambda-\mu]\, |||_{1/2-s} \\ &\leqslant C\{\, \|\, G[(\sigma-\beta)+\tau(\lambda-\mu)_{ss}]\, \|_{-1-s} + |\lambda-\mu|_{-1/2-s}\,\} \\ &\leqslant \{\, |\sigma-\beta|_{-5/2-s} + |\lambda-\mu|_{-1/2-s}\,\}.\end{aligned}$$

The lemma follows by combining all these results.

THEOREM 5.1 : *Suppose* $f \in H^m(\Omega)$,

$$(\lambda, \sigma) \in H^{l+2}(\Gamma) \times H^l(\Omega) \cap H^{\dot{l}+2}(\Gamma) \times H^{\dot{l}}(\Gamma),$$

and $(\lambda_k, \sigma_k) \in \dot{S}_k \times \dot{S}_k$ *are the respective solutions of Problems* (P) *and* (P_h^k). *Then for* $h \leqslant \varepsilon k$ *with* ε *sufficiently small, there exists a constant* C *independent of* h, k, σ, λ, *and* f *such that if* $\dot{S}_k \subset H^n(\Gamma)$, $n \geqslant 3/2$

$$\begin{aligned}&|\lambda-\lambda_k|_{-1/2-s} + |\sigma-\sigma_k|_{-5/2-s} \\ &\quad \leqslant C\{\, h^{m+3/2-i}\, k^{i+s+3/2}\, \|f\|_m + h^{l+1-i}\, k^{i+s+3/2}\, [|\sigma|_l + |\lambda|_{l+2}] \\ &\qquad + k^{\dot{l}+5/2+s}[|\sigma|_{\dot{l}} + |\lambda|_{\dot{l}+2}]\,\}\end{aligned}$$

for all $-1/2 \leqslant l \leqslant r-7/2$, $-1/2 \leqslant \dot{l} \leqslant \dot{r}-2$, $-2 \leqslant s \leqslant \min(r-3, \dot{r}-3/2)$, *and* $-1 \leqslant m \leqslant r-4$, *where* $i = \max(-n, -3/2-s)$.

Proof : Let $\pi_k \lambda$ and $\pi_k \sigma \in \dot{S}_k$ be approximations to λ and σ respectively which satisfy (3.4). Using the linearity of $u_h(\lambda, \sigma)$ and $M_h(\lambda, \sigma)$ and Theorem 4.1 we get for $0 \leqslant s \leqslant \min(r - 3, \dot{r} - 3/2)$

$$| \lambda_k - \pi_k \lambda |_{-1/2-s} + | \sigma_k - \pi_k \sigma |_{-5/2-s}$$
$$\leqslant C \{ | P_0[u_h(\lambda_k, \sigma_k) - u_h(\pi_k \lambda, \pi_k \sigma)] |_{1/2-s} + | P_0[M_h(\lambda_k, \sigma_k) - M_h(\pi_k \lambda, \pi_k \sigma)] |_{-3/2-s} \} .$$

Using (2.7), (2.8), (4.9), (4.10), (4.13), and (4.14) we obtain

$$P_0\, u_h(\lambda_k, \sigma_k) - P_0\, u_h(\pi_k \lambda, \pi_k \sigma) = - P_0\, T_h^2 f - P_0\, u_h(\pi_k \lambda, \pi_k \sigma)$$
$$= P_0(T^2 - T_h^2) f + P_0[u(\lambda, \sigma) - u_h(\pi_k \lambda, \pi_k \sigma)]$$

and

$$P_0\, M_h(\lambda_k, \sigma_k) - P_0\, M_h(\pi_k \lambda, \pi_k \sigma) =$$
$$= - P_0 \{ \tau K \alpha T_h^2 f - \tau (T_h^2 f)_{ss} - T_h f \} - P_0\, M_h(\pi_k \lambda, \pi_k \sigma)$$
$$= P_0 \{ \tau K \alpha [T^2 - T_h^2] f - \tau([T^2 - T_h^2] f)_{ss} - [T - T_h] f \}$$
$$+ P_0(M(\lambda, \sigma) - M_h(\pi_k \lambda, \pi_k \sigma)) .$$

Combining these results and using Lemma 3.3 and the triangle inequality we obtain for $\dot{S}_k \subset H^n(\Gamma)$, $0 \leqslant s \leqslant \min(r - 3, \dot{r} - 3/2)$ and

$$i = \max(-n, -3/2 - s)$$

that

$$| \lambda_k - \pi_k \lambda |_{-1/2-s} + | \sigma_k - \pi_k \sigma |_{-5/2-s}$$
$$\leqslant C \{ | [T^2 - T_h^2] f |_{1/2-s} + k^{i+s+3/2} | [T^2 - T_h^2] f |_{i+2}$$
$$+ | u(\lambda, \sigma) - u_h(\pi_k \lambda, \pi_k \sigma) |_{1/2-s} + k^{i+s+3/2} | u(\lambda, \sigma) - u_h(\pi_k \lambda, \pi_k \sigma) |_{i+2}$$
$$+ | [T - T_h] f |_{-3/2-s} + k^{i+s+3/2} | [T - T_h] f |_i$$
$$+ | M(\lambda, \sigma) - M_h(\pi_k \lambda, \pi_k \sigma) |_{-3/2-s} + k^{i+s+3/2} | M(\lambda, \sigma) - M_h(\pi_k \lambda, \pi_k \sigma) |_i \} .$$

From (4.13) and (4.14) we observe that

$$M(\lambda, \sigma) - M_h(\pi_k \lambda, \pi_k \sigma) = w_h(\pi_k \lambda, \pi_k \sigma) - w(\lambda, \sigma)$$
$$+ \tau [u_h(\pi_k \lambda, \pi_k \sigma) - u(\lambda, \sigma)]_{ss} + \tau K [\pi_k \lambda - \lambda] + \alpha \tau K [u(\lambda, \sigma) - u_h(\pi_k \lambda, \pi_k \sigma)] .$$

Inserting this result in the previous inequality we obtain :

$$\begin{aligned} |\lambda_k-\pi_k\lambda|_{-1/2-s}+|\sigma_k-\pi_k\sigma|_{-5/2-s} \\ \leqslant C\{ & |[T^2-T_h^2]f|_{1/2-s}+k^{i+s+3/2}|[T^2-T_h^2]f|_{i+2} \\ & +|u(\lambda,\sigma)-u_h(\pi_k\lambda,\pi_k\sigma)|_{1/2-s}+k^{i+s+3/2}|u(\lambda,\sigma)-u_h(\pi_k\lambda,\pi_k\sigma)|_{i+2} \\ & +|[T-T_h]f|_{-3/2-s}+k^{i+s+3/2}|[T-T_h]f|_i \\ & +|w_h(\pi_k\lambda,\pi_k\sigma)-w(\lambda,\sigma)|_{-3/2-s}+k^{i+s+3/2}|w_h(\pi_k\lambda,\pi_k\sigma)-w(\lambda,\sigma)|_i \\ & +|\pi_k\lambda-\lambda|_{-3/2-s}+k^{i+s+3/2}|\pi_k\lambda-\lambda|_i\}. \end{aligned}$$

To estimate the above we first observe that by (3.4) for $-1/2 \leqslant \dot{l} \leqslant \dot{r}-2$

$$|\pi_k\lambda-\lambda|_{-3/2-s}+k^{i+s+3/2}|\pi_k\lambda-\lambda|_i \leqslant Ck^{\dot{l}+7/2+s}|\lambda|_{\dot{l}+2}.$$

Using (2.1), Lemma 3.1, and Corollary 3.1, we have for all $f \in H^m(\Omega)$ with $-1 \leqslant m \leqslant r-4$ and all $0 \leqslant s \leqslant r-3$ that

$$|[T-T_h]f|_{-3/2-s}+k^{i+s+3/2}|[T-T_h]f|_i \leqslant Ch^{m+3/2-i}k^{i+s+3/2}\|f\|_m$$

and

$$|[T^2-T_h^2]f|_{1/2-s}+k^{i+s+3/2}|[T^2-T_h^2]f|_{1/2-s} \leqslant Ch^{m+3/2-i}k^{i+s+3/2}\|f\|_m.$$

Now from Lemma 5.1 with $(\mu, \dot{\beta}) = (\pi_k\lambda, \pi_k\sigma)$ and (3.4) we have for all $-1/2 \leqslant l \leqslant r-7/2$, $-1/2 \leqslant \dot{l} \leqslant r-2$, and $0 \leqslant s \leqslant r-3$ that

$$\begin{aligned} & |u(\lambda,\sigma)-u_h(\pi_k\lambda,\pi_k\sigma)|_{1/2-s}+k^{i+s+3/2}|u(\lambda,\sigma)-u_h(\pi_k\lambda,\pi_k\sigma)|_{i+2} \\ & \quad +|w(\lambda,\sigma)-w_h(\pi_k\lambda,\pi_k\sigma)|_{-3/2-s}+k^{i+s+3/2}|w(\lambda,\sigma)-w_h(\pi_k\lambda,\pi_k\sigma)|_i \\ & \leqslant C\{h^{l+1-i}k^{i+s+3/2}[|\sigma|_l+|\lambda|_{l+2}] \\ & \quad +h^{1/2-i}k^{i+s+3/2}[|\sigma-\pi_k\sigma|_{-1/2}+|\lambda-\pi_k|_{3/2}] \\ & \quad +|\sigma-\pi_k\sigma|_{-5/2-s}+|\lambda-\pi_k\lambda|_{-1/2-s} \\ & \quad +k^{i+s+3/2}[|\sigma-\pi_k\sigma|_{i-1}+|\lambda-\pi_k|_{i+1}]\} \\ & \leqslant C\{h^{l+1-i}k^{i+s+3/2}[|\sigma|_l+|\lambda|_{l+2}]+k^{\dot{l}+5/2+s}[|\sigma|_{\dot{l}}+|\lambda|_{\dot{l}+2}]\} \end{aligned}$$

(since $h \leqslant \varepsilon k$, $\varepsilon < 1$).

The theorem now follows easily for $0 \leqslant s \leqslant \min(r-3, \dot{r}-3/2)$ by combining these results and using (3.4) and the triangle inequality, and then for $-2 \leqslant s \leqslant 0$ using (3.3).

Using Lemma 5.1, Corollary 3.1, and Theorem 5.1, we now prove our main result.

THEOREM 5.2 : *Suppose the hypotheses of Theorem* 5.1 *are satisfied. Then if* $(\tilde{u}, \tilde{w}, \lambda, \sigma)$ *and* $(\tilde{u}_h, \tilde{w}_h, \lambda_k, \sigma_k)$ *are the respective solutions of Problems* $(\tilde{P})$ *and* $(\tilde{P}_h^k)$, *we have for* $i = \max(-n, -3/2 - s)$ *and all* $-1/2 \leqslant l \leqslant r - 7/2$, $-1/2 \leqslant \dot{l} \leqslant \dot{r} - 2$ *and* $-1 \leqslant m \leqslant r - 4$ *that*

$$| \tilde{w}-\tilde{w}_h |_{-3/2-s}+\| \tilde{w}-\tilde{w}_h \|_{-1-s} \leqslant C\{ h^{m+3/2-i} k^{i+s+3/2} \| f \|_m + h^{l+1-i} k^{i+s+3/2}[| \sigma |_l+| \lambda |_{l+2}]+k^{\dot{l}+5/2+s}[| \sigma |_{\dot{l}}+| \lambda |_{\dot{l}+2}] \}$$

when $-2 \leqslant s \leqslant \min(r-3, \dot{r}-3/2)$ *and*

$$| \tilde{u}-\tilde{u}_h |_{1/2-s}+\| \tilde{u}-\tilde{u}_h \|_{1-s} \leqslant C\{ h^{m+3/2-i} k^{i+s+3/2} \| f \|_m + h^{l+1-i} k^{i+s+3/2}[| \sigma |_l+| \lambda |_{l+2}]+k^{i+5/2+s}[| \sigma |_i+| \lambda |_{i+2}] \}$$

when $0 \leqslant s \leqslant \min(r-3, \dot{r}-3/2)$.

Proof : From the definitions of $\tilde{u}$, $\tilde{w}$, $\tilde{u}_h$ and $\tilde{w}_h$ we have

$$\tilde{w} - \tilde{w}_h = [T - T_h]f + w(\lambda, \sigma) - w_h(\lambda_k, \sigma_k)$$

and

$$\tilde{u} - \tilde{u}_h = [T^2 - T_h^2]f + u(\lambda, \sigma) - u_h(\lambda_k, \sigma_k) .$$

Again denoting $| \phi |_s + \| \phi \|_{s+1/2}$ by $||| \phi |||_s$ we have by the triangle inequality that

$$||| \tilde{w}-\tilde{w}_h |||_{-3/2-s} \leqslant ||| [T-T_h]f |||_{-3/2-s}+||| w(\lambda, \sigma)-w_h(\lambda_k, \sigma_k) |||_{-3/2-s}$$

and

$$||| \tilde{u}-\tilde{u}_h |||_{1/2-s} \leqslant |||[T^2-T_h^2]f |||_{1/2-s}+||| u(\lambda, \sigma)-u_h(\lambda_k, \sigma_k) |||_{1/2-s} .$$

It then follows directly from (2.1), Lemmas 3.1, 5.1, and Corollary 3.1 that for all $-1 \leqslant m \leqslant r-2$, $-1/2 \leqslant l \leqslant r-3/2$, and $-2 \leqslant s \leqslant r-3$

$$||| \tilde{w}-\tilde{w}_h |||_{-3/2-s} \leqslant C\{ h^{m+3+s} \| f \|_m + h^{l+5/2+s}[| \sigma |_l+| \lambda |_{l+2}]+h^{2+s}[| \sigma-\sigma_k |_{-1/2}+| \lambda-\lambda_k |_{3/2}] + | \sigma-\sigma_k |_{-5/2-s}+| \lambda-\lambda_k |_{-1/2-s} \}$$

and for all $-1 \leqslant m \leqslant r-4$, $-1/2 \leqslant l \leqslant r-7/2$ and $0 \leqslant s \leqslant r-3$ that

$$||| \tilde{u}-\tilde{u}_h |||_{1/2-s} \leqslant C\{ h^{m+3+s} \| f \|_m + h^{l+3/2+s}[| \sigma |_l+| \lambda |_{l+2}] + h^{2+s}[| \sigma-\sigma_k |_{-1/2}+| \lambda-\lambda_k |_{3/2}]+| \sigma-\sigma_k |_{-5/2-s}+| \lambda-\lambda_k |_{-1/2-s} \} .$$

The theorem now follows directly from Theorem 5.1.

We now consider some applications of the error estimates in Theorem 5.2. Suppose $r \leqslant \dot{r} + 3/2$, $f \in H^{r-4}(\Omega)$ and $(\sigma, \lambda) \in H^{r}(\Gamma) \times H^{\dot{r}-2}(\Gamma)$. Then if $\dot{S}_k \subset H^{3/2}(\Gamma)$ and $r \geqslant 3$ we have

$$\| \tilde{u} - \tilde{u}_h \|_1 \leqslant C \{ h^{r-1} [\| f \|_{r-4} + | \sigma |_{r-7/2} + | \lambda |_{r-3/2}] + k^{\dot{r}+1/2} [| \sigma |_{\dot{r}-2} + | \lambda |_{\dot{r}}] \}$$

and if $\dot{S}_k \subset H^{5/2}(\Gamma)$ and $r \geqslant 4$ we have

$$\| \tilde{u} - \tilde{u}_h \|_0 \leqslant C \{ h^{r} [\| f \|_{r-4} + | \sigma |_{r-7/2} + | \lambda |_{r-3/2}] + k^{\dot{r}+3/2} [| \sigma |_{\dot{r}-2} + | \lambda |_{\dot{r}} \} .$$

In particular if we take S_h to be continuous piecewise cubics, Hermite cubics, or cubic splines and $\dot{S}_k$ to be cubic splines defined on Γ as a function of arclength then we are in the case $r = 4$, $\dot{r} = 4$ and $\dot{S}_k \subset H^{5/2}(\Gamma)$. Hence if $f \in L^2(\Omega)$ and $(\sigma, \lambda) \in H^2(\Gamma) \cap H^4(\Gamma)$ we obtain

$$\| \tilde{u} - \tilde{u}_h \|_0 \leqslant C \{ h^4 [\| f \|_0 + | \sigma |_{1/2} + | \lambda |_{5/2}] + k^{11/2} [| \sigma |_2 + | \lambda |_4] \} .$$

To balance these terms we could choose $h = k^{11/8}$ so that for k sufficiently small the condition $h \leqslant \varepsilon k$ is automatically satisfied.

In the next three sections of this paper we shall consider the case where Ω has strictly positive curvature K and analyze a finite element method based on the variational formulation Problem $(\tilde{P}^*)$ given in Section 1.

6. SOME FURTHER PRELIMINARIES

Using the definitions of T and G given in Section 2, we see from (1.12) that

$$\tilde{w} = Tf + G\sigma \tag{6.1}$$

and from (1.13) that

$$\tilde{u} = T\tilde{w} - G\left[\frac{1}{\tau K}\tilde{w}\right] = T^2 f + TG\sigma - G\left[\frac{1}{\tau K}Tf\right] - G\left[\frac{1}{\tau K}G\sigma\right]. \tag{6.2}$$

Let us now define

$$w(\sigma) = G\sigma \tag{6.3}$$

and

$$u(\sigma) = TG\sigma - G\left[\frac{1}{\tau K}G\sigma\right]. \tag{6.4}$$

Then

$$\tilde{w} = Tf + w(\sigma)$$

and

$$\tilde{u} = T^2 f - G\left[\frac{1}{\tau K} Tf\right] + u(\sigma)$$

so that Problem ($\tilde{P}^*$) can be restated in the form :

Problem $(P)^*$: Find $\sigma \in H^{-1/2}(\Gamma)$ such that

$$u(\sigma) = -T^2 f + G\left[\frac{1}{\tau K} Tf\right]. \tag{6.5}$$

We now establish some results about the function $u(\sigma)$ which will be needed in the analysis of the finite element approximation of Problem (P^*). From the definitions of T and G we first observe that $u(\sigma)$ is the solution of the biharmonic problem :

Problem (Q^*) : Given $\sigma \in H^{-1/2}(\Gamma)$ find $u \in H^2(\Omega)$ satisfying :

$$\Delta^2 u = 0 \quad \text{in} \quad \Omega \tag{6.6}$$

$$-\frac{\partial}{\partial n}\Delta u - \alpha\,\Delta u = \sigma \quad \text{on} \quad \Gamma \tag{6.7}$$

and

$$-\Delta u + \tau K[u_n + \alpha u] = 0 \quad \text{on} \quad \Gamma\,. \tag{6.8}$$

We then have the following a priori estimate.

THEOREM 6.1 : *There exist positive constants C_1 and C_2 independent of σ such that for all $s \geqslant 0$*

$$C_1 \mid \sigma \mid_{-3/2-s} \leqslant \mid u(\sigma) \mid_{1/2-s} \leqslant C_2 \mid \sigma \mid_{-3/2-s}$$

where $u(\sigma)$ is the solution of Problem (Q^*).

Proof : Using (6.4), (2.1), and (2.2) we have

$$\begin{aligned} \mid u(\sigma) \mid_{1/2-s} &\leqslant \mid TG\sigma \mid_{1/2-s} + \left| G\left[\frac{1}{\tau K} G\sigma\right]\right|_{1/2-s} \\ &\leqslant C\left[\| G\sigma \|_{-1-s} + \left|\frac{1}{\tau K} G\sigma\right|_{-1/2-s}\right] \\ &\leqslant C[\mid \sigma \mid_{-5/2-s} + \mid \sigma \mid_{-3/2-s}] \leqslant C \mid \sigma \mid_{-3/2-s}\,. \end{aligned}$$

To prove the first inequality we use the fact (*cf.* [7]) that

$$\mid \sigma \mid_{-3/2-s} \leqslant C \mid G\sigma \mid_{-1/2-s}\,.$$

But from (6.6)-(6.8) it follows that $G\sigma = -\Delta u = -\tau K[u_n + \alpha u]$ and so we get

$$| \sigma |_{-3/2-s} \leqslant C[| u_n |_{-1/2-s} + | u |_{-1/2-s}] .$$

Applying Lemma 2.4 we get that

$$| u_n |_{-1/2-s} \leqslant C[| u |_{1/2-s} + | M(u) |_{-3/2-s}]$$

where $M(u) = \Delta u - \tau(u_{ss} + Ku_n)$. Now since u satisfies (6.8),

$$M(u) = \tau K\alpha u - \tau u_{ss} .$$

Hence $| u_n |_{-1/2-s} \leqslant C | u |_{1/2-s}$ and so

$$| \sigma |_{-3/2-s} \leqslant C | u |_{1/2-s} .$$

To establish our next result, we will need the following lemma.

LEMMA 6.1 : *If w is a harmonic function in Ω, then*

$$\| w \|_0^2 \leqslant \left\langle \frac{w}{K}, w \right\rangle .$$

Proof : Let $z \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfy $\Delta z = w$. Then

$$(w, w) = (w, \Delta z) = \left\langle w, \frac{\partial z}{\partial n} \right\rangle - (\nabla w, \nabla z) = \left\langle w, \frac{\partial z}{\partial n} \right\rangle$$

(since w is harmonic). Hence

$$\| w \|_0^2 = \left\langle \frac{1}{K^{1/2}} w , K^{1/2} \frac{\partial z}{\partial n} \right\rangle \leqslant \left\langle \frac{1}{K} w, w \right\rangle^{1/2} \left\langle K \frac{\partial z}{\partial n}, \frac{\partial z}{\partial n} \right\rangle^{1/2}$$

Now since $z = 0$ on Γ, we have the identity (*cf.* [5], equation (5.4))

$$\sum_{i,j=1}^{2} \left\| \frac{\partial^2 z}{\partial x_i \, \partial x_j} \right\|_0^2 + \left\langle K \frac{\partial z}{\partial n}, \frac{\partial z}{\partial n} \right\rangle = \| \Delta z \|_0^2 .$$

Hence $\left\langle K \frac{\partial z}{\partial n}, \frac{\partial z}{\partial n} \right\rangle^{1/2} \leqslant \| \Delta z \|_0 = \| w \|_0$ which easily gives

$$\| w \|_0 \leqslant \left\langle \frac{1}{K} w, w \right\rangle .$$

LEMMA 6.2 : *There exist positive constants C_1 and C_2 independent of σ such that*

$$C_1 | \sigma |_{-1}^2 \leqslant | \langle u(\sigma), \sigma \rangle | \leqslant C_2 | \sigma |_{-1}^2 .$$

Proof : Using (2.1), (2.2), and (6.4) we get

$$\begin{aligned} | u(\sigma) |_1 &\leqslant | TG\sigma |_1 + \left| G\left[\frac{1}{\tau K} G\sigma\right]\right|_1 \\ &\leqslant C\left[\| G\sigma \|_{-1/2} + \left|\frac{1}{\tau K} G\sigma\right|_0\right] \\ &\leqslant C[| \sigma |_{-2} + | \sigma |_{-1}] \leqslant C | \sigma |_{-1} . \end{aligned}$$

Hence $| \langle u(\sigma), \sigma \rangle | \leqslant | u(\sigma) |_1 | \sigma |_{-1} \leqslant C | \sigma |_{-1}^2$.

To prove the first inequality, we observe that it follows easily from the definitions of $u(\sigma)$ and $w(\sigma)$ that $(\sigma, u(\sigma), w(\sigma))$ satisfy the variational equations

$$A_\alpha(w(\sigma), v) = \langle \sigma, v \rangle \quad \text{for all} \quad v \in H^1(\Omega) \tag{6.9}$$

and

$$A_\alpha(u(\sigma), z) = (w(\sigma), z) - \left\langle \frac{w(\sigma)}{\tau K}, z \right\rangle \quad \text{for all} \quad z \in H^1(\Omega) . \tag{6.10}$$

Hence

$$\langle \sigma, u(\sigma) \rangle = \| w(\sigma) \|_0^2 - \left\langle \frac{w(\sigma)}{\tau K}, w(\sigma) \right\rangle . \tag{6.11}$$

Since $w(\sigma)$ is harmonic we may apply Lemma 6.1 to obtain

$$| \langle \sigma, u(\sigma) \rangle | \geqslant \left[\frac{1}{\tau} - 1\right]\left\langle \frac{w(\sigma)}{K}, w(\sigma) \right\rangle \geqslant C | w(\sigma) |_0^2 \tag{6.12}$$

since $0 < \tau < 1$ and $K > 0$.

Now let v satisfy

$$\begin{aligned} - \Delta v &= 0 \quad \text{in} \quad \Omega \\ v &= \psi \quad \text{on} \quad \Gamma . \end{aligned}$$

Then a standard a priori estimate gives

$$\| v \|_{3/2} \leqslant C | \psi |_1$$

and since v is harmonic, we have the estimate

$$\left| \frac{\partial v}{\partial n} \right|_0 \leqslant C \| v \|_{3/2} .$$

Then

$$\langle \sigma, \psi \rangle = \langle \sigma, v \rangle = A_\alpha(w(\sigma), v)$$
$$= \left\langle w(\sigma), \frac{\partial v}{\partial n} + \alpha v \right\rangle \leqslant | w(\sigma) |_0 \left[\left| \frac{\partial v}{\partial n} \right|_0 + \alpha | v |_0 \right]$$
$$\leqslant C \, | w(\sigma) |_0 \, | \psi |_1 .$$

Hence

$$| \sigma |_{-1} = \sup_{\psi \in H^1(\Gamma)} \frac{\langle \sigma, \psi \rangle}{| \psi |_1} \leqslant C \, | w(\sigma) |_0 .$$

Combining results we get

$$| \sigma |_{-1}^2 \leqslant C \, | \langle \sigma, u(\sigma) \rangle | .$$

7. THE FINITE ELEMENT APPROXIMATION SCHEME FOR THE CASE $K > 0$

Based on the variational formulation of the simply supported plate problem given in Problem ($\tilde{P}^*$) we now consider the following finite element approximation scheme. The approximating subspaces are those described in Section 3, except now we only assume that $\dot{S}_k \subset H^n(\Gamma)$, $n \geqslant 1/2$, $\dot{r} \geqslant 2$, and $r \geqslant 2$.

Problem ($\tilde{P}_h^{k*}$) : Find $(\tilde{u}_h, \tilde{w}_h, \sigma_k) \in S_h \times S_h \times \dot{S}_k$ such that

$$A_\alpha(\tilde{w}_h, v_h) = (f, v_h) + \langle \sigma_k, v_h \rangle \quad \text{for all} \quad v_h \in S_h \tag{7.1}$$

$$A_\alpha(\tilde{u}_h, z_h) = (\tilde{w}_h, z_h) - \left\langle \frac{\tilde{w}_h}{\tau K}, z_h \right\rangle \quad \text{for all} \quad z_h \in S_h \tag{7.2}$$

and

$$\langle \tilde{u}_h, \beta_k \rangle = 0 \quad \text{for all} \quad \beta_k \in \dot{S}_k . \tag{7.3}$$

Using the operators T_h and G_h we can also rewrite Problem ($\tilde{P}_h^{k*}$) in a form analogous to Problem (P^*). From (7.1) we have that

$$\tilde{w}_h = T_h f + G_h \sigma_k \tag{7.4}$$

and from (7.2) that

$$\tilde{u}_h = T_h \tilde{w}_h - G_h\left[\frac{1}{\tau K} \tilde{w}_h\right] = T_h^2 f - T_h G_h \sigma_k - G_h\left[\frac{1}{\tau K} T_h f\right] - G_h\left[\frac{1}{\tau K} G_h \sigma_k\right]. \tag{7.5}$$

We now define for $\sigma \in H^{-1/2}(\Gamma)$

$$w_h(\sigma) = G_h \sigma \tag{7.6}$$

and

$$u_h(\sigma) = T_h G_h \sigma - G_h\left[\frac{1}{\tau K} G_h \sigma\right]. \tag{7.7}$$

Then $\tilde{w}_h = T_h f + w_h(\sigma_k)$
and

$$\tilde{u}_h = T_h^2 f - G_h\left[\frac{1}{\tau K} T_h f\right] + u_h(\sigma_k)$$

so that Problem $(\tilde{P}_h^k *)$ can be restated in the form :

Problem $(P_h^k *)$: Find $\sigma_k \in \dot{S}_k$ such that

$$P_0\, u_h(\sigma_k) = -P_0\, T_h^2 f + P_0\, G_h\left[\frac{1}{\tau K} T_h f\right]. \tag{7.8}$$

Our aim now is to study the function $u_h(\sigma_k)$ and prove results analogous to those of Theorem 6.1 and Lemma 6.2. We first note that from the definitions of T_h and G_h it easily follows that $u_h(\sigma_k)$, $w_h(\sigma_k)$ is the solution of :

Problem $(Q_h^k *)$: Given $\sigma_k \in \dot{S}_k$ find $(u_h, w_h) \in S_h \times S_h$ satisfying

$$A_\alpha(w_h, v_h) = \langle \sigma_k, v_h \rangle \quad \text{for all} \quad v_h \in S_h$$

and

$$A_\alpha(u_h, z_h) = (w_h, z_h) - \left\langle \frac{w_h}{\tau K}, z_h \right\rangle \quad \text{for all} \quad z_h \in S_h .$$

To simplify the proof of the main result of this section and also the derivation of the error estimates in Section 8, it will be convenient to have the following result.

LEMMA 7.1 : *Let $u(\sigma)$ and $u_h(\sigma)$ be defined by (6.4) and (7.7) respectively. Then if $\sigma \in H^l(\Gamma)$ we have for $-1/2 \leqslant l \leqslant r - 5/2$ and $0 \leqslant s \leqslant r - 2$ that*

$$| u(\sigma) - u_h(\sigma) |_{1/2-s} + \| u(\sigma) - u_h(\sigma) \|_{1-s} \leqslant C h^{l+s+3/2} | \sigma |_l . \tag{7.9}$$

Proof : From (6.4) and (7.7) we have

$$u(\sigma) - u_h(\sigma) = [TG - T_h G_h]\sigma - \left[G\left(\frac{1}{\tau K} G\right) - G_h\left(\frac{1}{\tau K} G_h\right)\right]\sigma .$$

Hence the result follows directly from Corollary 3.2, Theorem 3.2, and the triangle inequality.

We are now ready to state the main result of this section.

THEOREM 7.1 : *For $h \leqslant \varepsilon k$, with ε sufficiently small, there exist positive constants C_1 and C_2 independent of σ, h, and k such that for all*

$$0 \leqslant s \leqslant \min(r - 2, \dot{r} + 1/2)$$

$$C_1 \mid \sigma \mid_{-3/2-s} \leqslant \mid P_0 u_h(\sigma) \mid_{1/2-s} \leqslant C_2 \mid \sigma \mid_{-3/2-s} \quad \textit{for all} \quad \sigma \in \dot{S}_k .$$

To simplify the proof of this theorem we first prove the following lemma which is a restatement of the theorem with $u_h(\sigma)$ replaced by $u(\sigma)$.

LEMMA 7.2 : *There exist positive constants C_1 and C_2 independent of σ and k such that for all $0 \leqslant s \leqslant \dot{r} + 1/2$*

$$C_1 \mid \sigma \mid_{-3/2-s} \leqslant \mid P_0 u(\sigma) \mid_{1/2-s} \leqslant C_2 \mid \sigma \mid_{-3/2-s}$$

for all $\sigma \in \dot{S}_k$, where $u(\sigma)$ is the solution of Problem (Q^*).

Proof : Using Theorem 6.1 and the triangle inequality we have

$$C_1 \mid \sigma \mid_{-3/2-s} - \mid (I - P_0) u(\sigma) \mid_{1/2-s} \leqslant \mid P_0 u(\sigma) \mid_{1/2-s}$$
$$\leqslant C_2 \mid \sigma \mid_{-3/2-s} + \mid (I - P_0) u(\sigma) \mid_{1/2-s} . \quad (7.10)$$

Applying Lemma 3.2 and a standard trace theorem we get for $0 \leqslant s \leqslant \dot{r} + 1/2$ that

$$\mid (I - P_0) u(\sigma) \mid_{1/2-s}^2 \leqslant C k^{2s+1} \mid u(\sigma) \mid_1^2 \leqslant C k^{2s+1} \| u(\sigma) \|_{3/2}^2 .$$

Now using (6.10) and elliptic regularity theory we have

$$\| u(\sigma) \|_{3/2} \leqslant C [\| w(\sigma) \|_{-1/2} + \mid w(\sigma) \mid_0] .$$

Applying Lemma 6.1 and (6.12) we get

$$\| u(\sigma) \|_{3/2}^2 \leqslant C \mid w(\sigma) \mid_0^2 \leqslant C \mid \langle u(\sigma), \sigma \rangle \mid$$

and so

$$\mid (I - P_0) u(\sigma) \mid_{1/2-s}^2 \leqslant C k^{2s+1} \mid \langle u(\sigma), \sigma \rangle \mid .$$

Since $\sigma \in \dot{S}_k$, we get using (3.3) that

$$\mid \langle u(\sigma), \sigma \rangle \mid = \mid \langle P_0 u(\sigma), \sigma \rangle \mid \leqslant \mid P_0 u(\sigma) \mid_{1/2} \mid \sigma \mid_{-1/2}$$
$$\leqslant C k^{-2s-1} \mid P_0 u(\sigma) \mid_{1/2-s} \mid \sigma \mid_{-3/2-s} .$$

Hence $|(I - P_0)\, u(\sigma)|^2_{1/2-s} \leqslant |P_0\, u(\sigma)|_{1/2-s}\, |\sigma|_{-3/2-s}$.

Combining this result with (7.10) and using the arithmetic-geometric mean inequality establishes the lemma.

Proof of Theorem 7.1 : Using Lemma 7.2 and the triangle inequality we have for $0 \leqslant s \leqslant \dot{r} + 1/2$ that

$$C_1\, |\sigma|_{-3/2-s} - |P_0[u(\sigma) - u_h(\sigma)]|_{1/2-s} \leqslant |P_0\, u_h(\sigma)|_{1/2-s}$$
$$\leqslant C_2\, |\sigma|_{-3/2-s} + |P_0[u(\sigma) - u_h(\sigma)|_{1/2-s}\,.$$

Hence to prove Theorem 7.1 we need only show that for

$$0 \leqslant s \leqslant \min(r - 2, \dot{r} + 1/2)$$

$$|P_0[u(\sigma) - u_h(\sigma)]|_{1/2-s} \leqslant \delta\, |\sigma|_{-3/2-s}$$

where δ is a constant which is small with $\varepsilon = h/k$.

Applying the triangle inequality, Lemmas 3.2 and 7.1 and 3.3 we get

$$|P_0[u(\sigma) - u_h(\sigma)]|_{1/2-s} \leqslant |u(\sigma) - u_h(\sigma)|_{1/2-s}$$
$$+ |(I - P_0)\,[u(\sigma) - u_h(\sigma)]|_{1/2-s}$$
$$\leqslant |u(\sigma) - u_h(\sigma)|_{1/2-s} + Ck^s\, |u(\sigma) - u_h(\sigma)|_{1/2}$$
$$\leqslant C[h^{s+1} + hk^s]\, |\sigma|_{-1/2} \leqslant C\left[\left(\frac{h}{k}\right)^{s+1} + \frac{h}{k}\right] |\sigma|_{-3/2-s}\,.$$

The result now follows for $h \leqslant \varepsilon k$ with ε sufficiently small.

In the discussion of the solution of the linear system of equations arising from Problem $(P_h^k{}^*)$, we shall need to make use of the following result, which is a discrete version of Lemma 6.2.

LEMMA 7.3 : *For $h \leqslant \varepsilon k$, with ε sufficiently small, there exist positive constants C_1 and C_2 independent of σ, h, and k such that*

$$C_1\, |\sigma|^2_{-1} \leqslant |\langle P_0\, u_h(\sigma), \sigma \rangle| \leqslant C_2\, |\sigma|^2_{-1} \quad \textit{for all} \quad \sigma \in \dot{S}_k\,.$$

Proof : Applying Lemma 6.2 and the triangle inequality, we have for all $\sigma \in \dot{S}_k$ that

$$C_1\, |\sigma|^2_{-1} - |\langle u(\sigma) - u_h(\sigma), \sigma \rangle| \leqslant |\langle P_0\, u_h(\sigma), \sigma \rangle|$$
$$\leqslant C_2\, |\sigma|^2_{-1} + |\langle u(\sigma) - u_h(\sigma), \sigma \rangle|\,.$$

Now using Lemma 7.1 and (3.3) we get

$$| \langle u(\sigma) - u_h(\sigma), \sigma \rangle | \leqslant | u(\sigma) - u_h(\sigma) |_{1/2} | \sigma |_{-1/2}$$

$$\leqslant Ch | \sigma |_{-1/2}^2 \leqslant \frac{Ch}{k} | \sigma |_{-1}^2 .$$

The result follows for $h \leqslant \varepsilon k$ and ε sufficiently small.

8. ERROR ESTIMATES FOR THE APPROXIMATION OF PROBLEM ($\tilde{P}^*$)

We begin this section by proving a preliminary lemma.

LEMMA 8.1 : *Suppose the hypotheses of Lemma 7.1 are satisfied. Then for all* $\beta \in \dot{S}_k$ *we have*

$$| u(\sigma) - u_h(\beta) |_{1/2-s} + \| u(\sigma) - u_h(\beta) \|_{1-s}$$
$$\leqslant C \{ h^{l+s+3/2} | \sigma |_l + h^{s+1} | \sigma - \beta |_{-1/2} + | \sigma - \beta |_{-3/2-s} \}$$

for $-1/2 \leqslant l \leqslant r - 5/2$ *and* $0 \leqslant s \leqslant r - 2$.

Proof : Applying the triangle inequality we have

$$||| u(\sigma) - u_h(\beta) |||_{1/2-s} \leqslant ||| u(\sigma) - u_h(\sigma) |||_{1/2-s}$$
$$+ ||| u_h(\sigma - \beta) - u(\sigma - \beta) |||_{1/2-s} + ||| u(\sigma - \beta) |||_{1/2-s} .$$

From Lemma 7.1 we get

$$||| u(\sigma) - u_h(\sigma) |||_{1/2-s} \leqslant Ch^{l+s+3/2} | \sigma |_l$$

and

$$||| u_h(\sigma - \beta) - u(\sigma - \beta) |||_{1/2-s} \leqslant Ch^{s+1} | \sigma - \beta |_{-1/2} .$$

Using (2.1), (2.2), and (6.4),

$$||| u(\sigma - \beta) |||_{1/2-s} \leqslant ||| TG(\sigma - \beta) |||_{1/2-s} + \left|\left|\left| G\left[\frac{1}{\tau K}\right] G(\sigma-\beta) \right|\right|\right|_{1/2-s}$$

$$\leqslant C[\| G(\sigma - \beta) \|_{-1-s} + | G(\sigma - \beta) |_{-1/2-s}]$$
$$\leqslant C[| \sigma-\beta |_{-5/2-s} + | \sigma-\beta |_{-3/2-s} \leqslant C | \sigma-\beta |_{-3/2-s}] .$$

The lemma follows by combining these results.

THEOREM 8.1 : *Suppose* $f \in H^m(\Omega)$ *and* $\sigma \in H^l(\Gamma) \cap H^{\dot{l}}(\Gamma)$ *and* $\sigma_k \in \dot{S}_k$ *are the respective solutions of Problems* (P^*) *and* (P_h^{k*}). *Then for* $h \leqslant \varepsilon k$ *with* ε *sufficiently small, there exists a constant* C *independent of* h, k, σ, *and* f *such that if* $\dot{S}_k \subset H^n(\Gamma)$, $n \geqslant 1/2$, *then*

$$| \sigma - \sigma_k |_{-3/2-s} \leqslant C \{ h^{m+5/2-i} k^{i+s-1/2} \| f \|_m + h^{l+2-i} k^{i+s-1/2} | \sigma |_l + k^{\dot{l}+s+3/2} | \sigma |_{\dot{l}} \}$$

for all $-1/2 \leqslant l \leqslant r - 5/2$, $-1/2 \leqslant \dot{l} \leqslant \dot{r}$, $-1 \leqslant s \leqslant \min(r-2, \dot{r}+1/2)$, *and* $-1 \leqslant m \leqslant r-3$, *where* $i = \max(-n, 1/2 - s)$.

Proof : Let $\pi_k \sigma \in \dot{S}_k$ be an approximation to σ satisfying (3.4). By the linearity of $u(\sigma)$ and Theorem 7.1 we get for $0 \leqslant s \leqslant \min(r-2, \dot{r}+1/2)$

$$| \sigma_k - \pi_k \sigma |_{-3/2-s} \leqslant C | P_0[u_h(\sigma_k) - u_h(\pi_k \sigma)] |_{1/2-s} .$$

Using (6.5) and (7.8) we have

$$\begin{aligned} P_0 u_h(\sigma_k) - P_0 u_h(\pi_k \sigma) &= -P_0 T_h^2 f + P_0 G_h\left[\frac{1}{\tau K} T_h f\right] - P_0 u_h(\pi_k \sigma) \\ &= P_0(T^2 - T_h^2) f - P_0 \left\{ G\left(\frac{1}{\tau K}\right) T - G_h\left(\frac{1}{\tau K}\right) T_h \right\} f + P_0[u(\sigma) - u_h(\pi_k \sigma)] . \end{aligned}$$

Now using Lemma 3.3, Corollary 3.2, Theorem 3.2, and the triangle inequality, we obtain for $\dot{S}_k \subset H^n(\Gamma)$, $0 \leqslant s \leqslant \min(r-2, \dot{r}+1/2)$, and $i = \max(-n, 1/2 - s)$ that

$$\begin{aligned} | \sigma_k - \pi_k \sigma |_{-3/2-s} \leqslant C \Big\{ & | [T^2 - T_h^2] f |_{1/2-s} + k^{i+s-1/2} | [T^2 - T_h^2] f |_i \\ & + \left| \left[G\left(\frac{1}{\tau K}\right) T - G_h\left(\frac{1}{\tau K}\right) T_h \right] f \right|_{1/2-s} \\ & + k^{i+s-1/2} \left| \left[G\left(\frac{1}{\tau K}\right) T - G_h\left(\frac{1}{\tau K}\right) T_h \right] f \right|_i \\ & + | u(\sigma) - u_h(\pi_k \sigma) |_{1/2-s} + k^{i+s-1/2} | u(\sigma) - u_h(\pi_k \sigma) |_i \Big\} \\ \leqslant C \{ & h^{m+5/2-i} k^{i+s-1/2} \| f \|_m \\ & + | u(\sigma) - u_h(\pi_k \sigma) |_{1/2-s} + k^{i+s-1/2} | u(\sigma) - u_h(\pi_k \sigma) |_i \} . \end{aligned}$$

Applying Lemma 8.1 with $\beta = \pi_k \sigma$ and (3.4) we have for $-1/2 \leqslant l \leqslant r-5/2$ and $-1/2 \leqslant \dot{l} \leqslant \dot{r}$ that

$$
\begin{aligned}
| u(\sigma) - u_h(\pi_k \sigma) |_{1/2-s} &+ k^{\iota+s-1/2} | u(\sigma) - u_h(\pi_k \sigma) |_\iota \\
&\leqslant C \{ h^{l+2-\iota} k^{\iota+s-1/2} | \sigma |_l + h^{3/2-\iota} k^{\iota+s-1/2} | \sigma - \pi_k \sigma |_{-1/2} \\
&\quad + | \sigma - \pi_k \sigma |_{-3/2-s} + k^{\iota+s-1/2} | \sigma - \pi_k \sigma |_{\iota-2} \} \\
&\leqslant C \{ h^{l+2-\iota} k^{\iota+s-1/2} | \sigma |_l + k^{3/2+s+\dot{l}} | \sigma |_{\dot{l}} \}
\end{aligned}
$$

(since $h \leqslant \varepsilon k$, $\varepsilon < 1$).

The theorem follows for $0 \leqslant s \leqslant \min(r-2, \dot{r}+1/2)$ by combining these results and (3.4) and for $-1 \leqslant s < 0$ using (3.3).

THEOREM 8.2 : *Suppose the hypotheses of Theorem* 8.1 *are satisfied. Then if* $(\tilde{u}, \tilde{w}, \sigma)$ *and* $(\tilde{u}_h, \tilde{w}_h, \sigma_k)$ *are the respective solutions of Problem* $(\tilde{P}^*)$ *and* $(\tilde{P}_h^{k*})$, *we have for* $i = \max(-n, 1/2-s)$, *and all* $-1/2 \leqslant l \leqslant r-5/2$,

$$-1/2 \leqslant \dot{l} \leqslant \dot{r}, \text{ and } -1 \leqslant m \leqslant r-3 \text{ that}$$

$$
\begin{aligned}
| \tilde{u} - \tilde{u}_h |_{1/2-s} &+ \| \tilde{u} - \tilde{u}_h \|_{1-s} \\
&\leqslant C \{ h^{m+5/2-\iota} k^{\iota+s-1/2} \| f \|_m \\
&\quad + h^{l+2-\iota} k^{\iota+s-1/2} | \sigma |_l + k^{\dot{l}+s+3/2} | \sigma |_{\dot{l}} \}
\end{aligned}
$$

for $0 \leqslant s \leqslant \min(r-2, \dot{r}+1/2)$
and

$$
\begin{aligned}
| \tilde{w} - \tilde{w}_h |_{-1/2-s} &+ \| \tilde{w} - \tilde{w}_h \|_{-s} \\
&\leqslant C \{ h^{m+5/2-\iota} k^{\iota+s-1/2} \| f \|_m \\
&\quad + h^{l+2-\iota} k^{\iota+s-1/2} | \sigma |_l + k^{\dot{l}+s+3/2} | \sigma |_{\dot{l}} \}
\end{aligned}
$$

for $-1 \leqslant s \leqslant \min(r-2, \dot{r}+1/2)$.

Proof : Using the definitions of $\tilde{u}$ and $\tilde{u}_h$ and the triangle inequality we get for $0 \leqslant s \leqslant \min(r-2, \dot{r}+1/2)$ that

$$
\begin{aligned}
||| \tilde{u} - \tilde{u}_h |||_{1/2-s} &\leqslant ||| [T^2 - T_h^2] f |||_{1/2-s} \\
&\quad + \left|\left|\left| \left[G\left(\frac{1}{\tau K}\right) T - G_h\left(\frac{1}{\tau K}\right) T_h \right] f \right|\right|\right|_{1/2-s} + ||| u(\sigma) - u_h(\sigma_k) |||_{1/2-s} .
\end{aligned}
$$

From Lemma 8.1 with $\beta = \sigma_k$ we have

$$|||\, u(\sigma) - u_h(\sigma_k) \,|||_{1/2-s} \leqslant$$
$$\leqslant C \{ h^{l+s+3/2} \,|\, \sigma \,|_l + h^{s+1} \,|\, \sigma - \sigma_k \,|_{-1/2} + |\, \sigma - \sigma_k \,|_{-3/2-s} \} .$$

The estimate for $|||\, \tilde{u} - \tilde{u}_h \,|||_{1/2-s}$ now follows immediately from Corollary 3.2, Theorem 3.2, and Theorem 8.1.
Using the definitions of $\tilde{w}$ and $\tilde{w}_h$ we get

$$\begin{aligned} |||\, \tilde{w} - \tilde{w}_h \,|||_{-1/2-s} &\leqslant |||\, [T - T_h] f \,|||_{-1/2-s} + |||\, G\sigma - G_h \sigma_k \,|||_{-1/2-s} \\ &\leqslant |||\, [T - T_h] f \,|||_{-1/2-s} + |||\, [G - G_h] \sigma \,|||_{-1/2-s} \\ &\quad + |||\, [G - G_h] (\sigma - \sigma_k) \,|||_{-1/2-s} + |||\, G(\sigma - \sigma_k) \,|||_{-1/2-s} . \end{aligned}$$

Applying Lemma 3.1 and estimates (2.1*b*) and (2.2*b*) we get for

$$-1 \leqslant s \leqslant \min(r - 2, \dot{r} + 1/2)$$

$$\begin{aligned} |||\, \tilde{w} - \tilde{w}_h \,|||_{-1/2-s} \leqslant C \{ & h^{m+s+2} \,\|\, f \,\|_m + h^{l+3/2+s} \,|\, \sigma \,|_l \\ & + h^{s+1} \,|\, \sigma - \sigma_k \,|_{-1/2} + |\, \sigma - \sigma_k \,|_{-3/2-s} \} . \end{aligned}$$

The result follows immediately from Theorem 8.1.

We now consider some applications of the error estimates in Theorem 8.2. Suppose $r \leqslant \dot{r} + 5/2, f \in H^{r-3}(\Omega)$, and $\sigma \in H^{\dot{r}}(\Gamma)$. Then if $\dot{S}_k \subset H^{1/2}(\Gamma)$ we have for $r \geqslant 2$ that

$$\|\, \tilde{u} - \tilde{u}_h \,\|_1 \leqslant C \{ h^{r-1} [\|\, f \,\|_{r-3} + |\, \sigma \,|_{r-5/2}] + k^{\dot{r}+3/2} \,|\, \sigma \,|_{\dot{r}} \}$$

and for $r \geqslant 3$ that

$$\|\, \tilde{u} - \tilde{u}_h \,\|_0 \leqslant C \{ h^r [\|\, f \,\|_{r-3} + |\, \sigma \,|_{r-5/2}] + k^{\dot{r}+5/2} \,|\, \sigma \,|_{\dot{r}} \} .$$

In particular if we use continuous piecewise cubics for S_h and continuous piecewise linear functions for $\dot{S}_k$, then $r = 4$, $\dot{r} = 2$ and we obtain the estimate

$$\|\, \tilde{u} - \tilde{u}_h \,\|_0 \leqslant C \{ h^4 [\|\, f \,\|_1 + |\, \sigma \,|_{3/2}] + k^{9/2} \,|\, \sigma \,|_2 \} .$$

To balance these terms we could choose $h = k^{9/8}$ so that for k sufficiently small the condition $h \leqslant \varepsilon k$ is automatically satisfied.

9. EFFICIENT SOLUTION OF PROBLEMS (P_h^k) and (P_h^{k*})

In this section we show how some ideas developed in [7] can be used to develop methods for the efficient solution of the linear systems of equations arising from Problems (P_h^k) and (P_h^{k*}). To describe these ideas we first define a discrete boundary Laplacian

$$l_k : \dot{S}_k \to \dot{S}_k$$

by

$$\langle l_k \phi, \theta \rangle = \langle \phi, \theta \rangle_1$$

where

$$\langle \phi, \theta \rangle_1 = \langle \phi, \theta \rangle + \langle \phi_s, \theta_s \rangle .$$

Now l_k is positive definite and symmetric and hence l_k^s may be defined in the usual way by taking powers of its eigenvalues.

The methods presented in this action depend heavily on the following property of the operator l_k^s.

LEMMA 9.1 : (cf. [7]). *Let $\dot{S}_k \subset H^1(\partial\Omega)$. Then for $| s | \leqslant 1$, there are constants C_1 and C_2 such that for $\phi \in \dot{S}_k$*

$$C_1 \mid \phi \mid_s^2 \leqslant \mid l_k^{s/2} \phi \mid_0^2 \leqslant C_2 \mid \phi \mid_s^2 .$$

We now show that this result is also valid for a larger range of values of s.

LEMMA 9.2 : *Let $\dot{S}_k \subset H^1(\partial\Omega)$. Then for $- \dot{r} \leqslant s \leqslant 1$, there are constants C_1 and C_2 such that for $\phi \in \dot{S}_k$*

$$C_1 \mid \phi \mid_s^2 \leqslant \mid l_k^{s/2} \phi \mid_0^2 \leqslant C_2 \mid \phi \mid_s^2 . \tag{9.1}$$

Proof : The proof is by induction. By Lemma 9.1 the result is true for $- 1 \leqslant s \leqslant 1$. We now show that assuming (9.1) holds for a value $s \leqslant 1$, it also holds for the value $s - 2$ (provided $s - 2 \geqslant - \dot{r}$). Since

$$C_1 \mid l^{-1} \phi \mid_s \leqslant \mid \phi \mid_{s-2} \leqslant C_1 \mid l^{-1} \phi \mid_s \quad (\text{for } s \leqslant 1),$$

we have by the triangle inequality that

$$C_1 \mid l_k^{-1} \phi \mid_s - C_1 \mid [l^{-1} - l_k^{-1}] \phi \mid_s \leqslant \mid \phi \mid_{s-2}$$
$$\leqslant C_2 \mid l_k^{-1} \phi \mid_s + C_2 \mid [l^{-1} - l_k^{-1}] \phi \mid_s .$$

By standard estimates analogous to those of Lemma (3.1) we have for $2 - \dot{r} \leqslant s \leqslant 1$ that

$$| [l^{-1} - l_k^{-1}] \phi |_s \leqslant C k^{2-s} | \phi |_0 .$$

Using the induction hypothesis we have that

$$C_1 | l_k^{-1} \phi |_s \leqslant | l_k^{3/2-1} \phi |_0 \leqslant C_2 | l_k^{-1} \phi |_s$$

so that after combining results we obtain

$$C_1 | l_k^{s/2-1} \phi |_0 \leqslant | \phi |_{s-2} + C k^{2-s} | \phi |_0$$

and

$$| \phi |_{s-2} \leqslant C_2 | l_k^{s/2-1} \phi |_0 + C k^{2-s} | \phi |_0 .$$

The lower inequality now follows directly from (3.3).

To get the upper inequality we note that it was proved in [7], Lemma 7.1 that

$$| \phi |_0 \leqslant C k^{-1} | l_k^{-1/2} \phi |_0 .$$

It therefore follows easily by induction and then interpolation that for $s \leqslant 1$

$$| \phi |_0 \leqslant C k^{s-2} | l_k^{s/2-1} \phi |_0 .$$

The upper inequality follows directly from this result.

We now consider the implications of this inequality for the solution of Problem (P_h^{k*}).

Combining Lemmas 7.3 and 9.2 we see that

$$C_1 | l_k^{-1/2} \sigma |_0^2 \leqslant | \langle P_0 u_h(\sigma), \sigma \rangle | \leqslant C_2 | l_k^{-1/2} \sigma |_0^2 .$$

Setting $\sigma = l_k^{1/2} \theta$ we further obtain

$$C_1 | \theta |_0^2 \leqslant | \langle P_0 u_h(l_k^{1/2} \theta), l_k^{1/2} \theta \rangle | \leqslant C_2 | \theta |_0^2 .$$

Using the definition (7.7) of $u_h(\sigma)$ we have

$$C_1 | \theta |_0^2 \leqslant \left\langle l_k^{1/2} P_0 \left[T_h G_h - G_h \left(\frac{1}{\tau K} \right) G_h \right] l_k^{1/2} \theta, \theta \right\rangle \leqslant C_2 | \theta |_0^2 .$$

This inequality means that the system with matrix induced by

$$l_k^{1/2} P_0 \left[T_h G_h - G_h \left(\frac{1}{\tau K} \right) G_h \right] l_k^{1/2}$$

has bounded condition number and hence we can obtain a solution θ to the equation

$$l_k^{1/2} P_0\left[T_h G_h - G_h\left(\frac{1}{\tau K}\right) G_h\right] l_k^{1/2} \theta = l_k^{1/2}\left[-P_0 T_h^2 f + P_0 G_h\left(\frac{1}{\tau K}\right) T_h f\right]$$

to within accuracy h^r by the conjugate gradient method (on some other iterative method) in $0(\ln 1/h)$ iterations. To apply such a method we need for each iteration to compute $P_0 g$ for $g \in H^{1/2}(\Gamma)$ and $\left[T_h G_h - G_h\left(\frac{1}{\tau K}\right) G_h\right] \sigma$ and $l_k \sigma$ for $\sigma \in \dot{S}_k$ (*cf.* [1]). All of these operations can be done by solving sparse systems of linear equations and will involve only back substitution at each iteration since the matrices do not change and hence require only an initial factorization.

We now turn our attention to the study of Problem (P_h^k). From Theorem 4.1 and Lemma 9.2 it easily follows that for $0 \leqslant s \leqslant \min(r - 3, \dot{r} - 3/2)$.

$$\begin{aligned} C_1[| l_k^{-1/4-s/2} \lambda |_0^2 + | l_k^{-5/4-s/2} \sigma |_0^2] &\leqslant | l_k^{1/4-s/2} P_0 u_h(\lambda, \sigma) |_0^2 + | l_k^{-3/4-s/2} P_0 M_h(\lambda, \sigma) |_0^2 \\ &\leqslant C_2[| l_k^{-1/4-s/2} \lambda |_0^2 + | l_k^{-5/4-s/2} \sigma |_0^2] . \end{aligned}$$

Letting

$$\lambda = l_k^{1/4+s/2} \lambda^* \quad \text{and} \quad \sigma = l_k^{5/4+s/2} \sigma^*$$

we get

$$\begin{aligned} C_1[| \lambda^* |_0^2 + | \sigma^* |_0^2] &\leqslant | l_k^{1/4-s/2} P_0 u_h(l_k^{1/4+s/2} \lambda^*, l_k^{5/4+s/2} \sigma^*) |_0^2 \\ &+ | l_k^{-3/4-5/2} P_0 M_h(l_k^{1/4+s/2} \lambda^*, l_k^{5/4+s/2} \sigma^*) |_0^2 \leqslant C_2[| \lambda^* |_0^2 + | \sigma^* |_0^2] . \end{aligned} \tag{9.2}$$

Using the definitions of $M_h(\lambda, \sigma)$ and $u_h(\lambda, \sigma)$ and the fact that $P_0 G_h$, $P_0 T_h G_h$, and l_k^s are self adjoint operators on $\dot{S}_k$, it is possible to find operators D_{11}, D_{12}, D_{21} $(= D_{12}^*)$, and D_{22} so that (9.2) can be rewritten in the form :

$$\begin{aligned} C_1[| \lambda^* |_0^2 + | \sigma^* |_0^2] &\leqslant \langle D_{11} \lambda^*, \lambda^* \rangle + \langle D_{12} \sigma^*, \lambda^* \rangle \\ &+ \langle D_{21} \lambda^*, \sigma^* \rangle + \langle D_{22} \sigma^*, \sigma^* \rangle \leqslant C_2[| \lambda^* |_0^2 + | \sigma^* |_0^2] . \end{aligned}$$

This inequality means that the matrix induced by the operator

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \text{ has bounded}$$

condition number. Hence if instead of solving the system (4.9), (4.10) we solve the well conditioned system :

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} \lambda^* \\ \sigma^* \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{9.3}$$

where $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$ is chosen so that (4.9)-(4.10) and (9.3) are equivalent, then the conjugate gradient method can be used to obtain the solution in $0(\ln 1/h)$ iterations.

One finds after calculation of the operators D_{ij} that if the conjugate gradient method is applied to this system in the untransformed variables (λ, σ) and $s = 1/2$, we need only compute the action of the operators T_h, G_h, P_0, $\partial^2/\partial s^2$, and integer powers of l_k. From the definitions of these quantities it follows that all these computations are quite easy.

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