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## A MIXED FINITE ELEMENT METHOD FOR A WEIGHTED ELLIPTIC PROBLEM (\*)

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**Abstract** — *In this paper, we study a Dirichlet's weighted problem, we give a mixed formulation which has a unique solution and we obtain error bounds in weighted Sobolev spaces for a mixed finite element approximation*

**Résumé** — *Cet article est consacré à l'étude d'un problème de Dirichlet avec poids, nous donnons une formulation mixte de ce problème admettant une solution unique, nous obtenons ensuite des estimations d'erreur dans des espaces de Sobolev avec poids pour une approximation par des éléments finis mixtes*

### I. INTRODUCTION

Let  $\Omega$  be a bounded domain of  $\mathbb{R}^2$  with a regular boundary  $\Gamma$ . We consider the Dirichlet's problem : Find a function  $u$  defined over  $\Omega$  such that :

$$\left. \begin{aligned} -\operatorname{div}(D^{-1} \overrightarrow{\operatorname{grad}} u) &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \Gamma, \end{aligned} \right\} \quad (1.1)$$

where  $D$  is a given function  $\in C^\infty(\overline{\Omega})$ , positive over  $\Omega$  and null over  $\Gamma$  such that there are two constants  $a_1, a_2 > 0$  such that

$$0 < a_1 \leq \frac{D(x)}{d(x, \Gamma)} \leq a_2 \quad \text{for } x \in \overline{\Omega} \text{ such that } d(x, \Gamma) \leq a \quad (a > 0). \quad (1.2)$$

This problem is occurring in oceanography ;  $D$  is the deepthness of the water, which is null on the shore ;  $f$  is the vorticity and the components of the hori-

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zontal velocity  $\vec{V}$  on the surface can be obtained from  $u$ . As a matter of fact, we have

$$\vec{V} = \begin{pmatrix} D^{-1} \frac{\partial u}{\partial y} \\ - D^{-1} \frac{\partial u}{\partial x} \end{pmatrix}.$$

This kind of problem is found at each time step for vorticity method. So, in order to calculate  $\vec{V}$ , we shall use a mixed finite element method.

An outline of the paper is as follows : in section 2, we introduce a mixed formulation of problem (1. 1) ; in section 3, we give a formulation of the finite element approximation. Section 4 will be devoted to the derivation of error bounds and section 5 is devoted to the proof of a result used in section 4.

**II. A MIXED FORMULATION OF PROBLEM 1.1**

Introduce the space :

$$H^1_{-1/2}(\Omega) = \{ v \in \mathcal{D}'(\Omega) / D^{-3/2} v \in L^2(\Omega) ; D^{-1/2} \overrightarrow{\text{grad}} v \in (L^2(\Omega))^2 \} \quad (2.1)$$

provided with the norm :

$$\| v \|_{1,-1/2,\Omega} = ( | D^{-3/2} v |^2_{L^2(\Omega)} + | D^{-1/2} \overrightarrow{\text{grad}} v |^2_{(L^2(\Omega))^2} )^{1/2}. \quad (2.2)$$

We have the following result (cf. Bolley-Camus [1]).

**THEOREM 2.1 :**  $\mathcal{D}(\Omega)$  is dense in  $H^1_{-1/2}(\Omega)$  and the semi-norm

$$| v |_{1,-1/2,\Omega} = | D^{-1/2} \overrightarrow{\text{grad}} v |_{(L^2(\Omega))^2} \quad (2.3)$$

is a norm over  $H^1_{-1/2}(\Omega)$  which is equivalent to the norm  $\| \cdot \|_{1,-1/2,\Omega}$ .

Then, a variational formulation of problem (1.1) is as follows : given a function  $f \in L^2_{3/2}(\Omega) = \{ f \in \mathcal{D}'(\Omega) / D^{3/2} f \in L^2(\Omega) \}$ , find a function  $u \in H^1_{-1/2}(\Omega)$  such that :

$$\int_{\Omega} D^{-1} \overrightarrow{\text{grad}} u \cdot \overrightarrow{\text{grad}} v \, dx = \int_{\Omega} f v \, dx \text{ for all } v \in H^1_{-1/2}(\Omega). \quad (2.4)$$

Clearly, from theorem 2.1, this problem is coercive in  $H^1_{-1/2}(\Omega)$  and then it has a unique solution  $u \in H^1_{-1/2}(\Omega)$ .

Introduce now the space :

$$H_{1/2}^2(\Omega) = \{ v \in \mathcal{D}'(\Omega) / D^{-3/2} v \in L^2(\Omega), D^{-1/2} \partial^\alpha v \in L^2(\Omega), |\alpha| = 1, \\ D^{1/2} \partial^\alpha v \in L^2(\Omega), |\alpha| = 2 \} \quad (2.5)$$

provided with the norm :

$$\| u \|_{2,1/2,\Omega} = \left( \sum_{|\alpha| \leq 2} | D^{-3/2+|\alpha|} \partial^\alpha v |_{L^2(\Omega)}^2 \right)^{1/2} \quad (2.6)$$

**THEOREM 2.2 :** *Problem (2.4) has a unique solution  $u$  which is in  $H_{1/2}^2(\Omega)$  if  $f$  is in  $L_{3/2}^2(\Omega)$ ; moreover, this solution satisfies :*

$$\| u \|_{2,1/2,\Omega} \leq C | D^{3/2} f |_{L^2(\Omega)} \quad (2.7)$$

where  $C$  is a positive constant depending only on  $\Omega$ .

*Proof :* The dual space of  $H_{-1/2}^1(\Omega)$  is given by :

$$(H_{-1/2}^1(\Omega))' = \left\{ f \in \mathcal{D}'(\Omega) / f = f_0 + \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}; \quad D^{3/2} f_0 \in L^2(\Omega), \right. \\ \left. D^{1/2} f_1 \in L^2(\Omega), \quad D^{1/2} f_2 \in L^2(\Omega) \right\}$$

and the problem : given a function  $f \in (H_{-1/2}^1(\Omega))'$ , find  $u \in H_{-1/2}^1(\Omega)$  such that :

$$\int_{\Omega} D^{-1} \overrightarrow{\text{grad}} u \overrightarrow{\text{grad}} v \, dx = \langle f, v \rangle_{(H_{-1/2}^1(\Omega))', H_{-1/2}^1(\Omega)} \quad \text{for all } v \in H_{-1/2}^1(\Omega)$$

has a unique solution.

Assume now that  $f$  is in  $L_{3/2}^2(\Omega)$ ; we note

$$\Psi = D \frac{\partial u}{\partial x_1}.$$

Then, we obtain

$$\text{div} (D^{-1} \overrightarrow{\text{grad}} \Psi) = \text{div} \left( D^{-1} \frac{\partial u}{\partial x_1} \overrightarrow{\text{grad}} D \right) + \frac{\partial}{\partial x_1} (\Delta u).$$

Besides,

$$\Delta u = D^{-1} \overrightarrow{\text{grad}} D \cdot \overrightarrow{\text{grad}} u - Df,$$

hence, we get

$$D^{+1/2} \Delta u \in L^2(\Omega) \quad \text{and} \quad \frac{\partial}{\partial x_1} (\Delta u) \in (H^1_{-1/2}(\Omega))'.$$

Similarly, we have

$$D^{1/2} \left( D^{-1} \frac{\partial u}{\partial x_1} \overrightarrow{\text{grad}} D \right) \in L^2(\Omega)$$

and then

$$\text{div} \left( D^{-1} \frac{\partial u}{\partial x_1} \overrightarrow{\text{grad}} D \right) \in (H^1_{-1/2}(\Omega))';$$

so, we get

$$\text{div} (D^{-1} \overrightarrow{\text{grad}} \Psi) \in (H^1_{-1/2}(\Omega))'.$$

Besides we have

$$D^{1/2} u \in H^2(\Omega) \quad (\Delta(D^{1/2} u) \in L^2(\Omega)) \quad \text{and} \quad D^{-1/2} u \in H^1(\Omega)$$

so, we get

$$D^{1/2} \frac{\partial u}{\partial x_1} \Big|_{\Gamma} \in L^2(\Gamma), \quad \text{hence } \Psi = 0 \text{ on } \Gamma.$$

Then, we obtain

$$\Psi \in H^1_{-1/2}(\Omega).$$

Hence, we get

$$D^{-1/2} \frac{\partial \Psi}{\partial x_1} = D^{1/2} \frac{\partial^2 u}{\partial x_1^2} + D^{-1/2} \frac{\partial D}{\partial x_1} \frac{\partial u}{\partial x_1} \in L^2(\Omega),$$

$$D^{-1/2} \frac{\partial \Psi}{\partial x_2} = D^{1/2} \frac{\partial^2 u}{\partial x_1 \partial x_2} + D^{-1/2} \frac{\partial D}{\partial x_2} \frac{\partial u}{\partial x_1} \in L^2(\Omega)$$

and then

$$D^{1/2} \frac{\partial^2 u}{\partial x_1^2} \in L^2(\Omega), \quad D^{1/2} \frac{\partial^2 u}{\partial x_1 \partial x_2} \in L^2(\Omega),$$

$$D^{1/2} \frac{\partial^2 u}{\partial x_2^2} = D^{1/2} \Delta u - D^{1/2} \frac{\partial^2 u}{\partial x_1^2} \in L^2(\Omega)$$

that is  $u \in H^2_{1/2}(\Omega)$  and we get easily the estimate (2.7).

To obtain a mixed formulation of problem (1.5), we introduce the space  $X$  defined by :

$$X = \{ \vec{q} \in (\mathcal{D}'(\Omega))^2 / D^{1/2} \vec{q} \in (L^2(\Omega))^2, D^{3/2} \operatorname{div} \vec{q} \in L^2(\Omega) \} \quad (2.8)$$

and provided with the norm :

$$\| \vec{q} \|_X = (\| D^{1/2} \vec{q} \|_{(L^2(\Omega))^2}^2 + | D^{3/2} \operatorname{div} \vec{q} |_{L^2(\Omega)}^2)^{1/2}. \quad (2.9)$$

**THEOREM 2.3 :**  $(\mathcal{D}(\Omega))^2$  is dense in  $X$ .

*Proof :* Let  $\vec{q} \in X$ . First, let us show that there exists a sequence of functions of  $X$ , with compact support in  $\Omega$  that tends to  $\vec{q}$ .

Let  $\theta$  be a function in  $C^\infty([0, +\infty[)$  such that :

$$\begin{cases} \theta(x) = 0, & 0 \leq x \leq 1; \\ 0 \leq \theta(x) \leq 1, & 1 \leq x \leq 2; \\ \theta(x) = 1, & x \geq 2. \end{cases}$$

We set

$$\begin{aligned} \theta_\lambda(x) &= \theta(\lambda D(x)), \quad \lambda \in \mathbb{R}^+, \quad x \in \bar{\Omega}; \\ \vec{q}_\lambda(x) &= \theta_\lambda(x) \vec{q}(x). \end{aligned}$$

Therefore, we have

$$| D^{1/2} (\vec{q}_\lambda - \vec{q}) |_{(L^2(\Omega))^2}^2 = \int_{\Omega \cap \{x/D(x) \leq 2/\lambda\}} D(x) (\theta_\lambda(x) - 1)^2 | \vec{q}(x) |^2 dx$$

and from Lebesgue's theorem, this term tends to 0 when  $\lambda \rightarrow +\infty$ .

Similarly, we get

$$\begin{aligned} | D^{3/2} \operatorname{div} (\vec{q}_\lambda - \vec{q}) |_{L^2(\Omega)}^2 &= \int_{\Omega \cap \{x/D(x) \leq 2/\lambda\}} D^3(x) (\theta_\lambda(x) - 1)^2 \times \\ &\times | \operatorname{div} \vec{q}(x) |^2 dx + \int_{\Omega} D^3(x) (\overrightarrow{\operatorname{grad}} \theta_\lambda(x) \cdot \vec{q}(x))^2 dx. \end{aligned}$$

But,

$$\begin{cases} \overrightarrow{\operatorname{grad}} \theta_\lambda(x) = 0 & \text{for } D(x) \leq \frac{1}{\lambda} \text{ or } D(x) \geq \frac{2}{\lambda}, \\ \overrightarrow{\operatorname{grad}} \theta_\lambda(x) = \lambda \theta'(\lambda D(x)) \overrightarrow{\operatorname{grad}} D(x), & \text{for } 1 \leq \lambda D(x) \leq 2. \end{cases}$$

Thus, we get

$$\int_{\Omega} D^3(x) (\overrightarrow{\text{grad}} \theta_{\lambda}(x) \cdot \vec{q}(x))^2 dx \leq C \int_{\Omega \cap \{x/\lambda \leq D(x) \leq \frac{2}{\lambda}\}} D(x) |q(x)|^2 dx$$

and from Lebesgue's theorem

$$\lim_{\lambda \rightarrow +\infty} |D^{3/2} \text{div}(\vec{q}_{\lambda} - \vec{q})|_{L^2(\Omega)} = 0.$$

Hence,  $\vec{q}_{\lambda}$  is a sequence with compact support in  $\Omega$  that tends to  $\vec{q}$  in  $X$ .

From the above, we can assume that  $\vec{q}$  has a compact support in  $\Omega$ ; then,  $\vec{q} \in (L^2(\Omega))^2$  and  $\text{div} \vec{q} \in L^2(\Omega)$ . Therefore there exists a sequence of functions  $\vec{q}_n \in (\mathcal{D}'(\Omega))^2$  such that

$$\lim_{n \rightarrow +\infty} (|\vec{q} - \vec{q}_n|_{(L^2(\Omega))^2}^2 + |\text{div}(\vec{q} - \vec{q}_n)|_{L^2(\Omega)}) = 0 \quad [7]$$

and then,

$$\lim_{n \rightarrow +\infty} \|\vec{q} - \vec{q}_n\|_X \leq C \lim_{n \rightarrow +\infty} (|\vec{q} - \vec{q}_n|_{(L^2(\Omega))^2}^2 + |\text{div}(\vec{q} - \vec{q}_n)|_{L^2(\Omega)})^{1/2} = 0.$$

LEMMA 2.1 · For any  $u \in H^1_{-1/2}(\Omega)$  and  $\vec{q} \in X$ , we have :

$$\int_{\Omega} (\overrightarrow{\text{grad}} u \cdot \vec{q} + u \cdot \text{div} \vec{q}) dx = 0. \tag{2.10}$$

This result follows immediately from theorems 2.1 and 2.3.

Now, we define a bilinear form  $a$  over  $(L^2_{1/2}(\Omega))^2$

$$\begin{aligned} (L^2_{1/2}(\Omega)) &= \{v \in \mathcal{D}'(\Omega) / D^{1/2} v \in L^2(\Omega)\} \\ a(\vec{p}, \vec{q}) &= \int_{\Omega} D(x) \vec{p}(x) \cdot \vec{q}(x) dx, \quad \forall \vec{p}, \vec{q} \in (L^2_{1/2}(\Omega))^2 \end{aligned} \tag{2.11}$$

Let us introduce the space  $M$

$$M = \{v \in \mathcal{D}'(\Omega) / D^{-3/2} v \in L^2(\Omega)\} \tag{2.12}$$

provided with the norm

$$\|v\|_M = |D^{-3/2} v|_{L^2(\Omega)}. \tag{2.13}$$

We define a bilinear form  $b$  over  $X \times M$  by

$$b(\vec{q}, v) = \int_{\Omega} v \cdot \operatorname{div} \vec{q} \, dx \tag{2.14}$$

and we consider the following problem :

Given a function  $f \in L^2_{3/2}(\Omega) = M'$ , find functions  $\vec{p} \in X, u \in M$  such that :

$$\left. \begin{aligned} \forall \vec{q} \in X, a(\vec{p}, \vec{q}) + b(\vec{q}, u) &= 0 \\ \forall v \in M, b(\vec{p}, v) + \int_{\Omega} f v \, dx &= 0. \end{aligned} \right\} \tag{2.15}$$

**THEOREM 2.4 :** *Problem (2.15) has a unique solution  $(\vec{p}, u) \in X \times M$ ; besides,  $u$  is the solution of problem (2.4) and  $\vec{p}$  is given by :*

$$\vec{p} = D^{-1} \overrightarrow{\operatorname{grad}} u. \tag{2.16}$$

*Proof :* From a result of Brezzi [2], the problem (2.15) has a unique solution if the following conditions are satisfied :

i) There is some positive constant  $\alpha > 0$  such that :

$$a(\vec{q}, \vec{q}) \geq \alpha \|\vec{q}\|_X^2; \quad \forall \vec{q} \in V = \{ \vec{q} \in X / b(\vec{q}, v) = 0 \quad \forall v \in M \}. \tag{2.17}$$

ii) There is some positive constant  $\beta > 0$  such that :

$$\sup_{\vec{q} \in X} \frac{b(\vec{q}, v)}{\|\vec{q}\|_X} \geq \beta \|v\|_M, \quad \forall v \in M. \tag{2.18}$$

Clearly,  $V$  is also defined by

$$V = \{ \vec{q} \in X / \operatorname{div} \vec{q} = 0 \}.$$

Therefore, if  $\vec{q} \in V$ , we have

$$\|\vec{q}\|_X = (a(\vec{q}, \vec{q}))^{1/2}$$

and hypothesis i) is satisfied with  $\alpha = 1$ .

Now, if  $v \in M$ , let us consider  $\varphi$  solution of the problem :

$$\begin{cases} \operatorname{div} (D^{-1} \overrightarrow{\operatorname{grad}} \varphi) = D^{-3} v, \text{ over } \Omega, \\ \varphi = 0 \text{ on } \Gamma. \end{cases}$$



From theorem 2.2,  $\varphi \in H_{1/2}^2(\Omega)$  and  $\|\varphi\|_{2,1/2,\Omega} \leq C \|D^{-3/2} v\|_{L^2(\Omega)}$ .  
We set,

$$\vec{q} = D^{-1} \overrightarrow{\text{grad}} \varphi.$$

Then,

$$\|\vec{q}\|_X \leq \|\varphi\|_{2,1/2,\Omega} \leq C \|v\|_M$$

and

$$b(\vec{q}, v) = \|v\|_M^2.$$

Thus, hypothesis ii) is satisfied with  $\beta = 1/C$ .

Besides, if  $u$  is the solution of problem (2.4) and  $\vec{p}$  defined by (2.16) we get :

$$-\text{div } \vec{p} = f$$

and

$$b(\vec{p}, v) = - \int_{\Omega} f v \, dx, \quad \forall v \in M;$$

$$\forall \vec{q} \in X a(\vec{p}, \vec{q}) + b(\vec{q}, u) = \int_{\Omega} (\overrightarrow{\text{grad}} u \cdot \vec{q} + \text{div } \vec{q} \cdot u) \, dx = 0$$

from lemma 2.1.

Therefore,  $(\vec{p}, u)$  is the unique solution of problem (2.15).

### III. A MIXED FINITE ELEMENT APPROXIMATION OF PROBLEM (2.15)

In order to approximate problem (2.15), we first construct a set  $\overline{\Omega}_h = \bigcup_{K \in \mathfrak{T}_h} K$  as a finite union of triangles with vertices in  $\overline{\Omega}$ . For any  $K \in \mathfrak{T}_h$ , we set

$$h(K) = \text{diameter of } K, \quad h(K) \leq h,$$

$$\rho(K) = \text{diameter of the inscribed sphere of } K.$$

We assume that this triangulation is regular, that is :

$$\sup_{K \in \mathfrak{T}_h} \frac{h(K)}{\rho(K)} \leq \sigma \quad (\sigma > 0). \quad (3.1)$$

We also assume that a triangle  $K$  has not three vertices on the boundary  $\Gamma$ .

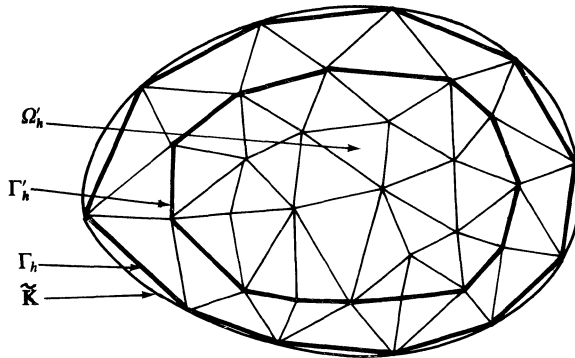
*Notations*

We denote :

- $\Gamma_h$  the boundary of  $\Omega_h$ ; it is clear that  $\Gamma_h$  does not coincide with  $\Gamma$ ,

- $\mathcal{T}'_h$  the union of the triangles of  $\mathcal{T}_h$  which have no vertices on  $\Gamma$
- $\Omega'_h = \bigcup_{K \in \mathcal{T}'_h} K$
- $\Gamma'_h$  is the boundary of  $\Omega'_h$
- $\partial\mathcal{T}_h = \mathcal{T}_h - \mathcal{T}'_h$ .

If  $K \in \partial\mathcal{T}_h$  has two vertices on the boundary  $\Gamma$ , let us denote by  $\tilde{K}$  the surface limited by the two sides of  $K$  which are in  $\Omega$  and the part of  $\Gamma$  between the two vertices of  $K$ .



Now, we define an approximation  $X_h$  of  $X$ .

If  $\hat{K}$  is the reference finite element with vertices  $(\hat{a}_1 = (1, 0), \hat{a}_2 = (0, 1), \hat{a}_3 = (0, 0))$ , we define the space  $\hat{X}$  by :

$$\hat{X} = \{ \hat{q} = (\hat{q}_1, \hat{q}_2) / \hat{q}_1 = \mu_1 + \alpha x_1, \hat{q}_2 = \mu_2 + \alpha x_2 ; \mu_1, \mu_2, \alpha \in P_0 \} \quad (3.2)$$

( $P_0$  is the space of the constants).

Let us denote by  $F$

$$F : \hat{x} \mapsto F(\hat{x}) = B\hat{x} + b, \quad B \in \mathcal{L}(\mathbb{R}^2), \quad b \in \mathbb{R}^2$$

an affine invertible mapping such that  $K = F(\hat{K})$ . To any scalar function  $\hat{\varphi}$  and any vectorial function  $\hat{q}$  defined over  $\hat{K}$ , we associate the functions  $\varphi$  and  $\vec{q}$  defined over  $K$  by :

$$\left. \begin{aligned} \varphi &= \hat{\varphi} \circ F^{-1} \\ \vec{q} \circ F &= J^{-1} B \hat{q}, \quad J = |\det B| = 2 \text{ mes } K. \end{aligned} \right\} \quad (3.3)$$

For any  $K \in \mathcal{T}_h$ , we set :

$$X_K = \{ \vec{q}_h / \hat{q}_h \in \hat{X} \}. \quad (3.4)$$

We have the equalities :

$$\operatorname{div} \vec{q}_h = J^{-1} \operatorname{div} \hat{\vec{q}}_h, \tag{3.5}$$

$$\vec{q}_h \cdot \vec{n}_K = J_n^{-1} \hat{\vec{q}}_h \cdot \hat{\vec{n}}_K, \quad J_n = J |B^{-1} \vec{n}| \tag{3.6}$$

where  $\vec{n}_K$  denotes the outward unit vector normal to the boundary  $\partial K$  of  $K$ .

Besides, for  $K \in \partial \mathcal{T}_h$ ,  $\vec{q}_h$  can be defined over  $\tilde{K}$  by the same expression as over  $K$ .

We note :

$$H(\operatorname{div}, \Omega) = \{ \vec{q} \in (L^2(\Omega))^2 / \operatorname{div} \vec{q} \in L^2(\Omega) \} \subset X$$

and we set

$$X_h = \{ \vec{q}_h \in H(\operatorname{div}; \Omega_h) / \forall K \in \mathcal{T}_h \vec{q}_{h|K} \in X_K \}. \tag{3.7}$$

Then,  $\vec{q}_h \in X_h$  if  $\vec{q}_{h|K} \in X_K, \forall K \in \mathcal{T}_h$  and satisfies :

$$\forall K_1, K_2 \in \mathcal{T}_h, \vec{q}_{h|K_1} \cdot \vec{n}_{K_1} + \vec{q}_{h|K_2} \cdot \vec{n}_{K_2} = 0 \quad \text{over} \quad \partial K_1 \cap \partial K_2 \quad [8]. \tag{3.8}$$

Let  $M_h$  be an approximation of the space  $M$  defined by .

$$M_h = \{ v_h \in L^2(\Omega_h) / \forall K \in \mathcal{T}_h, v_{h|K} \in P_0 \}. \tag{3.9}$$

Then  $M_h \not\subset M$ ; but to any  $v_h \in M_h$ , we associate  $\tilde{v}_h$  defined, over  $\Omega$  by :

$$\forall K \in \mathcal{T}_h, \tilde{v}_{h|K} = D^{3/2}(x) D^{-3/2}(a_K) v_{h|K} \quad (K' = K \text{ if } K \in \mathcal{T}'_h \\ K' = \tilde{K} \text{ if } K \in \partial \mathcal{T}_h) \tag{3.10}$$

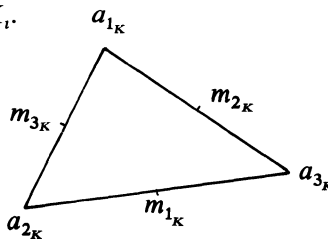
where  $a_K$  is the barycenter of  $K$  if  $a_K$  lies in  $\Omega$ , otherwise  $a_K$  is the vertex of  $K$  inside  $\Omega$ .

*Notations*

We denote by  $a_{i_K}, i = 1, 2, 3$  the vertices of  $K$ ;  $a_K$ , the barycenter of  $K$  if  $a_K$  lies in  $\Omega$ ; otherwise  $a_K$  is the vertex of  $K$  inside  $\Omega$   $\partial K$  the boundary of  $K$

$$\partial K_i = [a_j, a_k] \quad j \neq k \neq i \quad i = 1, 2, 3$$

$m_{i_K}$  the midpoint of  $\partial K_i$ .



To define the discrete problem, we shall use numerical integration.

Denote by  $a_h(\vec{p}_h, \vec{q}_h)$  the bilinear form defined over  $X_h \times X_h$  by

$$a_h(\vec{p}_h, \vec{q}_h) = \sum_{K \in \mathfrak{T}_h} \frac{(\text{mes } K)}{3} \sum_{i=1}^3 D(m_{iK}) \vec{p}_h(m_{iK}) \cdot \vec{q}_h(m_{iK}) \quad (3 \ 11)$$

(if  $m_{iK} \notin \Omega$ , we set  $D(m_{iK}) = 0$ ),  $b_h$  defined over  $X_h \times M_h$  by

$$b_h(\vec{q}_h, v_h) = \sum_{K \in \mathfrak{T}_h} (\text{mes } K) v_h|_K \cdot \text{div } \vec{q}_h|_K \quad (3 \ 12)$$

and  $f_h(v_h)$  is defined over  $M_h$  by

$$f_h(v_h) = \sum_{K \in \mathfrak{T}_h} (\text{mes } K) v_h|_K f(a_K) \quad (3 \ 13)$$

Then, the approximate problem is the following Find a pair  $(\vec{p}_h, u_h) \in X_h \times M_h$  such that

$$\left. \begin{aligned} \forall \vec{q}_h \in X_h, a_h(\vec{p}_h, \vec{q}_h) + b_h(\vec{q}_h, u_h) &= 0 \\ \forall v_h \in M_h, b_h(\vec{p}_h, v_h) + f_h(v_h) &= 0 \end{aligned} \right\} \quad (3 \ 14)$$

**THEOREM 3 1** *The approximate problem (3 14) has a unique solution  $(\vec{p}_h, u_h) \in X_h \times M_h$*

*Proof* It is sufficient to prove the unicity of the homogeneous problem Find a pair  $(\vec{p}_h, u_h) \in X_h \times M_h$  such that

$$\left. \begin{aligned} \forall \vec{q}_h \in X_h, a_h(\vec{p}_h, \vec{q}_h) + b_h(\vec{q}_h, u_h) &= 0 \\ \forall v_h \in M_h, b_h(\vec{p}_h, v_h) &= 0 \end{aligned} \right\}$$

Then, we get

$$a_h(\vec{p}_h, \vec{p}_h) = 0,$$

and  $\forall K \in \mathfrak{T}_h, D(m_{iK}) \vec{p}_h^2(m_{iK}) = 0, \quad i = 1, 2, 3.$

But  $D(m_{iK}) \neq 0$  for at least two points  $m_{iK}$ , so, we get :

$$\vec{p}_h = 0 \quad \text{and,} \quad \forall \vec{q}_h \in X_h, \quad b_h(\vec{q}_h, u_h) = 0.$$

Therefore,  $u_h = 0$  (cf Thomas [8])

IV. ERROR ESTIMATES

a) General error bounds

We provide the spaces  $M_h$  and  $X_h$  with the norms :

$$\| v_h \|_{M_h} = \left( \sum_{K \in \mathcal{T}_h} (\text{mes } K') D^{-3}(a_K) (v_{h|K})^2 \right)^{1/2}, \tag{4.1}$$

then

$$\| v_h \|_{M_h} = \| \tilde{v}_h \|_M ;$$

$$| \vec{q}_h |_h = (a_h(\vec{q}_h, \vec{q}_h))^{1/2} = \left( \sum_{K \in \mathcal{T}_h} \frac{(\text{mes } K)}{3} \sum_{i=1}^3 D(m_{iK}) (\vec{q}_h(m_{iK}))^2 \right)^{1/2} \tag{4.2}$$

$$| \text{div } \vec{q}_h |_h = \left( \sum_{K \in \mathcal{T}_h} (\text{mes } K) D^3(a_K) (\text{div } q_{h|K})^2 \right)^{1/2} \tag{4.3}$$

$$\| \vec{q}_h \|_h = (| \vec{q}_h |_h^2 + | \text{div } \vec{q}_h |_h^2)^{1/2} \tag{4.4}$$

We note :  $| \vec{q} |_X = | D^{1/2} \vec{q} |_{(L^2(\Omega))^2}$ .

LEMMA 4.1 : *The mapping :  $\vec{q}_h \in X_h \rightarrow | \vec{q}_h |_h$  is a norm over  $X_h$  and there are two positive constants  $C_1, C_2$  such that*

$$C_1 | \vec{q} |_X \leq | \vec{q}_h |_h \leq C_2 | \vec{q} |_X . \tag{4.5}$$

*Proof* : We have :

$$| \vec{q}_h |_h^2 = \sum_{K \in \mathcal{T}_h} \frac{\text{mes } K}{3} \sum_{i=1}^3 D(m_{iK}) (\vec{q}_h(m_{iK}))^2 ,$$

$$\vec{q}_{h|K} \in X_K \quad \text{and} \quad \hat{\vec{q}}_{h|K} \in \hat{X} .$$

So we get,

$$\forall K \in \mathcal{T}_h, \quad \sum_{i=1}^3 D(m_{iK}) (\vec{q}_h(m_{iK}))^2 \leq J^{-2} \| B \|^2 \left( \sup_{x \in K} D(x) \right) \sum_{i=1}^3 (\hat{\vec{q}}_h(\hat{m}_i))^2 .$$

But,  $\hat{\vec{q}}_h \in X$  which is a space of finite dimension, so there is some constant  $C'_2$  such that :

$$\sum_{i=1}^3 (\hat{\vec{q}}_h(\hat{m}_i))^2 \leq C'_2 \| \hat{\vec{q}}_h \|_{(L^2(K))^2}^2$$

and then, by using (3.1)

$$\forall K \in \mathfrak{C}_h, \frac{(\text{mes } K)}{3} \sum_{i=1}^3 D(m_{iK}) (\vec{q}_h(m_{iK}))^2 \leq C_2' \left( \sup_{x \in K} D(x) \right) \| \vec{q}_h \|_{(L^2(K))^2}^2 \leq C_2 \| D^{1/2} \vec{q}_h \|_{(L^2(K))^2}^2 .$$

Further, we have

$$\forall K \in \mathfrak{C}_h, D(m_{iK}) \neq 0 \text{ for at least two points } m_i ,$$

then  $\forall K \in \mathfrak{C}_h, \sum_{i=1}^3 D(m_{iK}) \vec{q}_h^2(m_{iK})$  is a norm over  $X_K$  and we have :

$$\frac{\text{mes } K}{3} \sum_{i=1}^3 D(m_{iK}) (\vec{q}_h(m_{iK}))^2 \geq \frac{1}{6} \| B \|^2 J^{-1} \left( \inf_{D(m_{iK}) \neq 0} D(m_{iK}) \right) \times \left( \sum_{\substack{i=1 \\ D(m_{iK}) \neq 0}}^3 \hat{q}_h^2(\hat{m}_{iK}) \right)^2 .$$

Hence, we get

$$\frac{\text{mes } K}{3} \sum_{i=1}^3 D(m_{iK}) (\vec{q}_h(m_{iK}))^2 \geq C_1 \| D^{1/2} \vec{q}_h \|_{(L^2(K))^2}^2, \quad \forall K \in \mathfrak{C}_h$$

and we deduce (4.5).

Let us denote by :

$$V_h = \{ \vec{q}_h \in X_h / \text{div } \vec{q}_h = 0 \} ,$$

$$\bar{V}_h = \{ \vec{q}_h \in X_h / \forall K \in \mathfrak{C}_h \text{ div } \vec{q}_h|_K + f(a_K) = 0 \} .$$

We may write for all  $\vec{q}_h \in \bar{V}_h$

$$a_h(\vec{p}_h - \vec{q}_h, \vec{p}_h - \vec{q}_h) = a(\vec{p} - \vec{q}_h, \vec{p}_h - \vec{q}_h) + a(\vec{q}_h, \vec{p}_h - \vec{q}_h) - a_h(\vec{q}_h, \vec{p}_h - \vec{q}_h) .$$

Thus, we get by using lemme 4.1

$$| \vec{p} - \vec{p}_h |_X \leq C \inf_{\vec{q}_h \in \bar{V}_h} \left[ | \vec{p} - \vec{q}_h |_X + \sup_{\vec{z}_h \in V_h} \frac{| a(\vec{q}_h, \vec{z}_h) - a_h(\vec{q}_h, \vec{z}_h) |}{| \vec{z}_h |_X} \right] \} \quad (4.6)$$

In order to evaluate these terms, we need the following theorem which shall be proved in section 5.

**THEOREM 4 1** *If  $\mathcal{T}_h$  is a regular triangulation of  $\Omega$ , there is some positive constant  $\beta$ , independent of  $h$  such that*

$$\forall v_h \in M_h \sup_{\vec{q}_h \in X_h} \frac{b_h(\vec{q}_h, v_h)}{\|\vec{q}_h\|_h} \geq \beta \|v_h\|_{M_h} \tag{4 7}$$

Then, we have the following theorem (Thomas [8])

**THEOREM 4 2** *There is some positive constant  $C$  depending only on  $\beta$  and  $C_1$  such that*

$$\left. \begin{aligned} |\vec{p} - \vec{p}_h|_X &\leq \inf_{\vec{q}_h \in X_h} \left[ \|\vec{p} - \vec{q}_h\|_X + \sup_{\vec{z}_h \in V_h} \frac{|a(\vec{q}_h, \vec{z}_h) - a_h(\vec{q}_h, \vec{z}_h)|}{\|\vec{z}_h\|_X} + \right. \\ &+ C \left( \sup_{v_h \in M_h} \frac{|b(\vec{p} - \vec{q}_h, \tilde{v}_h)|}{\|v_h\|_{M_h}} + \sup_{v_h \in M_h} \frac{|b(\vec{q}_h, \tilde{v}_h) - b_h(\vec{q}_h, v_h)|}{\|v_h\|_{M_h}} \right. \\ &\left. \left. + \sup_{v_h \in M_h} \frac{|f_h(v_h) - (f, \tilde{v}_h)|}{\|v_h\|_{M_h}} \right) \right] \end{aligned} \right\} \tag{4 8}$$

and,

$$\left. \begin{aligned} \|u - \tilde{u}_h\|_M &\leq C \left[ |\vec{p} - \vec{p}_h|_X \right. \\ &+ \inf_{\vec{q}_h \in X_h} \left( |\vec{p} - \vec{q}_h|_X + \sup_{\vec{z}_h \in X_h} \frac{|a(\vec{q}_h, \vec{z}_h) - a_h(\vec{q}_h, \vec{z}_h)|}{\|\vec{z}_h\|_h} \right) \\ &\left. + \inf_{v_h \in M_h} \left( \|u - \tilde{v}_h\|_M + \sup_{\vec{z}_h \in X_h} \frac{|b(\vec{z}_h, \tilde{v}_h) - b_h(\vec{z}_h, v_h)|}{\|\vec{z}_h\|_h} \right) \right] \end{aligned} \right\} \tag{4 9}$$

**b) Consistency error bounds**

We estimate the consistency errors

$$\begin{aligned} &\sup_{\vec{z}_h \in X_h} \frac{a(\vec{q}_h, \vec{z}_h) - a_h(\vec{q}_h, \vec{z}_h)}{\|\vec{z}_h\|_h}, \quad \sup_{z_h \in X_h} \frac{b(\vec{z}_h, \tilde{v}_h) - b_h(\vec{z}_h, v_h)}{\|\vec{z}_h\|_h}, \\ &\sup_{v_h \in M_h} \frac{b(\vec{q}_h, \tilde{v}_h) - b_h(\vec{q}_h, v_h)}{\|v_h\|_{M_h}}, \quad \sup_{v_h \in M_h} \frac{|f_h(v_h) - (f, \tilde{v}_h)|}{\|v_h\|_{M_h}} \end{aligned}$$

These are the object of the next three lemmas

LEMMA 4.2 : *There is a positive constant C independent of h such that*

$$\left. \begin{aligned} &\forall \vec{q}_h, \vec{z}_h \in X_h \\ &| a(\vec{q}_h, \vec{z}_h) - a_h(\vec{q}_h, \vec{z}_h) | \leq Ch | \vec{z}_h |_X ( | \vec{q}_h |_X + | D^{1/2} \operatorname{div} \vec{q}_h |_{L^2(\Omega)} ). \end{aligned} \right\} \quad (4.10)$$

*Proof :* We set :

$$E_K(\varphi) = \int_K \varphi(x) \, dx - \sum_{K \in \mathcal{T}_h} (\operatorname{mes} K) \sum_{i=1}^3 \varphi(m_{iK}) .$$

Let  $\tilde{\Omega}$  be an open set such that  $\Omega \subset \tilde{\Omega}$  and  $\Omega_h \subset \tilde{\Omega}$  for any  $h$ .  $D$  is a regular function defined over  $\Omega$  : then there is an extension  $\tilde{D}$  of  $D$  on  $\tilde{\Omega}$  such that :

$$\tilde{D} \in W^{2,\infty}(\tilde{\Omega}), \quad \| \tilde{D} \|_{2,\infty,\tilde{\Omega}} \leq C \| D \|_{2,\infty,\Omega} .$$

We have :

$$\begin{aligned} a(\vec{q}_h, \vec{z}_h) - a_h(\vec{q}_h, \vec{z}_h) &= \sum_{K \in \mathcal{T}_h} E_K(\tilde{D}\vec{q}_h, \vec{z}_h) + \\ &+ \sum_{K \in \partial \mathcal{T}_h} \left[ \int_K D(x) \vec{q}_h(x) \vec{z}_h(x) \, dx - \int_K \tilde{D}(x) \vec{q}_h(x) \vec{z}_h(x) \, dx \right] . \end{aligned}$$

But, we have the estimate (Ciarlet [4]) :

$$| E_K(\tilde{D}\vec{q}_h, \vec{z}_h) | \leq Ch^2 \| D \|_{2,\infty,K} \| \vec{q}_h \|_{1,K} \| \vec{z}_h \|_{L^2(K)} .$$

Besides,  $\vec{q}_h \in X_h$ , so, we get,

$$| \vec{q}_h |_{1,K} \leq C | \operatorname{div} \vec{q}_h |_{L^2(K)} .$$

Thus, we obtain :

$$| E_K(\tilde{D}\vec{q}_h, \vec{z}_h) | \leq Ch ( | D^{1/2} \vec{q}_h |_{(L^2(K))^2}^2 + | D^{1/2} \operatorname{div} \vec{q}_h |_{L^2(K)}^2 )^{1/2} | D^{1/2} \vec{z}_h |_{L^2(K)}$$

and

$$\begin{aligned} \left| \int_{\tilde{K}} D(x) \vec{q}_h(x) \vec{z}_h(x) \, dx - \int_K \tilde{D}(x) \vec{q}_h(x) \vec{z}_h(x) \, dx \right| &\leq \\ &\leq Ch | D^{1/2} \vec{q}_h |_{L^2(\tilde{K})} | D^{1/2} \vec{z}_h |_{L^2(\tilde{K})} . \end{aligned}$$

Hence, we can deduce (4.10).



LEMMA 4 3 · There is some positive constant  $C$  independent of  $h$  such that

$$\forall \vec{q}_h \in X_h, \quad \forall v_h \in M_h, \\ |b(\vec{q}_h, \tilde{v}_h) - b_h(\vec{q}_h, v_h)| \leq Ch \|v_h\|_{M_h} |D^{1/2} \operatorname{div} \vec{q}_h|_{L^2(\Omega)} \quad (4 11)$$

and

$$|b(\vec{q}_h, \tilde{v}_h) - b_h(\vec{q}_h, v_h)| \leq Ch |D^{3/2} \operatorname{div} \vec{q}_h|_{L^2(\Omega)} \left( \sum_{K \in \mathcal{T}_h} (\operatorname{mes} K) D^{-5}(a_K) v_{h|K} \right)^{1/2} \quad (4 12)$$

*Proof* We set

$$E_K(\varphi) = \int_K \varphi(x) dx - (\operatorname{mes} K) \varphi(a)$$

Then, we have

$$b(\vec{q}_h, \tilde{v}_h) - b_h(\vec{q}_h, v_h) = \sum_{K \in \mathcal{T}_h} D^{-3/2}(a_K) E_K(\tilde{D}^{3/2}) \operatorname{div} \vec{q}_{h|K} v_{h|K} + \\ + \sum_{K \in \mathcal{T}_h} \left( \int_{\tilde{K}} D^{3/2}(x) dx - \int_K \tilde{D}^{3/2}(x) dx \right) \operatorname{div} \vec{q}_{h|K} v_{h|K} D^{-3/2}(a_K)$$

Besides,

$$|E_K(D^{3/2})| \leq Ch(\operatorname{mes} K) \|D^{1/2}\|_{0 \infty K} \\ \left| \int_K D^{3/2}(x) dx - \int_K \tilde{D}^{3/2}(x) dx \right| \leq Ch^3 \|D^{3/2}\|_{0 \infty K}$$

Hence, we get

$$|b(\vec{q}_h, \tilde{v}_h) - b_h(\vec{q}_h, v_h)| \leq Ch \|v_h\|_{M_h} \left( \sum_{K \in \mathcal{T}_h} (\operatorname{mes} K) D(a_K) \operatorname{div}^2 \vec{q}_{h|K} \right)^{1/2}$$

or

$$|b(\vec{q}_h, \tilde{v}_h) - b_h(\vec{q}_h, v_h)| \leq Ch |D^{3/2} \operatorname{div} \vec{q}_h|_{L^2(\Omega)} \left( \sum_{K \in \mathcal{T}_h} (\operatorname{mes} K) D^{-5}(a_K) v_{h|K}^2 \right)^{1/2}$$

LEMMA 4 4 There is some positive constant  $C$  independent of  $h$  such that

$$\forall v_h \in M_h, \quad |(f, \tilde{v}_h) - f_h(v_h)| \leq Ch \|v_h\|_{M_h} [|D^{1/2} \vec{f}|_{L^2(\Omega)} + |D^{3/2} \overrightarrow{\operatorname{grad}} \vec{f}|_{(L^2(\Omega))^2}] \quad (4 13)$$

*Proof* : We set, as in lemma 4.3,

$$E_K(\varphi) = \int_K \varphi(x) dx - (\text{mes } K) \varphi(a_K).$$

Then, we get,

$$\begin{aligned} (f, \tilde{v}_h) - f_h(v_h) &= \sum_{K \in \mathcal{T}_h} D^{-3/2}(a_K) E_K(D^{3/2} f) v_{h|K} + \\ &+ \sum_{K \in \partial \mathcal{T}_h} \left( \int_{\tilde{K}} D^{3/2}(x) f(x) dx - \int_K \tilde{D}^{3/2}(x) f(x) dx \right) D^{-3/2}(a_K) v_{h|K}. \end{aligned}$$

Besides,

$$|E_K(D^{3/2} f)| \leq Ch^2 | \overrightarrow{\text{grad}}(D^{3/2} f) |_{(L^2(K))^2}$$

and

$$\left| \int_{\tilde{K}} D^{3/2}(x) f(x) dx - \int_K D^{3/2}(x) f(x) dx \right| \leq Ch^{7/2} |D^{1/2} f|_{L^2(K)}.$$

Hence, we deduce 4.13.

**c) Interpolation error bounds**

It remains to estimate :

$$\inf_{\vec{q}_h \in X_h} \| \vec{p} - \vec{q}_h \|_X \quad \text{and} \quad \inf_{v_h \in M_h} \| u - \tilde{v}_h \|_M.$$

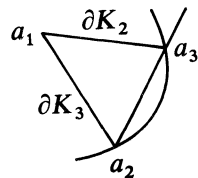
We define an interpolate  $\overrightarrow{\rho_K \vec{q}}$  of  $\vec{q}$  with  $\overrightarrow{\rho_K \vec{q}} \in X_h$  as follows :

1. If  $K \in \mathcal{C}'_h$ ,  $\vec{q} \in H(\text{div}; K)$ , and if we also assume that  $\vec{q} \in (H^1(K))^2$ , we can define the interpolate  $\overrightarrow{\rho_K \vec{q}}$  of  $\vec{q}$  by [8] :

$$\left. \begin{aligned} &\overrightarrow{\rho_K \vec{q}} \in X_K \\ &\forall i = 1, 2, 3 \int_{\partial K_i} \overrightarrow{\rho_K \vec{q}} \cdot \vec{n}_i d\gamma = \int_{\partial K_i} \vec{q} \cdot \vec{n}_i d\gamma. \end{aligned} \right\} \quad (4.14)$$

2. If  $K \in \partial \mathcal{C}_h$  and has two vertices  $a_2^K$  and  $a_3^K$  on  $\Gamma$ , if  $\vec{q} \in X_K$  such that  $D^3 \vec{q} \cdot \vec{n} \in L^1(\partial K_i)$  ( $i = 2, 3$ ), then we define the interpolate  $\overrightarrow{\rho_K \vec{q}}$  by :

$$\left. \begin{aligned} &\overrightarrow{\rho_K \vec{q}} \in X_K \\ &\forall i = 2, 3 \int_{\partial K_i} D^3 \overrightarrow{\rho_K \vec{q}} \cdot \vec{n}_i d\gamma = \int_{\partial K_i} D^3 \vec{q} \cdot \vec{n}_i d\gamma \\ &\int_{\tilde{K}} D^3 \text{div} \overrightarrow{\rho_K \vec{q}} dx = \int_{\tilde{K}} D^3 \text{div} \vec{q} dx. \end{aligned} \right\} \quad (4.15)$$



3 Therefore, because  $\overline{\rho_h \vec{q}}$  must be in  $H(\text{div}, \Omega)$ , the interpolate  $\overline{\rho_K \vec{q}}$  of  $q$  on a triangle  $K$  which has one vertex  $a_1$  on  $\Gamma$  is defined by

$$\left. \begin{aligned} \overline{\rho_K \vec{q}} \in X \\ \int_{\partial K_1} \overline{\rho_K \vec{q}} \cdot \vec{n}_1 \, d\gamma = \int_{\partial K_1} \vec{q} \cdot \vec{n}_1 \, d\gamma \\ \forall i = 2, 3 \int_{\partial K_i} D^3 \overline{\rho_K \vec{q}} \cdot \vec{n}_i \, d\gamma = \int_{\partial K_i} D^3 \vec{q} \cdot \vec{n}_i \, d\gamma \end{aligned} \right\} \begin{array}{l} \text{Diagram of triangle } K \text{ with vertices } a_1, a_2, a_3 \text{ and edges } \partial K_1, \partial K_2, \partial K_3. \end{array} \quad (4.16)$$

Thus  $\overline{\rho_h \vec{q}}$  is in  $X_h$  and is defined for any  $\vec{q} \in X$  such that  $(D^{3/2} \vec{q}) \in (H^1(\Omega))^2$ . We shall note

$$\overline{\text{grad}} \vec{q} = \left( \frac{\partial q_1}{\partial x_1}, \frac{\partial q_1}{\partial x_2}, \frac{\partial q_2}{\partial x_1}, \frac{\partial q_2}{\partial x_2} \right)$$

**THEOREM 4.3** *Let the triangulation  $\mathcal{T}_h$  be regular. Then, there is some positive constant  $C$  independent of  $h$  such that*

$$\begin{aligned} \forall \vec{q} \in X, D^{1/2} \overline{\text{grad}} \vec{q} &\in (L^2(\Omega))^4 \\ |D^{1/2}(\vec{q} - \overline{\rho_h \vec{q}})|_{(L^2(\Omega))^2} &\leq Ch(|D^{1/2} \vec{q}|_{(L^2(\Omega))^2} + |D^{1/2} \overline{\text{grad}} \vec{q}|_{(L^2(\Omega))^4}) \quad (4.17) \\ \forall \vec{q} \in X, D^{1/2} \overline{\text{grad}} \vec{q} &\in (L^2(\Omega))^4, \quad D^{3/2} \overline{\text{grad}}(\text{div} \vec{q}) \in (L^2(\Omega))^2 \\ |D^{3/2} \text{div}(\vec{q} - \overline{\rho_h \vec{q}})|_{L^2(\Omega)} &\leq Ch(|D^{1/2} \vec{q}|_{L^2(\Omega)^2} + |D^{1/2} \overline{\text{grad}} \vec{q}|_{(L^2(\Omega))^4} + \\ &\quad + |D^{3/2} \overline{\text{grad}}(\text{div} \vec{q})|_{(L^2(\Omega))^2}) \quad (4.18) \end{aligned}$$

*Proof* If  $K \in \mathcal{T}_h$ , we have (cf Thomas [8])

$$\begin{aligned} |D^{1/2}(\vec{q} - \overline{\rho_K \vec{q}})|_{(L^2(K))^2} &\leq Ch |D^{1/2} \overline{\text{grad}} \vec{q}|_{(L^2(K))^4} \\ |D^{3/2} \text{div}(\vec{q} - \overline{\rho_K \vec{q}})|_{L^2(K)} &\leq Ch |D^{3/2} \overline{\text{grad}}(\text{div} \vec{q})|_{(L^2(K))^2} \end{aligned}$$

Now, if  $K \in \hat{\partial} \mathcal{T}_h$  and has two vertices on  $\Gamma$ ,  $\text{div} \overline{\rho_h \vec{q}}$  is the orthogonal projection of  $\text{div} \vec{q}$  on the space of constants with respect to the scalar product  $(D^{3/2}, D^{3/2})$  and then we get

$$|D^{3/2} \text{div}(\vec{q} - \overline{\rho_K \vec{q}})|_{L^2(K)} \leq Ch |D^{3/2} \overline{\text{grad}}(\text{div} \vec{q})|_{(L^2(\tilde{K}))^2}$$

Besides, there is a positive constant  $C$  independent of  $K$  such that

$$|D^{1/2} \overline{\rho_K \vec{q}}|_{(L^2(\tilde{K}))^2} \leq C(|D^{1/2} \vec{q}|_{L^2(\tilde{K})} + |D^{3/2} \overline{\text{grad}} \vec{q}|_{(L^2(\tilde{K}))^4})$$

Then, we get

$$|D^{1/2}(\vec{q} - \overline{\rho_K \vec{q}})|_{L^2(\tilde{K})} \leq C \inf_{\vec{a} \in (\rho_0)^2} |D^{1/2}(\vec{q} - \vec{a})|_{(L^2(\tilde{K}))^2} \leq Ch |D^{1/2} \overrightarrow{\text{grad}} \vec{q}|_{(L^2(\tilde{K}))^4}.$$

It remains to estimate  $|D^{1/2}(\vec{q} - \overline{\rho_K \vec{q}})|_{(L^2(K))^2}$  and  $|D^{3/2} \text{div}(\vec{q} - \overline{\rho_K \vec{q}})|_{L^2(K)}$  when  $K$  has one vertex  $a_1$  on  $\Gamma$ .

Let us denote by  $\overline{\tilde{\rho}_K \vec{q}}$  the function of  $X_K$  defined as follows

$$\left. \begin{aligned} \overline{\tilde{\rho}_K \vec{q}} \cdot \vec{n}_i &= \overline{\rho_K \vec{q}} \cdot \vec{n}_i, \quad i = 2, 3; \\ \int_K D^3 \text{div} \overline{\tilde{\rho}_K \vec{q}} \, dx &= \int_K D^3 \text{div} \vec{q} \, dx. \end{aligned} \right\} \quad (4.19)$$

Hence, we get

$$\begin{aligned} |D^{1/2}(\vec{q} - \overline{\tilde{\rho}_K \vec{q}})|_{(L^2(K))^2} &\leq Ch |D^{1/2} \overrightarrow{\text{grad}} \vec{q}|_{(L^2(K))^2}, \\ |D^{3/2} \text{div}(\vec{q} - \overline{\tilde{\rho}_K \vec{q}})|_{L^2(K)} &\leq Ch |D^{3/2} \overrightarrow{\text{grad}}(\text{div} \vec{q})|_{L^2(K)}. \end{aligned}$$

Further, we have :

$$\begin{aligned} |D^{1/2}(\overline{\rho_K \vec{q}} - \overline{\tilde{\rho}_K \vec{q}})|_{(L^2(K))^2} &\leq Ch^{3/2} |\overline{\rho_K \vec{q}} \cdot \vec{n}_1 - \overline{\tilde{\rho}_K \vec{q}} \cdot \vec{n}_1|; \\ |D^{3/2} \text{div}(\overline{\rho_K \vec{q}} - \overline{\tilde{\rho}_K \vec{q}})|_{L^2(K)} &\leq Ch^{3/2} |\overline{\rho_K \vec{q}} \cdot \vec{n}_1 - \overline{\tilde{\rho}_K \vec{q}} \cdot \vec{n}_1|. \end{aligned}$$

We have the equality :

$$\begin{aligned} \int_{\partial K_1} D^3(\overline{\tilde{\rho}_K \vec{q}} \cdot \vec{n}_1 - \overline{\rho_K \vec{q}} \cdot \vec{n}_1) \, d\gamma &= \int_K (D^3 \text{div} \overline{\tilde{\rho}_K \vec{q}} + \overrightarrow{\text{grad}} D^3 \cdot \overline{\tilde{\rho}_K \vec{q}}) \, dx - \\ &\quad - \int_{\partial K_2 \cup \partial K_3} D^3 \overline{\tilde{\rho}_K \vec{q}} \cdot \vec{n} \, d\gamma - \int_{\partial K_1} D^3 \overline{\rho_K \vec{q}} \cdot \vec{n}_1 \, d\gamma \end{aligned}$$

and from (4.16) and (4.19), we get

$$\begin{aligned} \int_{\partial K_1} D^3(\overline{\tilde{\rho}_K \vec{q}} \cdot \vec{n}_1 - \overline{\rho_K \vec{q}} \cdot \vec{n}_1) \, d\gamma &= \int_K \overrightarrow{\text{grad}} D^3(\overline{\tilde{\rho}_K \vec{q}} - \vec{q}) \, dx + \\ &\quad + \int_{\partial K_1} D^3(\vec{q} - \overline{\rho_K \vec{q}}) \cdot \vec{n}_1 \, d\gamma. \end{aligned}$$

Hence, we deduce

$$| \overrightarrow{\rho_K \bar{q}} \cdot \vec{n}_1 - \overrightarrow{\rho_K \bar{q}} \cdot \vec{n}_1 | \leq C(h^{-3/2} | \vec{q} - \overrightarrow{\rho_K \bar{q}} |_X + h^{-1/2} | \vec{q} \cdot \vec{n}_1 - \overrightarrow{\rho_K \bar{q}} \cdot \vec{n}_1 |_{L^2(\partial K_1)}).$$

Thus, we get

$$\begin{aligned} | \vec{q} - \overrightarrow{\rho_K \bar{q}} |_X &\leq C(h | D^{1/2} \overrightarrow{\text{grad}} \vec{q} |_{(L^2(K))} + h | \vec{q} \cdot \vec{n}_1 - \overrightarrow{\rho_K \bar{q}} \cdot \vec{n}_1 |_{L^2(\partial K_1)}) \\ | D^{3/2} \text{div} (\vec{q} - \overrightarrow{\rho_K \bar{q}}) |_{L^2(K)} &\leq Ch[ | D^{1/2} \overrightarrow{\text{grad}} \vec{q} |_{(L^2(K))} + \\ &+ | D^{3/2} \overrightarrow{\text{grad}} (\text{div} \vec{q}) |_{(L^2(K))} + | \vec{q} \cdot \vec{n}_1 - \overrightarrow{\rho_K \bar{q}} \cdot \vec{n}_1 |_{L^2(\partial K_1)} ] \end{aligned}$$

and, finally,

$$\begin{aligned} | D^{1/2} (\vec{q} - \overrightarrow{\rho_h \bar{q}}) |_{(L^2(\Omega))^2} &\leq Ch( | D^{1/2} \overrightarrow{\text{grad}} \vec{q} |_{(L^2(\Omega))^4} + | \vec{q} \cdot \vec{n} - \overrightarrow{\rho_h \bar{q}} \cdot \vec{n} |_{L^2(\Gamma_h)}). \\ | D^{3/2} \text{div} (\vec{q} - \overrightarrow{\rho_h \bar{q}}) |_{L^2(\Omega)} &\leq Ch( | D^{1/2} \overrightarrow{\text{grad}} \vec{q} |_{(L^2(\Omega))^4} + \\ &+ | D^{3/2} \overrightarrow{\text{grad}} (\text{div} \vec{q}) |_{(L^2(\Omega))^2} + | \vec{q} \cdot \vec{n} - \overrightarrow{\rho_h \bar{q}} \cdot \vec{n} |_{L^2(\Gamma_h)}). \end{aligned}$$

Further, we have [5]

$$| \vec{q} \cdot \vec{n} - \overrightarrow{\rho_h \bar{q}} \cdot \vec{n} |_{L^2(\Gamma_h)} \leq Ch^{1/2} \| \vec{q} \|_{H^{1/2}(\Gamma_h)} \leq Ch^{1/2} \| \vec{q} \|_{H^1(\Omega_h)}.$$

Then, we deduce (4.17) and (4.18).

Now we consider the  $M_h$ -interpolation operator  $\pi_h$ .

Given a function  $v \in M$ , the  $M_h$ -interpolant  $\pi_h v$  is the unique function which satisfies :

$$\left. \begin{aligned} \pi_h v &\in M_h \\ \forall K \in \mathcal{T}'_h, \pi_h v|_K &= \pi_K v = \frac{1}{\text{mes } K} \int_K v \, dx \\ \forall K \in \partial \mathcal{T}_h, \pi_h v &= 0. \end{aligned} \right\} \quad (4.20)$$

**THEOREM 4.4 :** *Let the triangulation  $\mathcal{T}_h$  be regular. Then, there is some positive constant  $C$  independent of  $h$  such that :*

$$\forall v \in H^{-1/2}(\Omega), \| v - \tilde{\pi}_h v \|_M \leq C \{ h[ | D^{-5/2} v |_{L^2(\Omega_h)} + | D^{-3/2} \overrightarrow{\text{grad}} v |_{L^2(\Omega_h)} ] + | D^{-1/2} \overrightarrow{\text{grad}} v |_{L^2(\Omega - \Omega_h)} \}. \quad (4.21)$$

*Proof :* If  $K \in \mathcal{T}'_h$ , we get

$$| D^{-3/2} (v - \tilde{\pi}_K v) |_{L^2(K)} = | D^{-3/2} v - D^{-3/2} (a) \pi_K v |_{L^2(K)}$$

and then,

$$| D^{-3/2} (v - \tilde{\pi}_K v) |_{L^2(K)} \leq Ch( | D^{-3/2} \overrightarrow{\text{grad}} v |_{L^2(K)} + | D^{-5/2} v |_{L^2(K)}).$$

Now, we assume that  $K \in \partial\mathcal{G}_h$

Then, we have the estimate (cf Bolley-Camus [1])

$$|D^{-3/2} v|_{L^2(\tilde{K})} \leq C |D^{-1/2} \overrightarrow{\text{grad}} v|_{L^2(\tilde{K})}$$

(where  $C$  is a constant independent of  $\tilde{K}$ )

Then we deduce (4 21)

**d) Error estimates**

**THEOREM 4 5** Assume that the solution  $(\vec{p}, u)$  of (2 15) satisfies the smoothness properties

$$\left. \begin{aligned} D^{1/2} \overrightarrow{\text{grad}} p \in (L^2(\Omega))^4, \quad D^{-3/2} (\text{Log } R/D)^{-1} \overrightarrow{\text{grad}} u \in (L^2(\Omega))^2 \\ D^{-5/2} (\text{Log } R/D)^{-1} u \in L^2(\Omega) \end{aligned} \right\} \quad (4 22)$$

with  $R > \max \left( \max_{x \in \bar{\Omega}} D(x), e^2 \right)$

Then, we have the estimates

$$|\vec{p} - \vec{p}_h|_X \leq Ch(|\vec{p}|_X + |D^{1/2} \overrightarrow{\text{grad}} \vec{p}|_{(L^2(\Omega))^4} + |D^{1/2} f|_{L^2(\Omega)} + |D^{3/2} \overrightarrow{\text{grad}} f|_{(L^2(\Omega))^2}) \quad (4 23)$$

$$\|u - \tilde{u}_h\|_M \leq C \{ h(|\vec{p}|_X + |D^{1/2} \overrightarrow{\text{grad}} \vec{p}|_{(L^2(\Omega))^4} + h \text{Log } h(|D^{-3/2} (\text{Log } R/D)^{-1} \overrightarrow{\text{grad}} u|_{(L^2(\Omega))^2} + |D^{-5/2} (\text{Log } R/D)^{-1} u|_{L^2(\Omega)}) \} \quad (4 24)$$

*Proof* In the estimate (4 8), we choose  $\vec{q}_h = \rho_h \vec{p}$ , then by using lemmas 4 2, 4 3, 4 4, we get

$$|\vec{p} - \vec{p}_h|_X \leq \|\vec{p} - \overline{\rho_h \vec{p}}\|_X + C \{ h(|\overline{\rho_h \vec{p}}|_X + |D^{1/2} \text{div } \overline{\rho_h \vec{p}}|_{L^2(\Omega)}) + |D^{3/2} \text{div } (\vec{p} - \overline{\rho_h \vec{p}})|_{L^2(\Omega)} + h(|D^{1/2} f|_{L^2(\Omega)} + |D^{3/2} \overrightarrow{\text{grad}} f|_{(L^2(\Omega))^2}) \}$$

Then, from theorem (4 3), we deduce (4 23)

In the estimate (4 9), we choose  $v_h = \pi_h u$ , then, we get

$$\|u - \tilde{u}_h\|_M \leq C \left\{ |p - \vec{p}_h|_X + |\vec{p} - \overline{\rho_h \vec{p}}|_X + h(|\overline{\rho_h \vec{p}}|_X + |D^{1/2} \text{div } \overline{\rho_h \vec{p}}|_{L^2(\Omega)}) + \|u - \tilde{\pi}_h u\|_M + h \left( \sum_{K \in \mathcal{T}_h} (\text{mes } K) D^{-5}(a_K) (\pi_K u)^2 \right)^{1/2} \right\}$$

and by using theorem (4 4), we obtain (4 24)

*Remark* : If we assume that  $D^{1/2} \vec{p} \in (L^2(\Omega))^2$ ,  $D^{1/2} \overrightarrow{\text{grad}} \vec{p} \in (L^2(\Omega))^4$ , then, we have the estimate (Bolley-Camus [1])

$$| D^{-1/2}(\text{Log } R/D)^{-1} \vec{p} |_{(L^2(\Omega))^2} \leq C | D^{1/2} \overrightarrow{\text{grad}} \vec{p} |_{(L^2(\Omega))^4},$$

that is

$$| D^{-3/2}(\text{Log } R/D)^{-1} \overrightarrow{\text{grad}} u |_{(L^2(\Omega))^2} \leq C | D^{1/2} \overrightarrow{\text{grad}} \vec{p} |_{(L^2(\Omega))^4}.$$

Similarly, we have

$$| D^{-5/2}(\text{Log } R/D)^{-1} u |_{L^2(\Omega)} \leq C | D^{-3/2}(\text{Log } R/D)^{-1} \overrightarrow{\text{grad}} u |_{(L^2(\Omega))^2}.$$

**V. PROOF OF THEOREM 4.1**

Before proving theorem 4.1, we need several results.

*Notations.* We define the spaces  $S_K$ ,  $\mathcal{M}_h$ ,  $\mathcal{H}_h$  by :

$$S_K = \{ \mu \in L^2(\partial K) / \mu |_{\partial K_i} \in P_0, \quad i = 1, 2, 3 \}$$

$$\mathcal{M}_h = \left\{ \mu_h \in \prod_{K \in \mathcal{T}_h} L^2(\partial K) / \mu_h |_{\partial K} \in S_K; \quad \forall K_1, K_2 \in \mathcal{T}_h, \mu_h |_{K_1} + \mu_h |_{K_2} = 0 \right.$$

$$\left. \text{over } K_1 \cap K_2 \right\}$$

$$\mathcal{H}_h = \{ \varphi_h \in L^2(\Omega_h) / \forall K \in \mathcal{T}_h, \varphi_h |_K \in P_1 \}.$$

If  $\Psi$  is a function of  $\prod_{K \in \mathcal{T}_h} L^2(\partial K)$ , let us denote by  $s_h \Psi$  the function such that  $s_h \Psi \in \prod_{K \in \mathcal{T}_h} L^2(\partial K)$  and  $s_h \Psi |_{\partial K_i}$  is the projection of  $\Psi$  onto  $P_0$ .

LEMMA 5.1 : *There exists a constant  $C > 0$  independent of  $h$  such that*

$$\left. \begin{aligned} \forall \varphi_h \in \mathcal{H}_h, \quad \forall \Psi \in H^1(\Omega'_h), \\ \sum_{K \in \mathcal{T}_h} D(a_K) \int_{\partial K} \varphi_h (\Psi - s_h \Psi) \, d\gamma \leq Ch \left( \sum_{K \in \mathcal{T}_h} D^{-1}(a_K) | \varphi_h |_{L^2(K)}^2 \right)^{1/2} \\ \left( | D^{3/2} \Psi |_{L^2(\Omega_h)}^2 + | D^{3/2} \overrightarrow{\text{grad}} \Psi |_{L^2(\Omega_h)}^2 \right)^{1/2}. \end{aligned} \right\} \quad (5.1)$$

*Proof* : Let  $\varphi_h \in \mathcal{H}_h$  and  $\Psi \in H^1(\Omega'_h)$ ; then we have the equality

$$\int_{\partial K} \varphi_h(\Psi - s_h \Psi) d\gamma = \int_{\partial K} (\varphi_h - s_h \varphi_h)(\Psi - s_h \Psi) d\gamma$$

and then

$$\begin{aligned} & \left| \sum_{K \in \mathcal{T}_h} D(a_K) \int_{\partial K} \varphi_h(\Psi - s_h \Psi) d\gamma \right| \\ & \leq C \sum_{K \in \mathcal{T}_h} D(a_K) |\varphi_h - s_h \varphi_h|_{L^2(\partial K)} |\Psi - s_h \Psi|_{L^2(\partial K)} \\ & \leq C \left( \sum_{K \in \mathcal{T}_h} D^{-1}(a_K) |\varphi_h - s_h \varphi_h|_{L^2(\partial K)}^2 \right)^{1/2} \left( \sum_{K \in \mathcal{T}_h} D^3(a_K) |\Psi - s_h \Psi|_{L^2(\partial K)}^2 \right)^{1/2}. \end{aligned}$$

Besides, we have the estimate [5]

$$|\Psi - s_h \Psi|_{L^2(\partial K)} \leq Ch^{1/2} \|\Psi\|_{H^{1/2}(\partial K)} \leq Ch^{1/2} \|\Psi\|_{H^1(K)}$$

hence, we get

$$\begin{aligned} & \left( \sum_{K \in \mathcal{T}_h} D^3(a_K) |\Psi - s_h \Psi|_{L^2(\partial K)}^2 \right)^{1/2} \leq \\ & \leq Ch^{1/2} (|D^{3/2} \Psi|_{L^2(\Omega_h)}^2 + |D^{3/2} \overrightarrow{\text{grad}} \Psi|_{L^2(\Omega_h)}^2)^{1/2}. \end{aligned}$$

Further, we have

$$|\varphi_h - s_h \varphi_h|_{L^2(\partial K)} \leq Ch |\overrightarrow{\text{grad}} \varphi_h|_{L^2(\partial K)} \leq Ch^{1/2} |\overrightarrow{\text{grad}} \varphi_h|_{L^2(K)}$$

and then

$$\begin{aligned} & \left( \sum_{K \in \mathcal{T}_h} D^{-1}(a_K) |\varphi_h - s_h \varphi_h|_{L^2(\partial K)}^2 \right)^{1/2} \leq \\ & \leq Ch^{+1/2} \left( \sum_{K \in \mathcal{T}_h} D^{-1}(a_K) |\overrightarrow{\text{grad}} \varphi_h|_{L^2(K)}^2 \right)^{1/2}. \end{aligned}$$

So, we obtain 5.1.

**THEOREM 5.1** : Let  $\varphi_h$  be a function in  $\mathcal{H}_h$  such that :

$$\forall \mu_h \in \mathcal{M}_h, \quad \sum_{K \in \mathcal{T}_h} D(a_K) \int_{\partial K} \varphi_h \mu_h d\gamma = 0. \tag{5.2}$$



Then, there is some positive constant  $C$  such that :

$$\sum_{K \in \mathfrak{T}_h} D^{-1}(a_K) |\varphi_h|_{L^2(K)}^2 \leq C \left( \sum_{K \in \mathfrak{T}_h} D(a_K) |\overrightarrow{\text{grad}} \varphi_h|_{L^2(K)}^2 \right). \quad (5.3)$$

*Proof* : Let  $\varphi_h \in \mathcal{H}_h$  satisfying (5.2) and consider the problem. Find  $u \in H_{-1/2}^1(\Omega)$  such that :

$$\left. \begin{aligned} \text{div}(D^{-1} \overrightarrow{\text{grad}} u) &= \overline{D}^{-2} \varphi_h, \\ u|_{\Gamma} &= 0 \end{aligned} \right\} \quad (5.4)$$

where  $\overline{D}$  is the function defined over  $\Omega$  by :

$$\overline{D}|_K = D(a_K).$$

The solution of (6.4) is in  $H_{1/2}^2(\Omega)$  and we have the estimate :

$$\|u\|_{2,1/2,\Omega} \leq C |D^{3/2} \overline{D}^{-2} \varphi_h|_{L^2(\Omega)} \leq C \left( \sum_{K \in \mathfrak{T}_h} D^{-1}(a_K) |\varphi_h|_{L^2(K)}^2 \right)^{1/2}. \quad (5.5)$$

Then, we may write :

$$\begin{aligned} \sum_{K \in \mathfrak{T}_h} D^{-1}(a_K) |\varphi_h|_{L^2(K)}^2 &= \sum_{K \in \mathfrak{T}_h} \int_K \text{div}(D^{-1} \overrightarrow{\text{grad}} u) D(a_K) \varphi_h \, dx + \\ &\quad + \sum_{K \in \partial \mathfrak{T}_h} D^{-1}(a_K) |\varphi_h|_{L^2(K)}. \end{aligned}$$

From (5.2), the function  $\overline{D}\varphi_h$  is continue in the midpoints of each side  $\partial K_i$  and it is null in the midpoint of a side of the boundary  $\Gamma_h$ ; then, we get easily

$$\begin{aligned} \sum_{K \in \partial \mathfrak{T}_h} D^{-1}(a_K) |\varphi_h|_{L^2(\partial K)}^2 &\leq C \sum_{K \in \partial \mathfrak{T}_h} h^2 D^{-1}(a_K) |\overrightarrow{\text{grad}} \varphi_h|_{L^2(K)}^2 \leq \\ &\leq C \sum_{K \in \partial \mathfrak{T}_h} D(a_K) |\overrightarrow{\text{grad}} \varphi_h|_{L^2(K)}. \end{aligned}$$

Besides, by using Green's formula over each  $K \in \mathfrak{T}'_h$ , we obtain :

$$\begin{aligned} \sum_{K \in \mathfrak{T}'_h} \int_K \text{div}(D^{-1} \overrightarrow{\text{grad}} u) D(a_K) \varphi_h \, dx &= \\ = - \sum_{K \in \mathfrak{T}'_h} \int_K D(a_K) D^{-1} \overrightarrow{\text{grad}} u \cdot \overrightarrow{\text{grad}} \varphi_h \, dx &+ \sum_{K \in \mathfrak{T}'_h} \int_{\partial K} D(a_K) D^{-1} \frac{\partial u}{\partial n} \, d\gamma. \end{aligned}$$

Further, we have

$$\left| \sum_{K \in \mathcal{T}_h} \int_K D(a_K) D^{-1} \overrightarrow{\text{grad}} u \overrightarrow{\text{grad}} \varphi_h dx \right| \leq C \left( \sum_{K \in \mathcal{T}_h} D(a_K) | \overrightarrow{\text{grad}} \varphi_h |_{L^2(K)}^2 \right)^{1/2} | D^{-1/2} \overrightarrow{\text{grad}} u |_{(L^2(\Omega))^2}$$

and from (5.5), we get

$$\left| \sum_{K \in \mathcal{T}_h} \int_K D(a_K) D^{-1} \overrightarrow{\text{grad}} u \overrightarrow{\text{grad}} \varphi_h dx \right| \leq C \left( \sum_{K \in \mathcal{T}_h} D(a_K) | \overrightarrow{\text{grad}} \varphi_h |_{L^2(K)}^2 \right)^{1/2} \left( \sum_{K \in \mathcal{T}_h} D^{-1}(a_K) | \varphi_h |_{L^2(K)}^2 \right)^{1/2}.$$

Consider now the term

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} D(a_K) D^{-1} \frac{\partial u}{\partial n} \varphi_h d\gamma.$$

We have

$$\frac{\partial u}{\partial n} = \sum_{i=1}^2 \frac{\partial u}{\partial x_i} n_i \quad (n_i \text{ are the components of } \vec{n}).$$

We note

$$\Psi_i(x) = D^{-1}(x) \frac{\partial u}{\partial x_i}(x), \quad i = 1, 2.$$

We may write

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} \int_{\partial K} D(a_K) D^{-1} \frac{\partial u}{\partial n} \varphi_h d\gamma &= \sum_{K \in \mathcal{T}_h} \sum_{i=1}^2 D(a_K) \int_{\partial K} \Psi_i \varphi_h n_i d\gamma = \\ &= \sum_{K \in \mathcal{T}_h} \sum_{i=1}^2 \int_{\partial K} D(a_K) \varphi_h (\Psi_i - s_h \Psi_i) n_i d\gamma + \\ &\quad + \sum_{K \in \mathcal{T}_h} \sum_{i=1}^2 \int_{\partial K} D(a_K) s_h \Psi_i \varphi_h n_i d\gamma \end{aligned}$$

and from lemma 5.1,

$$\begin{aligned} \left| \sum_{K \in \mathcal{T}_h} \sum_{i=1}^2 \int_{\partial K} D(a_K) \varphi_h(\Psi_i - s_h \Psi_i) n_i d\gamma \right| &\leq \\ &\leq C \left( \sum_{K \in \mathcal{T}_h} D(a_K) |\overrightarrow{\text{grad}} \varphi_h|_{L^2(K)}^2 \right)^{1/2} \\ &\times \sum_{i=1}^2 \left( |D^{3/2} \Psi_i|_{L^2(\Omega_h)}^2 + |D^{3/2} \overrightarrow{\text{grad}} \Psi_i|_{L^2(\Omega_h)}^2 \right)^{1/2}. \end{aligned}$$

Then, by using (5.5) we get

$$\begin{aligned} \left| \sum_{K \in \mathcal{T}_h} \int_{\partial K} D(a_K) D^{-1} \frac{\partial u}{\partial n} \varphi_h d\gamma \right| &\leq C \left( \sum_{K \in \mathcal{T}_h} D(a_K) |\overrightarrow{\text{grad}} \varphi_h|_{L^2(K)}^2 \right)^{1/2} \times \\ &\times \left( \sum_{K \in \mathcal{T}_h} D^{-1}(a_K) |\varphi_h|_{L^2(K)}^2 \right)^{1/2} \end{aligned}$$

Now, we estimate

$$\sum_{K \in \mathcal{T}_h} \sum_{i=1}^2 \int_{\partial K} D(a_K) s_h \Psi_i n_i \varphi_h d\gamma$$

Using hypothesis (5.2), we may write

$$\sum_{K \in \mathcal{T}_h} \sum_{i=1}^2 \int_{\partial K} D(a_K) s_h \Psi_i n_i \varphi_h d\gamma = - \sum_{K \in \partial \mathcal{T}_h} \sum_{i=1}^2 \int_{\Gamma_h \cap \partial K} D(a_K) s_h \Psi_i n_i \varphi_h d\gamma$$

and therefore,

$$\begin{aligned} \left| \sum_{K \in \mathcal{T}_h} \sum_{i=1}^2 \int_{\partial K} D(a_K) s_h \Psi_i n_i \varphi_h d\gamma \right| &\leq \left( \sum_{K \in \partial \mathcal{T}_h} D^{-1}(a_K) |\varphi_h|_{L^2(\partial K)}^2 \right)^{1/2} \times \\ &\times \left( \sum_{K \in \partial \mathcal{T}_h} \sum_{i=1}^2 D^3(a_K) |s_h \Psi_i|_{L^2(\partial K \cap \Gamma_h)}^2 \right)^{1/2}. \end{aligned}$$

But,

$$\begin{aligned} \left( \sum_{K \in \partial \mathcal{T}_h} D^3(a_K) |s_h \Psi_i|_{L^2(\partial K \cap \Gamma_h)}^2 \right)^{1/2} &\leq \\ &\leq Ch^{1/2} \{ |D^{3/2} \Psi_i|_{L^2(\Omega_h)}^2 + |D^{3/2} \overrightarrow{\text{grad}} \Psi_i|_{L^2(\Omega_h)}^2 \}^{1/2} \\ &\leq Ch^{1/2} \left( \sum_{K \in \mathcal{T}_h} D^{-1}(a_K) |\varphi_h|_{L^2(K)}^2 \right)^{1/2} \end{aligned}$$

and

$$\left( \sum_{K \in \partial \mathcal{T}_h} \cdot D^{-1}(a_K) | \varphi_h |_{L^2(\partial K)}^2 \right)^{1/2} \leq Ch^{-1/2} \left( \sum_{K \in \partial \mathcal{T}_h} D(a_K) | \overrightarrow{\text{grad}} \varphi_h |_{(L^2(K))^2}^2 \right)^{1/2} .$$

Then

$$\left| \sum_{K \in \mathcal{T}_h} \sum_{i=1}^2 \int_{\partial K} D(a_K) s_h \Psi_i n_i \varphi_h d\gamma \right| \leq C \left( \sum_{K \in \mathcal{T}_h} D^{-1}(a_K) | \varphi_h |_{L^2(K)}^2 \right)^{1/2} \times \left( \sum_{K \in \partial \mathcal{T}_h} D(a_K) | \overrightarrow{\text{grad}} \varphi_h |_{L^2(K)}^2 \right)^{1/2} .$$

Thus, we obtain (5.3).

Let us provide the space  $\mathcal{M}_h$  with the norm :

$$||| \mu_h |||_h = \left( \sum_{K \in \mathcal{T}_h} h_K D(a_K) \| \mu_h \|_{L^2(\partial K)}^2 \right)^{1/2}$$

and the space  $\mathcal{H}_h$  with the norm :

$$||| \Psi_h |||_{\mathcal{H}_h} = \sum_{K \in \mathcal{T}_h} (D^{-3}(a_K) | \Psi_h |_{L^2(K)}^2 + D^{-1}(a_K) | \overrightarrow{\text{grad}} \Psi_h |_{(L^2(K))^2}^2)^{1/2} .$$

LEMMA 5.2 : We have the estimate :

$$\forall \mu_h \in \mathcal{M}_h, \quad ||| \mu_h |||_h \leq C \sup_{\Psi_h \in \mathcal{H}_h} \frac{\sum_{K \in \mathcal{T}_h} \int_{\partial K} \Psi_h \mu_h d\gamma}{||| \Psi_h |||_{\mathcal{H}_h}} . \tag{5.6}$$

*Proof* : Let us denote by :

$$\hat{\Psi}_h = \Psi_h \circ F, \quad \hat{\mu}_h = J_n \mu_h \circ F \quad (J_n = J | B^{-1} \hat{n} |) .$$

Then,

$$\int_{\partial K} \Psi_h \mu_h d\gamma = \int_{\partial \hat{K}} \hat{\Psi}_h \hat{\mu}_h d\hat{\gamma}$$

and the mapping

$$\hat{\mu}_h \mapsto \sup_{\hat{\Psi}_h \in P_1} \frac{\int_{\hat{K}} \hat{\Psi}_h \hat{\mu}_h d\hat{\gamma}}{\| \hat{\Psi}_h \|_{H^1(\hat{K})}}$$

is a norm over  $S_K$ ; then, there is a positive constant  $C$  such that

$$\|\hat{\mu}_h\|_{L^2(\partial K)} \leq C \sup_{\Psi_h \in P_1} \frac{\int_{\partial \hat{K}} \Psi_h \hat{\mu}_h \, d\hat{\gamma}}{\|\hat{\Psi}_h\|_{H^1(\hat{K})}}.$$

Besides, we have [6]

$$\|\mu_h\|_{L^2(\partial K)} \leq Ch^{-1/2} \|\hat{\mu}_h\|_{L^2(\partial \hat{K})}$$

and

$$\|\hat{\Psi}_h\|_{H^1(\hat{K})} \geq \frac{C}{h} (\|\Psi_h\|_{L^2(K)}^2 + h^2 |\overrightarrow{\text{grad}} \Psi_h|_{(L^2(K))^2}^2)^{1/2}.$$

Then, we get

$$\|\hat{\Psi}_h\|_{H^1(\hat{K})} \geq CD^{1/2}(a_K) (D^{-3}(a_K) |\Psi_h|_{L^2(K)}^2 + D^{-1}(a_K) |\overrightarrow{\text{grad}} \Psi_h|_{(L^2(K))^2}^2)^{1/2}.$$

Hence, we deduce

$$\begin{aligned} |\mu_h|_{L^2(\partial K)} &\leq Ch^{-1/2} D^{-1/2}(a_K) \times \\ &\times \sup_{\Psi_h \in P_1} \frac{\int_{\partial K} \Psi_h \mu_h \, d\gamma}{(D^{-3}(a_K) |\Psi_h|_{L^2(K)}^2 + D^{-1}(a_K) |\overrightarrow{\text{grad}} \Psi_h|_{(L^2(K))^2}^2)^{1/2}}. \end{aligned}$$

But, since the space  $\mathcal{H}_h$  is isomorphic to the space  $\prod_{K \in \mathcal{T}_h} P_1(K)$ , the local estimate implies the global estimate.

LEMMA 5.3 : *There is a linear operator  $\phi_h$  from  $L^2(\Omega_h)$  into  $\mathcal{M}_h$  such that if  $\mu_h = \phi_h v$*

$$\forall K \in \mathcal{T}_h \int_{\partial K} \mu_h \, d\gamma = \int_K v \, dx \tag{5.7}$$

and

$$\|\mu_h\|_h \leq C \left( \sum_{K \in \mathcal{T}_h} D^3(a_K) |v|_{L^2(K)}^2 \right)^{1/2}. \tag{5.8}$$

*Proof* : Let us consider the problem : given  $v \in L^2(\Omega_h)$ , find a pair

$$(\varphi_h, \mu_h) \in \mathcal{H}_h \times \mathcal{M}_h$$

such that

$$\left. \begin{aligned} \forall \Psi_h \in \mathcal{H}_h, \sum_{K \in \mathcal{T}_h} \int_K \overrightarrow{\text{grad}} \varphi_h \overrightarrow{\text{grad}} \Psi_h \, dx - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \Psi_h \mu_h \, d\gamma = \\ = - \sum_{K \in \mathcal{T}_h} \int_K v \Psi_h \, dx \\ \forall \mu'_h \in \mathcal{M}_h, \sum_{K \in \mathcal{T}_h} D(a_K) \int_K \varphi_h \mu'_h \, d\gamma = 0. \end{aligned} \right\} \quad (5.9)$$

This problem has a unique solution (Thomas [8]).

Besides, from lemma 5.2, we have

$$\| \mu_h \|_h \leq C \sup_{\Psi_h \in \mathcal{H}_h} \frac{\sum_{K \in \mathcal{T}_h} \int_{\partial K} \Psi_h \mu_h \, d\gamma}{\| \Psi_h \|_{\mathcal{H}_h}}$$

and

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \Psi_h \mu_h \, d\gamma = \sum_{K \in \mathcal{T}_h} \int_K \overrightarrow{\text{grad}} \varphi_h \overrightarrow{\text{grad}} \Psi_h \, dx - \sum_{K \in \mathcal{T}_h} \int_K v \Psi_h \, dx.$$

Then, we obtain

$$\| \mu_h \|_h \leq \left( \sum_{K \in \mathcal{T}_h} D(a_K) | \overrightarrow{\text{grad}} \varphi_h |_{L^2(K)}^2 \right)^{1/2} + \left( \sum_{K \in \mathcal{T}_h} D^3(a_K) | v |_{L^2(K)}^2 \right)^{1/2}.$$

Now, if in (5.9), we choose  $\Psi_h = \bar{D}\varphi_h$ , we obtain

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} D(a_K) | \overrightarrow{\text{grad}} \varphi_h |_{L^2(K)}^2 &= - \sum_{K \in \mathcal{T}_h} \int_K D(a_K) v \varphi_h \, dx \leq \\ &\leq \left( \sum_{K \in \mathcal{T}_h} D^3(a_K) | v |_{L^2(K)}^2 \right)^{1/2} \left( \sum_{K \in \mathcal{T}_h} D^{-1}(a_K) | \varphi_h |_{L^2(K)}^2 \right)^{1/2} \end{aligned}$$

and, then from lemma 5.1, we get

$$\sum_{K \in \mathcal{T}_h} D(a_K) | \overrightarrow{\text{grad}} \varphi_h |_{L^2(K)}^2 \leq C \sum_{K \in \mathcal{T}_h} D^3(a_K) | v |_{L^2(K)}^2.$$

Finally, we obtain (5.8).

LEMMA 5.4 : *There is a linear operator  $\theta_h$  from  $M_h$  into  $X_h$  such that if  $\theta_h \vec{v}_h = \vec{q}_h$*

$$\operatorname{div} \vec{q}_h = v_h \tag{5.10}$$

and

$$\|\vec{q}_h\|_h \leq C \sum_{K \in \mathcal{T}_h} (\operatorname{mes} K) D^3(a_K) (v_{h|K})^2. \tag{5.11}$$

*Proof* : Let  $v_h$  be in  $M_h$  ; we note

$$\mu_h = \phi_h v_h.$$

Then, there is  $\vec{q}_h$  in  $X_h$  such that [8]

$$\begin{cases} \vec{q}_h \cdot \vec{n}_K = \mu_h \\ \operatorname{div} \vec{q}_h = v_h. \end{cases}$$

Besides

$$\begin{aligned} |\vec{q}_h|_h^2 &= \sum_{K \in \mathcal{T}_h} \frac{\operatorname{mes} K}{3} \sum_{i=1}^3 D(m_i) (\vec{q}_h(m_i))^2 \\ \vec{q}_h(x) &= J^{-1} B \hat{q}_h(\hat{x}). \end{aligned}$$

Hence, we get

$$\begin{aligned} |\vec{q}_h|_h^2 &\leq C \sum_{K \in \mathcal{T}_h} \frac{\operatorname{mes} K}{3} \frac{D(a_K)}{h^2} \sum_{i=1}^3 |\vec{q}_h \cdot \vec{n}_i|^2 \\ &\leq C' \sum_{K \in \mathcal{T}_h} \frac{\operatorname{mes} K}{3} D(a_K) \sum_{i=1}^3 |\vec{q}_h \cdot \vec{n}_i|^2. \end{aligned}$$

Thus, we obtain

$$|\vec{q}_h|_h \leq C \|\mu_h\|_h$$

and from lemma 5.3, we deduce

$$\|\vec{q}_h\|_h \leq C \sum_{K \in \mathcal{T}_h} (\operatorname{mes} K) D^3(a_K) (v_{h|K})^2.$$

*Proof of theorem 4.1* : Let  $v_h$  be in  $M_h$  ; we note

$$\begin{aligned} w_h &= \bar{D}^{-3} v_h \quad (\text{i.e. } w_{h|K} = D^{-3}(a_K) v_{h|K}) \\ \vec{q}_h &= \theta_h w_h. \end{aligned}$$

Then,

$$\begin{aligned} b_h(\vec{q}_h, v_h) &= \sum_{K \in \mathcal{T}_h} (\text{mes } K) v_{h|K} \operatorname{div} \vec{q}_h = \sum_{K \in \mathcal{T}_h} (\text{mes } K) D^{-3}(a_K) (v_{h|K})^2 \\ &= \|v_h\|_{M_h}^2. \end{aligned}$$

Further, we have

$$\|\vec{q}_h\|_h \leq C \sum_{K \in \mathcal{T}_h} D^3(a_K) (w_{h|K})^2 = \|v_h\|_{M_h}^2.$$

Thus, we obtain

$$\frac{b_h(\vec{q}_h, v_h)}{\|\vec{q}_h\|_h} \geq C \|v_h\|_{M_h}.$$

#### BIBLIOGRAPHY

1. P. BOLLEY et J. CAMUS, *Quelques résultats sur les espaces de Sobolev avec poids*. Publications des Séminaires de Mathématiques, Rennes, 1968-1969.
2. F. BREZZI, *On the Existence, Uniqueness and Approximation of Saddle-point Problems arising from Lagrangian Multipliers*. R.A.I.R.O., R 2, pp. 129-151 (1974).
3. P. G. CIARLET, P. A. RAVIART, *General Lagrange and Hermite Interpolation in  $\mathbb{R}^n$  with Applications to Finite Element Methods*. Arch. Rat. Mech. Anal., vol. 46, 1972, pp. 177-199.
4. P. G. CIARLET, *The finite element method for elliptic problems*. North Holland, 1978.
5. M.-N. LE ROUX, *Méthode d'éléments finis pour la résolution numérique de problèmes extérieurs en dimension 2*. R.A.I.R.O., Analyse numérique/Numerical Analysis, vol. 11, n° 1, 1977, pp. 27 à 60.
6. P. A. RAVIART, J. M. THOMAS, *A mixed finite element method for 2nd order elliptic problems*. Proceedings of the Symposium on the Mathematical Aspects of the Finite Element Methods. Rome, décembre 1975.
7. R. TEMAM, *Navier-Stokes equations*. North Holland, 1979.
8. J. M. THOMAS, *Sur l'analyse numérique des méthodes d'éléments finis hybrides et mixtes*. Thesis, Paris 6, 1977.