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# FINITE ELEMENT SUBSPACES WITH OPTIMAL RATES OF CONVERGENCE FOR THE STATIONARY STOKES PROBLEM (*) (**)

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Abstract — *When finite element methods are used to solve the stationary Stokes problem there is a compatibility condition between the subspaces used to approximate the velocity $\underline{u}$ and the pressure p which must be satisfied to obtain optimal rates of convergence Finite element subspaces of arbitrary degree are constructed which have optimal rates of convergence for the stationary Stokes problem These results include regions with curved boundaries where elements similar to isoparametric elements are used*

Résumé — *Lorsqu'on utilise des méthodes d'éléments finis pour résoudre le problème de Stokes stationnaire, les sous-espaces utilisés pour l'approximation de la vitesse $\underline{u}$ et de la pression p doivent satisfaire une condition de compatibilité afin d'obtenir des taux optimaux de convergence On construit ici des espaces d'éléments finis de degre arbitraire qui conduisent à des taux optimaux de convergence pour le problème de Stokes stationnaire Ces résultats s'appliquent en particulier à des régions à frontiere courbe, où l'on utilise des éléments finis analogues aux éléments finis isoparamétriques*

## 1. INTRODUCTION

When finite element methods are used to solve the stationary Stokes problem there is a compatibility condition between the subspace $\underline{V}^h$ used to approximate the velocity $\underline{u}$ and the subspace $P^h$ used to approximate the pressure $p$ which must be satisfied to obtain optimal rates of convergence. One usually approximates the pressure by piecewise polynomials of degree $k - 1$, and chooses the subspace $\underline{V}^h = (V^h)^N$, $N = 2, 3$, so that the compatibility condition is satisfied. Several examples of triangular finite element subspaces in two dimensions and tetrahedral finite element subspaces in three dimensions are given in [7]. The purpose of this note is to extend the quadratic and cubic

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conforming subspaces given in [7] to subspaces of arbitrary degree. The balance between accuracy and ease of use may very well indicate that the lowest order finite element subspaces which we give, namely those already given in [7], are the most practical. However, it seems worthwhile to show how to extend these subspaces to subspaces of arbitrary degree so that the construction of appropriate subspaces seems less ad hoc.

When the boundary of the domain is curved the usual procedure is to use isoparametric elements on boundary triangles or boundary tetrahedra. In Section 5 we show how to extend the idea of isoparametric elements to the context of the Stokes problem where the variables $\underline{u}$ and $p$ are approximated by different types of finite element spaces. We also show that the optimal rate of convergence can be preserved when isoparametric elements are used.

The same compatibility condition between $\underline{V}^h$ and $P^h$ which arises when finite element methods are used to solve the stationary Stokes problem also arises when finite element methods are used to solve the stationary Navier-Stokes equations for incompressible fluid flow, and so considerations regarding the choice of appropriate finite element subspaces are the same for both problems. An analysis of finite element methods for the stationary Navier-Stokes equations at low Reynolds numbers is given in [9].

## 2. PRELIMINARY ANALYSIS

Let $\Omega$ be a bounded domain of $R^N$ ($N = 2, 3$) with boundary $\Gamma$. The stationary Stokes problem for an incompressible viscous fluid confined in $\Omega$ consists of finding functions $\underline{u} = (u_1, ..., u_N)$ and $p$ defined over $\Omega$ such that

$$\begin{aligned} -\nu\,\Delta\underline{u} + \nabla p &= f \quad \text{in} \quad \Omega\,, \\ \operatorname{div}\underline{u} &= 0 \quad \text{in} \quad \Omega\,, \\ \underline{u} &= 0 \quad \text{on} \quad \Gamma\,, \end{aligned} \tag{2.1}$$

where $\underline{u}$ is the fluid velocity, $p$ is the pressure, $\underline{f}$ are the body forces and $\nu > 0$ is the viscosity. It is known that the velocity $\underline{u}$ is uniquely determined by (2.1) while the pressure $p$ is only determined up to an arbitrary constant.

Given any integer $m \geqslant 0$, let

$$H^m(\Omega) = \{ v \mid v \in L^2(\Omega), \partial^\alpha v \in L^2(\Omega), |\alpha| \leqslant m \}$$

be the usual Sobolev space provided with the norm

$$\| v \|_{m,\Omega} = \left( \sum_{|\alpha| \leqslant m} \| \partial^\alpha v \|^2_{L^2(\Omega)} \right)^{1/2}.$$

We shall also need the seminorm

$$| v |_{m,\Omega} = \left( \sum_{|\alpha|=m} \| \partial^\alpha v \|^2_{L^2(\Omega)} \right)^{1/2} .$$

Let

$$H_0^1(\Omega) = \{ v \mid v \in H^1(\Omega), v \mid_\Gamma 0 \} .$$

Consider also the quotient space $L^2(\Omega)/R$ provided with the quotient norm

$$\| v \|_{L^2(\Omega)/R} = \inf_{c \in R} \| v + c \|_{L^2(\Omega)} .$$

The problem (2.1) may be expressed in weak form as : find functions $\underline{u} \in (H_0^1(\Omega))^N$, $p \in L^2(\Omega)/R$ such that

$$\nu(\nabla \underline{u} \cdot \nabla \underline{v}) - (p, \operatorname{div} \underline{v}) = (f \cdot \underline{v}) , \quad \text{all } \underline{v} \in (H_0^1(\Omega))^N , \tag{2.2}$$

$$(\operatorname{div} \underline{u}, q) = 0 , \quad \text{all } q \in L^2(\Omega)/R . \tag{2.3}$$

To approximate $\underline{u}$ and $p$ by the finite element method, we construct a triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ with nondegenerate $N$-simplices $T$ (i.e. triangles if $N = 2$ or tetrahedra if $N = 3$) with diameters $\leqslant h$. For any $T \in \mathcal{T}_h$, let

$$h(T) = \text{diameter of } T ,$$

$$\rho(T) = \text{diameter of the inscribed sphere of } T .$$

We assume that

$$\sigma(T) = \frac{h(T)}{\rho(T)} \geqslant \alpha > 0 , \quad \text{all } T \in \mathcal{T}_h , \tag{2.4}$$

where $\alpha$ is an absolute constant. We are assuming here that $\Omega$ is a polyhedral domain. More general domains are considered in Section 5.

Let $P^h \subset L^2(\Omega)/R$ and $\underline{V}^h = (V^h)^N \subset (H_0^1(\Omega))^N$ be finite dimensional subspaces of piecewise polynomials over $\mathcal{T}_h$. The approximate problem is : find $\underline{u}^h \in \underline{V}^h$, $p^h \in P^h$ such that

$$\nu(\nabla \underline{u}^h \cdot \nabla \underline{v}^h) - (p^h, \operatorname{div} \underline{v}^h) = (\underline{f} \cdot \underline{v}^h) , \quad \text{all } \underline{v}^h \in \underline{V}^h , \tag{2.5}$$

$$(\operatorname{div} \underline{u}^h, q^h) = 0 , \quad \text{all } q^h \in P^h . \tag{2.6}$$

We shall need to assume that the following *compatibility condition* holds between the subspaces $P^h$ and $\underline{V}^h$ : for any $\phi^h \in P^h$, there is a function $\underline{w}^h \in \underline{V}^h$ such that

$$(\operatorname{div} \underline{w}^h, q^h) = (\phi^h, q^h), \quad \text{all } q^h \in P^h, \tag{2.7}$$

$$\| \underline{w}^h \|_{1,\Omega} \leqslant C \| \phi^h \|_{0,\Omega}, \tag{2.8}$$

where $C$ is an absolute constant.

THEOREM 2.1 : *Let $\tilde{\underline{u}}^h$, $\tilde{p}^h$ be arbitrary elements of $\underline{V}^h$ and $P^h$ respectively, and let $(\underline{u}^h, p^h)$ solve* (2.5)-(2.6) *where $\underline{V}^h$ and $P^h$ satisfy* (2.7)-(2.8). *Let $(\underline{u}, p)$ solve* (2.2)-(2.3). *If $\nu \leqslant 1$, then*

$$2\nu \| \nabla(\underline{u}^h - \tilde{\underline{u}}^h) \|_{0,\Omega}^2 + \frac{1}{C^2} \| p^h - \tilde{p}^h \|_{0,\Omega}^2 \leqslant$$

$$\leqslant 16(C^2 + 2\nu) \| \underline{u} - \tilde{\underline{u}}^h \|_{1,\Omega}^2 + 16\left(1 + \frac{1}{\nu}\right) \| p - \tilde{p}^h \|_{L^2(\Omega)/R}^2 . \tag{2.9}$$

*Proof* : Letting $\underline{v}$ and $\underline{v}^h$ equal $\underline{u}^h - \tilde{\underline{u}}^h - \frac{1}{C^2}\underline{w}^h$ in (2.2) and (2.5) where $\underline{w}^h$ satisfies

$$(\operatorname{div} \underline{w}^h, q^h) = (p^h - \tilde{p}^h, q^h), \quad \text{all } q^h \in P^h,$$

$$\| \underline{w}^h \|_{1,\Omega} \leqslant C \| p^h - \tilde{p}^h \|_{0,\Omega},$$

along with the fact that $(\operatorname{div}(\underline{u} - \underline{u}^h), q^h) = 0$, all $q^h \in P^h$, gives

$$\nu \| \nabla(\underline{u}^h - \tilde{\underline{u}}^h) \|_{0,\Omega}^2 + \frac{1}{C^2} \| p^h - \tilde{p}^h \|_{0,\Omega}^2 =$$

$$= (p^h - \tilde{p}^h, \operatorname{div}(\underline{u} - \tilde{\underline{u}}^h)) - (p - \tilde{p}^h, \operatorname{div}(\underline{u}^h - \tilde{\underline{u}}^h)) +$$

$$+ \nu(\nabla(\underline{u} - \underline{u}^h)\cdot\nabla(\underline{u}^h - \tilde{\underline{u}}^h)) + \frac{1}{C^2}(p - \tilde{p}^h, \operatorname{div} \underline{w}^h)$$

$$+ \frac{\nu}{C^2}(\nabla(\underline{u}^h - \tilde{\underline{u}}^h)\cdot\nabla\underline{w}^h) - \frac{\nu}{C^2}(\nabla(\underline{u} - \tilde{\underline{u}}^h), \nabla\underline{w}^h)$$

$$\leqslant \frac{1}{8C^2} \| p^h - \tilde{p}^h \|_{0,\Omega}^2 + 2C^2 \| \underline{u} - \tilde{\underline{u}}^h \|_{1,\Omega}^2 + \frac{2}{\nu} \| p - \tilde{p}^h \|_{0,\Omega}^2$$

$$+ \frac{\nu}{8} \| \nabla(\underline{u}^h - \tilde{\underline{u}}^h) \|_{0,\Omega}^2 + 2\nu \| \underline{u} - \underline{u}^h \|_{1,\Omega}^2 + \frac{\nu}{8} \| \nabla(\underline{u}^h - \tilde{\underline{u}}^h) \|_{0,\Omega}^2$$

$$+ 2 \| p - \tilde{p}^h \|_{0,\Omega}^2 + \frac{1}{8C^2} \| p^h - \tilde{p}^h \|_{0,\Omega}^2 + \frac{\nu}{2} \| \nabla(\underline{u}^h - \tilde{\underline{u}}^h) \|_{0,\Omega}^2$$

$$+ \frac{\nu}{2C^2} \| p^h - \tilde{p}^h \|_{0,\Omega}^2 + 2\nu \| \underline{u} - \underline{\tilde{u}}^h \|_{1,\Omega}^2 + \frac{\nu}{8C^2} \| p^h - \tilde{p}^h \|_{0,\Omega}^2$$

$$= \frac{1}{C^2}\left(\frac{5\nu}{8} + \frac{1}{4}\right) \| p^h - \tilde{p}^h \|_{0,\Omega}^2 + \frac{3\nu}{4} \| \nabla(\underline{u}^h - \underline{\tilde{u}}^h) \|_{0,\Omega}^2$$

$$+ (2C^2 + 4\nu) \| \underline{u} - \underline{\tilde{u}}^h \|_{1,\Omega}^2 + \left(\frac{2}{\nu} + 2\right) \| p - \tilde{p}^h \|_{0,\Omega}^2 ,$$

from which (2.9) follows if $\nu \leqslant 1$.

Suppose that the subspaces $\underline{V}^h$ and $P^h$ have the following approximation properties. There is some $\rho_1 \geqslant 2$ and a constant $C_A$ independent of $h$ such that if $\underline{v} \in (H^r(\Omega))^N \cap (H_0^1(\Omega))^N$, $2 \leqslant r \leqslant \rho_1$, then there is some $\underline{\tilde{v}}^h \in \underline{V}^h$ such that

$$\| \underline{v} - \underline{\tilde{v}}^h \|_{m,\Omega} \leqslant C_A h^{r-m} \| \underline{v} \|_{r,\Omega} , \qquad m = 0, 1 . \tag{2.10}$$

There is some $\rho_2 \geqslant 1$ such that if $q \in H^s(\Omega)$, $1 \leqslant s \leqslant \rho_2$, then there is some $\tilde{q}^h \in P^h$ such that

$$\| q - \tilde{q}^h \|_{0,\Omega} \leqslant C_A h^s \| q \|_{s,\Omega} . \tag{2.11}$$

COROLLARY : *Let* $(\underline{u}, p)$ *solve* (2.2)-(2.3) *and suppose* $\underline{u} \in (H^r(\Omega))^N \cap (H_0^1(\Omega))^N$, $p \in H^s(\Omega)/R$. *Let* $(\underline{u}^h, p^h)$ *solve* (2.5)-(2.6). *Let* $\underline{V}^h$ *and* $P^h$ *satisfy the compatibility condition* (2.7)-(2.8) *along with* (2.10) *and* (2.11). *Then*

$$2\nu \| \nabla(\underline{u} - \underline{u}^h) \|_{0,\Omega}^2 + \frac{1}{C^2} \| p - p^h \|_{0,\Omega}^2 \leqslant$$

$$\leqslant C_1 h^{2(r-1)} \| \underline{u} \|_{r,\Omega}^2 + C_2 h^{2s} \| p \|_{s,\Omega}^2 . \tag{2.12}$$

Assumptions (2.10) and (2.11) are satisfied if $\underline{V}^h$ contains all polynomials of degree $r - 1$ or less and $P^h$ contains all polynomials of degree $s - 1$ or less. From (2.12) one should choose $s = r - 1$ provided one can satisfy (2.7)-(2.8). The purpose of the next two sections is to give spaces $\underline{V}^h$ and $P^h$ for which (2.7)-(2.8) are satisfied and for which $s = r - 1$.

If $\Omega$ is convex, we have the regularity property

$$\| \underline{u} \|_{2,\Omega} + | p |_{1,\Omega} \leqslant C_R \| \underline{f} \|_{0,\Omega} , \tag{2.13}$$

where $(\underline{u}, p)$ is the solution to (2.2)-(2.3). $L^2$ error estimates for $\underline{u} - \underline{u}^h$ may be obtained using (2.13) along with a duality argument. The following is proved in [7].

THEOREM 2.2 : *Assume the region* $\Omega$ *is convex. Let* $(\underline{u}, p)$ *solve* (2.2)-(2.3) *and* $(\underline{u}^h, p^h)$ *solve* (2.5)-(2.6). *Then*

$$\| \underline{u} - \underline{u}^h \|_{0,\Omega} \leqslant C_D \, h(\| \underline{u} - \underline{u}^h \|_{1,\Omega} + \| p - p^h \|_{0,\Omega}) \,. \tag{2.14}$$

Our analysis is similar to that in [7]. We reduce their hypotheses, one of which is equivalent to (2.7)-(2.8) and the other that there is an element $\underline{\tilde{v}}^h$ which satisfies (2.10) along with

$$(\operatorname{div} \underline{\tilde{v}}^h, \phi^h) = (\operatorname{div} \underline{v}, \phi^h) \,, \quad \text{all } \phi^h \in P^h \,, \tag{2.15}$$

to the single hypothesis (2.7)-(2.8). Our method of demonstrating (2.7)-(2.8), however, will essentially involve the construction of an element satisfying (2.15).

Alternatively, one can obtain similar results by replacing our condition (2.7)-(2.8) by the assumption

$$\sup_{\underline{v}^h \in \underline{V}^h} \frac{(\operatorname{div} \underline{v}^h, q^h)}{\| \underline{v}^h \|_{1,\Omega}} \geqslant \beta \, \| q^h \|_{0,\Omega} \,, \quad \text{all } q^h \in P^h \,, \tag{2.16}$$

for $\beta > 0$, and require (2.15) along with (2.16) as was done in [8]. It is not hard to show that (2.7)-(2.8) implies (2.16).

## 3. SUBSPACES WITH OPTIMAL ACCURACY FOR $N = 2$

We take for $P^h$ a set of piecewise polynomials of degree $k - 1$ such that $P^h \subset L^2(\Omega)/R$. Since no inter-element continuity is required, it has been common in the mathematical literature [1], [7] to choose piecewise polynomials with discontinuities across element boundaries. Since (2.7) only requires that one be able to find a $\underline{w}^h \in \underline{V}^h$ such that $\operatorname{div} \underline{w}^h = q^h$ holds weakly, i.e. that (2.7) holds, there is no necessity to do this, and the dimension of $P^h$ is reduced with no loss in order of convergence if $P^h \in C(\Omega)$. In this section we construct subspaces $\underline{V}^h$ which contain all polynomials of degree $k$ or less and which satisfy (2.7) for $P^h$ consisting of piecewise polynomials of degree $k - 1$. Elements of $\underline{V}^h = V^h \times V^h$ will be obtained by piecing together polynomials defined over triangles $T \in \mathcal{T}_h$ to obtain piecewise polynomials which are in $C(\Omega)$.

Suppose the triangle $T \in \mathcal{T}_h$ has vertices $\underline{a}_i$, $i = 1, 2, 3$. Let $\lambda_i(x, y)$ denote the barycentric coordinates of a point $(x, y) \in \bar{R}^2$ with respect to the vertices of $T$. Equivalently, suppose the edge $e_i$ of $T$ opposite the vertex $\underline{a}_i$ has the equation $\lambda_i(x, y) = 0$ normalized so that $\lambda_i(\underline{a}_i) = 1$. Let $\Pi_T$ be the space of polynomials spanned by the set of polynomials of degree $k$ along with the polynomials

$\lambda_1 \lambda_2 \lambda_3 x^i y^j$, $i + j = k - 2$. It is interesting to note that $\Pi_T$ consists of all those polynomials which are polynomials of degree $k$ along parallels to the edges $e_i$ of $T$ and which are polynomials of degree $k + 1$ or less. Thus for $k = 2s - 1$, $\Pi_T$ is a subset of the set $\mathcal{C}_{2s-1}(T)$ of polynomials which are of degree $2s - 1$ or less along parallels to the edges of $T$, introduced in the cubic case and called tricubic polynomials in [2], and used in another context in [3] and [10].

We let $\underline{V}^h = V^h \times V^h$ where

$$V^h = \{ v^h \mid v^h \in \Pi_T \text{ on each } T \in \mathcal{C}_h, v^h \in H_0^1(\Omega) \} . \tag{3.1}$$

THEOREM 3.1 : *With* $\underline{V}^h = V^h \times V^h$ *where* $V^h$ *is defined by* (3.1), *and* $P^h$ *consisting of piecewise polynomials of degree* $k - 1$ *or less, given* $q^h \in P^h$, *there exists a* $\underline{v}^h \in \underline{V}^h$ *such that* (2.7)-(2.8) *holds.*

*Proof* : Given $q^h \in P^h$, by lemma 6 of [7], there exists a function $\underline{v} \in (H_0^1(\Omega))^2$ such that

$$\operatorname{div} \underline{v} = q^h ,$$
$$\| \underline{v} \|_{1,\Omega} \leqslant c_1 \| q^h \|_{0,\Omega} .$$

Let $\underline{w}^h$ be the orthogonal projection in $(H_0^1(\Omega))^2$ of $\underline{v}$ on $\underline{V}^h$. Let $\underline{z} = \underline{v} - \underline{w}^h$, and define $\underline{z}^h \in \underline{V}^h$ by

$$\underline{z}^h(a_i) = 0 , \qquad i = 1, 2, 3 , \tag{i}$$

$$\int_{e_j} \underline{z}^h \lambda_{j+1}^i \, d\sigma = \int_{e_j} \underline{z} \lambda_{j+1}^i \, d\sigma , \quad j = 1, 2, 3 , \quad i = 0, \dots, k - 2 , \tag{ii}$$

$$\int_T \underline{z}^h x^r y^s \, dx \, dy = \int_T \underline{z} x^r y^s \, dx \, dy , \quad 0 \leqslant r + s \leqslant k - 3 , \tag{iii}$$

$$\int_T r x^{r-1} y^s z_1^h \, dx \, dy - \int_{\partial T} \nu_1 x^r y^s z_1^h \, d\sigma = \int_T r x^{r-1} y^s z_1 \, dx \, dy -$$
$$- \int_{\partial T} \nu_1 x^r y^s z_1 \, d\sigma , \quad r + s = k - 1 , \quad r \geqslant 1, s \geqslant 1 , \tag{iv}$$

$$\int_T (k - 1) x^{k-2} z_1^h \, dx \, dy - \int_{\partial T} x^{k-1} \underline{z}^h \cdot \nu \, d\sigma =$$
$$= \int_T (k - 1) x^{k-2} z_1 \, dx \, dy - \int_T x^{k-1} \underline{z} \cdot \nu \, d\sigma ,$$

$$\int_T s x^r y^{s-1} z_2^h \, dx\, dy - \int_{\partial T} \nu_2 x^r y^s z_2^h \, d\sigma = \int_T s x^r y^{s-1} z_2 \, dx\, dy -$$

$$- \int_{\partial T} \nu_2 x^r y^s z_2 \, d\sigma \quad r + s = k - 1 , \quad r \geqslant 1, s \geqslant 1 , \quad \text{(v)}$$

$$\int_T (k-1) y^{k-2} z_2^h \, dx\, dy - \int_{\partial T} y^{k-1} \underline{z}^h \cdot \nu \, d\sigma =$$

$$= \int_T (k-1) y^{k-2} z_2 \, dx\, dy - \int_{\partial T} y^{k-1} \underline{z} \cdot \nu \, d\sigma ,$$

on each $T \in \mathcal{T}_h$, where $\nu = (\nu_1, \nu_2)$ is the outward normal on $\partial T$. In (ii) and below subscripts on $\lambda$ are to be taken cylically, so $\lambda_{j+1}$ for $j = 3$ is $\lambda_1$. If $k = 2$, the conditions (iii) are absent.

Using the identity

$$\int_T x^r y^s \operatorname{div} \underline{z}^h \, dx\, dy = \int_{\partial T} x^r y^s \underline{z}^h . \nu \, d\sigma - \int_T (r x^{r-1} y^s z_1^h + s x^r y^{s-1} z_2^h) \, dx\, dy ,$$

it is straightforward to show that if $\underline{z}^h$ satisfies (i)-(v) on each $T \in \mathcal{T}$, then

$$(\operatorname{div} \underline{z}^h, \phi^h) = (\operatorname{div} \underline{z}, \phi^h) , \quad \text{all } \phi^h \in P^h .$$

We show that $\underline{z}^h$ is uniquely defined on each $T \in \mathcal{T}_h$ by (i)-(v). On each $T \in \mathcal{T}_h$, we can write $\underline{z}^h$ as

$$\underline{z}^h = \lambda_1 \lambda_2 \left( \sum_{i=0}^{k-2} \underline{\alpha}_i^3 \lambda_1^i \right) + \lambda_2 \lambda_3 \left( \sum_{i=0}^{k-2} \underline{\alpha}_i^1 \lambda_2^i \right) + \lambda_3 \lambda_1 \left( \sum_{i=0}^{k-2} \underline{\alpha}_i^2 \lambda_3^i \right) +$$

$$+ \lambda_1 \lambda_2 \lambda_3 \left( \sum_{i+j \leqslant k-3} \underline{\beta}_{ij} x^i y^j \right) + \lambda_1 \lambda_2 \lambda_3 \left( \sum_{i+j=k-2} \underline{\beta}_{ij} (x^i y^j + \text{l.o. terms}) \right), \tag{3.3}$$

where the $\underline{\alpha}_i^j$ and $\underline{\beta}_{ij}$ are coefficients. The lower order terms in the sum in (3.3) can be chosen so that

$$\int_T \lambda_1 \lambda_2 \lambda_3 (x^i y^j + \text{l.o. terms}) \, x^m y^n \, dx\, dy = 0 ,$$

$$i + j = k - 2 , \quad 0 \leqslant m + n < k - 2 .$$

See Stroud [12, p. 67, ff.]. On $e_j, j = 1, 2, 3$,

$$\int_{e_j} \underline{z}^h \lambda_{j+1}^l \, d\sigma = \int_{e_j} \lambda_{j+1} \lambda_{j+2} \left( \sum_{i=0}^{k-2} \underline{\alpha}_i^j \lambda_{j+1}^i \right) \lambda_{j+1}^l \, d\sigma , \quad l = 0, \dots, k-2 . \tag{3.4}$$

The coefficients $\underline{\alpha}_i^j$ can be uniquely determined to satisfy (ii) since the coefficient matrix for the linear system obtained by substituting (3.4) into (ii) is the Gram matrix for the least squares problem with inner product

$$\int_{e_j} \lambda_{j+1} \lambda_{j+2} \, fq \, d\sigma .$$

Here $\lambda_{j+1} \lambda_{j+2}$ is a non-negative weight function. The coefficients $\underline{\beta}_{ij}$, $0 \leqslant i + j \leqslant k - 3$ can be uniquely determined from

$$\int_T \lambda_1 \lambda_2 \lambda_3 \left( \sum_{i+j \leqslant k-3} \underline{\beta}_{ij} \, x^i y^j \right) x^m y^n \, dx \, dy = \int_T \underline{z} x^m y^n \, dx \, dy - \\ - \int_T \underline{p}_e \, x^m y^n \, dx \, dy , \quad 0 \leqslant m + n \leqslant k - 3$$

where $\underline{p}_e$ denotes the sum of the first three terms in (3.3), since the coefficient matrix is the Gram matrix for the least squares problem with inner product

$$\int_T \lambda_1 \lambda_2 \lambda_3 \, fg \, dx \, dy .$$

Finally the coefficients $\underline{\beta}_{ij}$, $i + j = k - 2$ can be determined from (iv) and (v) since the resulting coefficient matrix is the Gram matrix with entries

$$\int_T \lambda_1 \lambda_2 \lambda_3 \, x^{k-2-i} y^i x^{k-2-j} y^j \, dx \, dy , \quad 0 \leqslant i, j \leqslant k - 2 .$$

Let $\hat{T}$ be a standard reference triangle with vertices $\underline{\hat{a}}_i$. There is a 1-1 affine transformation which maps $\hat{T}$ onto $T$. In (3.3) and in the conditions (iii)-(v) we could just as well written $\lambda_1^i \lambda_2^j$ in place of $x^i y^j$. Then the transform

$\hat{\underline{z}}^h$ of $\underline{z}^h$ to $\hat{T}$ is given by

$$\hat{\underline{z}}^h = \hat{\lambda}_1 \hat{\lambda}_2 \left( \sum_{i=0}^{k-2} \underline{\alpha}_i^3 \hat{\lambda}_1^i \right) + \hat{\lambda}_2 \hat{\lambda}_3 \left( \sum_{i=0}^{k-2} \underline{\alpha}_i^1 \hat{\lambda}_2^i \right) + \hat{\lambda}_3 \hat{\lambda}_1 \left( \sum_{i=0}^{k-2} \underline{\alpha}_i^2 \hat{\lambda}_3^i \right) +$$
$$+ \hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_3 \left( \sum_{i+j \leqslant k-3} \underline{\beta}_{ij} \hat{\lambda}_1 \hat{\lambda}_2 \right) + \hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_3 \left( \sum_{i+j=k-2} \underline{\beta}_{ij} (\hat{\lambda}_1 \hat{\lambda}_2 + \text{l.o. terms}) \right) \tag{3.5}$$

where $\hat{\lambda}_i(\hat{x}, \hat{y}) = 0$ on $\hat{e}_i$, the edge opposite $\hat{a}_i$, and $\hat{\lambda}_i(\hat{a}_i) = 1$. The functions $\hat{\lambda}_i$ can be bounded uniformly. Thus

$$\| \hat{\lambda}_i \|_{1,\hat{T}} \leqslant C_1 .$$

Since

$$\int_{e_j} \lambda_{j+1} \lambda_{j+2} \left( \sum_{i=0}^{k-2} \underline{\alpha}_i^j \lambda_{j+1}^i \right) \lambda_{j+1}^l \, d\sigma = \int_{e_j} \underline{z} \lambda_{j+1}^l \, d\sigma , \quad 0 \leqslant l \leqslant k-2$$

or

$$| e_j | \int_{\hat{e}_j} \hat{\lambda}_{j+1} \hat{\lambda}_{j+2} \left( \sum_{i=0}^{k-2} \underline{\alpha}_i^j \hat{\lambda}_{j+1}^i \right) \hat{\lambda}_{j+1}^l \, d\hat{\sigma} = | e_j | \int_{\hat{e}_j} \hat{\underline{z}} \hat{\lambda}_{j+1} \, d\hat{\sigma} ,$$

the coefficient matrix and right hand side of the linear system which determines the $\underline{\alpha}_i^j$ can be bounded independently of the geometry of $T$. Thus

$$| \underline{\alpha}_i^j | \leqslant C_2 \| \hat{\underline{z}} \|_{1,T} , \quad 0 \leqslant i \leqslant k-2 , \quad j = 1, 2, 3 .$$

Similarly

$$| \underline{\beta}_{ij} | \leqslant C_3 \| \hat{\underline{z}} \|_{1,\hat{T}} , \quad 0 \leqslant i + j \leqslant k-2 .$$

Thus

$$\| \underline{z}^h \|_{1,T} \leqslant C_4 \frac{| J |^{1/2}}{h} \| \hat{\underline{z}}^h \|_{1,\hat{T}} \leqslant C_5 \frac{| J |^{1/2}}{h} (| \hat{\underline{z}} |_{1,\hat{T}} + \| \hat{\underline{z}} \|_{0,\hat{T}})$$
$$\leqslant C_6 (| \underline{z} |_{1,T} + h^{-1} \| \underline{z} \|_{0,T})$$

so that

$$\| \underline{z}^h \|_{1,\Omega}^2 \leqslant \tfrac{1}{2} C_6^2 (| \underline{z} |_{1,\Omega}^2 + h^{-2} \| \underline{z} \|_{0,\Omega}) \leqslant C_7^2 \| \underline{z} \|_{1,T}^2 ,$$

where $| J |$ is the Jacobian of the mapping from $\hat{T}$ to $T$ and the last inequality follows from the Nitsche duality argument.

Finally, let $\underline{v}^h = \underline{z}^h + \underline{w}^h$. Then $\underline{v}^h$ satisfies

$$(\operatorname{div} \underline{v}^h, \phi^h) = (\operatorname{div} \underline{v}, \phi^h)\,, \quad \text{all } \phi^h \in P^h\,,$$

$$\| \underline{v}^h \|_{1,\Omega} \leqslant \| v \|_{1,\Omega}\,,$$

which proves the theorem.

If conditions (iv)-(v) are omitted, the proof of theorem 3.1 shows that (2.7)-(2.8) holds if $\underline{v}^h$ consists of $C^0$-piecewise polynomials of degree $k$ or less, and $P^h$ consists of piecewise polynomials of degree $k - 2$ or less. It seems necessary to augment $\underline{V}^h$ as we have done in order to satisfy (2.7)-(2.8) for $P^h$ containing piecewise polynomials of degree $k - 1$. These additional functions all have support only on one triangle, and so shouldn't add very much to the cost of solving the resulting algebraic systems since condensation techniques can be used. It was shown in [7] that one can satisfy (2.7)-(2.8) with $\underline{V}^h$ consisting of piecewise polynomials of degree $k$ and $P^h$ consisting of piecewise polynomials of degree $k - 1$ if non-conforming elements are used for $\underline{V}^h$. Actually, this doesn't reduce the dimension of $\underline{V}^h$ as much as it might first appear, if at all, since basis functions corresponding to vertices are replaced by basis functions corresponding to points along edges; and there are roughly three times as many edges as vertices, see [3, p. 543].

The conditions (i)-(v) given in the proof of theorem 3.1 can be used to define an interpolant $\tilde{\underline{v}}^h$ to $\underline{v} \in H^r$, $1 \leqslant r \leqslant k + 1$, with the property that

$$(\operatorname{div} \tilde{\underline{v}}^h, \phi^h) = (\operatorname{div} \underline{v}, \phi^h)\,, \quad \text{all } \phi^h \in P^h\,.$$

Since $\tilde{\underline{v}}^h = \underline{v}$ for all polynomials of degree $k$ or less, one can use the usual finite element error techniques to conclude that the approximation property (2.10) holds with $r = k + 1$. However, since interpolation schemes are also used to provide suitable bases for computation, and since we don't believe that a basis derived from (i)-(v) above is the most practical to use, we give the following alternative interpolation scheme.

LEMMA 3.1 : *There exists a unique polynomial $q \in \Pi_T$ which has given values for*

$$q(\underline{a}_i)\,, i = 1, 2, 3\,, \tag{i}$$

$q(\underline{a}_{i,j})$, $j = 1, \ldots, k - 1$, *on each edge $e_i$ of $T$, where the points $\underline{a}_{i,j}$ divide $e_i$ into $k$ equal parts,* (ii)

$\dfrac{\partial^{r+s} q(\underline{c})}{\partial x^r\, \partial y^s}$, $0 \leqslant r + s \leqslant k - 2$, *at the center of gravity $\underline{c}$ of $T$.* (iii)

*Furthermore there exists a unique* $\tilde{v}^h \in V^h$ *which interpolates*

$$v \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$$

*with respect to the conditions* (i)-(iii) *on each triangle* $T \in \mathcal{C}_h$ *and*

$$\| v - \tilde{v}^h \|_{m,\Omega} \leqslant C_A h^{k+1-m} \| v \|_{k+1,\Omega}, \quad m = 0, 1. \tag{3.6}$$

*Proof* : The number of conditions in (i)-(iii) is equal to the dimension of $\Pi_T$. Suppose $q \in \Pi_T$ has the conditions in (i)-(iii) all zero. Then (i)-(ii) imply that $q$ is zero on the edges $e_i$ of $T$. Thus $q = \lambda_1 \lambda_2 \lambda_3 p_{k-2}$, where $p_{k-2}$ is a polynomial of degree $k - 2$ or less. Since $\lambda_i(\underline{c}) \neq 0$, $i = 1, 2, 3$, the conditions (iii) imply that $p_{k-2} \equiv 0$. Thus $q \equiv 0$. Let the piecewise polynomial $\tilde{v}^h$ interpolate $v \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$ with respect to (i)-(iii) on each triangle $T \in \mathcal{C}_h$. Then $\tilde{v}^h \in C(\Omega)$ and $\tilde{v}^h |_\Gamma = 0$; so $\tilde{v}^h \in V^h$. Since $\tilde{v}^h = v$ for $v$ a polynomial of degree $k$ or less, the bound (3.6) follows from well-known finite element error techniques, see [4] and [5], for example.

## 4. SUBSPACES WITH OPTIMAL ACCURACY FOR $N = 3$

Again we take for $P^h$ a set of piecewise polynomials of degree $k - 1$ such that $P^h \subset L^2(\Omega)/R$. It would seem to be more practical in that the dimension reduced for no loss in the order of convergence if $P^h \subset C(\Omega)$. For each tetrahedran $T \in \mathcal{C}_h$ with vertices $\underline{a}_i$, $i = 1, 2, 3, 4$, let $\Pi_T$ be the space of polynomials spanned by the set of polynomials of degree $k$ or less along with the polynomials

$$\lambda_i \lambda_{i+1} \lambda_{i+2} x_1^l x_2^m x_3^n, \quad l + m + n = k - 2, \quad i = 1, 2, 3, 4,$$

plus the polynomials $\lambda_1 \lambda_2 \lambda_3 \lambda_4 x_1^l x_2^m x_3^n$, $l+m+n = k-2$. Here $\lambda_i(x_1, x_2, x_3)$ denotes the barycentric coordinates of a point $(x_1, x_2, x_3) \in R^3$ with respect to the vertices of $T$. As in the previous section we assume the subscripts of the $\lambda_i$ are augmented cyclically. Equivalently $\Pi_T$ consists of all polynomials which are of degree $k$ along parallels to the edges of $T$, are of degree $k + 1$ on parallels to the faces of $T$, and are of degree at most $k + 2$. Thus for $k = 3$, $\Pi_T$ is a subset of the set of tetracubic polynomials introduced in [11, p. 149]. We let $\underline{V}^h = (V^h)^3$ where

$$V^h = \{ v^h \mid v^h \in \Pi_T \text{ on each } T \in \mathcal{C}_h, v^h \in H_0^1(\Omega) \}. \tag{4.1}$$

THEOREM 4.1 : *With* $\underline{V}^h = (V^h)^3$ *where* $V^h$ *is defined by* (4.1), *and* $P^h$ *consisting of piecewise polynomials of degree* $k - 1$ *or less, given* $q^h \in P^h$, *there exists a unique* $\underline{v}^h \in \underline{V}^h$ *such that* (2.7)-(2.8) *holds.*

*Proof* : As in the proof of theorem 3.1, given $q^h \in P^h$, there exists a function $\underline{v} \in (H_0^1(\Omega))^3$ such that

$$\operatorname{div} \underline{v} = q^h ,$$
$$\| \underline{v} \|_{1,\Omega} \leqslant c_1 \| q^h \|_{0,\Omega} .$$

Let $\underline{w}^h$ be the orthogonal projection in $(H_0^1(\Omega))^3$ of $\underline{v}$ on $\underline{V}^h$. Let $\underline{z} = \underline{v} - \underline{w}^h$, and define $\underline{z}^h \in \underline{V}^h$ by

$$\underline{z}^h(\underline{a}_i) = 0 , \quad i = 1, 2, 3, 4 , \tag{i}$$

$$\underline{z}^h(\underline{a}_{i,j}) = 0 , \quad j = 1, 2, ..., k - 1, \text{ on each edge } e_i, i = 1, 2, ..., 6, \text{ where the points } \underline{a}_{i,j} \text{ divide } e_i \text{ into } k \text{ equal parts,} \tag{ii}$$

$$\int_{F_j} \underline{z}^h \lambda_{j+1}^r \lambda_{j+2}^s \, d\sigma = \int_{F_j} \underline{z} \lambda_{j+1}^r \lambda_{j+2}^s , \quad 0 \leqslant r + s \leqslant k - 2 , \tag{iii}$$

on each face $F_j, j = 1, 2, 3, 4$,

$$\int_T \underline{z}^h x_1^r x_2^s x_3^t \, d\underline{x} = \int_T \underline{z} x_1^r x_2^s x_3^t \, d\underline{x} , \quad 0 \leqslant r + s + t \leqslant k - 3 , \tag{iv}$$

$$\int_T r x_i^{r-1} x_{i+1}^s x_{i+2}^t z_i^h \, d\underline{x} - \int_{\partial T} \nu_i x_i^r x_{i+1}^s x_{i+2}^t z_i^h \, d\sigma = \int_T r x_i^{r-1} x_{i+1}^s x_{i+2}^t z_i \, d\underline{x} -$$
$$- \int_{\partial T} \nu_i x_i^r x_{i+1}^s x_{i+2}^t z_i \, d\sigma , \quad r+s+t=k-1 \quad r \geqslant 1, s \geqslant 1, t \geqslant 1 , \quad i=1,2,3 \tag{v}$$

$$\int_T r x_i^{r-1} x_{i+1}^s z_i^h \, d\underline{x} - \int_{\partial T} \left\{ \nu_i x_i^r x_{i+1}^s z_i^h + \frac{1}{2} \nu_{i+2} x_i^r x_{i+1}^s z_{i+2}^h \right\} d\sigma =$$
$$= \int_T r x_i^{r-1} x_{i+1}^s z_i \, d\underline{x} - \int_{\partial T} \{ \nu_i x_i^r x_{i+1}^s z_i + 1/2 \, \nu_{i+2} x_i^r x_{i+1}^s z_{i+2} \} \, d\sigma ,$$
$$r + s = k - 1 , \quad r \geqslant 1, s \geqslant 1 , \quad i = 1, 2, 3 ,$$

$$\int_T r x_i^{r-1} x_{i+2}^s z_i^h \, d\underline{x} - \int_{\partial T} \{ \nu_i x_i^r x_{i+2}^s z_i^h + 1/2 \, \nu_{i+1} x_i^r x_{i+2}^s z_{i+1}^h \} \, d\sigma =$$
$$= \int_T r x_i^{r-1} x_{i+2}^s z_i \, d\underline{x} - \int_{\partial T} \{ \nu_i x_i^r x_{i+2}^s z_i + 1/2 \, \nu_{i+1} x_i^r x_{i+2}^s z_{i+1}^h \} \, d\sigma ,$$
$$r + s = k - 1 , \quad r \geqslant 1 , \quad s \geqslant 1 , \quad i = 1, 2, 3 ,$$

$$\int_T (k-1)\, x_i^{k-2}\, z_i^h\, d\underline{x} - \int_{\partial T} x_i^{k-1}\, \underline{z}^h \cdot \nu\, d\sigma =$$

$$= \int_T (k-1)\, x_i^{k-2}\, z_i^h\, d\underline{x} - \int_{\partial T} x_i^{k-1}\, \underline{z} \cdot \nu\, d\sigma\,, \quad i = 1, 2, 3\,,$$

on each $T \in \mathcal{C}_h$, where again $\nu = (\nu_1, \nu_2, \nu_3)$ is the outward normal on $\partial T$, and the subscripts on the $x_i$ are to be taken cyclically so that for $j = 4$, $x_{j+1} = x_1$ If $k = 2$, the conditions (iii) are absent

Again it is straightforward to show that if $\underline{z}^h$ satisfies (i)-(v) on each $T \in \mathcal{C}_h$, then

$$(\operatorname{div} \underline{z}^h, \phi^h) = (\operatorname{div} \underline{z}, \phi^h)\,, \quad \text{all } \phi^h \in P^h$$

By writing $z^h$ similarly to (3 3), it can be shown in the same way as in the proof theorem 3 1 that $\underline{z}^h$ is uniquely determined on each $T \in \mathcal{C}^h$ by (i)-(v) In addition, the interpolation conditions in (iii)-(v) correspond to bounded linear functionals in $(H_0^1(\Omega))^3$ This enables us to show that

$$\| \underline{z}^h \|_{1\,\Omega} \leqslant C_1 \| \underline{z} \|_{1\,\Omega}$$

Finally, let $\underline{v}^h = \underline{z}^h + \underline{w}^h$ Then $\underline{v}^h$ satisfies

$$(\operatorname{div} \underline{v}^h, \phi^h) = (\operatorname{div} \underline{v}, \phi^h), \text{ all } \phi^h \in P^h\,,$$

$$\| \underline{v}^h \|_{1\,\Omega} \leqslant \| v \|_{1\,\Omega}\,,$$

which proves the theorem

If the conditions (v) are omitted, the proof of theorem 4 1 shows that (2 7)-(2 8) holds if $\underline{V}^h$ consists of $C^0$-piecewise polynomials of degree $k$ or less and $P^h$ consists of piecewise polynomials of degree $k-3$ or less Similarly to when $N = 2$, we have the following alternative interpolation scheme which furnishes a more practical set of basis functions than the interpolation conditions of (i)-(v) in the proof of theorem 4 1 Let $\partial/\partial\tau_{i,j}$, $j = 1, 2$, denote directional differentiation in two specified nonparallel directions on the face $F_i$ of $T$

LEMMA 4 1 *There exists a unique polynomial* $q \in \Pi_T$ *which has given values for*

$q(\underline{a}_i)\,, \quad i = 1, 2, 3, 4$ (i)

$q(\underline{a}_{i\,j})\,, \quad j = 1, 2, \quad , k-1$ *on each edge* $e_i$, $i = 1, 2, \quad , 6$, *where the points* $\underline{a}_{ij}$ *divide* $e_i$ *into* $k$ *equal parts,* (ii)

$$\frac{\partial^{r+s} q(\underline{c}_i)}{\partial \tau_{i,1}^r \, \partial \tau_{i,2}^s}, \quad 0 \leqslant r + s \leqslant k - 2, \text{ at the center of gravity } \underline{c}_i \text{ of each face } F_i \text{ of } T, \tag{iii}$$

$$\frac{\partial^{r+s+t} q(\underline{c})}{\partial x_1^r \, \partial x_2^s \, \partial x_3^t}, \quad 0 \leqslant r + s \leqslant k - 2, \text{ at the center of gravity } \underline{c} \text{ of } T. \tag{iv}$$

*Futhermore there exists a unique* $\tilde{v}^h \in V^h$ *which interpolates*

$$v \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$$

*with respect to the conditions* (i)-(iv) *on each tetrahedron* $T \in \mathcal{C}_h$ *and*

$$\| v - \tilde{v}^h \|_{m,\Omega} \leqslant C_A \, h^{k+1-m} \, \| v \|_{k+1,\Omega}, \quad m = 0, 1 .$$

## 5. ISOPARAMETRIC ELEMENTS

We assume the region $\Omega$ has been triangulated with boundary triangles or boundary tetrahedra having a curved edge or curved face. As isoparametric elements have been used by engineers the curved triangles or tetrahedra are straightened by a change of coordinates which is described by the same class of polynomials that are used as finite elements in the approximation of the variables involved. Here we shall separate the coordinate transformation from the definition of the finite elements.

The coordinate change may be described as follows. Let $T$ be a boundary triangle or tetrahedron, and let $\hat{T}$ be a standard reference triangle or tetrahedron. Let $\{ \underline{a}_i \}_{i=1}^M$ be a set of distinct points in $T$ and let $\{ \hat{\underline{a}}_i \}_{i=1}^M$ be a set of distinct points in $\hat{T}$. Let $\{ \hat{q}_i \}_{i=1}^M$ be a finite dimensional set of functions such that

$$\hat{q}_i(\hat{\underline{a}}_j) = \delta_{ij}, \qquad 1 \leqslant i, j \leqslant M ,$$

and let $\hat{Q}$ be the space of functions spanned by the $\hat{q}_i$. Let $F$ be the mapping

$$F = \sum_{i=1}^M \hat{q}_i \, \underline{a}_i \, . \tag{5.1}$$

Note that $F(\hat{\underline{a}}_i) = \underline{a}_i$. The mapping $F$ has been shown to be 1-1 for sufficiently refined triangulations in [6]. It is usual to take $\hat{Q}$ to be a space of polynomials, and here we take them to be the set of all polynomials of degree $k$ or less.

The boundary triangle or tetrahedron $T$ is defined by $T = F(\hat{T})$. The edges or surfaces of $T$ which are along the boundary of $\Omega$ will not coincide with $\Gamma$ but will be a polynomial approximation to $\Gamma$. Thus in this procedure the region $\Omega$ is replaced by a region $\Omega_h$ with piecewise polynomial boundary $\Gamma_h$. For boundary conditions $\underline{u}\,|_\Gamma = 0$, it is easy to show in the same manner as in [6, Section 1] that the inequality (2.9) holds when the boundary is approximated and isoparametric elements are used except that all norms are over the approximate region $\Omega_h$ rather than over $\Omega$. It is necessary, however, that $\Gamma_h$ consist of piecewise polynomials of degree $k$ in order that an interpolant $\underline{\tilde{u}}^h$ to $\underline{u}$, the solution to (2.2)-(2.3), can be constructed such that $\tilde{u}_i^h \in H_0^1(\Omega_h)$ and $(\bar{2}.10)$ holds for $r = k + 1$.

On boundary triangles or tetrahedra, we suppose elements of $P^h$ are given by functions of the form $\eta^h = \hat{\eta}^h . F^{-1}$, where $F$ is the mapping defined by (5.1) and $\hat{\eta}^h$ is a polynomial of degree $k - 1$ or less defined on $\hat{T}$. For $N = 2$, for boundary triangles $T$ let $\Pi_T$ consist of all functions of the form $v^h = \hat{v}^h \cdot F^{-1}$ where $\hat{v}^h$ is a polynomial of degree $k$ or less defined on $\hat{T}$ along with certain additional functions similar to those included in $\Pi_T$ in Section 2. To define these additional functions, let $\hat{\lambda}_i$, $i = 1, 2, 3$, denote the barycentric coordinates of a point $(\hat{x}, \hat{y}) \in R^2$ with respect to the vertices of $\hat{T}$, and let $\lambda_i = \hat{\lambda}_i \cdot F^{-1}$. In the first component of $\underline{V}^h$, we include in $\Pi_T$ all functions of the form $\lambda_1 \lambda_2 \lambda_3$ $\psi$ where $\psi = \frac{\partial}{\partial x}((\hat{x}^r \cdot F^{-1})(\hat{y}^s \cdot F^{-1}))$, $r + s = k - 1$, and in the second component, we include all functions of the form $\lambda_1 \lambda_2 \lambda_3 \psi$ where

$$\psi = \frac{\partial}{\partial y}((\hat{x}^r \cdot F^{-1})(\hat{y}^s \cdot F^{-1})), \quad r + s = k - 1 .$$

THEOREM 5.1 : *Let $\underline{V}^h = V^h \times V^h$, where $V^h$ is defined by* (3.1), *where $\Pi_T$ for curved boundary triangles is defined above, and let $P^h$ consist of locally defined functions which on each triangle are given by $\eta^h = \hat{\eta}^h \cdot F^{-1}$, where $F$ is the mapping defined by* (5.1) *and $\hat{\eta}^h$ is a polynomial of degree $k - 1$ or less on the reference triangle $\hat{T}$. Then given $q^h \in P^h$, there exists a function $\underline{v}^h \in \underline{V}^h$ such that* (2.7)-(2.8) *holds.*

*Proof* : For $q^h \in P^h$, let the functions $\underline{v}$, $\underline{z}$, and $\underline{u}^h$ be defined as in the proof of theorem 3.1. We define a function $\underline{z}^h \in \underline{V}^h$ by (i)-(ii) in the proof of theorem 3.1 along with

$$\int_T \underline{z}^h(\hat{x}_1^r \cdot F^{-1})(\hat{x}_2^s \cdot F^{-1})\, dx_1\, dx_2 = \int_T \underline{z}(\hat{x}_1^r \cdot F^{-1})(\hat{x}_2^s \cdot F^{-1})\, dx_1\, dx_2 ,$$

$$0 \leqslant r + s \leqslant k - 3 , \quad \text{(iii)}$$

$$\int_T \frac{\partial}{\partial x_i}((\hat{x}_1^r \cdot F^{-1})(\hat{x}_2^s \cdot F^{-1}))\, z_i^h\, dx_1\, dx_2 - \int_{\partial T} \nu_i(\hat{x}_1^r \cdot F^{-1})(\hat{x}_2^s \cdot F^{-1})\, z_i^h\, d\sigma =$$

$$= \int_T \frac{\partial}{\partial x_i}(\hat{x}_1^r \cdot F^{-1})(\hat{x}_2^s \cdot F^{-1})\, z_i\, dx_1\, dx_2 - \int_{\partial T} \nu_i(\hat{x}_1^r \cdot F^{-1})(\hat{x}_2^s \cdot F^{-1})\, z_i\, d\sigma$$

$$r + s = k - 1 , \quad i = 1, 2 .$$

By mapping to the reference triangle $\hat{T}$ it can be shown in the same manner as in the proof of theorem 3.1 that $\underline{z}^h$ is uniquely determined on each triangle $T \in \mathcal{T}_h$ by (i)-(v) and that

$$(\operatorname{div} \underline{z}^h, \phi^h) = (\operatorname{div} \underline{z}, \phi^h), \quad \text{all } \phi^h \in P^h .$$

Similarly $\| \underline{z}^h \|_{1,\Omega} \leqslant C_1 \| \underline{z} \|_{1,\Omega}$ and so $\underline{v}^h = \underline{z}^h + \underline{w}^h$ satisfies (2.7)-(2.8) for the given $q^h \in P^h$.

If on each triangle $T$, $P^h$ consists of transforms of polynomials of degree $k - 2$ on less on $\hat{T}$, (2.7)-(2.8) will be satisfied for $\underline{V}^h$ consisting on each triangle of transforms of polynomials of degree $k$ or less. Thus in this case the situation is identical for both straight and curved triangles.

For $\underline{V}^h$ in the case of straight triangles and $k = 2$, the set of piecewise quadratic polynomials was augmented by adding the single function $\lambda_1 \lambda_2 \lambda_3$ for each triangle. In the above, for curved triangles we had to add the two functions which are transforms from the reference triangle $\hat{T}$ of

$$\hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_3 \frac{\partial y}{\partial \hat{y}}, \qquad \hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_3 \frac{\partial y}{\partial \hat{x}}, \tag{5.3a}$$

in the first component and

$$\hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_3 \frac{\partial x}{\partial \hat{y}}, \qquad \hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_3 \frac{\partial x}{\partial \hat{x}}, \tag{5.3b}$$

in the second component.

For $N = 3$, for boundary tetrahedra let $\Pi_T$ consists of all functions of the form $v^h = \hat{v}^h \cdot F^{-1}$, where $\hat{v}^h$ is a polynomial of degree $k$ or less on $\hat{T}$ or a polynomial of the form $\lambda_i \lambda_{i+1} \lambda_{i+2} \hat{x}_1^l \hat{x}_2^m \hat{x}_3^n$, $l + m + n = k - 2$, $i = 1, 2, 3, 4$, along with, in the ith component of $\underline{V}^h$ $\lambda_1 \lambda_2 \lambda_3 \lambda_4 \psi$ where

$$\psi = \frac{\partial}{\partial x_i} ((\hat{x}_1^l \cdot F^{-1}) (\hat{x}_2^m \cdot F^{-1}) (\hat{x}_3^n \cdot F^{-1})), \quad l + m + n = k - 1 .$$

As for $N = 2$ for triangles, one can prove for $N = 3$ for tetrahedra.

THEOREM 5.2 : *Let $\underline{V}^h = (V^h)^3$, where $V^h$ is defined by* (4.1) *where $\Pi_T$ for curved boundary tetrahedra is defined above, and let $P^h$ consist of locally defined functions which on each tetrahedron are given by $\phi^h = \hat{\phi}^h \cdot F^{-1}$ where $F$ is the mapping defined by* (5.1) *and $\hat{\phi}^h$ is a polynomial of degree $k - 1$ or less on the reference tetrahedron. Then given $q^h \in P^h$, there exists a function $\underline{v}^h \in \underline{V}^h$ such that* (2.7)-(2.8) *holds.*

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