

RAIRO. ANALYSE NUMÉRIQUE

J. R. ROCHE

On the sensitivity of the matrix exponential problem

RAIRO. Analyse numérique, tome 15, n° 3 (1981), p. 249-255

http://www.numdam.org/item?id=M2AN_1981__15_3_249_0

© AFCET, 1981, tous droits réservés.

L'accès aux archives de la revue « RAIRO. Analyse numérique » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

*Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques*
<http://www.numdam.org/>

ON THE SENSITIVITY OF THE MATRIX EXPONENTIAL PROBLEM (*)

par J R ROCHE ⁽¹⁾

Communique par F ROBERT

*Resume — On discute le probleme de comparer les applications $\text{Exp}(At)$ et $\text{Exp}((A + B)t)$ ou la matrice B est consideree comme une perturbation de A
On montre que ce probleme est fortement lie a la multiplicité des valeurs propres de A et $A + B$
En conclusion, on montre que l'application $\text{Exp}(At)$ est moins affectee par les perturbations de A , si le spectre de A est simple*

*Abstract — We discuss the problem of comparing the mapping $\text{Exp}(At)$ and $\text{Exp}((A + B)t)$ where the square matrix B is considered as a perturbation of A
We show that this problem is strongly related to the multiplicity of eigenvalues of A and $A + B$
In conclusion, we set that the matrices A , for which $\text{Exp}(At)$ is less sensitive to perturbations, are those which have a simple spectrum*

I. INTRODUCTION

Many models of physical, biological and economic processes involve systems of linear, constant coefficient ordinary differential equations

$$\begin{aligned} \dot{X}(t) &= AX(t) \\ X(0) &= I_{n \times n}, t \geq 0 \end{aligned} \tag{1}$$

where A is a fixed square matrix, of dimension n

The solution is given by $X(t) = \text{Exp}(At)$, where $\text{Exp}(At)$ can be formally defined by

$$\text{Exp}(At) = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}, t \geq 0, A^0 = I$$

(*) Reçu le 5 novembre 1980

(¹) Laboratoire IMAG, Grenoble

The subject of this paper concerns the sensitivity of the quantity $\text{Exp}(At)$ with respect to a perturbation of A .

Van Loan [4] has suggested that the problem under consideration is related to the behaviour of the function :

$$\theta(t) = \frac{\| \text{Exp}((A + B) t) - \text{Exp}(At) \|}{\| \text{Exp}(At) \|}$$

as t tends to infinity.

We are going to show that $\theta(t)$ is a quantity related not only to the structure of A , but also to the structure of B .

It follows that it is not possible to characterize those A for which $\text{Exp}(At)$ is very sensitive to changes in A .

Then we study the quantity :

$$\phi(t) = \frac{\| \text{Exp}(At) - \text{Exp}(Dt) \|}{\text{Max} \{ \| \text{Exp}(Dt) \|, \| \text{Exp}(At) \| \}}$$

when t tends to infinity.

A characterization of $\phi(t)$ is given as a function of the structure of A and D .

II. NOTATIONS AND SOME PRELIMINARY LEMMAS

Let us note $\sigma(A)$ the spectrum of A ,

$$\rho(A) = \mathbb{C} - \sigma(A), \tag{2.1}$$

$$\alpha(A) = \text{Max} \{ \text{Re}(\lambda)/\lambda \in \sigma(A) \}, \tag{2.2}$$

$$A^* = (\bar{a}_{ij}) \tag{2.3}$$

$\text{Det}(A)$ the determinant of A .

We shall work exclusively with the 2-norms :

$$\| x \| = \left[\sum_{i=1}^n |x_i|^2 \right]^{1/2}, \| A \| = \max_{\|x\|=1} \| Ax \|. \tag{2.4}$$

LEMMA 1 : *Let A be a matrix $n \times n$ and $\sigma(A)$ its spectrum. Let Γ be a closed jordan curve in \mathbb{C} around $\sigma(A)$ which contains no point of $\sigma(A)$. Then*

$$\text{Exp}(At) = \frac{1}{2\pi i} \int_{\Gamma} (zI - A)^{-1} e^{zt} dz. \tag{2.5}$$

Proof [2].

LEMMA 2 [Souriau's form] : Let A be a matrix of dimension n . If :

$$\text{Det}(z) = \text{determinant of } (zI - A)$$

$A_0 = I$ the identity of dimension n ,

$$c_{n-k} = - \text{trace}(A_{k-1} * A)$$

$$A_k = A_{k-1} * A - c_{n-k} I; k = 1, \dots, n - 1.$$

Then the resolvent

$$(zI - A)^{-1} = \sum_{k=0}^{n-1} \frac{z^{n-k-1}}{\text{Det}(z)} A_k. \tag{2.6}$$

Proof[1].

LEMMA 3 : Let $f : \mathbb{C} \times \mathbb{R} \rightarrow \mathbb{C}$ be the function defined by

$$f(z, t) = \frac{z^l e^{zt}}{\prod_{i=1}^n (z - \lambda_i)} ; \lambda_i \in \mathbb{C}. \tag{2.7}$$

Then $\frac{d^k}{dz^k} f(z, t) = e^{zt} p(z, t)$, where $p(z, t)$ is a polynomial of degree k in t , with coefficient of t^k equal to $z^l \left/ \prod_{i=1}^n (z - \lambda_i) \right.$.

Proof [3].

III. THE ANALYSIS OF $\theta(t)$

Van Loan [4] has concluded that the bounds of $\theta(t)$ for normal matrices are as small as it can be expected. Furthermore, when A is normal the $\text{Exp}(At)$ problem is « well conditioned ».

We are going to give an example of a normal matrix such that for different choices of B , $\theta(t)$ behaves as a constant or an exponential when t tends to infinity.

Let A be a square normal matrix.

Let $\sigma(A) = \{ \lambda_i \}$ and

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_n.$$

Let B be a square matrix such that $\sigma(A + B) = \{ \mu_i \}$ is real and simple.

By lemmas 1, 2, 3 we have

$$\theta(t) = \frac{1}{e^{\lambda_N t}} \left\| \sum_{k=0}^{n-1} D_k \sum_{p=1}^n \frac{\mu_p^{n-k-1} e^{\mu_p t}}{\prod_{i \neq p} (\mu_p - \mu_i)} - \sum_{k=1}^{n-1} A_k \sum_{p=1}^n \frac{\lambda^{n-k-1} e^{\lambda_p t}}{\prod_{i \neq p} (\lambda_p - \lambda_i)} \right\| \tag{3.1}$$

where $D_k = (A + B)_k$ in the Souriau's form.

It is easy to show that :

$$\left\| \sum_{k=1}^{n-1} A_k \sum_{p=1}^n \lambda_p^{n-k-1} \frac{e^{(\lambda_p - \lambda_n)t}}{\prod_{i \neq p} (\lambda_p - \lambda_i)} \right\| \tag{3.2}$$

converges to :

$$\left\| \sum_{k=0}^{n-1} \frac{\lambda^{n-k-1} A_k}{\prod_{i \neq p} (\lambda_p - \lambda_i)} \right\|, \text{ as } t \text{ tends to infinity.} \tag{3.3}$$

If $\lambda_n < \mu_n$ then

$$\left\| \sum_{k=0}^{n-1} D_k \sum_{p=1}^n \mu_p^{n-k-1} \frac{e^{(\mu_p - \lambda_n)t}}{\prod_{i \neq p} (\mu_p - \mu_i)} \right\| \tag{3.4}$$

tends to the infinity like $e^{(\mu_n - \lambda_n)t}$ as t tends to the infinity. If $0 < \mu_i < \lambda_i$; $i = 1, \dots, n$ then (3.4) tends to

$$\sum_{k=0}^{n-1} D_k \sum_{p=1}^n \frac{\mu_p^{n-k-1}}{\prod_{i \neq p} (\mu_p - \mu_i)}, \text{ as } t \text{ tends to infinity.} \tag{3.5}$$

Then according to the structure of B , $\theta(t)$ may converge to infinity as e^{ct} , $c > 0$, or to a constant.

This exemple shows that the structure of A is not enough to characterize the behaviour of $\theta(t)$.

IV. THE MAIN THEOREM

In this section we introduce a function $\phi(t)$ which enables us to study the sensitivity of the problem $\text{Exp}(At)$. This function is symmetrical with respect to A and $A + B$.

If we note $D = A + B$:

$$\phi(t) = \frac{\| \text{Exp}(Dt) - \text{Exp}(At) \|}{\text{Max} \{ \| \text{Exp}(Dt) \|, \| \text{Exp}(At) \| \}}; \quad t \geq 0. \tag{4.1}$$

The main theorem is the following :

THEOREM : *Let A and D be two square matrices of dimension n and $\{ \lambda_1, \dots, \lambda_r \}$ equals $\sigma(A) \cup \sigma(D)$.*

If $\lambda_i \in \sigma(A) \cup \sigma(D) - \sigma(A) \cap \sigma(D)$ let m_i be the corresponding multiplicity of λ_i .

If $\lambda_i \in \sigma(A) \cap \sigma(D)$, let m_i be the sum of the multiplicity of λ_i as eigenvalue of A plus the multiplicity of λ_i as eigenvalue of D .

If $m = \max_{1 \leq i \leq r} (m_i)$, then $\phi(t) \leq \|D - A\| p(t)$ where $p(t)$ is a polynomial in t of degree less than m .

The proof of the theorem : By lemma 1 we have

$$\text{Exp}(At) - \text{Exp}(Dt) = \frac{1}{2\pi i} \int_{\Gamma} ((zI - A)^{-1} - (zI - D)^{-1}) e^{zt} dz, \quad (4.2)$$

where Γ is a closed Jordan curve in \mathbb{C} around $\sigma(A) \cup \sigma(D)$ which contains no point of $\sigma(A) \cup \sigma(D)$.

It follows that

$$\text{Exp}(At) - \text{Exp}(Dt) = \frac{1}{2\pi i} \int_{\Gamma} (zI - D)^{-1} (D - A) (zI - A)^{-1} e^{zt} dz. \quad (4.3)$$

If we set $c_1(z) = \text{Det}(zI - A)$ and $c_2(z) = \text{Det}(zI - D)$ then by lemma 2

$$(zI - A)^{-1} = \sum_{l=0}^{n-1} \frac{z^{n-l-1}}{c_1(z)} A_l, \quad (4.4)$$

and

$$(zI - D)^{-1} = \sum_{k=0}^{n-1} \frac{z^{n-k-1}}{c_2(z)} D_k.$$

This yields

$$\text{Exp}(At) - \text{Exp}(Dt) = \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} D_k (D - A) A_l \sum_{p=1}^r \text{Res} \left(\frac{z^{2n-l-k-2} e^{zt}}{\prod_{i=1}^r (z - \lambda_i)^{m_i}}, \lambda_p \right). \quad (4.6)$$

If λ_p is of multiplicity m_p

$$\text{Res} \left(\frac{z^{2n-l-k-2}}{\prod_{i=1}^r (z - \lambda_i)^{m_i}}, \lambda_p \right) = \frac{1}{(m_p - 1)!} \frac{d^{m_p-1}}{dz^{m_p-1}} \left(\frac{z^{2n-l-k-2} e^{zt} (z - \lambda_p)^{m_p}}{\prod_{i=1}^r (z - \lambda_i)^{m_i}} \right)_{z=\lambda_p}. \quad (4.7)$$

Then by lemma 3 we have :

$$\text{Exp}(At) - \text{Exp}(Dt) = \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} D_k(D - A) A_l \sum_{p=1}^r \frac{e^{\lambda_p t}}{(m_p - 1)!} p_{l,k}(\lambda_p t), \quad (4.8)$$

where $p_{l,k}(\lambda_p t)$ is a polynomial of degree less than or equal to $m_p - 1$, and the coefficient of $(m_p - 1)$ th power of t is :

$$\frac{\lambda_p^{2n-k-l-2}}{\prod_{\substack{i=1 \\ i \neq p}}^r (\lambda_p - \lambda_i)^{m_i}}. \quad (4.9)$$

If we note $p_{l,k}(\lambda_p t) = \sum_{i=0}^s n_i t^i$, we write

$$q_{l,k}(\lambda_p t) = \sum_{i=0}^s |n_i| t^i. \quad (4.10)$$

Then if $t \geq 0$,

$$|p_{l,k}(\lambda_p t)| \leq q_{l,k}(\lambda_p t). \quad (4.11)$$

It follows that

$$\|\text{Exp}(At) - \text{Exp}(Dt)\| \leq \|D - A\| \sum_{p=1}^r \frac{|e^{\lambda_p t}|}{(m_p - 1)!} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \|D_k\| \|A_l\| q_{l,k}(\lambda_p t). \quad (4.12)$$

But $|e^{\lambda_p t}| \leq \text{Max} \{ \|\text{Exp}(Dt)\|, \|\text{Exp}(At)\| \}$, then

$$\phi(t) \leq \|D - A\| \sum_{p=1}^r \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \|D_k\| \|A_l\| q_{l,k}(\lambda_p t). \quad (4.13)$$

If we set

$$p(t) = \sum_{p=1}^r \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \|D_k\| \|A_l\| q_{l,k}(\lambda_p t). \quad (4.14)$$

Then

$$\phi(t) \leq \|D - A\| p(t) \quad (4.15)$$

where $p(t)$ is a polynomial of degree at most $m - 1$.

We can remark that if a non zero eigenvalue of A with multiplicity m , exist then $p(t)$ is a polynomial of degree exactly $m - 1$.

If all the eigenvalues of A and D are simple and $\sigma(A) \cap \sigma(D) = \emptyset$ then $\phi(t)$ is bounded by a constant.

V. CONCLUSION

We have shown that the function $\theta(t)$ is insufficient to characterize the matrices for which the mapping $\text{Exp}(At)$ is sensitive to changes in A .

We have introduced a function $\phi(t)$ which measures the relative distance between $\text{Exp}(At)$ and $\text{Exp}((A + B)t)$. In the main theorem we show that the behaviour of the bound of $\phi(t)$ depends on the multiplicity of the eigenvalues of A and $A + B$. Another factor is the distance between two different eigenvalues, but it's a secondary factor as it modifies the coefficients of $p(t)$ but not the degree.

This fact agrees with the conclusion obtained in section 3 by a formal development of $\text{Exp}(At)$.

The analysis of the (4.15) bound of $\phi(t)$ lead us to conclude that if A in a matrix with a simple spectrum, the mapping $\text{Exp}(At)$ is less sensitive to change on A , because the degree of $p(t)$ may be at most n .

REFERENCES

- [1] N. GASTINEL, *Analyse Numérique Linéaire*. Hermman, Paris.
- [2] C. C. MAC DUFFEE, *The Theory of matrices*. Chelsea Publishing Company. New York (1956).
- [3] J. R. ROCHE, Thèse 3^e cycle, U.S.M.G. Grenoble (1980).
- [4] Charles VAN LOAN, *The sensitivity of the matrix exponential*. S.I.A.M. Journal Numer. Anal. Vol. 14, n° 6 (1977).