RAIRO. ANALYSE NUMÉRIQUE

CLAUDIO CANUTO

A hybrid finite element method to compute the free vibration frequencies of a clamped plate

RAIRO. Analyse numérique, tome 15, nº 2 (1981), p. 101-118

<http://www.numdam.org/item?id=M2AN_1981__15_2_101_0>

R A I R O Analyse numérique/Numerical Analysis
(vol 15, n° 2, 1981, p 101 à 118)

A HYBRID FINITE ELEMENT METHOD TO COMPUTE THE FREE VIBRATION FREQUENCIES OF A CLAMPED PLATE (*) (**)

by Claudio CANUTO (1)

Communicated by E MAGENES

Abstract — *An " assumed stresses " hybrid method is studied to provide (external) approximations to the plate vibration problem, variationally formulated on some space which is orthogonal to all selfequilibrating stresses Convergence results and error estimates are derived by a Rayleigh principle*

Resumé — *On utilise une méthode d'elements finis « hybrides duaux » pour approcher le problème des vibrations libres d'une plaque encastrée, le problème a été formulé à l'aide d'un principe variationnel sur un espace orthogonal aux tenseurs auto-équilibrés On demontre la convergence de l'approximation et on déduit les estimations d'erreur par un principe du type Rayleigh*

0. INTRODUCTION

This paper deals with the approximation of the vibration frequencies and modes of a clamped plate, via a " dual hybrid " finite element method first introduced by Tabarrok [19] and based on a variational principle due to Toupin [20].

The corresponding method for the static case, proposed by Pian and Tong (see e.g. [15]) and analyzed by Brezzi [4] and Brezzi-Marini [5], essentially involves the tensors « selfequilibrating » over each element; on the contrary the natural space for the dynamical case is some orthogonal complement to the tensors selfequilibrating over the whole domain, in that way we can reduce our problem to a spectral problem for a compact selfadjoint operator T, whose finite dimensional approximation is however not of inner type.

(*) Received in December 1979
AMS (MOS) subject classification (1980) Primary 65N15, 65N25, 65N30
(**) Work partially supported by the G N I M -C N R and the L A N -C N R (Pavia)
(1) Istituto di Matematica Applicata, Università di Pavia, 27100 Pavia, Italie

Convergence and error estimates are then derived by a Rayleigh-Ritz principle; since we cannot prove the uniform convergence of the approximate operators T_h to T, we check an approximability property over suitable finite dimensional spaces, similar to Property P_2 by Rappaz [17] (actually, our approach is analogous to the one he uses for a second order problem; see also Rappaz [16], and Anselone [1] whose notion of " collective compactness ", however, does not apply in the present case). Spectral approximations by methods of hybrid type are investigated by Kolata [11], Mercier-Rappaz [13], Mercier-Osborn-Rappaz-Raviart [14], who all assume $T_h \to T$ in norm.

From a computational point of view, Brandt [2] has carried out calculations of eigenvalues for a square (cantilever or simply-supported) plate using polynomial tensors of rather high degree; a good accuracy can be achieved using polynomials of suitable lower degree, according to the theory here presented (see Gilardi [21]).

1. THE EIGENVALUE PROBLEM

Let Ω be a convex bounded polygon in $\mathbb{R}^2$, whose boundary will be denoted by Γ; Ω will represent the mean section of a thin elastic plate clamped along its edges, whose mass density and flexural rigidity are assumed, for the sake of simplicity, to be constant and equal to 1.

We define the Hilbert space of stresses

$$\mathscr{S} = \{ \underset{\sim}{v} \mid \underset{\sim}{v} = \{v_{ij}\}_{1 \leq i,j \leq 2} \quad \text{with} \quad v_{ij} \in L^2(\Omega) \quad \text{and} \quad v_{12} = v_{21} \}$$

with inner product

$$(\underset{\sim}{u}, \underset{\sim}{v}) = \int_\Omega u_{ij}\, v_{ij}\, dx \tag{1}$$

and associated norm $| \underset{\sim}{v} |^2 = (\underset{\sim}{v}, \underset{\sim}{v})$, and the subspace

$$S = \{ \underset{\sim}{v} \in \mathscr{S} \mid \operatorname{Div} \underset{\sim}{v} \in L^2(\Omega) \}$$

where $\operatorname{Div} \underset{\sim}{v} = v_{ij/ij}$ (here and in the following repeated indices imply summation over 1, 2, and $\cdot/i$ means differentiation with respect to the x_i-variable in the distributional sense). S is equipped with the graph norm. Then, according to Toupin's complementary energy principle [20], $\omega \neq 0$ is a free vibration frequency for our plate iff the functional

$$J_\omega(\underset{\sim}{v}) = \frac{1}{2} \int_\Omega (v_{ij/ij})^2\, dx - \frac{\omega^2}{2} \int_\Omega v_{ij}\, v_{ij}\, dx$$

has a non-zero stationary point over S.

Following Tabarrok [19], we can modify the functional J_ω in order to weaken the regularity required on assumed stresses. To this end we establish a decomposition $\mathcal{T}_h$ of Ω into convex subdomains, to which we associate the Hilbert space

$$V(\mathcal{T}_h) = \{ \underset{\sim}{v} \in \mathcal{S} \mid \mathrm{Div}\,(\underset{\sim}{v}\,|_K) \in L^2(K)\,, \quad \forall K \in \mathcal{T}_h \}$$

with norm $\| \underset{\sim}{v} \|^2 = | \underset{\sim}{v} |^2 + e(\underset{\sim}{v}, \underset{\sim}{v})$ where

$$e(\underset{\sim}{u}, \underset{\sim}{v}) = \sum_{K \in \mathcal{T}_h} \int_K \mathrm{Div}\, \underset{\sim}{u}\; \mathrm{Div}\, \underset{\sim}{v}\; dx\,.$$

We also define the continuous bilinear form on $V(\mathcal{T}_h) \times H_0^2(\Omega)$:

$$b(\underset{\sim}{v}, \varphi) = \sum_{K \in \mathcal{T}_h} \int_K (v_{ij/ij}\, \varphi - v_{ij}\, \varphi/ij)\, dx$$

$$= \sum_{K \in \mathcal{T}_h} \int_{\partial K} \left(Q(\underset{\sim}{v})\, \varphi - M_n(\underset{\sim}{v}) \frac{\partial \varphi}{\partial \underline{n}} - M_{nt}(\underset{\sim}{v}) \frac{\partial \varphi}{\partial \underline{t}} \right) d\tau \tag{2}$$

where $\underline{n} = (n_1, n_2)$ and $\underline{t} = (t_1, t_2)$ are the outward normal and counter-clockwise tangent vectors to ∂K, and the quantities

$$M_n(\underset{\sim}{v}) = v_{ij}\, n_i\, n_j\,, \quad M_{nt}(\underset{\sim}{v}) = v_{ij}\, n_i\, t_j\,, \quad Q(\underset{\sim}{v}) = v_{ij/i}\, n_j$$

can be rigorously defined on ∂K for $\underset{\sim}{v} \in V(\mathcal{T}_h)$ (see Brezzi [4]). We define the closed subspace of $H_0^2(\Omega)$

$$W(\mathcal{T}_h) = \{ \varphi \in H_0^2(\Omega) \mid \Delta^2 \varphi = 0 \text{ in each } K \in \mathcal{T}_h \}$$

and recall the following properties of b :

$$\underset{\sim}{v} \in S \text{ iff } \underset{\sim}{v} \in V(\mathcal{T}_h) \quad \text{and} \quad b(\underset{\sim}{v}, \varphi) = 0 \quad \text{for every } \varphi \in W(\mathcal{T}_h); \tag{3}$$

$$\sup_{\underset{\sim}{v} \in V(\mathcal{T}_h) - \{\underset{\sim}{0}\}} b(\underset{\sim}{v}, \varphi)/\| \underset{\sim}{v} \| \geqslant \| \varphi \|_{H_0^2(\Omega)} \qquad \forall \varphi \in W(\mathcal{T}_h)\,. \tag{4}$$

The theory of Lagrange multipliers assures that $\underset{\sim}{u} \in S$ is a stationary point of J_ω over S iff there exists a « multiplier » $\psi \in W(\mathcal{T}_h)$ such that the functional

$$\mathcal{L}_\omega(\underset{\sim}{v}, \varphi) = J_\omega(\underset{\sim}{v}) + b(\underset{\sim}{v}, \varphi)$$

has a saddle point over $V(\mathcal{T}_h) \times W(\mathcal{T}_h)$ at $(\underset{\sim}{u}, \psi)$. Therefore, setting $\omega^2 = \lambda$ we are led to consider the following eigenvalue problem :

(EP) *Find the eingenvalues* $0 \neq \lambda \in \mathbb{R}$ *and the corresponding eigenvectors* $(\underset{\sim}{u}, \psi) \in V(\mathcal{T}_h) \times W(\mathcal{T}_h)$ *such that*

$$\left.\begin{aligned} e(\underset{\sim}{u}, \underset{\sim}{v}) + b(\underset{\sim}{v}, \psi) &= \lambda(\underset{\sim}{u}, \underset{\sim}{v}) \quad \forall \underset{\sim}{v} \in V(\mathcal{T}_h) \\ b(\underset{\sim}{u}, \varphi) &= 0 \qquad\qquad \forall \varphi \in W(\mathcal{T}_h) \end{aligned}\right\}. \tag{5}$$

Note that formulation (5) admits the null eigenvalue with infinite multiplicity; namely its eigenspace is $S^0 \times \{0\}$, where

$$S^0 = \{ \underset{\sim}{v} \in \mathcal{S} \mid \mathrm{Div}\, \underset{\sim}{v} = 0 \quad \text{in}\, \Omega \}$$

is the closed subspace of $\mathcal{S}$ of the stresses " self-equilibrating " in Ω; on the other hand every $\underset{\sim}{u}$ in the eigenspace of a non-zero eigenvalue of (EP) is orthogonal to S^0 in the inner product of $\mathcal{S}$. So it is natural to consider the orthogonal complement of S^0 in each space we have just defined, by setting

$$\mathcal{S}(\perp) = \text{orthogonal complement of } S^0 \text{ in } \mathcal{S}\,;$$

$$S(\perp) = \mathcal{S}(\perp) \cap S\,; \quad V(\mathcal{T}_h\,; \perp) = \mathcal{S}(\perp) \cap V(\mathcal{T}_h)$$

and reduce problem (EP) to finding the eigenvalues $\lambda \in \mathbb{R}$ and the eigenvectors $(\underset{\sim}{u}, \psi) \in V(\mathcal{T}_h\,; \perp) \times W(\mathcal{T}_h)$ such that

$$\left.\begin{aligned} e(\underset{\sim}{u}, \underset{\sim}{v}) + b(\underset{\sim}{v}, \psi) &= \lambda(\underset{\sim}{u}, \underset{\sim}{v}) \quad \forall \underset{\sim}{v} \in V(\mathcal{T}_h\,; \perp) \\ b(\underset{\sim}{u}, \varphi) &= 0 \qquad\qquad \forall \varphi \in W(\mathcal{T}_h) \end{aligned}\right\}. \tag{6}$$

The following characterization will be used throughout the paper :

PROPOSITION 1.1 : *The map* $M : \varphi \mapsto \{ \varphi_{/ij} \}$ *maps* $H_0^2(\Omega)$ *onto* $\mathcal{S}(\perp)$ *isometrically.*

Proof : The result is consequence of the Closed Range Theorem since $M : H_0^2(\Omega) \to \mathcal{S}$ is the adjoint operator of $\mathrm{Div} : \mathcal{S} \to H^{-2}(\Omega)$. ■

If $\underset{\sim}{v} \in S(\perp)$, then $\underset{\sim}{v} = M\varphi$ with $\varphi \in H_0^2(\Omega)$, $\Delta^2 \varphi \in L^2(\Omega)$, hence $\varphi \in H^3(\Omega)$ at least (see Grisvard [10] or Kondrat'ev [12]) with $\| \varphi \|_{H^3(\Omega)} \leqslant C \| \underset{\sim}{v} \|$. In particular the inclusion $S(\perp) \subseteq \mathcal{S}(\perp)$ is compact, and $S(\perp)$ is dense in $\mathcal{S}(\perp)$ since $\mathcal{D}(\Omega)$ is dense in $H_0^2(\Omega)$. Moreover $e(\underset{\sim}{v}, \underset{\sim}{v}) = \int_\Omega (\Delta^2 \varphi)^2\, dx$ for $\underset{\sim}{v} \in S(\perp)$, so the form e is $V(\mathcal{T}_h)$-coercive over $S(\perp)$ (note that e is not coercive over $V(\mathcal{T}_h\,; \perp)$, since $e(M\varphi, M\varphi) = 0$ for every $\varphi \in W(\mathcal{T}_h)$); finally the form b is also « coercive » with respect to $V(\mathcal{T}_h\,; \perp)$, that is

$$\sup_{\underset{\sim}{v} \in V(\mathcal{T}_h\,;\perp) - \{\underset{\sim}{0}\}} b(\underset{\sim}{v}, \varphi)/\| \underset{\sim}{v} \| \geqslant \| \varphi \|_{H_0^2(\Omega)} \quad \forall \varphi \in W(\mathcal{T}_h) \tag{7}$$

since $\underset{\sim}{v} = M\varphi \in V(\mathcal{T}_h ; \perp)$. Therefore problem (EP) is equivalent to :

Find $\lambda \in \mathbb{R}$ *and* $\underset{\sim}{u} \in S(\perp)$ *such that*

$$e(\underset{\sim}{u}, \underset{\sim}{v}) = \lambda(\underset{\sim}{u}, \underset{\sim}{v}) \qquad \forall \underset{\sim}{v} \in S(\perp), \tag{8}$$

which in turn is equivalent to an eigenvalue problem for a compact selfadjoint positive operator T over $\mathcal{S}(\perp)$. Actually, by replacing e by the coercive form on $V(\mathcal{T}_h)$ $\tilde{e}(\underset{\sim}{u}, \underset{\sim}{v}) = e(\underset{\sim}{u}, \underset{\sim}{v}) + (\underset{\sim}{u}, \underset{\sim}{v})$ (i.e. shifting by 1 every eigenvalue of (EP)), we can apply theorem 1.1 by Brezzi [3] to get — for every $\underset{\sim}{g} \in \mathcal{S}$ — unique solutions of the following problems :

Find $(\underset{\sim}{u}, \psi) \in V(\mathcal{T}_h) \times W(\mathcal{T}_h)$ *such that*

$$\left.\begin{array}{ll} \tilde{e}(\underset{\sim}{u}, \underset{\sim}{v}) + b(\underset{\sim}{v}, \psi) = (\underset{\sim}{g}, \underset{\sim}{v}) & \forall \underset{\sim}{v} \in V(\mathcal{T}_h) \\ b(\underset{\sim}{u}, \varphi) = 0 & \forall \varphi \in W(\mathcal{T}_h) \end{array}\right\}. \tag{9}$$

Find $(\underset{\sim}{u}', \psi') \in V(\mathcal{T}_h ; \perp) \times W(\mathcal{T}_h)$ *such that*

$$\left.\begin{array}{ll} \tilde{e}(\underset{\sim}{u}', \underset{\sim}{v}) + b(\underset{\sim}{v}, \psi') = (\underset{\sim}{g}, \underset{\sim}{v}) & \forall \underset{\sim}{v} \in V(\mathcal{T}_h ; \perp) \\ b(\underset{\sim}{u}', \varphi) = 0 & \forall \varphi \in W(\mathcal{T}_h) \end{array}\right\}. \tag{10}$$

Find $(\underset{\sim}{u}'', \psi'') \in V(\mathcal{T}_h)^0 \times W(\mathcal{T}_h)$ *such that*

$$\left.\begin{array}{ll} (\underset{\sim}{u}'', \underset{\sim}{v}) + b(\underset{\sim}{v}, \psi'') = (\underset{\sim}{g}, \underset{\sim}{v}) & \forall \underset{\sim}{v} \in V(\mathcal{T}_h)^0 \\ b(\underset{\sim}{u}'', \varphi) = 0 & \forall \varphi \in W(\mathcal{T}_h) \end{array}\right\} \tag{11}$$

where

$$V(\mathcal{T}_h)^0 = \{ \underset{\sim}{v} \in V(\mathcal{T}_h) \mid \operatorname{Div} \underset{\sim}{v} = 0 \text{ in each } K \in \mathcal{T}_h \}.$$

Clearly we are interested in the second problem, which defines the operator T : $\mathcal{S} \to \mathcal{S}$ as $Tg = u'$; since an inner finite dimensional approximation of problem (10) is not easy to achieve, we introduced problems (9) and (11) which are easier to deal with. On the other hand we shall show in the following property 1.1 that solving (10) can de reduced to solving (9) and (11).

For any $\varphi \in H_0^2(\Omega)$ we define $\tilde{\varphi}$ to be the unique function in $W(\mathcal{T}_h)$ satisfying

$$\left.\begin{array}{l} \tilde{\varphi} = \varphi \\ \tilde{\varphi}_{/i} = \varphi_{/i}, \quad i = 1, 2 \end{array}\right\} \text{ on } \bigcup_{K \in \mathcal{T}_h} \partial K ; \tag{12}$$

moreover we split $\underset{\sim}{u}$ into its orthogonal components

$$\underset{\sim}{u} = \underset{\sim}{u}^{\perp} + \underset{\sim}{u}^{0} \in S(\perp) \oplus S^{0}$$

where $\underset{\sim}{u}^{\perp} = M\Psi$, with $\Psi \in H_0^2(\Omega)$, $\Delta^2\Psi \in L^2(\Omega)$.

PROPERTY 1.1 : *The following relations hold* :

$$\begin{aligned} &\text{i) } \underset{\sim}{u}' = \underset{\sim}{u}^{\perp} \quad \textit{and} \quad \psi' = \psi \\ &\text{ii) } \underset{\sim}{u}'' = \underset{\sim}{u}^{0} \quad \textit{and} \quad \psi'' = \psi - \widetilde{\Psi}\,. \end{aligned} \tag{13}$$

Proof : By taking any $\underset{\sim}{v} \in V(\mathscr{T}_h; \perp)$ in (9.1) we check that $(\underset{\sim}{u}^{\perp}, \psi)$ is the solution of (10). On the other hand, by taking $\underset{\sim}{v} \in S^0 \subseteq V(\mathscr{T}_h)^0$ in (9) and (11) and substracting, we get $(\underset{\sim}{u} - \underset{\sim}{u}'', \underset{\sim}{v}) = 0 \ \forall \underset{\sim}{v} \in S^0$, i.e. $\underset{\sim}{u}''$ is the projection of $\underset{\sim}{u}$ over S^0, so $\underset{\sim}{u}'' = \underset{\sim}{u}^0$. Now if we substract (11) from (9) we have

$$(\underset{\sim}{u}^{\perp}, \underset{\sim}{v}) + b(\underset{\sim}{v}, \psi - \psi'') = 0 \qquad \forall \underset{\sim}{v} \in V(\mathscr{T}_h)^0\,.$$

But according to (1) and (2), $(\underset{\sim}{u}^{\perp}, \underset{\sim}{v}) = -\,b(\underset{\sim}{v}, \Psi) = -\,b(v, \widetilde{\Psi})$ hence

$$b(\underset{\sim}{v}, \psi - \psi'' - \widetilde{\Psi}) = 0 \quad \forall \underset{\sim}{v} \in V(\mathscr{T}_h)^0\,.$$

Picking $\underset{\sim}{v} = M(\psi - \psi'' - \widetilde{\Psi})$ we finish the proof. ∎

Splitting $\underset{\sim}{g} \in \mathscr{S}$ into its orthogonal components $\underset{\sim}{g} = \underset{\sim}{g}^{\perp} + \underset{\sim}{g}^{0} \in \mathscr{S}(\perp) \oplus S^0$ with $\underset{\sim}{g}^{\perp} = M\Phi$, $\Phi \in H_0^2(\Omega)$, we obtain the following characterization :

PROPERTY 1.2 :

$$\begin{aligned} &\text{i) } \Psi \in H_0^2(\Omega) \textit{ is the solution of the problem } \Delta^2\Psi + \Psi = \Phi \textit{ in } \Omega\,; \\ &\text{ii) } \underset{\sim}{u}^0 = \underset{\sim}{g}^0\,; \\ &\text{iii) } \psi = \widetilde{\Psi} - \tilde{\Phi}\,; \\ &\text{iv) } \psi'' = \tilde{\Phi}. \end{aligned} \tag{14}$$

Proof : Integrating by parts in (10.1) (where we can replace $\underset{\sim}{g}$ by $\underset{\sim}{g}^{\perp}$) we get

$$\sum_{K \in \mathscr{T}_h} \int_K (\Delta^2\Psi + \Psi - \Phi)\, v_{ij/ij}\, dx + b(\underset{\sim}{v}, \psi - (\Psi - \Phi)) = 0 \quad \forall \underset{\sim}{v} \in V(\mathscr{T}_h; \perp)$$

which gives i) and iii), while ii) and iv) are consequences of (12.ii)). ∎

2. FINITE DIMENSIONAL APPROXIMATION

Let $\{ \mathscr{T}_h \}_{h>0}$ be a regular family of triangulations of Ω, each $K \in \mathscr{T}_h$ being a triangle whose diameter is less than h (of course we may consider quadrilaterals as well). For any integer $m \geqslant 2$ we set

$$V_h = \{ \underset{\sim}{v} \in V(\mathcal{T}_h) \mid v_{ij|K} \in P_m \quad \forall K \in \mathcal{T}_h \} \tag{15}$$

where P_m is the space of polynomials in x_1, x_2 of degree $\leqslant m$. For any integer $r \geqslant 3$, $s \geqslant 1$ we set

$$W_h = W_h(r, s) = \left\{ \varphi \in W(\mathcal{T}_h) \mid \forall K \in \mathcal{T}_h, \varphi|_{\partial K} \in P_r(\partial K) \text{ and } \left.\frac{\partial \varphi}{\partial \underline{n}}\right|_{\partial K} \in P_s(\partial K) \right\}$$

where $P_p(\partial K)$ denotes the space of the functions defined on ∂K which are polynomials of degree $\leqslant p$ on each side. We consider the following approximation of problem (EP) :

$(\mathrm{EP})_h$ *Find* $0 \neq \lambda_h \in \mathbb{R}$ *and* $(\underset{\sim}{u}_h, \psi_h) \in V_h \times W_h$ *such that*

$$\left.\begin{aligned} e(\underset{\sim}{u}_h, \underset{\sim}{v}_h) + b(\underset{\sim}{v}_h, \psi_h) &= \lambda_h(\underset{\sim}{u}_h, \underset{\sim}{v}_h) \quad && \forall v_h \in V_h \\ b(\underset{\sim}{u}_h, \varphi_h) &= 0 && \forall \varphi_h \in W_h \end{aligned}\right\} . \tag{16}$$

If we introduce the finite dimensional approximations of the spaces defined in section 1 :

$$\begin{aligned} S_h &= \{ \underset{\sim}{v}_h \in V_h \mid b(\underset{\sim}{v}_h, \varphi_h) = 0 \quad \forall \varphi_h \in W_h \} \\ S_h^0 &= V(\mathcal{T}_h)^0 \cap S_h \\ V_h(\perp) &= \text{orthogonal complement of } S_h^0 \text{ in } V_h \\ S_h(\perp) &= V_h(\perp) \cap S_h\,, \quad V_h^0 = V(\mathcal{T}_h)^0 \cap V_h \end{aligned}$$

then $S_h^0 \times \{0\}$ is the null eigenspace for $(\mathrm{EP})_h$, while the eigenspace of any eigenvalue $\lambda_h \neq 0$ is contained in $S_h(\perp) \times W_h$. Note that $V_h(\perp) \not\subseteq V(\mathcal{T}_h; \perp)$. According to Brezzi-Marini [5], we choose the parameters m, r, s satisfying $m \geqslant \max(r - 2, s - 1)$ (unless $r - 2 = s - 1$ and s is even, in which case $m \geqslant s$), so that the following estimate holds

$$\sup_{\underset{\sim}{v}_h \in V_h^0 - \{\underset{\sim}{0}\}} b(\underset{\sim}{v}_h, \varphi_h)/| \underset{\sim}{v}_h | \geqslant \gamma \| \varphi_h \|_{H_0^2(\Omega)} \qquad \forall \varphi_h \in W_h$$

with $\gamma > 0$ independent of h. For any $\varphi_h \in W_h - \{0\}$, let $\bar{\underset{\sim}{v}}_h \in V_h - \{\underset{\sim}{0}\}$ be an element on which the « sup » is attained ; splitting $\bar{\underset{\sim}{v}}_h$ as

$$\bar{\underset{\sim}{v}}_h = \underset{\sim}{v}_h^0 + \underset{\sim}{v}_h^\perp \in S_h^0 \oplus V_h(\perp) \quad \text{we have} \quad b(\underset{\sim}{v}_h^\perp, \varphi_h) = b(\bar{\underset{\sim}{v}}_h, \varphi_h) \neq 0\,,$$

hence $\underset{\sim}{v}_h^\perp \neq 0$ and $\| \underset{\sim}{v}_h^\perp \| = | \underset{\sim}{v}_h^\perp | \leqslant | \bar{\underset{\sim}{v}}_h |$, so that

$$\sup_{\underset{\sim}{v}_h \in V_h(\perp) - \{\underset{\sim}{0}\}} b(\underset{\sim}{v}_h, \varphi_h)/\| \underset{\sim}{v}_h \| \geqslant \gamma \| \varphi_h \|_{H_0^2(\Omega)} \quad \forall \varphi_h \in W_h\,. \tag{18}$$

In particular, for any $\underset{\sim}{g} \in \mathcal{S}$ we can uniquely solve the following problems, which approximate respectively (9), (10), (11) :

Find $(\underset{\sim}{u}_h, \psi_h) \in V_h \times W_h$ *such that*

$$\left.\begin{array}{ll}\tilde{e}(\underset{\sim}{u}_h, \underset{\sim}{v}_h) + b(\underset{\sim}{v}_h, \psi_h) = (\underset{\sim}{g}, \underset{\sim}{v}_h) & \forall \underset{\sim}{v}_h \in V_h \\ b(\underset{\sim}{u}_h, \varphi_h) = 0 & \forall \varphi_h \in W_h\end{array}\right\} . \tag{19}$$

Find $(\underset{\sim}{u}'_h, \psi'_h) \in V_h(\perp) \times W_h$ *such that*

$$\left.\begin{array}{ll}\tilde{e}(\underset{\sim}{u}'_h, \underset{\sim}{v}_h) + b(\underset{\sim}{v}_h, \psi'_h) = (\underset{\sim}{g}, \underset{\sim}{v}_h) & \forall \underset{\sim}{v}_h \in V_h(\perp) \\ b(\underset{\sim}{u}'_h, \varphi_h) = 0 & \forall \varphi_h \in W_h\end{array}\right\} . \tag{20}$$

Find $(\underset{\sim}{u}''_h, \psi''_h) \in V_h^0 \times W_h$ *such that*

$$\left.\begin{array}{ll}(\underset{\sim}{u}''_h, \underset{\sim}{v}_h) + b(\underset{\sim}{v}_h, \psi''_h) = (\underset{\sim}{g}, \underset{\sim}{v}_h) & \forall \underset{\sim}{v}_h \in V_h^0 \\ b(\underset{\sim}{u}''_h, \varphi_h) = 0 & \forall \varphi_h \in W_h\end{array}\right\} . \tag{21}$$

As in the continuous case, if $\underset{\sim}{u}_h = \underset{\sim}{u}_h^\perp + \underset{\sim}{u}_h^0 \in S_h(\perp) \oplus S_h^0$ we have $\underset{\sim}{u}'_h = \underset{\sim}{u}_h^\perp$, $\underset{\sim}{u}''_h = \underset{\sim}{u}_h^0$, $\psi'_h = \psi_h$, and we define a linear compact operator $T_h : \mathcal{S} \to \mathcal{S}$ by setting $T_h \underset{\sim}{g} = \underset{\sim}{u}_h^\perp$.

In the following we are interested in the error between $\underset{\sim}{u}^\perp$ and $\underset{\sim}{u}_h^\perp$. Since $V_h(\perp) \not\subset V(\mathcal{T}_h ; \perp)$, (20) is not an inner approximation of (10). However we can use the estimates by Brezzi [3] for each pair of problems (9)-(19) and (11)-(21), obtaining

$$\| \underset{\sim}{u}^\perp - \underset{\sim}{u}_h^\perp \| \leqslant \| \underset{\sim}{u} - \underset{\sim}{u}_h \| + | \underset{\sim}{u}^0 - \underset{\sim}{u}_h^0 | \tag{22}$$

with

$$\| \underset{\sim}{u} - \underset{\sim}{u}_h \| + \| \psi - \psi_h \|_{H_0^2(\Omega)} \leqslant c_1 \left(\inf_{\underset{\sim}{v}_h \in V_h} \| \underset{\sim}{u} - \underset{\sim}{v}_h \| + \inf_{\varphi_h \in W_h} \| \psi - \varphi_h \|_{H_0^2(\Omega)} \right) \tag{23}$$

and

$$| \underset{\sim}{u}^0 - \underset{\sim}{u}_h^0 | \leqslant c_2 \left(\inf_{\underset{\sim}{v}_h \in V_h^0} | \underset{\sim}{u}^0 - \underset{\sim}{v}_h | + \inf_{\varphi_h \in W_h} \| \psi'' - \varphi_h \|_{H_0^2(\Omega)} \right) \tag{24}$$

c_1 and c_2 being independent of the decomposition. Now if $\underset{\sim}{u}^\perp$ and $\underset{\sim}{u}^0$ are regular enough, there exist constants c_3 and c_4 positive and independent of h such that

$$\inf_{\underset{\sim}{v}_h \in V_h} \| \underset{\sim}{u} - \underset{\sim}{v}_h \| \leqslant c_3 h^{\mu - 1} (| \underset{\sim}{u} |_{\mu - 1, \Omega} + | \operatorname{Div} \underset{\sim}{u} |_{\mu - 1, \Omega}), \quad 1 \leqslant \mu \leqslant m \tag{25}$$

(see Canuto [7]) and

$$\inf_{\underset{\sim}{v}_h \in V_h^0} | \underset{\sim}{u}^0 - \underset{\sim}{v}_h | \leqslant c_4 \, h^{\mu-1} \, | \underset{\sim}{u}^0 |_{\mu-1,\Omega}, \quad 1 \leqslant \mu \leqslant m$$

(see Brezzi-Marini [5]); while it can be shown, following Brezzi-Marini [5], proof of theorem 4.2, that for any $\varphi \in H_0^2(\Omega)$ regular enough

$$\inf_{\varphi_h \in W_h} \| \tilde{\varphi} - \varphi_h \|_{H_2^0(\Omega)} \leqslant c_5 \, h^\gamma \, \| \varphi \|_{H^{\gamma+2}(\Omega)}, \quad 1 \leqslant \gamma \leqslant q \tag{26}$$

with $q = \min(r - 1, s)$. Such estimates, together with property 1.1 and a standard density argument, yield

$$\| \underset{\sim}{u}^\perp - \underset{\sim}{u}_h^\perp \| \to 0 \quad \text{as} \quad h \to 0 \tag{27}$$

for arbitrary $\underset{\sim}{g} \in \mathscr{S}$; however, by property 1.1 again, no uniform convergence of T_h to T can be expected.

3. CONVERGENCE OF THE APPROXIMATIONS AND ERROR ESTIMATES

Let us denote by $\{ \lambda_l \}_{l=1}^\infty$ the increasing sequence of the nonzero eigenvalues of (EP), repeated according to their multiplicity, and by

$$\{ \underset{\sim}{u}_l, \psi_l \}_{l=1}^\infty \in S(\perp) \times W(\mathscr{T}_h)$$

a family of corresponding eigenvectors, such that $\{ \underset{\sim}{u}_l \}_{l=1}^\infty$ be orthonormal with respect to the inner product of $\mathscr{S}(\{ \underset{\sim}{u}_l \}_{l=1}^\infty$ is complete in $\mathscr{S}(\perp))$. We set $\tilde{\lambda}_l = \lambda_l + 1$, $\tilde{\mu}_l = 1/\tilde{\lambda}_l$ and we recall that $(\tilde{\lambda}_l ; \underset{\sim}{u}_l, \psi_l)$ satisfies

$$\left. \begin{aligned} \tilde{e}(\underset{\sim}{u}_l, \underset{\sim}{v}) + b(\underset{\sim}{v}, \psi_l) &= \tilde{\lambda}_l(\underset{\sim}{u}_l, \underset{\sim}{v}) \quad && \forall \underset{\sim}{v} \in V(\mathscr{T}_h) \\ b(\underset{\sim}{u}_l, \varphi) &= 0 && \forall \varphi \in W(\mathscr{T}_h) \end{aligned} \right\} . \tag{28}$$

E_l will be the space spanned by the first l eigenfunctions $\{ \underset{\sim}{u}_i \}_{i=1}^l$. Similarly, for each h, $\{ \lambda_l^h \}_{l=1}^{N_h}$ will denote the increasing sequence of the nonzero eigenvalues for $(\text{EP})_h$ (N_h = dimension of $S_h(\perp)$), and $\{ \underset{\sim}{u}_l^h, \psi_l^h \}_{l=1}^{N_h} \in S_h(\perp) \times W_h$ a set of corresponding eigenvectors, with $\{ \underset{\sim}{u}_l^h \}$ orthonormal in $V_h(\perp)$. Setting $\tilde{\lambda}_l^h = \lambda_l^h + 1$, $\tilde{\mu}_l^h = 1/\tilde{\lambda}_l^h$, then $(\tilde{\lambda}_l^h ; \underset{\sim}{u}_l^h, \psi_l^h)$ satisfies

$$\left. \begin{aligned} \tilde{e}(\underset{\sim}{u}_l^h, \underset{\sim}{v}_h) + b(\underset{\sim}{v}_h, \psi_l^h) &= \tilde{\lambda}_l^h(\underset{\sim}{u}_l^h, \underset{\sim}{v}_h) \quad && \forall \underset{\sim}{v}_h \in V_h \\ b(\underset{\sim}{u}_l^h, \varphi_h) &= 0 && \forall \varphi_h \in W_h \end{aligned} \right\} . \tag{29}$$

E_l^h will be the space spanned by the first l eigenfunctions $\{ \underset{\sim}{u}_i^h \}_{i=1}^l$.

Since T restricts to a bijection on the finite dimensional subspace E_l and T_h

converges to T strongly in $\mathscr{S}$, T_h restricts to a bijection between E_l and $T_h E_l$ for any h small enough. If $\underset{\sim}{u} = T\underset{\sim}{g} \in E_l$, define

$$P_h \underset{\sim}{u} = T_h \underset{\sim}{g}\ ,\quad P_h \psi = \text{the Lagrange multiplier associated with } P_h \underset{\sim}{u}\,.$$

In particular

$$\left.\begin{aligned} \tilde{e}(P_h \underset{\sim}{u}_l, \underset{\sim}{v}_h) + b(\underset{\sim}{v}_h, P_h \psi_l) &= \tilde{\lambda}_l(\underset{\sim}{u}_l, \underset{\sim}{v}_h) && \forall \underset{\sim}{v}_h \in V_h(\perp) \\ b(P_h \underset{\sim}{u}_l, \varphi_h) &= 0 && \forall \varphi_h \in W_h \end{aligned}\right\}. \tag{30}$$

Since $T_h E_l$ is an l-dimensional subspace of $S_h(\perp)$, we can apply the min-max principle (see e.g. Strang-Fix [16]) to get

$$\lambda_l^h \leqslant \max_{\substack{P_h\underset{\sim}{u} \in T_h E_l \\ |P_h \underset{\sim}{u}| = 1}} e(P_h \underset{\sim}{u}, P_h \underset{\sim}{u})$$

whence

$$\lambda_l^h \leqslant \lambda_l + \max_{\substack{P_h\underset{\sim}{u} \in T_h E_l \\ |P_h \underset{\sim}{u}| = 1}} \left| e(P_h \underset{\sim}{u}, P_h \underset{\sim}{u}) - e(\underset{\sim}{u}, \underset{\sim}{u})/| \underset{\sim}{u} |^2 \right|.$$

By (27) we easily obtain

$$\limsup_{h \to 0} \lambda_l^h \leqslant \lambda_l\,, \quad \forall l \in \mathbb{N} - \{ 0 \}\,. \tag{31}$$

In order to obtain a bound from below, we use the following estimate, whose proof will be given in the Appendix; we denote by $\pi : V(\mathscr{T}_h) \to S(\perp)$ the orthogonal projection over $S(\perp)$ with respect to the inner product of $V(\mathscr{T}_h)$.

PROPERTY 3.1 : *There exists a constant c independent of h such that*

$$\| \underset{\sim}{g}_h - \pi \underset{\sim}{g}_h \| \leqslant ch \| \underset{\sim}{g}_h \|\,, \quad \forall \underset{\sim}{g}_h \in S_h(\perp)\,. \quad \blacksquare \tag{32}$$

Then, for any $\underset{\sim}{g}_h \in E_l^h$ with $| \underset{\sim}{g}_h | = 1$ we have

$$\begin{aligned} (T_h \underset{\sim}{g}_h, \underset{\sim}{g}_h) &= (T\pi\underset{\sim}{g}_h, \pi\underset{\sim}{g}_h) + (T\pi\underset{\sim}{g}_h, \underset{\sim}{g}_h - \pi\underset{\sim}{g}_h) + (T_h \underset{\sim}{g}_h - T\pi\underset{\sim}{g}_h, \underset{\sim}{g}_h) \\ &= \frac{(T\pi\underset{\sim}{g}_h, \pi\underset{\sim}{g}_h)}{| \pi\underset{\sim}{g}_h |^2} + \frac{(T\pi\underset{\sim}{g}_h, \pi\underset{\sim}{g}_h)}{| \pi\underset{\sim}{g}_h |^2} (| \pi\underset{\sim}{g}_h |^2 - | \underset{\sim}{g}_h |^2) + \\ &\quad + (T\pi\underset{\sim}{g}_h, \underset{\sim}{g}_h - \pi\underset{\sim}{g}_h) + (T_h \underset{\sim}{g}_h - T_h \pi\underset{\sim}{g}_h, \underset{\sim}{g}_h) + (T_h \pi\underset{\sim}{g}_h - T\pi\underset{\sim}{g}_h, \underset{\sim}{g}_h) \\ &= \frac{(T\pi\underset{\sim}{g}_h, \pi\underset{\sim}{g}_h)}{| \pi\underset{\sim}{g}_h |^2} + \varepsilon_1(h) + \varepsilon_2(h) + \varepsilon_3(h) + \varepsilon_4(h)\,. \end{aligned} \tag{33}$$

Now

$$\varepsilon_1(h) \leqslant 2 \| T \| \| \underset{\sim}{g}_h \| | \underset{\sim}{g}_h - \pi \underset{\sim}{g}_h | \leqslant 2c \| T \| h \| \underset{\sim}{g}_h \|^2$$

$$\varepsilon_2(h) \leqslant c \| T \| h \| \underset{\sim}{g}_h \|^2$$

$$\varepsilon_3(h) \leqslant \| T_h \| | \underset{\sim}{g}_h - \pi \underset{\sim}{g}_h | \leqslant c' h \| \underset{\sim}{g}_h \|$$

since $\| T_h \|$ is bounded independently of h, due to (18). Finally

$$\varepsilon_4(h) \leqslant \| T \pi \underset{\sim}{g}_h - T_h \pi \underset{\sim}{g}_h \|$$

and according to proposition 1.1 there exists $\chi \in H_0^2(\Omega)$ with $\Delta^2 \chi \in L^2(\Omega)$ such that $\pi \underset{\sim}{g}_h = M\chi$ and $\| \chi \|_{H^3(\Omega)} \leqslant c \| \underset{\sim}{g}_h \|$; so we can apply estimates (22)-(24) to obtain

$$\varepsilon_4(h) \leqslant ch \| \underset{\sim}{g}_h \| .$$

We observe that estimate (32) implies $\pi \underset{\sim}{g}_h \neq 0$ if $\underset{\sim}{g}_h \neq 0$, at least for small h; moreover $e(\underset{\sim}{g}_h, \underset{\sim}{g}_h) \leqslant \lambda_l^h$, $\underset{\sim}{g}_h$ being a linear combination of approximate eigenfunctions, so that by (31) $\| \underset{\sim}{g}_h \|$ is bounded independently of h. Taking the minimum of both sides of (33) over $\{ \underset{\sim}{g}_h \in E_l^h \mid | \underset{\sim}{g}_h | = 1 \}$ and using the min-max principle (here in the form of max-min principle for T over $\mathscr{S}(\perp)$), we get

$$\tilde{\mu}_l^h \leqslant \tilde{\mu}_l + O(h) .$$

Together with (31), this proves the following :

PROPERTY 3.2 : *For each $l \in \mathbb{N} - \{ 0 \}$, λ_l^h converges to λ_l as h tends to* 0. ■

As a consequence, we can estimate the L^2-norm error for the eigenfunctions.

PROPERTY 3.3 : *Let λ_l be an eigenvalue of multiplicity m, i.e.*

$$\lambda_l = \lambda_{l+1} = \cdots = \lambda_{l+m-1} ,$$

and set $2 d_l = \min(\lambda_l - \lambda_{l-1}, \lambda_m - \lambda_l)$. Then there exists a constant c independent of l and h such that for h small enough

$$| \underset{\sim}{\hat{u}}_{l+i} - \underset{\sim}{u}_{l+i}^h | \leqslant c \left(1 + \frac{\tilde{\lambda}_l}{d_l} \right) \sum_{k=0}^{m-1} | \underset{\sim}{u}_{l+k} - P_h \underset{\sim}{u}_{l+k} | \tag{34}$$

for $i = 0, \ldots, m-1$, where $\{ \underset{\sim}{\hat{u}}_{l+i} \}_{i=0}^{m-1}$ is a suitable orthonormal basis of the eigenspace of λ_l.

Proof : Since $\{ \underset{\sim}{u}_j^h \}_{j=1}^{N_h}$ is an orthonormal basis of $S_h(\perp)$, we can write

$$P_h \underset{\sim}{u}_{l+k} = \sum_{j=1}^{N_h} (P_h \underset{\sim}{u}_{l+k}, \underset{\sim}{u}_j^h) \underset{\sim}{u}_j^h .$$

Now

$$\tilde{\lambda}_j^h(u_j^h, P_h u_{l+k}) = \tilde{e}(u_j^h, P_h u_{l+k}) = \tilde{\lambda}_l(u_{l+k}, u_j^h)$$

so that

$$(P_h u_{l+k}, u_j^h) = \tilde{\lambda}_l(\lambda_l - \lambda_j^h)^{-1} (u_{l+k} - P_h u_{l+k}, u_j^h).$$

We conclude the proof as in Strang-Fix [18] or Canuto [6] by property 3.2. ■

For the eigenvalues, we can prove the following estimate :

PROPERTY 3.4 : *For each* $l \in \mathbb{N} - \{0\}$

$$|\lambda_l - \lambda_l^h| \leqslant c(\tilde{\lambda}_l |u_l - u_l^h| \; |u_l - P_h u_l| + \| u_l - P_h u_l \|^2 + \\ + 2 \| u_l - P_h u_l \| \, \| \psi_l - P_h \psi_l \|) \quad (35)$$

where c^{-1} *is a lower bound for* $(u_l^h, P_h u_l)$ *for each h small enough.*

Proof : By (28) with $v = u_l^h$ we get

$$\tilde{\lambda}_l = \frac{\tilde{e}(u_l, u_l^h) + b(u_l^h, \psi_l)}{(u_l, u_l^h)}$$

while by (29) with $v_h = P_h u_l$ we get

$$\tilde{\lambda}_l^h = \frac{\tilde{e}(u_l^h, P_h u_l) + b(P_h u_l, \psi_l^h)}{(u_l^h, P_h u_l)}.$$

Now (30) yields

$$\tilde{e}(u_l^h, P_h u_l) + b(P_h u_l, \psi_l^h) = \tilde{e}(P_h u_l, u_l^h) + b(u_l^h, P_h \psi_l) \\ = \tilde{e}(u_l, u_l^h) + b(u_l^h, \psi_l)$$

since $b(P_h u_l, \psi_l^h) = b(u_l^h, P_h \psi_l) = 0$. Hence, assuming $(u_l^h, P_h u_l) \neq 0$ we obtain

$$\lambda_l^h - \lambda_l = \tilde{\lambda}_l(u_l^h, P_h u_l)^{-1} (u_l - P_h u_l, u_l^h).$$

But

$$(u_l - P_h u_l, u_l^h) = (u_l, - P_h u_l \; u_l^h - u_l) + (u_l - P_h u_l, u_l)$$

with

$$\tilde{\lambda}_l(u_l, u_l - P_h u_l) = \tilde{e}(u_l, u_l - P_h u_l) + b(u_l - P_h u_l, \psi_l) \\ = \tilde{e}(u_l - P_h u_l, u_l - P_h u_l) + \tilde{e}(P_h u_l, u_l - P_h u_l) + \\ + b(u_l - P_h u_l, \psi_l)$$

and

$$\tilde{e}(\underset{\sim}{u}_l - P_h \underset{\sim}{u}_l, P_h \underset{\sim}{u}_l) = - b(P_h \underset{\sim}{u}_l, \psi_l - P_h \psi_l) = b(u_l - P_h u_l, \psi_l)$$
$$= b(\underset{\sim}{u}_l - P_h \underset{\sim}{u}_l, \psi_l - P_h \psi_l) . \quad \blacksquare$$

At last, we look for a bound for $\| \underset{\sim}{u}_l - \underset{\sim}{u}_l^h \|$ and $\| \psi_l - \psi_l^h \|_{H_0^2(\Omega)}$. According to a result by Brezzi ([3], property 1.1), we have

$$\| P_h \underset{\sim}{u}_l - \underset{\sim}{u}_l^h \| + \| P_h \psi_l - \psi_l^h \|_{H_0^2(\Omega)} \leqslant c \,| \tilde{\lambda}_l \underset{\sim}{u}_l - \tilde{\lambda}_l^h \underset{\sim}{u}_l^h |$$

where c is independent of h due to (18).

PROPERTY 3.5 : *The following estimates hold* :

$$\| \underset{\sim}{u}_l - \underset{\sim}{u}_l^h \| \leqslant \| \underset{\sim}{u}_l - P_h \underset{\sim}{u}_l \| + c(\tilde{\lambda}_l \,| \underset{\sim}{u}_l - \underset{\sim}{u}_l^h | + | \lambda_l - \lambda_l^h |)$$
$$\| \psi_l - \psi_l^h \|_{H_0^2(\Omega)} \leqslant \| \psi_l - P_h \psi_l \|_{H_0^2(\Omega)} + c(\tilde{\lambda}_l \,| \underset{\sim}{u}_l - \underset{\sim}{u}_l^h | + | \lambda_l - \lambda_l^h |) . \quad \blacksquare$$

We are now able to compute the order of convergence for approximate eigenvalues and eigenvectors with respect to the parameter h; for the sake of simplicity in the notations we only consider the case of a simple eigenvalue λ_l, since the extension to the case of a multiple eigenvalue is obvious. If $\underset{\sim}{u}_l = M \Psi_l$, then according to property 1.2 $\Psi_l \in H_0^2(\Omega)$ is an eigenfunction for the biharmonic problem, and $\psi_l = - \lambda_l \tilde{\Psi}_l$. We assume that $\Psi_l \in H^{\rho}(\Omega)$; well known regularity results yield $\rho > 3$. Hence

$$\inf_{\underset{\sim}{v}_h \in V_h} \| \underset{\sim}{u} - \underset{\sim}{v}_h \| \leqslant c h^{\bar{m}-2} (| \Psi_l |_{m,\Omega} + \lambda_l \,| \Psi_l |_{m-2,\Omega})$$

with $\bar{m} = \min (m + 1, \rho)$, while by (26) we obtain

$$\inf_{\varphi_h \in W_h} \| \psi_l - \varphi_h \|_{H_0^2(\Omega)} \leqslant c \lambda_l h^{\bar{q}-2} \| \Psi_l \|_{H^{\bar{q}-2}(\Omega)}$$

$$\inf_{\varphi_h \in W_h} \| \psi_l'' - \varphi_h \|_{H_0^2(\Omega)} \leqslant c(1 + \lambda_l) h^{\bar{q}-2} \| \Psi_l \|_{H^{\bar{q}-2}(\Omega)}$$

with $\bar{q} = \min (q + 2, \rho)$, $q = \min (r - 1, s)$. By using the bounds (22)-(24) we derive by properties 3.3, 3.4 and 3.5 the following estimates :

$$| \lambda_l - \lambda_l^h | \leqslant c_l h^{2(\bar{p}-2)}$$

$$\| \underset{\sim}{u}_l - \underset{\sim}{u}_l^h \| + \| \psi_l - \psi_l^h \|_{H_0^2(\Omega)} \leqslant c_l h^{\bar{p}-2}$$

where $\bar{p} = \min (\bar{q}, \bar{m})$ and c_l depends on l explicitly through λ_l and the norm $\| \Psi_l \|_{H^{\bar{p}}(\Omega)}$. Using polynomials of the lowest degree ($m = 2$, $r = 3$, $s = 1$) we have the minimum order of convergence

$$| \lambda_l - \lambda_l^h | = 0(h^2) , \quad \| \underset{\sim}{u}_l - \underset{\sim}{u}_l^h \| + \| \psi_l - \psi_l^h \|_{H_0^2(\Omega)} = 0(h) .$$

4. APPENDIX : PROOF OF PROPERTY 3.1

For each $\underset{\sim}{v} \in V(\mathscr{T}_h)$, we denote by $\widetilde{\mathrm{Div}}\, \underset{\sim}{v}$ the $L^2(\Omega)$-function such that

$$(\widetilde{\mathrm{Div}}\, \underset{\sim}{v})|_K = \mathrm{Div}\,(\underset{\sim}{v}|_K)$$

for each $K \in \mathscr{T}_h$ (recall that globally $\mathrm{Div}\, \underset{\sim}{v} \in H^{-2}(\Omega)$).
Let $\underset{\sim}{g}_h \in S_h(\perp)$ be fixed ; denote by $\Phi \in H_0^2(\Omega)$ and by

$$\Psi \in X(\Omega) = H_0^2(\Omega) \cap H^3(\Omega)$$

respectively the solutions of the problems $\Delta^2\Phi = \mathrm{Div}\, \underset{\sim}{g}_h$ and $\Delta^2\Psi = \widetilde{\mathrm{Div}}\, \underset{\sim}{g}_h$ in Ω ; note that $\underset{\sim}{g}_h$ splits as $\underset{\sim}{g}_h = \underset{\sim}{g}_h^0 + M\Phi \in S^0 \oplus V(\mathscr{T}_h ; \perp)$. Proposition 1.1 yields $\pi \underset{\sim}{g}_h = M\chi$, where $\chi \in X(\Omega)$ satisfies

$$\Delta^2\chi + \chi = \widetilde{\mathrm{Div}}\, \underset{\sim}{g}_h + \Phi\,, \tag{36}$$

since by definition of π

$$\int_\Omega (\Delta^2\chi\, \Delta^2\varphi + M\chi\, M\varphi)\, dx = \int_\Omega (\widetilde{\mathrm{Div}}\, \underset{\sim}{g}_h\, \Delta^2\varphi + M\Phi\, M\varphi)\, dx$$

for every $\varphi \in H_0^2(\Omega)$ with $\Delta^2\varphi \in L^2(\Omega)$. We write

$$\| \underset{\sim}{g}_h - \pi \underset{\sim}{g}_h \| \leqslant \| \underset{\sim}{g}_h - M\Psi \| + \| M\Psi - M\chi \|$$

and estimate separately the two terms on the right.

i) *Estimate of* $\| M\Psi - M\chi \|$. It is known that for every $\eta \in H_0^2(\Omega)$ solution of $\Delta^2\eta = f \in H^{-2}(\Omega)$, the bound $\| \eta \|_{L^2(\Omega)} \leqslant c \| f \|_{X(\Omega)'}$ holds. Taking $\eta = \Phi - \Psi$ we have

$$\| \Phi - \Psi \| \leqslant c \| \mathrm{Div}\, \underset{\sim}{g}_h - \widetilde{\mathrm{Div}}\, \underset{\sim}{g}_h \|_{X(\Omega)'} = c \sup_{\varphi \in X(\Omega) - \{0\}} b(\underset{\sim}{g}_h, \varphi)/\| \varphi \|_{H^3(\Omega)}$$

$$= c \sup_{\varphi \in X(\Omega) - \{0\}} \inf_{\varphi_h \in W_h} b(\underset{\sim}{g}_h, \tilde{\varphi} - \varphi_h)/\| \varphi \|_{H^3(\Omega)} \leqslant ch \| \underset{\sim}{g}_h \|$$

since $\underset{\sim}{g}_h \in S_h$. (36) yields $(\Delta^2 + I)\,(\chi - \Psi) = \Phi - \Psi$, by which we get the estimate

$$\| M\chi - M\Psi \| \leqslant ch \| \underset{\sim}{g}_h \| \,.$$

ii) *Estimate of* $\| \underset{\sim}{g}_h - M\Psi \|$. Define on $V(\mathscr{T}_h) \times (W(\mathscr{T}_h) \times L^2(\Omega))$ the bilinear form

$$B(\underset{\sim}{v} ; \varphi, \psi) = b(\underset{\sim}{v}, \varphi) + \beta(\underset{\sim}{v}, \psi)$$

where $\beta(\underset{\sim}{v}, \psi) = \sum_{K \in \mathscr{T}_h} \int_K \operatorname{Div} \underset{\sim}{v} \psi \, dx$, and consider the following problem of mixed type : *Find* $(\underset{\sim}{u} ; \overline{\varphi}, \overline{\psi}) \in V(\mathscr{T}_h) \times (W(\mathscr{T}_h) \times L^2(\Omega))$ *such that*

$$\left. \begin{aligned} (\underset{\sim}{u}, \underset{\sim}{v}) + B(\underset{\sim}{v} ; \overline{\varphi}, \overline{\psi}) &= 0 & \forall \underset{\sim}{v} \in V(\mathscr{T}_h) \\ B(\underset{\sim}{u} ; \varphi, \psi) &= -\beta(\underset{\sim}{g}_h, \psi) & \forall (\varphi, \psi) \in W(\mathscr{T}_h) \times L^2(\Omega) \end{aligned} \right\} . \tag{37}$$

Existence and uniqueness are assured by theorem 1.1 in Brezzi [3], since the form $(\underset{\sim}{u}, \underset{\sim}{v})$ is $V(\mathscr{T}_h)$-coercive on $V(\mathscr{T}_h)^0$ and for any $(\varphi, \psi) \in W(\mathscr{T}_h) \times L^2(\Omega)$ one has $B(\overline{\underset{\sim}{v}} ; \varphi, \psi) \geqslant c(\| \varphi \|_{H_0^2(\Omega)} + \| \psi \|_{L^2(\Omega)}) \| \overline{\underset{\sim}{v}} \|$ with $\overline{\underset{\sim}{v}} = -M\varphi - M\xi$, $\xi \in H_0^2(\Omega)$ being the solution of $\Delta^2 \xi = \psi$ in Ω. It is easily checked that the triple $(M\Psi, \widetilde{\Psi}, \Psi)$ satisfies equations (37) (recall that $\widetilde{\Psi}$ is obtained by Ψ as in (12)). Setting

$$L_h = \{ \psi \in L^2(\Omega) \mid \psi_{|K} \in P_{m-2}(K) , \quad \forall K \in \mathscr{T}_h \}$$

problem (37) can be approximated as follows :

Find $(\underset{\sim}{u}_h ; \overline{\varphi}_h, \overline{\psi}_h) \in V_h \times (W_h \times L_h)$
such that

$$\left. \begin{aligned} (\underset{\sim}{u}_h, \underset{\sim}{v}_h) + B(\underset{\sim}{v}_h ; \overline{\varphi}_h, \overline{\psi}_h) &= 0 & \forall \underset{\sim}{v}_h \in V_h \\ B(\underset{\sim}{u}_h ; \varphi_h, \psi_h) &= -\beta(\underset{\sim}{g}_h, \psi_h) & \forall (\varphi_h, \psi_h) \in W_h \times L_h . \end{aligned} \right\} . \tag{38}$$

We prove that B satisfies

$$\sup_{\underset{\sim}{v}_h \in V_h - \{\underset{\sim}{0}\}} B(\underset{\sim}{v}_h ; \varphi_h, \psi_h)/\| \underset{\sim}{v}_h \| \geqslant \sigma(\| \varphi_h \|_{H_0^2(\Omega)} + \| \psi_h \|_{L^2(\Omega)}) \tag{39}$$

for each $(\varphi_h, \psi_h) \in W_h \times L_h$, with $\sigma > 0$ independent of h. To this end, let $\eta \in H_0^2(\Omega)$ be the solution of $\Delta^2 \eta = \psi_h$ in Ω ; setting $\underset{\sim}{w} = M\eta$, let $\underset{\sim}{w}_h \in V_h$ be the interpolate of $\underset{\sim}{w}$ defined in Canuto [7] ; we have $\operatorname{Div} \underset{\sim}{w}_h = \operatorname{Div} \underset{\sim}{w} = \psi_h$, since on each $K \in \mathscr{T}_h$ $\operatorname{Div} \underset{\sim}{w}_h |_K$ is the projection of $\operatorname{Div} \underset{\sim}{w} |_K$ onto $P_{m-2}(K)$; moreover $| \underset{\sim}{w} - \underset{\sim}{w}_h | \leqslant ch \| \underset{\sim}{w} \|_{(H^1(\Omega))^4}$ so that

$$\| \underset{\sim}{w} - \underset{\sim}{w}_h \| \leqslant ch \| \psi_h \|_{L^2(\Omega)} . \tag{40}$$

Therefore

$$\beta(\overline{\underset{\sim}{w}}_h, \psi_h) \geqslant c_1 \| \psi_h \|_{L^2(\Omega)}$$

with c_1 independent of h and $\overline{\underset{\sim}{w}}_h = \underset{\sim}{w}_h/\| \underset{\sim}{w}_h \|$. On the other hand by (17) there exists $\overline{\underset{\sim}{z}}_h \in V_h^0$ such that

$$\| \overline{\underset{\sim}{z}}_h \| = 1 \quad \text{and} \quad b(\overline{\underset{\sim}{z}}_h, \varphi_h) \geqslant c_2 \| \varphi_h \|_{H_0^2(\Omega)} .$$

Setting $\bar{v}_h = \bar{z}_h - \bar{w}_h$ we have

$$B(\bar{v}_h ; \varphi_h, \psi_h) = b(\bar{z}_h, \varphi_h) + \beta(\bar{w}_h, \psi_h) - b(\bar{w}_h, \varphi_h)$$

where

$$| b(w_h, \varphi_h) | = | b(w - w_h, \varphi_h) | \leqslant c' h \| \varphi_h \|_{H_0^2(\Omega)} \| w_h \|$$

since $w \in S$. (39) is then easily verified. Note that if $m \geqslant 3$, then $w_h \in S$ (Canuto [7], proposition 4.1), hence $b(w_h, \varphi_h) = 0$.

In particular, problem (38) has a unique solution, and one easily checks that $u_h = g_h$; actually (38.1) yields $u_h \in V_h(\perp)$, hence $u_h - g_h \in V_h(\perp)$, while (38.2) is equivalent to the condition $u_h - g_h \in S_h^0$.

Applying the known error estimates for saddle point problems, using (40) relative to $\psi_h = \widetilde{\mathrm{Div}}\, g_h = \mathrm{Div}\, M\Psi$ and (26) we obtain

$$\| g_h - M\Psi \| \leqslant ch \| g_h \| . \qquad \blacksquare$$

5. NUMERICAL RESULTS

We report here some numerical results obtained for the model square plate $\Omega = (0,1) \times (0,1)$ by P. G. Gilardi. Ω is divided into four equal triangles by the lines $x + y = 1$ and $x - y = 0$; then each triangle obtained in this way is divided into equal triangles by three families of equidistant straight lines parallel to its edges. $N = h^{-1}$ denotes the number of elements on each side. The choice for W_h is $r = 4$ and $s = 3$, and the degrees of freedom for a function $\varphi \in W_h$ are the values of φ and grad φ at each vertex of an element, and the values of φ, $\partial\varphi/\partial \underline{n}$ and $\partial^2\varphi/\partial \underline{n}\, \partial \underline{t}$ at the middle point of each side of an element.

We present here some results relative to the first and fourth eigenvalue of Ω, for different choices of the degree m of the tensors and different values of the mesh parameter $N = h^{-1}$. We recall that Fichera obtained extremely accurate bounds, both from above and from below, for the eigenvalues of a large class of compact operators, by the « orthogonal invariants » method (refer to the book [9] for this theory and for a great number of related numerical results); in our table, λ_l^* is precisely the mean value between Fichera's upper and lower bound for the lth eigenvalue of the clamped plate. In each cell, we report the computed eigenvalue $\lambda_l^{(\mathrm{comp})}$ (above), and the relative error

$$| \lambda_l^* - \lambda_l^{(\mathrm{comp})} | / | \lambda_l^* | \quad \text{(below)} .$$

The discussion of the algorithm employed and other numerical results — even relative to different boundary conditions — can be found in Gilardi [21].

	$\lambda_1^* = 0\ 129493E+4$			$\lambda_4^* = 0.117103E+5$		
$N \backslash m$	2	3	4	2	3	4
1	0 223200E+4 0 72E−0	0 133690E+4 0 32E−1	0 133648E+4 0 32E−1	— —	0 279138E+5 0 14E+1	0 170824E+5 0 46E 0
2	0 141065E+4 0 89E−1	0 130882E+4 0 11E−1	0 129797E+4 0 23E−2	0 364448E+5 0 21E+1	0 121603E+5 0 38E−1	0 121098E+5 0 34E−1
3	0 134978E+4 0 42E−1	0 129809E+4 0 24E−2	0 129528E+4 0 27E−3	0.158046E+5 0 35E 0	0 119553E+5 0 21E−1	0 117498E+5 0 34E−2
4	0 132725E+4 0 25E−1	0 129593E+4 0 77E−3	0.129500E+4 0 55E−4	0.138154E+5 0 18E 0	0 118025E+5 0 79E−2	0 117185E+5 0 70E−3
5	0 131603E+4 0 16E−1	0 129534E+4 0 32E−3	0.129495E+4 0 16E−4	0 130234E+5 0 11E 0	0 117494E+5 0.33E−2	0 117130E+5 0 23E−3

REFERENCES

1 P M. ANSELONE, *Collectively Compact Operator Approximation Theory to Integral Equations*, Prentice Hall, Englewood Cliffs, N.J., 1971.

2 K. BRANDT, *Calculation of vibration frequencies by a hybrid element method based on a generalized complementary energy principle*, Int. J num. Meth. Engng., v. 12, 1977, pp. 231-246.

3. F. BREZZI, *On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers*, R.A.I.R.O , R-2, 1974, pp 129-151.

4 F. BREZZI, *Sur la méthode des éléments finis hybrides pour le problème biharmonique*, Num. Math. v 24, 1975, pp. 103-131.

5. F BREZZI and L. D. MARINI, *On the numerical solution of plate bending problems by hybrid methods*, R.A.I.R.O., R-3, 1975, pp. 5-50.

6 C CANUTO, *Eigenvalue approximations by mixed methods*, R.A.I.R.O. Anal Num., v. 12, 1978, pp. 27-50

7. C. CANUTO, *A finite element to interpolate symmetric tensors with divergence in L^2* (To appear on Calcolo).

8. P G CIARLET, *The finite element method for elliptic problems*, North-Holland, Amsterdam-New York-Oxford, 1978.

9. G. FICHERA, *Numerical and Quantitative Analysis*, Pitman, London-San Francisco-Melbourne, 1978.

10. P. GRISVARD, *Singularité des solutions du problème de Stokes dans un polygone* (To appear)

11 W. G. KOLATA, *Eigenvalue approximation by the finite element method : the method of Lagrange multipliers* (To appear)

12 V. A. KONDRAT'EV, *Boundary problems for elliptic equations in domains with conical or angular points*, Trans Moscow Math Soc., v 16, 1976, pp 227-313.

13. B. MERCIER and J. RAPPAZ, *Eigenvalue approximation via nonconforming and hybrid finite elements methods*, Rapport Interne du Centre de Mathématiques Appliquées de l'École Polytechnique, n° 33, 1978

14. B. MERCIER, J. OSBORN, J. RAPPAZ and P.-A. RAVIART, *Eigenvalue approximation by mixed and hybrid methods* (To appear).

15. T. H. H. PIANG and P. TONG, *The basis of finite element methods for solid continua*, Int. J. num. Meth. Engng., v. 1, 1969, pp. 3-28.

16. J. RAPPAZ, *Approximation of the spectrum of a non-compact operator given by the magnetohydrodynamic stability of a plasma*, Num. Math., v. 28, 1977, pp. 15-24.

17. J. RAPPAZ, *Spectral approximation by finite elements of a problem of MHD-stability of a plasma, The Mathematics of Finite Elements and Applications III*, MAFELAP 1978 (Ed. J. R. Whiteman), Academic Press, London-New York-San Francisco, 1979, pp. 311-318.

18. G. STRANG and G. FIX, *An Analysis of the Finite Element Method*, Prentice Hall, Englewood Cliffs, N.J., 1973.

19. B. TABARROK, *A variational principle for the dynamic analysis of continua by hybrid finite element method*, Int. J. Solids Struct., v. 7, 1971, pp. 251-268.

20. R. A. TOUPIN, *A variational principle for the mesh-type analysis of the mechanical system*, Trans. Am. Soc. Mech. Engngs., v. 74, 1952, pp. 151-152.

21. P. G. GILARDI (To appear).