RAIRO. ANALYSE NUMÉRIQUE

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RAIRO. Analyse numérique, tome 14, nº 3 (1980), p. 249-277
<http://www.numdam.org/item?id=M2AN_1980__14_3_249_0>

R.A.I.R.O. Analyse numérique/Numerical Analysis
(vol. 14, n° 3, 1980, p. 249 à 277)

ERROR ESTIMATES FOR MIXED METHODS (*)

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Communicated by P. A. RAVIART

Abstract. — *This paper presents abstract error estimates for mixed methods for the approximate solution of elliptic boundary value problems. These estimates are then applied to obtain quasi-optimal error estimates in the usual Sobolev norms for four examples: three mixed methods for the biharmonic problem and a mixed method for second order elliptic problems.*

Résumé. — *Dans cet article, on présente des estimations d'erreur abstraites pour des méthodes mixtes appliquées à la résolution approchée de problèmes aux limites elliptiques. On applique ensuite ces estimations afin d'obtenir des estimations d'erreur quasi-optimales, dans les normes de Sobolev habituelles, dans quatre exemples : Trois méthodes mixtes pour le problème biharmoniques, et une méthode mixte pour les problèmes elliptiques du second ordre.*

1. INTRODUCTION

In [5] Brezzi studied Ritz-Galerkin approximation of saddle-point problems arising in connection with Lagrange multipliers. These problems have the form:

Given $f \in V'$ and $g \in W'$, find $(u, \psi) \in V \times W$ satisfying

$$\left.\begin{aligned} a(u, v)+b(v, \psi)&=(f, v), \quad \forall v \in V, \\ b(u, \varphi)&=(g, \varphi), \quad \forall \varphi \in W, \end{aligned}\right\} \tag{1.1}$$

where V and W are real Hilbert spaces, and $a(.,.)$ and $b(.,.)$ are bounded bilinear forms on $V \times V$ and $V \times W$, respectively.

Given finite dimensional spaces $V_h \subset V$ and $W_h \subset W$, $0<h<1$, the Ritz-Galerkin approximation (u_h, ψ_h) to (u, ψ) is the solution of the following problem:

Find $(u_h, \psi_h) \in V_h \times W_h$ satisfying

$$\left.\begin{aligned} a(u_h, v)+b(v, \psi_h)&=(f, v), \quad \forall v \in V_h, \\ b(u_h, \varphi)&=(g, \varphi), \quad \forall \varphi \in W_h. \end{aligned}\right\} \tag{1.2}$$

(*) Received May 1979.
The work of the first author was partially supported by NSF grant MCS78-02737 and by the United States Army under Contract No. DAAG29-75-C-0024, and the second author by NSF Grant MCS78-02851 and by the United States Army under Contract No. DAAG29-75-C-0024.

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The major assumptions in Brezzi's results are

$$\sup_{v \in Z_h} \frac{|a(u, v)|}{\|v\|_V} \geqq \gamma_0 \|u\|_V, \qquad \forall u \in Z_h \text{ and } \forall h, \tag{1.3}$$

where $\gamma_0 > 0$ is a constant independent of h and

$$Z_h \equiv \{ v \in V_h : b(v, \varphi) = 0, \forall \varphi \in W_h \},$$

and

$$\sup_{v \in V_h} \frac{|b(v, \varphi)|}{\|v\|_V} \geqq k_0 \|\varphi\|_W, \qquad \forall \varphi \in W_h \text{ and } \forall h, \tag{1.4}$$

where $k_0 > 0$ is independent of h.

Using (1.3) and (1.4) Brezzi proves the following error estimate for the approximation method determined by (1.2):

$$\|u - u_h\|_V + \|\psi - \psi_l\|_W \leqq C(\inf_{\chi \in V_h} \|u - \chi\|_V + \inf_{\eta \in W_h} \|\psi - \eta\|_W), \qquad \forall h. \tag{1.5}$$

In [1, 2] Babuška studied Ritz-Galerkin approximation of general variationally posed problems. The main result of [1, 2], applied to (1.1) and (1.2), is that (1.5) holds provided

$$\left.\begin{aligned} &\sup_{(v, \varphi) \in V_h \times W_h} \frac{|a(u, v) + b(v, \psi) + b(u, \varphi)|}{\|v\|_V + \|\varphi\|_W} \geqq \tau_0(\|u\|_V + \|\psi\|_W), \\ &\qquad \forall (u, \psi) \in V_h \times W_h \text{ and } \forall h, \end{aligned}\right\} \tag{1.6}$$

where $\tau_0 > 0$ is independent of h.

It is clear from [1, 2, 5] that (1.3) and (1.4) hold if and only if (1.6) holds. (1.3)-(1.4) or, equivalently, (1.6) is referred to as the stability condition for this approximation method.

The results of [1, 2, 5] can be viewed as a strategy for analyzing these approximation methods: the approximation method is characterized by certain bilinear forms, norms (spaces), and families of finite dimensional approximating spaces, and if the method can be shown to be stable with respect to the chosen norms, then the error estimates in these norms follow in a simple manner provided the bilinear forms are bounded and the approximation properties of V_h and W_h are known in these norms. These results can be used to analyze, for example, certain hybrid methods for the biharmonic problem [5, 6] and the stationary Stokes problem [10]. The results of [1, 2] have also been used to analyze a variety of variationally posed problems that are not of form (1.1).

There are other problems of a similar nature, however, where attempts at using the ideas of [1, 2, 5] were not entirely successful since not all of the abstract hypotheses were satisfied: specifically the Brezzi condition (1.3) or, equivalently, the Babuška condition (1.6), is not satisfied with the usual choice of norms, i.e., the approximation methods for these problems are not stable with respect to the usual norms. This is the case, for example, in the analysis in [7] of the Hermann-Miyoshi [14, 15, 19] mixed method for the biharmonic problem. In the analysis of this method a natural choice for both $\| \cdot \|_V$ and $\| \cdot \|_W$ is the 1st order Sobolev norm; however this method is not stable with respect to this choice. As a result of this difficulty, the error estimates obtained in [7] are not quasi-optimal. A similar difficulty arises in the analysis of the Hermann-Johnson [14, 15, 16] and Ciarlet-Raviart [9] mixed methods for the biharmonic problem. In later work of Scholz [23] and Rannacher [21] quasi-optimal error estimates were obtained for the mixed methods of Ciarlet-Raviart and Hermann-Miyoshi, although the systematic approach of Brezzi and Babuška was abandoned.

In a forthcoming paper of Babuška, Osborn, and Pitkäranta [3] quasi-optimal error estimates for mixed methods for the biharmonic problem are derived by an application of the results of Brezzi and Babuška. In this work a new family of (mesh dependent) norms are introduced with respect to which the above mentioned mixed methods (Ciarlet-Raviart, Hermann-Miyoshi, Hermann-Johnson) are stable. Error estimates in these norms then follow directly from the results of Brezzi and Babuška, once the approximation properties of the subspaces V_h and W_h have been determined in these new norms. Error estimates in the more standard norms are then obtained by using the usual duality argument.

It is the intent of this paper to provide an abstract approach to the analysis of mixed methods which *leads to quasi-optimal error estimates*, *uses only standard norms*, and *is systematic*. We shall assume that existence and uniqueness for the continuous (infinite dimensional) problem has been established and develop an abstract framework under which quasi-optimal error estimates can be derived for a variety of examples which do not fit within the convergence theory of Brezzi and Babuska using the usual norms.

Section 2 contains the abstract convergence results of the paper. In section 3 we present four examples previously analyzed in the literature and show how error estimates can be derived from the theorems in section 2. Three of these methods are mixed methods for the biharmonic problem and the fourth is a mixed method for a second order problem analyzed by Raviart-Thomas [22, 25].

It is interesting to note that in this last example the results of Brezzi and Babuska apply with the choice of spaces used by Raviart-Thomas, but fail to yield quasi-optimal error estimates in all cases due to the way in which the variables are tied together in the error estimates. In our analysis the error estimates for the two variables are separated and quasi-optimal error estimates are obtained. For the three mixed methods for the biharmonic problem that are analyzed in section 3 the results of the present paper and those obtained in [3], using different techniques, are the same. For additional results on mixed methods see Oden [20]. Finally we note that some basic ideas in the analysis in this paper are similar to those employed in Scholz [23, 24].

Throughout this paper, we shall use the Sobolev spaces $W^{m,p}(\Omega)$, where Ω is a convex polygon in the plane, m is a nonnegative integer, and $1 \leqq p < \infty$. On these spaces we have the seminorms and norms

$$|v|_{m,p,\Omega} = \left(\int_\Omega \sum_{|\alpha|=m} |D^\alpha v|^p \, dx \right)^{1/p}$$

and

$$\|v\|_{m,p,\Omega} = \left(\int_\Omega \sum_{|\alpha|\leqq m} |D^\alpha v|^p \, dx \right)^{1/p}.$$

When $p=2$, we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and write

$$|v|_{m,2,\Omega} = |v|_{m,\Omega}$$

and

$$\|v\|_{m,2,\Omega} = \|v\|_{m,\Omega}.$$

We will further denote by $W_0^{1,p}(\Omega)$ the subspace of $W^{1,p}(\Omega)$ of functions that vanish on $\Gamma = \partial\Omega$ and by $H_0^2(\Omega)$ the subspace of $H^2(\Omega)$ of functions that vanish together with their normal derivatives on Γ. For $m=1$ and 2 we will also use the spaces $H^{-m}(\Omega) = [H_0^m(\Omega)]'$ [the dual space of $H_0^m(\Omega)$] with the norm on $H^{-m}(\Omega)$ taken to be the usual dual norm. To further simplify notation we often drop the use of the subscript Ω in the norm when the context is clear.

2. ABSTRACT RESULTS

Let V, W, and H be three real Banach spaces with norms $\|\cdot\|_V$, $\|\cdot\|_W$, and $\|\cdot\|_H$ respectively. We assume $V \subset H$ with a continuous imbedding. Let $a(.,.)$ and $b(.,.)$ be continuous bilinear forms on $H \times H$ and $V \times W$, respectively:

$$|a(u,v)| \leqq \|a\| \cdot \|u\|_H \|v\|_H, \qquad \forall u, v \in H, \tag{2.1}$$

$$|b(u,\psi)| \leqq \|b\| \cdot \|u\|_V \|\psi\|_W, \qquad \forall u \in V, \quad \forall \psi \in W. \tag{2.2}$$

We consider the following problem, which we refer to as problem P:

Given $f \in V'$ and $g \in W'$, find $(u, \psi) \in V \times W$ satisfying:

$$a(u, v) + b(v, \psi) = (f, v), \qquad \forall v \in V, \tag{2.3}$$

$$b(u, \varphi) = (g, \varphi), \qquad \forall \varphi \in W, \tag{2.4}$$

where $(., .)$ denotes the pairing between V and V' or W and W'.

We shall be interested in this problem for a subclass of data, i.e., for $(f, g) \in D$, where D is a subclass of $V' \times W'$. We shall assume that:

(H1) For $(f, g) \in D$, P has a unique solution.

In the analysis of problem P we will also consider the adjoint problem:

Given $d \in G'$, where G is a Banach space satisfying $W \subset G$ with a continuous imbedding, find $(y, \lambda) = (y_d, \lambda_d) \in V \times W$ satisfying

$$a(v, y) + b(v, \lambda) = 0, \qquad \forall v \in V, \tag{2.5}$$

$$b(y, \varphi) = (d, \varphi), \qquad \forall \varphi \in W. \tag{2.6}$$

We shall assume that:

(H2) Problem (2.5)-(2.6) has a unique solution for each $d \in G'$.

Throughout this paper we shall be concerned with the problem of approximating the solution (u, ψ) of P. Toward this end, we suppose we are given finite dimensional spaces $V_h \subset V$ and $W_h \subset W$. We then consider the following approximate problem, which we refer to as problem P_h:

Find $(u_h, \psi_h) \in V_h \times W_h$ satisfying:

$$a(u_h, v) + b(v, \psi_h) = (f, v), \qquad \forall v \in V_h, \tag{2.7}$$

$$b(u_h, \varphi) = (g, \varphi), \qquad \forall \varphi \in W_h. \tag{2.8}$$

We will then view u_h as an approximation to u and ψ_h as an approximation to ψ. In this section we obtain estimates for $u - u_h$ and $\psi - \psi_h$.

We now state several further assumptions which we will require in the proofs of our main results.

(H3) There is a constant $\alpha > 0$ (α independent of h) such that

$$a(v, v) \geqq \alpha \| v \|_H^2, \qquad \forall v \in Z_h,$$

where $Z_h = \{ v \in V_h : b(v, \varphi) = 0, \ \forall \varphi \in W_h \}$.

(H4) $S(h)$ is a number satisfying

$$\| v \|_V \leqq S(h) \| v \|_H, \qquad \forall v \in V_h.$$

(H5) There is an operator $\pi_h : Y \to V_h$ satisfying

$$b(y - \pi_h y, \varphi) = 0, \qquad \forall y \in Y \quad \text{and} \quad \forall \varphi \in W_h,$$

where $Y = \text{span}(\{y_d\}_{d \in G'}, u)$, (u, ψ) is the solution of problem P, and (y_d, λ_d) is the solution of (2.5)-(2.6) corresponding to $d \in G'$.

For the examples treated in section 3 the existence and uniqueness of the approximate solution (u_h, ψ_h) can be established in various ways. We now give a proof based on the assumptions made above.

THEOREM 1: *Assume that hypotheses* (H2), (H3) *and* (H5) *are valid. Then problem P_h has a unique solution.*

Proof: Since V_h and W_h are finite dimensional, it suffices to show that if $(u_h, \psi_h) \in V_h \times W_h$ satisfies

$$a(u_h, v) + b(v, \psi_h) = 0, \qquad \forall v \in V_h, \tag{2.9}$$

$$b(u_h, \varphi) = 0, \qquad \forall \varphi \in W_h, \tag{2.10}$$

then $u_h = \psi_h = 0$. Choosing $v = u_h$ in (2.9) and $\varphi = -\psi_h$ in (2.10) and adding the equations, we get $a(u_h, u_h) = 0$.

Noting from (2.10) that $u_h \in Z_h$ and using (H3) we have $\| u_h \|_H = 0$. Hence $u_h = 0$.

Setting $u_h = 0$ in (2.9) we obtain:

$$b(v, \psi_h) = 0, \qquad \forall v \in V_h. \tag{2.11}$$

Now

$$\| \psi_h \|_G = \sup_{d \in G'} \frac{(d, \psi_h)}{\| d \|_{G'}}. \tag{2.12}$$

By (H2), for each $d \in G'$, there exists $y_d \in V$ such that for all $\varphi \in W$:

$$(d, \varphi) = b(y_d, \varphi).$$

Thus

$$(d, \psi_h) = b(y_d, \psi_h) = b(\pi_h y_d, \psi_h) \quad \text{[applying (H5)]} \quad = 0 \quad \text{[using (2.11)]}.$$

Equation (2.12) then implies $\psi_h = 0$.

Our main result in this section are theorems 2 and 3 which present abstract estimates for the errors $u - u_h$ and $\psi - \psi_h$.

THEOREM 2: *Suppose hypotheses* (H1)-(H5) *are valid and that* (u, ψ) *and* (u_h, ψ_h) *are the respective solutions of problems* P *and* P_h. *Then* [*with* π_h *defined by* (H5)],

$$\|u-u_h\|_H \leqq \frac{1}{\alpha}[\|b\| S(h) \|\psi-\varphi\|_W + (\|a\|+\alpha)\|u-\pi_h u\|_H] \quad \text{for all } \varphi \in W_h \quad (2.13)$$

and

$$\|u-u_h\|_V \leqq \|u-\pi_h u\|_V + \frac{S(h)}{\alpha} \times [\|b\| S(h) \|\psi-\varphi\|_W + \|a\| \|u-\pi_h u\|_H] \quad \text{for all } \varphi \in W_h. \quad (2.14)$$

If in addition

(H6) $$Z_h \subset Z,$$

where

$$Z=\{v \in V : b(v, \varphi)=0, \forall \varphi \in W\},$$

then

$$\|u-u_h\|_H \leqq \left[1+\frac{\|a\|}{\alpha}\right]\|u-\pi_h u\|_H \quad (2.15)$$

and

$$\|u-u_h\|_V \leqq \|u-\pi_h u\|_V + \frac{S(h)\|a\|}{\alpha}\|u-\pi_h u\|_H. \quad (2.16)$$

Proof: Using (2.3) we see that

$$a(\pi_h u, v)+b(v, \psi)=a(u, v)+b(v, \psi)+a(\pi_h u-u, v) = (f, v)+a(\pi_h u-u, v), \quad \forall v \in V_h, \quad (2.17)$$

and from (2.4) and (H5) we see that

$$b(\pi_h u, \varphi)=(g, \varphi), \quad \forall \varphi \in W_h. \quad (2.18)$$

Subtracting (2.7) from (2.17) we find

$$a(\pi_h u-u_h, v)+b(v, \psi-\psi_h)=a(\pi_h u-u, v), \quad \forall v \in V_h, \quad (2.19)$$

and subtracting (2.8) from (2.18) we obtain:

$$b(\pi_h u-u_h, \varphi)=0, \quad \forall \varphi \in W_h. \quad (2.20)$$

Choosing $v=\pi_h u-u_h$ in (2.19) we have

$$a(\pi_h u-u_h,\ \pi_h u-u_h)+b(\pi_h u-u_h,\ \psi-\psi_h)=a(\pi_h u-u,\ \pi_h u-u_h).$$

Applying (2.20) we get

$$a(\pi_h u-u_h,\ \pi_h u-u_h)=a(\pi_h u-u,\ \pi_h u-u_h)\\ +b(u_h-\pi_h u,\ \psi-\varphi)\quad \text{for all } \varphi\in W_h. \tag{2.21}$$

Using (2.1), (2.2), (H3), (H4) and noting from (2.20) that $\pi_h u-u_h\in Z_h$, we then obtain:

$$\alpha\|\pi_h u-u_h\|_H^2\leqq\|a\|\,.\,\|\pi_h u-u\|_H\,\|\pi_h u-u_h\|_H\\ +\|b\|\,S(h)\,\|u_h-\pi_h u\|_H\,\|\psi-\varphi\|_W$$

and hence

$$\left.\begin{aligned}\|\pi_h u-u_h\|_H\leqq\frac{1}{\alpha}[\|a\|\,.\,\|u-\pi_h u\|_H+\|b\|\,S(h)\,\|\psi-\varphi\|_W]\\ \text{for all } \varphi\in W_h.\end{aligned}\right\} \tag{2.22}$$

Thus

$$\|u-u_h\|_H\leqq\|u-\pi_h u\|_H+\|\pi_h u-u_h\|_H\\ \leqq\frac{1}{\alpha}[\|b\|\,S(h)\,\|\psi-\varphi\|_W+(\|a\|+\alpha)\|u-\pi_h u\|_H]$$

for all $\varphi\in W_h$. This proves (2.13).

In order to prove (2.14) we first note that

$$\|u-u_h\|_V\leqq\|u-\pi_h u\|_V+\|\pi_h u-u_h\|_V\ \leqq\|u-\pi_h u\|_V+S(h)\,\|\pi_h u-u_h\|_H.$$

(2.14) now follows from (2.22).

To prove (2.15) we observe that (2.20) together with $Z_h\subset Z$ implies that

$$b(\pi_h u-u_h,\ \varphi)=0,\qquad \forall\,\varphi\in W. \tag{2.23}$$

Hence (2.21) simplifies to

$$a(\pi_h u-u_h,\ \pi_h u-u_h)=a(\pi_h u-u,\ \pi_h u-u_h). \tag{2.24}$$

Applying (2.1) and (H3) to (2.24) yields

$$\|\pi_h u-u_h\|_H\leqq\frac{\|a\|}{\alpha}\|\pi_h u-u\|_H. \tag{2.25}$$

(2.15) follows by the triangle inequality.

To establish (2.16) we write

$$\|u-u_h\|_V \leqq \|u-\pi_h u\|_V + \|\pi_h u-u_h\|_V$$
$$\leqq \|u-\pi_h u\|_1 + S(h)\|\pi_h u-u_h\|_H \quad \text{[by (H4)]}$$
$$\leqq \|u-\pi_h u\|_V + S(h)\frac{\|a\|}{\alpha}\|u-\pi_h u\|_H \quad \text{[using (2.25)]}.$$

COROLLARY: *Inequality* (2.15) *holds without assumption* (H4).

THEOREM 3: (*a*) *Suppose hypotheses* (H1), (H2), (H3), *and* (H5) *are valid and that* (u, ψ) *and* (u_h, ψ_h) *are the respective solutions of problems* P *and* P_h. *Then* (*with* (y_d, λ_d) *and* π_h *as defined in* (H2) *and* (H5), *respectively*),

$$\|\psi-\psi_h\|_G = \sup_{d\in G'}\{b(y_d-\pi_h y_d, \psi-\varphi)+a(u_h-u, \pi_h y_d-y_d)$$
$$+b(u-u_h, \lambda_d-\eta)\}/\|d\|_{G'} \quad \text{for all } \varphi, \ \eta\in W_h. \tag{2.26}$$

(*b*) *If in addition* (H6) *holds* $(Z_h\subset Z)$, *then*

$$b(u-u_h, \lambda_d-\eta)=b(u-\pi_h u, \lambda_d-\eta), \qquad \forall\eta\in W_h. \tag{2.27}$$

(*c*) *If we further have that:*

(H7) *There is an operator* $\Sigma_h:\Lambda\to W_h$ *satisfying* $b(v, \Sigma_h\lambda-\lambda)=0$ *for all* $v\in V_h$ *and all* $\lambda\in\Lambda$, *where* $\Lambda=\operatorname{span}(\{\lambda_d\}_{d\in G'}, \psi)$, (u, ψ) *is the solution of problem* P, *and* (y_d, λ_d) *is the solution of* (2.5)-(2.6) *corresponding to* $d\in G'$, *then*

$$b(y_d-\pi_h y_d, \psi-\Sigma_h\psi)=(d, \psi-\Sigma_h\psi) \tag{2.28}$$

and

$$b(u-u_h, \lambda_d-\Sigma_h\lambda_d)=(g, \lambda_d-\Sigma_h\lambda_d). \tag{2.29}$$

Proof: From (2.6) we have

$$\|\psi-\psi_h\|_G=\sup_{d\in G'}(d, \psi-\psi_h)/\|d\|_{G'} = \sup_{d\in G'} b(y_d, \psi-\psi_h)/\|d\|_{G'}. \tag{2.30}$$

Subtraction of (2.7) from (2.3) and (2.8) from (2.4) yields

$$a(u-u_h, v)+b(v, \psi-\psi_h)=0, \qquad \forall v\in V_h \tag{2.31}$$

and

$$b(u-u_h, \eta)=0, \qquad \forall\eta\in W_h. \tag{2.32}$$

Now, combining (2.5), (H5), (2.31), and (2.32) we obtain:

$$\begin{aligned}
b(y_d, \psi-\psi_h) &= b(y_d-\pi_h y_d, \psi-\psi_h)+b(\pi_h y_d, \psi-\psi_h)\\
&= b(y_d-\pi_h y_d, \psi-\varphi)+a(u_h-u, \pi_h y_d)\\
&= b(y_d-\pi_h y_d, \psi-\varphi)+a(u_h-u, \pi_h y_d-y_d)+a(u_h-u, y_d)\\
&= b(y_d-\pi_h y_d, \psi-\varphi)+a(u_h-u, \pi_h y_d-y_d)+b(u-u_h, \lambda_d)\\
&= b(y_d-\pi_h y_d, \psi-\varphi)+a(u_h-u, \pi_h y_d-y_d)\\
&\qquad +b(u-u_h, \lambda_d-\eta) \quad \text{for all } \varphi, \eta\in W_h.
\end{aligned}$$

Substitution of this identity in (2.30) yields (2.26).

If $Z_n \subset Z$ then

$$b(\pi_h u-u_h, \varphi)=0, \qquad \forall \varphi\in W \quad [\textit{see } (2.23) \text{ above}]$$

and so (2.27) follows immediately.

Now, if in addition (H7) holds, then

$$b(y_d-\pi_h y_d, \psi-\Sigma_h\psi)=b(y_d, \psi-\Sigma_h\psi)=(d, \psi-\Sigma_h\psi) \quad [\text{by } (2.6)].$$

and

$$b(u-u_h, \lambda_d-\Sigma_h\lambda_d)=b(u, \lambda_d-\Sigma_h\lambda_d)=(g, \lambda_d-\Sigma_h\lambda_d).$$

Thus (2.28) and (2.29) are established.

Remark: Note that inequality (2.15) in theorem 2 and all the results of theorem 3 hold without assumption (2.2). This observation is used in subsection 3 *c*.

We end this section with several remarks on the hypotheses (H3)-(H7). We assume here that V and W are Hilbert spaces.

1) It is clear that if

$$a(v, v)\geqq\gamma_0\|v\|_V^2 \quad \text{for all } v\in Z_h, \tag{2.33}$$

then hypotheses (1.3) in Brezzi's theorem is valid. In the applications we consider in section 3, (2.33) is not true (with γ_0 independent of h) but is valid when $\|v\|_V$ is replaced by $\|v\|_H$. This accounts for (H3) [and (H4)].

2) In hypotheses (H5)-(H7) it appears that we are not making use of conditions similar to (1.4). In fact, in applications the operator π_h described in (H5) is often constructed in order to verify (1.4). A more precise relationship is given below in propositions 1 and 2. For further ideas in this direction, consult the work of Fortin [11].

PROPOSITION 1: *Suppose*

$$\sup_{v \in V_h} \frac{|b(v, \varphi)|}{\|v\|_V} \geqq k_0 \|\varphi\|_W, \qquad \forall \varphi \in W_h \quad \text{and} \quad \forall h, \tag{2.34}$$

where $k_0 > 0$. *Then there is an operator* $\pi_h : V \to V_h$ *that satisfies*

$$b(v - \pi_h v, \varphi) = 0, \qquad \forall v \in V \quad \text{and} \quad \forall \varphi \in W_h$$

and

$$\|\pi_h v\|_V \leqq \frac{\|b\|}{k_0} \|v\|_V, \qquad \forall v \in V.$$

Proof: Consider $b(v, \varphi)$ on $Z_h^\perp \times W_h$, where $Z_h^\perp = \{ v \in V_h : v \text{ is } V\text{-orthogonal to } Z_h \}$. We immediately see that

$$\sup_{v \in Z_h^\perp} \frac{|b(v, \varphi)|}{\|v\|_V} \geqq k_0 \|\varphi\|_W, \qquad \forall \varphi \in W_h$$

and

$$\sup_{\varphi \in W_h} |b(v, \varphi)| > 0, \qquad \forall 0 \neq v \in Z_h^\perp.$$

It thus follows from [1, 2] that for each $v \in V$ there is a unique $\pi_h v \in Z_h^\perp$ satisfying

$$b(\pi_h v, \varphi) = b(v, \varphi), \qquad \forall \varphi \in W_h.$$

Furthermore,

$$\|\pi_h \mathrm{v}\|_V \leqq \frac{\|b\|}{k_0} \|v\|_V.$$

This proves proposition 1.

We also note that it follows from [1, 2] that for each $\varphi \in W$ there exists a unique $\Sigma_h \varphi \in W_h$ satisfying

$$b(v, \Sigma_h \varphi) = b(v, \varphi), \qquad \forall v \in Z_h^\perp.$$

Furthermore,

$$\|\Sigma_h \varphi\|_W \leqq \frac{\|b\|}{k_0} \|\varphi\|_W.$$

PROPOSITION 2: *Suppose*

$$\sup_{v \in V} \frac{|b(v, \varphi)|}{\|v\|_V} \geqq k \|\varphi\|_W, \qquad \forall \varphi \in W, \tag{2.35}$$

where $k>0$, *and suppose there is an operator* $\pi_h: V \to V_h$ *satisfying*

$$b(v-\pi_h v, \varphi)=0, \qquad \forall \varphi \in W_h$$

and

$$\|\pi_h v\|_V \leqq C\|v\|_V, \qquad \forall v \in V.$$

Then (2.34) *holds.*

Proof: Clearly we have

$$\sup_{v \in V_h} \frac{|b(v, \varphi)|}{\|v\|_V} \geqq \sup_{v \in V} \frac{|b(\pi_h v, \varphi)|}{\|\pi_h v\|_V}$$

$$= \sup_{v \in V} \frac{|b(v, \varphi)|}{\|v\|_V} \frac{\|v\|_V}{\|\pi_h v\|_V} \geqq \frac{k}{C}\|\varphi\|_W, \qquad \forall \varphi \in W_h,$$

i.e., (2.34) holds with $k_0=k/C$.

Thus we see that (H5) is closely related to (2.34), which is the same as (1.4).

3) Hypotheses (H6) and (H7) are also closed related as we see by the following result.

PROPOSITION 3: $\Sigma_h: W \to W_h$, *as defined in remark* 2, *satisfies*

$$b(v, \Sigma_h \varphi)=b(v, \varphi), \qquad \forall v \in V_h, \tag{2.36}$$

if and only if $Z_h \subset Z$.

Proof: Suppose (2.36) holds. Let $v \in Z_h$. Then

$$b(v, \varphi)=b(v, \Sigma_h \varphi)=0, \qquad \forall \varphi \in W,$$

i.e., $v \in Z$. Thus $Z_h \subset Z$.

Now suppose $Z_n \subset Z$. Then, if $v \in Z_h^{\perp}$ we have (2.36) by the definition of Σ_h, and, if $v \in Z_h$ we have (2.36) since both terms are zero. Since $V_h=Z_n \oplus Z_h^{\perp}$ we obtain (2.36) for all $v \in V_h$.

3. APPLICATIONS

In this section we apply the results of section 2 to several examples.

a) Ciarlet-Raviart method

Consider the biharmonic problem

$$\left.\begin{aligned} \Delta^2 \psi = g \quad \text{in } \Omega, \\ \psi = \frac{\partial \psi}{\partial n} = 0 \quad \text{on } \Gamma = \partial\Omega, \end{aligned}\right\} \tag{3.1}$$

where Ω is a convex polygon and g is a given function. If $g \in H^{-2}(\Omega)$ then there is a unique solution $\psi \in H_0^2(\Omega)$ of (3.1). In addition the following regularity result is known for this problem: If $g \in H^{-1}(\Omega)$, then $\psi \in H^3(\Omega) \cap H_0^2(\Omega)$ and there is a constant C such that

$$\|\psi\|_3 \leqq C\|g\|_{-1}, \qquad \forall g \in H^{-1}(\Omega). \tag{3.2}$$

Using the well-known correspondence between the biharmonic problem and the Stokes problem, this regularity result can be deduced from the regularity result for the Stokes problem proved in [17] (*cf.* also [13]).

We now seek an approximation to the solution ψ of (3.1) by a mixed method, i.e., we introduce an auxiliary variable ($u \equiv -\Delta\psi$ for the method of this subsection), write (3.1) as a lower order system, cast this system in variational form, and then consider the Ritz-Galerkin method corresponding to this variational formulation. In particular, the mixed method we study will be based on the following variational formulation of (3.1), first considered by Glowinski [12] and Mercier [18] and further developed by Ciarlet and Raviart [9]:

Given $g \in H^{-1}(\Omega)$, find $(u, \psi) \in H^1(\Omega) \times H_0^1(\Omega)$ satisfying

$$\left.\begin{aligned} \int_\Omega uv\,dx - \int_\Omega \nabla v \cdot \nabla\psi\,dx &= 0, \qquad \forall v \in H^1(\Omega), \\ -\int_\Omega \nabla u \cdot \nabla\varphi\,dx &= -\int_\Omega g\,\varphi\,dx, \qquad \forall \varphi \in H_0^1(\Omega). \end{aligned}\right\} \tag{3.3}$$

Using the regularity result (3.2) it is not difficult to show (*see* theorem 1 of [9]) that if ψ is the solution to (3.1) and $u = -\Delta\psi$, then (u, ψ) is a solution of (3.3), and if (u, ψ) is a solution of (3.3), then ψ is a solution of (3.1) and $u = -\Delta\psi$.

It is clear that (3.3) is an example of problem P of section 2 with

$$V = H^1(\Omega), \qquad W = H_0^1(\Omega), \qquad H = L_2(\Omega),$$

$$a(u, v) = \int_\Omega uv\,dx \qquad \text{and} \qquad b(u, \psi) = -\int_\Omega \nabla u \cdot \nabla\psi\,dx$$

(and with g replaced by $-g$). Here the subclass D of data for which (H1) is satisfied is given by $D = 0 \times W'$. Since the form a is symmetric, the adjoint problem (2.5), (2.6), with $G = W = H_0^1(\Omega)$, is the same as problem P and thus is uniquely solvable for all $d \in W'$. Hence (H2) is satisfied. Using (3.2) we also have

$$\|y_d\|_1 + \|\lambda_d\|_3 \leqq C\|d\|_{-1}. \tag{3.4}$$

Next we discuss the finite dimensional subspaces used in the approximation scheme. For $0 < h < 1$, let τ_h be a triangulation of $\overline{\Omega}$ with triangles T of diameter less than or equal to h. We assume the family $\{\tau_h\}$ satisfies the minimal angle condition, i.e., there is a constant $\sigma > 0$ such that

$$\max_{T \in \tau_h} \frac{h_T}{\rho_T} \leqq \sigma, \qquad \forall h,$$

where h_T is the diameter of T and ρ_T is the diameter of the largest circle contained in T, and is quasi-uniform, i.e., there is a constant $\tau > 0$ such that

$$\frac{\max_T h_T}{\min_T h_T} \leqq \tau, \qquad \forall h.$$

For $k \geqq 1$ a fixed integer we define

$$S_h = \{ v \in C^0(\overline{\Omega}) : v|_T \in P_k, \forall T \in \tau_h \}, \tag{3.5}$$

where P_k is the space of polynomials of degree k or less in the variables x_1 and x_2. We then consider the approximate problem P_h with $V_h = S_h$ and $W_h = S_h \cap H_0^1(\Omega)$. Note that this scheme yields direct approximations to ψ and $u = -\Delta\psi$ (the stream function and vorticity in hydrodynamical problems).

To apply our theorems we must check that hypotheses (H3)-(H5) are valid. (H3) is clearly valid with $\alpha = 1$ and since our family of triangulations is quasi-uniform, (H4) is satisfied with $S(h) = C/h$ for some constant C. It remains to check (H5). For $v \in H^1(\Omega)$ define $\pi_h v$ by:

$$\pi_h v \in V_h,$$

$$\int_\Omega \nabla(\pi_h v) \cdot \nabla\varphi \, dx = \int_\Omega \nabla v \cdot \nabla\varphi \, dx, \qquad \forall \varphi \in V_h$$

and

$$\int_\Omega \pi_h v \, dx = \int_\Omega v \, dx, \quad \text{i.e.,}$$

let $\pi_h v$ be the Neumann projection of v into V_h. Then (H5) is satisfied, and in addition standard approximability results imply that if $v \in H^{r-2}(\Omega)$, $r \geqq 3$, then

$$\left.\begin{array}{c} \| v - \pi_h v \|_j \leqq C h^{l-j} \| v \|_l, \\ j = 0, 1 \quad \text{and} \quad 1 \leqq l \leqq \min(k+1, r-2). \end{array}\right\} \tag{3.6}$$

We are now ready to apply theorems 2 and 3. Suppose $\psi \in H^r(\Omega)$, $r \geqq 3$ and that $k \geqq 2$. Then, using (2.13), (3.6), and standard approximability results, we have

$$\|u-u_h\|_0 \leqq C(h^{-1} \inf_{\varphi \in W_h} \|\psi-\varphi\|_1 + \|u-\pi_h u\|_0)$$
$$\leqq C(h^{-1} h^{s-1} \|\psi\|_s + h^{s-2} \|u\|_{s-2})$$
$$\leqq Ch^{s-2} \|\psi\|_s \quad (\text{since } u=-\Delta\psi), \quad (3.7)$$

where $s=\min(r, k+1)$.

From (2.14) we find in a similar fashion that

$$\|u-u_h\|_1 \leqq Ch^{s-3} \|\psi\|_s, \quad (3.8)$$

where $s=\min(r, k+1)$.

Finally, from (2.1), (2.2), (2.26), (3.4), (3.7), and (3.8), we have

$$\|\psi-\psi_h\|_1 \leqq C \sup_{d \in H^{-1}(\Omega)} \{\|y_d-\pi_h y_d\|_1 \inf_{\varphi \in W_h} \|\psi-\varphi\|_1$$
$$+ \|u-u_h\|_0 \|y_d-\pi_h y_d\|_0 + \|u-u_h\|_1 \inf_{\eta \in W_h} \|\lambda_d-\eta\|_1\}/\|d\|_{-1}$$
$$\leqq C \sup_{d \in H^{-1}(\Omega)} \{\|y_d\|_1 h^{s-1} \|\psi\|_s + h^{s-2} \|\psi\|_s h \|y_d\|_1$$
$$+ h^{s-3} \|\psi\|_s h^2 \|\lambda_d\|_3\}/\|d\|_{-1} \leqq Ch^{s-1} \|\psi\|_s, \quad (3.9)$$

where $s=\min(r, k+1)$.

Since (3.7)-(3.9) are valid only for $k \geqq 2$, the methods of this paper do not yield error estimates for the case $k=1$ in this example. For this case the reader is referred to Scholz [24]. The estimates (3.7)-(3.9) improve on those in Ciarlet and Raviart [9]. Scholz [23] obtained (3.7) under the assumption that Γ is smooth. (3.7) and (3.9) were obtained by Babuška, Osborn, and Pitkäranta [3].

We remark that theorem 3 could also be used to obtain an error estimate for $\|\psi-\psi_h\|_0$ [by choosing $G=L_2(\Omega)$] when $\psi \in H^4(\Omega)$. However in order to get quasi-optimal results we would require the regularity result that $d \in L_2(\Omega)$ implies $\lambda_d \in H^4(\Omega)$, which is not valid on a convex polygon.

b) Hermann-Miyoshi method

We consider in this subsection another mixed method for the approximate solution of (3.1). In this method the auxiliary variable is the vector of second partial derivatives of ψ.

Let

$$V=\{\mathbf{v}=(v_{ij}),\ 1 \leqq i, j \leqq 2 : v_{12}=v_{21},\ v_{ij} \in H^1(\Omega)\}$$

(with the usual product norm), and

$$W=H_0^1(\Omega).$$

Then the mixed method we study will be based on the following variational formulation of (3.1).

Given $g \in H^{-1}(\Omega)$, find $(\mathbf{u}, \psi) \in V \times W$ satisfying

$$\left.\begin{aligned} &\sum_{i,j=1}^{2} \int_\Omega u_{ij} v_{ij}\, dx + \sum_{i,j=1}^{2} \int_\Omega \frac{\partial v_{ij}}{\partial x_j} \frac{\partial \psi}{\partial x_i}\, dx = 0, \\ &\qquad\qquad \forall \mathbf{v} \in V, \\ &\sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_{ij}}{\partial x_j} \frac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega g\,\varphi\, dx, \qquad \forall \varphi \in W. \end{aligned}\right\} \tag{3.10}$$

Using the regularity result (3.2) it is not difficult to show that if ψ is a solution of (3.1) and $\mathbf{u}=(u_{ij})$ is defined by $u_{ij}=\partial^2\psi/\partial x_i \partial x_j$, then $(\mathbf{u}, \psi)$ is a solution of (3.10), and if $(\mathbf{u}, \psi)$ is a solution of (3.10), then ψ is a solution of (3.1) and $u_{ij}=\partial^2\psi/\partial x_i \partial x_j$.

We easily observe that (3.10) is an example of problem P with V and W as above,

$$H=\{\mathbf{v}=(v_{ij}),\ 1 \leqq i, j \leqq 2 : v_{12}=v_{21},\ v_{ij} \in L_2(\Omega)\}$$

(with the usual product norm),

$$a(\mathbf{u}, \mathbf{v})=\sum_{i,j=1}^{2} \int_\Omega u_{ij} v_{ij}\, dx$$

and

$$b(\mathbf{u}, \psi)=\sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_{ij}}{\partial x_j} \frac{\partial \psi}{\partial x_i}\, dx.$$

As in subsection 3 a the subclass D of data for which (H1) is satisfied is given by $D=0 \times W'$, and since a is again symmetric (H2) is satisfied with $G=W=H_0^1(\Omega)$.

Letting S_h be as defined in (3.5), we then consider the approximate problem P_h with

$$V_h=\{\mathbf{v}=(v_{ij}) : v_{12}=v_{21},\ v_{ij} \in S_h\}$$

and

$$W_h=S_h \cap H_0^1(\Omega).$$

With this choice for the forms a and b and the spaces V_h and W_h, problem P_h now describes the Hermann-Miyoshi method [14, 15, 19] for the approximation of the biharmonic problem. Note that with this method we obtain direct approximations to ψ and $\partial^2\psi/\partial x_i \partial x_j$ (the displacement and the moments in elasticity problems).

As in subsection 3 *a* we have hypothesis (H3) satisfied with $\alpha=1$ and (H4) satisfied with $S(h)=C/h$ for some constant C. (H5) for this example is contained in lemma 2 in [7]. Moreover, by a minor modification of the proof of lemma 2 in [7] we obtain the existence of $\pi_h : V \to V_h$ satisfying:

$$b(\mathbf{v}-\pi_h \mathbf{v}, \varphi)=0, \qquad \forall \varphi \in W_h$$

and for $\mathbf{v} \in V \cap [H^{r-2}(\Omega)]^4$, $r \geqq 3$, the estimate

$$\|\mathbf{v}-\pi_h \mathbf{v}\|_j \leqq C h^{l-j} \|\mathbf{v}\|_l, \qquad j=0, 1$$

and

$$1 \leqq l \leqq \min(k+1, r-2). \tag{3.11}$$

We can now apply theorems 2 and 3 in the same way as in subsection 3 *a*. Combining these theorems with (3.11) and standard approximability results, we obtain for $\psi \in H^r(\Omega)$, with $r \geqq 3$ and $k \geqq 2$:

$$\|\mathbf{u}-\mathbf{u}_h\|_0 \leqq C h^{s-2} \|\psi\|_s, \tag{3.12}$$

$$\|\mathbf{u}-\mathbf{u}_h\|_1 \leqq C h^{s-3} \|\psi\|_s \tag{3.13}$$

and

$$\|\psi-\psi_h\|_1 \leqq C h^{s-1} \|\psi\|_s, \tag{3.14}$$

where $s=\min(r, k+1)$. Estimates (3.12)-(3.14) improve those in Brezzi-Raviart [7]. Rannacher [21] recently proved these estimates for $k=2$. Babuška, Osborn, and Pitkäranta [3] proved (3.12) and (3.14).

c) Hermann-Johnson method

We consider here a further mixed method for the solution of (3.1) in which the auxiliary variable is the vector of second partial derivatives of ψ, as in section 3 *b*.

Given a triangle $T \in \tau_h$ and a function $\mathbf{v}=(v_{ij})$ with $v_{ij} \in H^1(T)$, $1 \leqq i, j \leqq 2$, and $v_{12}=v_{21}$ we define

$$M_\nu(\mathbf{v})=\sum_{i,j=1}^{2} v_{ij} \nu_j \nu_i$$

and

$$M_{\nu\tau}(\mathbf{v})=\sum_{i,j=1}^{2} v_{ij} \nu_j \tau_i,$$

where $\mathbf{v}=(\nu_1, \nu_2)$ is the unit outward normal and $\tau=(\tau_1, \tau_2)=(\nu_2, -\nu_1)$ is the unit tangent along ∂T. Let

$$V=V(h)=\{\mathbf{v}=(v_{ij}) : v_{ij} \in L_2(\Omega),\ v_{12}=v_{21},$$
$$v_{ij}|_T \in H^1(T),\ \forall T \in \tau_h,$$

and $M_{\nu}(\mathbf{v})$ is continuous at the interelement boundaries$\}$

$$\text{with } \|\mathbf{v}\|_V^2 = \sum_{i,j=1}^{2} \sum_{T \in \tau_h} \|v_{ij}\|_{1,T}^2$$

and

$$W = W_0^{1,p}(\Omega),$$

where p is some number larger than 2.

The mixed method we study in this subsection will be based on the following variational formulation of (3.1).

Given $g \in H^{-1}(\Omega)$, find $(u, \psi) \in V \times W$ satisfying

$$\left.\begin{aligned} &\sum_{i,j=1}^{2} \int_{\Omega} u_{ij} v_{ij}\, dx + \sum_{T \in \tau_h} \left\{ \sum_{i,j=1}^{2} \int_T \frac{\partial v_{ij}}{\partial x_j} \frac{\partial \psi}{\partial x_i}\, dx - \int_{\partial T} M_{\nu\tau}(\mathbf{v}) \frac{\partial \psi}{\partial \tau}\, ds \right\} = 0, \\ &\qquad \forall \mathbf{v} \in V, \\ &\sum_{T \in \tau_h} \left\{ \sum_{i,j=1} \int_T \frac{\partial u_{ij}}{\partial x_j} \frac{\partial \psi}{\partial x_i}\, dx \right. \\ &\left. - \int_{\partial T} M_{\nu\tau}(\mathbf{u}) \frac{\partial \varphi}{\partial \tau}\, ds \right\} = - \int_{\Omega} g\, \varphi\, dx, \qquad \forall \varphi \in W. \end{aligned}\right\} \quad (3.15)$$

The correspondence between (3.1) and (3.15) is the same as the correspondence between (3.1) and (3.10), i.e., if ψ is the solution of (3.1), then $([\partial^2 \psi / \partial x_i \partial x_j], \psi)$ is a solution of (3.15), and if $((u_{ij}), \psi)$ is a solution of (3.15), then ψ is the solution of (3.1) and $u_{ij} = \partial^2 \psi / \partial x_i \partial x_j$.

One easily sees that (3.15) is an example of problem P with V and W as above, H as in section 3 b,

$$a(\mathbf{u}, \mathbf{v}) = \sum_{i,j=1}^{2} \int_{\Omega} u_{ij} v_{ij}\, dx$$

and

$$b(\mathbf{u}, \psi) = \sum_{T \in \tau_h} \left\{ \sum_{i,j=1}^{2} \int_T \frac{\partial u_{ij}}{\partial x_j} \frac{\partial \psi}{\partial x_i}\, dx - \int_{\partial T} M_{\nu\tau}(\mathbf{u}) \frac{\partial \psi}{\partial \tau}\, ds \right\}.$$

As in the previous subsections, a subclass D of data for which (H1) is satisfied is given by $D = 0 \times H^{-1}(\Omega)$, and since a is again symmetric, (H2) is satisfied for $G = H_0^1(\Omega)$. We note that in this example the space $V = V(h)$ depends on h. For each h the form $b(\mathbf{u}, \psi)$ is bounded on $V(h) \times W$ (where $W = W_0^{1,p}$, $p > 2$) with a bound b that depends on h. In the error estimates in this subsection we do not require that this bound be independent of h. *Cf.* the remark following theorem 3.

Letting S_h be as defined by (3.5), we then consider the approximate problem P_h with

$$V_h=\{\mathbf{v}\in V : v_{ij}|_T \in P_{k-1}, \forall T\in\tau_h\}$$

and

$$W_h=S_h\cap H_0^1(\Omega).$$

With this choice for the forms a and b and the spaces V_h and W_h, we have the method of Hermann-Johnson [14, 15, 16].

As in the previous subsections, hypothesis (H3) is satisfied with $\alpha=1$. We now consider (H5).

For $\mathbf{v}\in V$ we define $\pi_h v\in V_h$ as in [7], section 4, i.e., $\pi_h \mathbf{v}$ is defined by the conditions

$$\left.\begin{aligned} &\int_{T'} M_\nu(\mathbf{v}-\pi_h\mathbf{v})\, q\, ds=0, \qquad \forall q\in P_{k-1} \\ &\qquad\qquad \text{and for all sides } T' \text{ of } \tau_h \\ \text{and}\quad &\int_T [v_{ij}-(\pi_h\mathbf{v})_{ij}]\, q\, dx=0, \qquad \forall q\in P_{k-2} \quad \text{and} \quad \forall T\in\tau_h. \end{aligned}\right\} \tag{3.16}$$

By lemma 3 in [7], $\pi_h\mathbf{v}$ is uniquely determined by (3.16). Since we can write

$$b(\mathbf{v}, \mu)=\sum_{T\in\tau_h}\left\{-\sum_{i,j=1}^{2}\int_T v_{ij}\frac{\partial^2\mu}{\partial x_i\partial x_j}\,dx+\int_{\partial T} M_\nu(\mathbf{v})\frac{\partial\mu}{\partial\nu}\,ds\right\},$$

(3.16) easily implies (H5). We note that by lemma 4 of [7] we also have for all $\mathbf{v}\in V\cap[H^{r-2}(\Omega)]^4$, $r\geqq 3$, that

$$\|\pi_h\mathbf{v}-\mathbf{v}\|_0\leqq Ch^l\|\mathbf{v}\|_l, \qquad 1\leqq l\leqq \min(k, r-2). \tag{3.17}$$

We next observe that by lemma 5 of [7], $Z_n\subset Z$, so that we are in the special cases of theorems 2 and 3. In particular, by the corollary to theorem 2, (H4) need not be satisfied in order to apply (2.15), so we shall not require $\{\tau_h\}$ to be quasi-uniform. Since we wish to apply theorem 3, part c, we now show that hypothesis (H7) is satisfied. As in the proof of lemma 5 in [7], for $\mathbf{v}\in V_h$ and $\mu\in W=W_0^{1,P}(\Omega)$ we can write

$$b(\mathbf{v}, \mu)=-\sum_{T\in\tau_h}\sum_{i,j=1}^{2}\int_T\frac{\partial^2 v_{ij}}{\partial x_i\partial x_j}\mu\,dx +\sum_{T'\in I_h}\int_{T'} A(T', \mathbf{v})\,\mu\,ds+\sum_{a\in J_h} B(a, \mathbf{v})\,\mu(a), \tag{3.18}$$

where I_h is the set of all sides of the triangulation τ_h, J_h is the set of all vertices of τ_h, and $A(T', \mathbf{v})$ is a polynomial of degree less than or equal to $k-2$ in the variable s.

For $\mu \in W$ we now choose $\Sigma_h \mu \in W_h$ so that

$$\int_T (\mu - \Sigma_h \mu)\, q dx = 0, \qquad \forall q \in P_{k-3} \text{ and } \forall T \in \tau_h, \tag{3.19}$$

$$\int_{T'} (\mu - \Sigma_h \mu)\, q\, ds = 0, \qquad \forall q \in P_{k-2} \text{ and } \forall T' \in I_h, \tag{3.20}$$

$$(\Sigma_h \mu - \mu)(a) = 0, \qquad \forall a \in J_h. \tag{3.21}$$

The unique solvability of this system is easily checked. Note that by the Sobolev imbedding theorem, $\mu \in W$ implies $\mu \in C^0(\overline{\Omega})$. Since for $\mathbf{v} \in V_h$ we have $\partial^2 v_{ij}/\partial x_i \partial x_j |_T \in P_{k-3}$ and $A(T', \mathbf{v}) \in P_{k-2}$, it follows from (3.18) that $\Sigma_h \mu$, as defined by (3.19)-(3.21), satisfies (H7). Furthermore, by a standard application of the Bramble-Hilbert lemma [4], we obtain for all $\mu \in W \cap H^r(\Omega)$:

$$\| \mu - \Sigma_h \mu \|_j \leqq C h^{l-j} \| \mu \|_l, \; j = 0, 1 \quad \text{and} \quad 1 \leqq l \leqq \min(r, k+1). \tag{3.22}$$

We are now ready to apply theorems 2 and 3. Suppose that $k \geqq 1$ and $\psi \in H^r(\Omega)$, $r \geqq 3$. From (2.15) and (3.17) we obtain:

$$\| \mathbf{u} - \mathbf{u}_h \|_0 \leqq C \| \mathbf{u} - \pi_h \mathbf{u} \|_0 \leqq C h^\delta \| \mathbf{u} \|_\delta \leqq C h^\delta \| \psi \|_{\delta+2}, \tag{3.23}$$

where $\delta = \min(k, r-2)$.

To obtain estimates for $\psi - \psi_h$ we shall apply theorem 3 in several different ways. Choosing $G = H_0^1(\Omega)$, $\varphi = \mathscr{I}_h \psi$, and $\eta = \mathscr{I}_h \lambda_d$ (where $\mathscr{I}_h \varphi$ denotes the standard Lagrange interpolant of φ in S_h), we get from theorem 3(*a*)-(*b*) that

$$\| \psi - \psi_h \|_1 = \sup_{d \in H^{-1}(\Omega)} \{ b(\mathbf{y}_d - \pi_h \mathbf{y}_d, \psi - \mathscr{I}_h \psi)$$
$$+ a(\mathbf{u}_h - \mathbf{u}, \pi_h \mathbf{y}_d - \mathbf{y}_d) + b(\mathbf{u} - \pi_h \mathbf{u}, \lambda_d - \mathscr{I}_h \lambda_d \} / \| d \|_{-1}.$$

To estimate the terms in the above expression, we introduce the affine transformation

$$x = F(\hat{x}) = B\hat{x} + b,$$

mapping the reference triangle $\hat{T}$ with vertices (0, 0), (1, 0) and (0, 1) onto T, and set

$$\hat{\mathbf{v}}(\hat{x}) = B^{-1} \mathbf{v} \circ F(\hat{x})\, (B^{-1})^T,$$

where

$$\hat{\mathbf{v}}=\begin{pmatrix}\hat{v}_{11} & \hat{v}_{12}\\ \hat{v}_{21} & \hat{v}_{22}\end{pmatrix} \quad \text{and} \quad \mathbf{v}=\begin{pmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{pmatrix},$$

and

$$\hat{\varphi}(\hat{x})=\varphi\circ F(\hat{x}).$$

Using the standard change of variables argument, we have that if $\mathbf{v}\in[H^l(T)]^4$ and $\varphi\in H^t(T)$, where $1\leqq l$ and $2\leqq t$, then

$$\int_T \sum_{i,j=1}^{2}\frac{\partial v_{ij}}{\partial x_j}\frac{\partial\varphi}{\partial x_i}\,dx=|\det B|\int_{\hat{T}}\sum_{i,j=1}^{2}\frac{\partial\hat{v}_{ij}}{\partial\hat{x}_j}\frac{\partial\hat{\varphi}}{\partial\hat{x}_i}\,d\hat{x}$$

and

$$\int_{\partial T} M_{\nu\tau}(\mathbf{v})\frac{\partial\varphi}{\partial\tau}\,ds\leqq\sum_{i=1}^{3}\|B^{-1}\tau\|^2\|B^T\nu\|\,|T'_i|\int_{\hat{T}'_i}\hat{\tau}^T B^T B\,\hat{v}(\hat{x})\,\hat{\nu}\frac{\partial\hat{\varphi}}{\partial\hat{\tau}}\,d\hat{s},$$

where $\hat{\tau}$ and $\hat{\nu}$ denote the unit tangent and unit outward normal to $\partial\hat{T}$, respectively, T'_i are the sides of T, and $|T'_i|$ = length of T'_i.

Since

$$\|B\|\leqq Ch,\qquad |\det B|\leqq Ch^2,\qquad \|B^{-1}\|\leqq C/h,$$

and

$$|T'_i|\leqq Ch\quad (cf.\ [8]),$$

it easily follows that

$$\left|\int_T\sum_{i,j=1}^{2}\frac{\partial v_{ij}}{\partial x_j}\frac{\partial\varphi}{\partial x_i}\,dx-\int_{\partial T}M_{\nu\tau}(\mathbf{v})\frac{\partial\varphi}{\partial\tau}\,ds\right|$$
$$\leqq Ch^2\left\{\int_{\hat{T}}\left|\sum_{i,j=1}\frac{\partial\hat{v}_{ij}}{\partial\hat{x}_j}\frac{\partial\hat{\varphi}}{\partial\hat{x}_i}\right|d\hat{x}+\int_{\partial\hat{T}}\left(\sum_{i,j=1}^{2}\hat{v}_{ij}^2\right)^{1/2}\left|\frac{\partial\hat{\varphi}}{\partial\hat{\tau}}\right|ds\right\}$$
$$\leqq Ch^2\|\hat{\mathbf{v}}\|_{1,\hat{T}}\|\hat{\varphi}\|_{2,\hat{T}}. \quad (3.24)$$

Now from lemma 4 of [7],

$$\left.\begin{aligned}&\widehat{\pi_h\mathbf{v}}-\hat{\mathbf{v}}=0\\ \text{if}\quad &\hat{\mathbf{v}}\in P_{k-1}^4(\hat{T})=\{\hat{\mathbf{v}}=(\hat{v}_{ij}),\ \hat{v}_{ij}\in P_{k-1}(\hat{T}),\ \hat{v}_{12}=\hat{v}_{21}\}\end{aligned}\right\}\quad (3.25)$$

and

$$\|\widehat{\pi_h\mathbf{v}}-\hat{\mathbf{v}}\|_{1,\hat{T}}\leqq C(\hat{T})\|\hat{\mathbf{v}}\|_{1,\hat{T}}. \quad (3.26)$$

Using the standard properties of the interpolant (*cf.* [8]) we also get

$$\mathscr{I}_h \hat{\varphi} - \hat{\varphi} = 0 \qquad \text{if} \quad \hat{\varphi} \in P_k(\hat{T}) \tag{3.27}$$

and

$$\| \mathscr{I}_h \hat{\varphi} - \hat{\varphi} \|_{2, \hat{T}} \leqq C(\hat{T}) \| \hat{\varphi} \|_{2, \hat{T}}. \tag{3.28}$$

Using (3.24)-(3.28) we easily obtain:

$$\begin{aligned} &\left| \int_T \sum_{i,j=1} \frac{\partial}{\partial x_j} (\mathbf{v} - \pi_h \mathbf{v})_{ij} \frac{\partial}{\partial x_i} (\varphi - \mathscr{I}_h \varphi)\, dx \right. \\ &\qquad \left. - \int_{\partial T} M_{\nu\tau}(\mathbf{v} - \pi_h \mathbf{v}) \frac{\partial}{\partial \tau} (\varphi - \mathscr{I}_h \varphi)\, ds \right| \\ &\qquad \leqq C h^2 \inf_{p \in P_{k-1}^4(\hat{T})} \| \hat{\mathbf{v}} - p \|_{1, \hat{T}} \inf_{q \in P_k(\hat{T})} \| \hat{\varphi} - q \|_{2, \hat{T}} \\ &\qquad \leqq C h^2 \inf_{p \in P_{k-1}^4(\hat{T})} \| \hat{\mathbf{v}} - p \|_{l, \hat{T}} \inf_{q \in P_k(\hat{T})} \| \hat{\varphi} - q \|_{t, \hat{T}} \\ &\qquad\qquad \leqq C h^2 | \hat{\mathbf{v}} |_{l, T} | \hat{\varphi} |_{t, \hat{T}} \end{aligned}$$

for $1 \leqq l \leqq k$ and $2 \leqq t \leqq k+1$. Changing back to the original variables we further obtain:

$$| \hat{\mathbf{v}} |_{l, \hat{T}} \leqq C h^{l-3} | \mathbf{v} |_{l, T}$$

and

$$| \hat{\psi} |_{t, \hat{T}} \leqq C h^{t-1} | \psi |_{t, T}.$$

Hence if $\mathbf{v} \in [H^l(\Omega)]^4$ and $\varphi \in H^t(\Omega)$ for $1 \leqq l \leqq k$ and $2 \leqq t \leqq k+1$, then

$$\begin{aligned} &| b(\mathbf{v} - \pi_h \mathbf{v}, \varphi - \mathscr{I}_h \varphi) | \\ &\qquad \leqq \sum_T \left| \int_T \sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} (\mathbf{v} - \pi_h \mathbf{v})_{ij} \frac{\partial}{\partial x_i} (\varphi - \mathscr{I}_h \varphi)\, dx \right. \\ &\qquad\qquad \left. - \int_{\partial T} M_{\nu\tau}(\mathbf{v} - \pi_h v) \frac{\partial}{\partial \tau} (\varphi - \mathscr{I}_h \varphi)\, ds \right| \\ &\qquad \leqq \sum_T C h^{t+l-2} | \mathbf{v} |_{l, T} | \psi |_{t, T} \leqq C h^{t+l-2} \| \mathbf{v} \|_l \| \psi \|_t. \end{aligned} \tag{3.29}$$

Choosing $\mathbf{v} = \mathbf{y}_d$, $\varphi = \psi$, $l = 1$ and $t = \min(r, k+1) \equiv s$ in (3.29) we get

$$| b(\mathbf{y}_d - \pi_h \mathbf{y}_d, \psi - \mathscr{I}_h \psi) | \leqq C h^{s-1} \| \mathbf{y}_d \|_1 \| \psi \|_s.$$

If $k \geqq 2$ we choose $\mathbf{v} = \mathbf{u}$, $\varphi = \lambda_d$, $l = s-2$, and $t = 3$ in (3.29) to obtain:

$$| b(\mathbf{u} - \pi_h \mathbf{u}, \lambda_d - \mathscr{I}_h \lambda_d) | \leqq C h^{s-1} \| \mathbf{u} \|_{s-2} \| \lambda_d \|_3 \leqq C h^{s-1} \| \psi \|_s \| \lambda_d \|_3.$$

If $k=1$ we choose $l=1$ and $t=2$ to obtain:

$$|b(\mathbf{u}-\pi_h\mathbf{u},\lambda_d-\mathscr{I}_h\lambda_d)|\leqq Ch\|\mathbf{u}\|_1\|\lambda_d\|_2\leqq Ch\|\psi\|_3\|\lambda_d\|_3.$$

Finally if $k\geqq 2$, from (2.1), (2.15), and (3.17) with $l=1$ and $l=s-2$ we have

$$\begin{aligned}|a(\mathbf{u}_h-\mathbf{u},\pi_h\mathbf{y}_d-\mathbf{y}_d)&\leqq C\|\mathbf{u}_h-\mathbf{u}\|_0\|\pi_h\mathbf{y}_d-\mathbf{y}_d\|_0\\&\leqq C\|\pi_h\mathbf{u}-\mathbf{u}\|_0\|\pi_h\mathbf{y}_d-\mathbf{y}_d\|_0\leqq Ch^{s-2}\|\mathbf{u}\|_{s-2}h\|\mathbf{y}_d\|_1\\&\leqq Ch^{s-1}\|\psi\|_s\|\mathbf{y}_d\|_1.\end{aligned}$$

If $k=1$ we choose $l=1$ in (3.17) to get

$$|a(\mathbf{u}_h-\mathbf{u},\pi_h\mathbf{y}_d-\mathbf{y}_d)\leqq Ch\|\mathbf{u}\|_1h\|\mathbf{y}_d\|_1\leqq Ch^2\|\psi\|_3\|\mathbf{y}_d\|_1.$$

Applying the regularity result

$$\|\mathbf{y}_d\|_1+\|\lambda_d\|_3\leqq C\|d\|_{-1}\tag{3.30}$$

and collecting terms, we get

$$\|\psi-\psi_h\|_1\leqq Ch^{s-1}\|\psi\|_s\quad\text{for}\quad k\geqq 2\quad\text{where } s=\min(r,k+1)\tag{3.31}$$

and

$$\|\psi-\psi_h\|_1\leqq Ch\|\psi\|_3\quad\text{for}\quad k=1.\tag{3.32}$$

We now derive estimates in $L_2(\Omega)$. First consider the case when $k=1$ and $\psi\in H^4(\Omega)$. Using theorem 3(*a*)-(*c*) with $G=L_2(\Omega)$, $\varphi=\Sigma_h\psi$, and $\eta=\Sigma_h\lambda_d$, and (2.1) we easily obtain:

$$\begin{aligned}\|\psi-\psi_h\|_0\leqq C\sup_{d\in L_2(\Omega)}\{&\|d\|_0\|\psi-\Sigma_h\psi\|_0+\|\mathbf{u}-\mathbf{u}_h\|_0\|\mathbf{y}_d-\pi_h\mathbf{y}_d\|_0\\&+\|g\|_0\|\lambda_d-\pi_h\lambda_d\|_0\}/\|d\|_0.\end{aligned}$$

From (3.22) we get

$$\|\psi-\Sigma_h\psi\|_0\leqq Ch^2\|\psi\|_2$$

and

$$\|\lambda_d-\Sigma_h\lambda_d\|_0\leqq Ch^2\|\lambda_d\|_2.$$

Using (2.15) and (3.17) we have

$$\|\mathbf{u}-\mathbf{u}_h\|_0\|\mathbf{y}_d-\pi_h\mathbf{y}_d\|_0\leqq C\|\mathbf{u}-\pi_h\mathbf{u}\|_0\|\mathbf{y}_d-\pi_h\mathbf{y}_d\|_0\leqq Ch^2\|\mathbf{u}\|_1\|\mathbf{y}_d\|_1.$$

Noting that $\|d\|_{-1}\leqq\|d\|_0$ and $\|g\|_0\leqq C\|\psi\|_4$, using (3.30) and combining terms we obtain:

$$\|\psi-\psi_h\|_0\leqq Ch^2\|\psi\|_4,\tag{3.33}$$

for $k=1$.

Next we consider the case $k \geqq 2$. Using theorems 3(*a*)-(*c*) with $G = L_2(\Omega)$, $\varphi = \Sigma_h \psi$, and $\eta = \mathscr{I}_h \lambda_d$, and (2.1) we have

$$\|\psi - \psi_h\|_0 \leqq C \sup_{d \in L_2(\Omega)} \{\|d\|_0 \|\psi - \Sigma_h \psi\|_0 + \|\mathbf{u} - \mathbf{u}_h\|_0 \|\mathbf{y}_d - \pi_h \mathbf{y}_d\|_0 + |b(\mathbf{u} - \pi_h \mathbf{u}, \lambda_d - \mathscr{I}_h \lambda_d)|\} / \|d\|_0.$$

From (3.22) with $l = \bar{s} = \min(r-1, k+1)$ we get

$$\|\psi - \Sigma_h \psi\|_0 \leqq C h^{\bar{s}} \|\psi\|_{\bar{s}}.$$

Using (2.15) and (3.17) with $l = \bar{s} - 1$ we see that

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leqq C h^{\bar{s}-1} \|\psi\|_{\bar{s}+1}$$

and using (3.17) with $l = 1$ and (3.30) we obtain:

$$\|\mathbf{y}_d - \pi_h \mathbf{y}_d\|_0 \leqq C h \|d\|_0.$$

Finally from (3.29) with $\mathbf{v} = \mathbf{u}$, $\varphi = \lambda_d$, $l = \bar{s} - 1$, and $t = 3$, and (3.31) we see that

$$|b(\mathbf{u} - \pi_h \mathbf{u}, \lambda_d - \mathscr{I}_h \lambda_d)| \leqq C h^{\bar{s}} \|\psi\|_{\bar{s}+1} \|d\|_0.$$

Combining these estimates we have

$$\|\psi - \psi_h\|_0 \leqq C h^{\bar{s}} \|\psi\|_{\bar{s}+1}, \tag{3.34}$$

where $\bar{s} = \min(r-1, k+1)$ and $k \geqq 2$.

Note that (3.34) gives an improvement over (3.31) only for $k+1 \leqq r-1$. Estimates (3.31) improve estimates in [7]. Babuška, Osborn, and Pitkäranta [3] have proved (3.23), (3.31)-(3.34).

d) Raviart-Thomas method

In our final example we study a mixed method for second order elliptic problems introduced by Raviart and Thomas [22, 25]. For $g \in L_2(\Omega)$, Ω a convex polygon in $\mathbb{R}^2$, we consider the model problem

$$\left.\begin{array}{l} -\Delta\psi = g \quad \text{in } \Omega \\ \psi = 0 \quad \text{on } \Gamma. \end{array}\right\} \tag{3.35}$$

Let $\mathbf{H}(\mathrm{div}; \Omega) = \{\mathbf{v} \in [L_2(\Omega)]^2 : \mathrm{div}\, \mathbf{v} \in L_2(\Omega)\}$ with the norm

$$\|\mathbf{v}\|_{\mathbf{H}(\mathrm{div};\, \Omega)} = (\|\mathbf{v}\|_0^2 + \|\mathrm{div}\, \mathbf{v}\|_0^2)^{1/2}.$$

The mixed method we study is based on the following variational formulation of (3.35).

Find $(\mathbf{u}, \psi) \in \mathbf{H}(\text{div}; \Omega) \times L_2(\Omega)$ such that

$$\int_\Omega \mathbf{u} \cdot \mathbf{v}\, dx + \int_\Omega \psi \,\text{div}\, \mathbf{v}\, dx = 0, \qquad \forall\, \mathbf{v} \in H(\text{div}; \Omega) \tag{3.36}$$

and

$$\int_\Omega \varphi\,(\text{div}\, \mathbf{u} + g)\, dx = 0, \qquad \forall\, \varphi \in L_2(\Omega). \tag{3.37}$$

In theorem 1 of [22] it is shown that problem (3.36)-(3.37) has a unique solution $(\mathbf{u}, \psi) \in \mathbf{H}(\text{div}; \Omega) \times L_2(\Omega)$, that ψ is the solution of problem (3.35), and $\mathbf{u} = \mathbf{grad}\, \psi$. In addition the following regularity result is known for this problem: If $g \in L_2(\Omega)$ then $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ and

$$\| \psi \|_2 \leqq C \| g \|_0, \qquad \forall\, g \in L_2(\Omega). \tag{3.38}$$

One easily sees that (3.36)-(3.37) is an example of problem P with $V = \mathbf{H}(\text{div}; \Omega)$, $H = W = L_2(\Omega)$, $[L_2(\Omega)]^2$,

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u} \cdot \mathbf{v}\, dx \qquad \text{and} \qquad b(\mathbf{u}, \psi) = \int_\Omega \psi \,\text{div}\, \mathbf{u}\, dx.$$

The subclass D of data for which (H1) is satisfied is given by $D = 0 \times W'$. Since a is symmetric, the adjoint problem (2.5), (2.6) with $G = W = L_2(\Omega)$ is the same as problem P and thus is uniquely solvable for all $d \in W'$. Hence (H2) is satisfied. Using (3.38) we also see that $\lambda_d \in H^2(\Omega) \cap H_0^1(\Omega)$, $\mathbf{y}_d = \mathbf{grad}\, \lambda_d$, and

$$\| \mathbf{y}_d \|_1 + \| \lambda_d \|_2 \leqq C \| d \|_0. \tag{3.39}$$

We now describe the finite dimensional subspaces used in the approximation scheme. Following [22] we begin by introducing the space $\hat{\mathbf{Q}}$ associated with the unit right triangle $\hat{T}$ in the (ξ, η)-plane whose vertices are $\hat{a}_1 = (1, 0)$, $\hat{a}_2 = (0, 1)$, $\hat{a}_3 = (0, 0)$. For $k \geqq 0$ an even integer, define $\hat{\mathbf{Q}}$ to be the space of all functions $\hat{\mathbf{q}}$ of the form

$$\hat{q}_1 = \text{pol}_k(\xi, \eta) + \alpha_0 \xi^{k+1} + \alpha_1 \xi^k \eta + \ldots + \alpha_{k/2} \xi^{k/2+1} \eta^{k/2},$$
$$\hat{q}_2 = \text{pol}_k(\xi, \eta) + \beta_0 \eta^{k+1} + \beta_1 \xi \eta^k + \ldots + \beta_{k/2} \xi^{k/2} \eta^{k/2+1}$$

with

$$\sum_{i=0}^{k/2} (-1)^i (\alpha_i - \beta_i) = 0,$$

where $\text{pol}_k(\xi, \eta)$ denotes any polynomial of degree k in the two variables ξ, η.

For $k \geqq 1$ an odd integer, define $\hat{\mathbf{Q}}$ to be the space of all functions $\hat{\mathbf{q}}$ of the form

$$\hat{q}_1 = \mathrm{pol}_k(\xi, \eta) + \alpha_0 \xi^{k+1} + \alpha_1 \xi^k \eta + \ldots + \alpha_{(k+1)/2} \xi^{(k+1)/2} \eta^{(k+1)/2},$$

$$\hat{q}_2 = \mathrm{pol}_k(\xi, \eta) + \beta_0 \eta^{k+1} + \beta_1 \xi \eta^k + \ldots + \beta_{(k+1)/2} \xi^{(k+1)/2} \eta^{(k+1)/2},$$

with

$$\sum_{i=0}^{(k+1)/2} (-1)^i \alpha_i = \sum_{i=0}^{(k+1)/2} (-1)^i \beta_i = 0.$$

Now consider any triangle T in the (x_1, x_2)-plane whose vertices are denoted by a_i, $1 \leqq i \leqq 3$. Let $F_T : \hat{x} \to F_T(\hat{x}) = B_T \hat{x} + b_T$, $B_T \in \mathscr{L}(\mathbb{R}^2)$, $b_T \in \mathbb{R}^2$ be the unique invertible affine mapping such that $F_T(\hat{a}_i) = a_i$, $1 \leqq i \leqq 3$. With each vector-valued function $\hat{\mathbf{v}} = (\hat{v}_1, \hat{v}_2)$ defined on $\hat{T}$, we associate the function $\mathbf{v}$ defined on T by

$$\mathbf{v} = \frac{1}{J_T} B_T \hat{\mathbf{v}} \circ F_T^{-1},$$

where

$$J_T = \det(B_T).$$

For $0 < h < 1$, assume that τ_h is a triangulation of $\bar{\Omega}$ made up of triangles T whose diameters are less than or equal to h which satisfy the minimal angle condition (*see* subsection 3 *a*). We finally consider problem P_h with

$$V_h = \{ \mathbf{v}_h \in \mathbf{H}(\mathrm{div}; \Omega) : \forall\, T \in \tau_h,\ \mathbf{v}_h|_T \in \mathbf{Q}_T \},$$

where

$$\mathbf{Q}_T = \{ \mathbf{v} \in H(\mathrm{div}; T) : \hat{\mathbf{v}} \in \hat{\mathbf{Q}} \}$$

and

$$W_h = \{ \varphi_h \in L_2(\Omega) : \forall\, T \in \tau_h,\ \varphi_h|_T \in P_k \}.$$

To apply our theorems we must check that the appropriate hypotheses are satisfied. Now (H3) is trivially satisfied with $\alpha = 1$. In the proof of theorem 3 of [22] it is essentially shown that there is an operator $\pi_h : [H^1(\Omega)]^2 \to V_h$ satisfying

$$b(v - \pi_h v, \varphi) = 0, \qquad \forall\, \mathbf{v} \in [H^1(\Omega)]^2 \quad \text{and} \quad \forall\, \varphi \in W_h.$$

Furthermore, for $\mathbf{v} \in [H^{r-1}(\Omega)]^2$, $r \geqq 2$, we have

$$\| \mathbf{v} - \pi_h \mathbf{v} \|_0 \leqq C h^l \| \mathbf{v} \|_l, \qquad 1 \leqq l \leqq \min(r-1, k+1), \tag{3.40}$$

and

$$\| \mathrm{div}(\mathbf{v} - \pi_h \mathbf{v}) \|_0 \leqq C h^m \| \mathrm{div}\, \mathbf{v} \|_m, \qquad 0 \leqq m \leqq \min(r-2, k+1). \tag{3.41}$$

Using the regularity results (3.38) and (3.39) we easily see that $Y \subset [H^1(\Omega)]^2$ so that (H5) is valid.

We next observe that for $\mathbf{v}_h \in V_h$, $\operatorname{div} \mathbf{v}_h|_T \in P_k$. Hence $\mathbf{v}_h \in Z_h$ easily implies $\operatorname{div} \mathbf{v}_h = 0$ and so $\mathbf{v}_h \in Z$. Thus $Z_h \subset Z$ and so we are in the special cases of theorems 2 and 3. Again by the corollary to theorem 2, (H4) need not be satisfied in order to apply (2.15), so we shall not require $\{\tau^h\}$ to be quasi-uniform.

We are now ready to derive the error estimates. Assume that $\psi \in H^r(\Omega)$, $r \geqq 2$. From (2.15) and (3.40) we obtain for $k \geqq 0$:

$$\|\mathbf{u}-\mathbf{u}_h\|_0 \leqq C\|\mathbf{u}-\pi_h \mathbf{u}\|_0 \leqq C h^t \|\mathbf{u}\|_t \leqq C h^t \|\psi\|_{t+1}, \tag{3.42}$$

where $t = \min(r-1, k+1)$.

Now applying theorem 3 (*a*)-(*b*) we get

$$\|\psi-\psi_h\|_0 = \sup_{d \in L_2(\Omega)} \{ b(\mathbf{y}_d - \pi_h \mathbf{y}_d, \psi - \varphi) + a(\mathbf{u}_h - \mathbf{u}, \pi_h \mathbf{y}_d - \mathbf{y}_d) + b(\mathbf{u} - \pi_h \mathbf{u}, \lambda_d - \eta) \} / \|d\|_0 \tag{3.43}$$

for all $\varphi, \eta \in W_h$. Using (3.41) and standard approximability properties of W_h, we have

$$\inf_{\eta \in W_h} |b(\mathbf{y}_d - \pi_h \mathbf{y}_d, \psi - \varphi)| \leqq \|\operatorname{div}(\mathbf{y}_d - \pi_h \mathbf{y}_d)\|_0 \inf_{\eta \in W_h} \|\psi - \varphi\|_0 \leqq C \|\operatorname{div} \mathbf{y}_d\|_0 h^{\mu} \|\psi\|_{\mu}, \tag{3.44}$$

where $\mu = \min(r, k+1)$, and choosing $m = \mu - 2$ in (3.32),

$$\inf_{\eta \in W_h} |b(\mathbf{u} - \pi_h \mathbf{u}, \lambda_d - \eta)| \leqq \|\operatorname{div}(\mathbf{u} - \pi_h \mathbf{u})\|_0 \inf_{\eta \in W_h} \|\lambda_d - \eta\|_0 \leqq C h^{\mu-2} \|\operatorname{div} \mathbf{u}\|_{\mu-2} h^2 \|\lambda_d\|_2, \tag{3.45}$$

provided $k \geqq 1$.

Using (2.1), (2.15), and (3.40) (with $l = \mu - 1$) we obtain for $k \geqq 1$:

$$|a(\mathbf{u}_h - \mathbf{u}, \pi_h \mathbf{y}_d - \mathbf{y}_d)| \leqq \|\mathbf{u}_h - \mathbf{u}\|_0 \|\pi_h \mathbf{y}_d - \mathbf{y}_d\|_0 \leqq C \|\mathbf{u} - \pi_h u\|_0 \|\pi_h \mathbf{y}_d - \mathbf{y}_d\|_0 \leqq C h^{\mu-1} \|\mathbf{u}\|_{\mu-1} h \|\mathbf{y}_d\|_1. \tag{3.46}$$

Now from (3.43), (3.44), (3.45), (3.46) and the regularity result (3.39) we obtain for $k \geqq 1$:

$$\|\psi - \psi_h\|_0 \leqq h^{\mu} \|\psi\|_{\mu}, \tag{3.47}$$

where $\mu = \min(r, k+1)$.

To obtain an estimate when $k=0$, we choose $m=0$ in (3 41) to obtain

$$\inf_{\eta\in W_h} |b(\mathbf{u}-\pi_h\mathbf{u}, \lambda_d-\eta)| \leqq \|\operatorname{div}(\mathbf{u}-\pi_h\mathbf{u})\|_0 \inf_{\eta\in W_h} \|\lambda_d-\eta\|_0$$

$$\leqq C\|\operatorname{div}\mathbf{u}\|_0 h\|\lambda_d\|_1 \quad (3\ 48)$$

and choose $l=1$ in (3 40) to obtain in the same manner as in (4 36) that

$$|a(\mathbf{u}_h-\mathbf{u}, \pi_h\mathbf{y}_d-\mathbf{y}_d)| \leqq C\|\mathbf{u}-\pi_h\mathbf{u}\|_0\|\pi_h\mathbf{y}_d-\mathbf{y}_d\|_0$$

$$\leqq Ch\|\mathbf{u}\|_1 h\|\mathbf{y}_d\|_1 \quad (3\ 49)$$

Combining (3 43), (3 44) with $k=0$, (3 48), (3 49), and the regularity result (3 39) we get

$$\|\psi-\psi_h\|_0 \leqq Ch\|\psi\|_2, \qquad k=0 \quad (3\ 50)$$

We note that estimate (3 42) was obtained in [25], IX-3 22 a, and that (3 47) gives an improvement over the result in [25], IX-3 22 a, in the case where $\psi\in H^r(\Omega)$, $\mathbf{u}\in H^{r-1}(\Omega)$, and $2\leqq r<k+1$

ACKNOWLEDGEMENT

The second author would like to thank I Babuška and J Pitkaranta for several helpful discussions on mixed methods

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