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BASIC COMPACTNESS PROPERTIES OF NONCONFORMING AND HYBRID FINITE ELEMENT SPACES (*)

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Communiqué par P. G. CIARLET

Abstract. — *A generalized Rellich theorem for nonconforming and hybrid finite element spaces is established. Thereby very general stability and convergence theorems for approximations of inhomogeneous elliptic variational equations and of elliptic eigenvalue problems are obtained.*

Résumé. — *On établit un théorème de Rellich généralisé pour les espaces d'éléments finis non conformes et hybrides. De cette façon, on obtient des théorèmes très généraux de stabilité et de convergence pour l'approximation d'équations elliptiques variationnelles non homogènes et de problèmes elliptiques de valeurs propres.*

INTRODUCTION

The paper generalizes the well-known Rellich compactness theorem to sequences of piecewise polynomial function spaces occuring in methods of nonconforming and hybrid finite elements. Basic assumptions are suitable weak continuity conditions at interelement boundaries together with the approximability condition and the validity of the generalized patch test. It is shown that the nonconforming finite elements of Wilson, Adini, Crouzeit-Raviart, Morley and de Veubecke satisfy these conditions. As applications, generalized Ehrling, Poincaré and Friedrichs inequalities are obtained as well as very general stability and convergence theorems for nonconforming and hybrid approximations of generalized elliptic variational equations and eigenvalue problems with variable, not necessarily smooth coefficients.

The present investigation, together with the papers [19, 20], continues our perturbation theory for Sobolev spaces $W^{m,p}(G)$, begun in [16, 17] by a study of perturbations of the boundary ∂G. Methods of nonconforming and hybrid finite

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elements are specified by sequences of piecewise polynomial function spaces $V_\iota \subset W^{m,p}(G_\iota)$, $\iota = 1, 2, \ldots$ The domains G_ι are the union of all open finite elements of the decompositions $\mathcal{K}_\iota$ of $\overline{G}$, the properties of which will be explained in sections 1.1, 1.2. As one easily sees the sequence of boundaries ∂G_ι converges to $\overline{G}$ in the Hausdorff metric for $\iota \to \infty$. Hence in this class of approximations at each point of the domain $\overline{G}$ essential perturbations can arise from subspaces V_ι lacking sufficient continuity properties on interelement boundaries (*see* Stummel [19]).

A first continuity condition in the mean is imposed by the requirement that the generalized patch test, stated in Stummel [20], is passed ensuring the closedness of the sequence $V_0, V_1, V_2, \ldots$ In section 1.3, a further weak continuity condition is introduced. The weak or the more special strong continuity condition, in general, do not entail the validity of the generalized patch test. For example, nonconforming approximations by Zienkiewicz triangles have the strong continuity property for arbitrary decompositions. It is well-known, however, that these elements do not converge for certain meshes and thus cannot pass the generalized patch test.

Sections 2.1, 2.2 deduce from the weak continuity condition the asymptotic equicontinuity in the L^p-mean of bounded sequences of functions $v_\iota \in V_\iota$, $\iota = 1, 2, \ldots$, and their derivatives up to the $(m-1)$-th order and thus the so-called weak discrete compactness of the sequence of natural embeddings of V_ι into $W^{m-1,p}(G_\iota)$, $\iota \in \mathbb{N}$. On this basis, the generalized Rellich theorem is easily established in section 2.3. Seemingly, such a general theorem for nonconforming and hybrid finite elements is not yet found in the literature. In the thesis of Thomas, from the continuity properties of hybrid finite elements only a generalized Poincaré-Friedrichs inequality is derived ([21, p. V-38]). By other methods, Rannacher [10] obtains a compactness theorem for a special class of nonconforming finite elements approximating $H^2(G)$.

Section 3.1 demonstrates that the nonconforming and hybrid finite elements named above, passing the generalized patch test [20], also possess the additional continuity properties required here. Thus the fundamental preconditions are valid for applying general functional analytic theorems (*see* Stummel [14, 15]) to this class of approximations. Using compactness arguments, a very general stability and convergence theorem is established in section 3.2 for approximations of inhomogeneous variational equations. In this way, particularly, the common assumption of V-ellipticity may be replaced by the weaker assumption of uniform coerciveness of the sequence of approximating problems. The theorem simultaneously states the solvability of almost all approximating equations of a properly posed inhomogeneous variational problem. Under corresponding general assumptions for generalized elliptic eigenvalue problems spectra and resolvent sets, eigenvalues and eigenspaces of the approximating eigenvalue problems converge as it is ascertained in section 3.3, using results of Grigorieff [6], Stummel [14, 15, 18]. Note hat the results of sections 3.2, 3.3 are valid as well in conforming finite element approximations.

1. DECOMPOSITIONS OF POLYHEDRAL DOMAINS AND CONTINUITY CONDITIONS

Methods of nonconforming and hybrid finite elements work with approximating spaces of piecewise polynomial functions being discontinuous at interelement boundaries. In order to ensure the convergence of such approximations, certain asymptotic continuity requirements in the mean over the interelement boundaries are necessary and sufficient which have been stated in form of a generalized patch test in our paper [20]. With regard to compactness properties of the approximations, this section introduces a further weak continuity condition ensuring the asymptotic equicontinuity of sequences of functions having uniformly bounded gradients. The weak continuity condition is valid, in particular, if the approximating functions possess at each $(n-1)$-dimensional interelement face F of the finite elements $K \in \mathscr{K}_\iota$ at least one point of continuity or a continuous mean value over the face.

1.1. Decompositions of polyhedral domains

This and the following section collect the basic assumptions regarding decompositions of polyhedral domains in $\mathbb{R}^n$ by convex polyhedra and deduce some special properties as far as they are needed in the paper. For basic concepts and properties of convex polyhedra we refer to [2, 7, 12]. The topological assumptions (K1), (K2) correspond to the conditions $(\mathscr{T}_h 1), \ldots, (\mathscr{T}_h 5)$ of Ciarlet [4, p. 38, 51]. The alternative $(\mathscr{T}_h 5)$ is here a consequence of (K1), (K2) what is proved in theorem (6). Further, the metric properties (K3), (K4) are needed. Assumption (K3) is the usual regularity condition for the finite elements together with the requirement of a continued refinement of the meshes for $\iota \to \infty$. In addition, the assumption (K4) is made, called "inverse assumption" by Ciarlet [4, p. 140], guaranteeing a certain uniformity of the decompositions. Note that this condition is also used by Thomas [21, p. III-24, V-38] in the context of hybrid finite elements.

Let $\overline{G}$ be a closed polyhedral domain in $\mathbb{R}^n$, not necessarily being bounded. For every $\iota = 1, 2, \ldots$, let $\mathscr{K}_\iota$ be a locally finite decomposition of $\overline{G}$ by bounded closed convex polyhedra $K \subset \bar{G}$ having nonvoid *interior* $\underline{K} = \operatorname{int}(K)$. The polyhedra K constitute the *finite elements* of the decompositions of $\bar{G}$. A subset F is said to be a *face* of K if there exists a supporting hyperplane H of K such that $F = H \cap K$. We assume that the decompositions $\mathscr{K}_\iota$, $\iota = 1, 2, \ldots$, have the topological properties

(K1) $$\bar{G} = \bigcup_{K \in \mathscr{K}_\iota} K.$$

(K2) *For every pair of distinct elements K, K' in $\mathscr{K}_\iota$ either $K \cap K'$ is empty or a face of both K and K'.*

The decompositions $\mathscr{K}_\iota$ define open subsets G_ι of $\bar{G}$ by

$$G_\iota = \bigcup_{K \in \mathscr{K}_\iota} \underline{K}, \qquad \iota = 1, 2, \ldots \tag{1}$$

The elements K are closed domains, that is, $K = \overline{(\underline{K})} = \operatorname{cl}(\operatorname{int}(K))$. Hence

$$K = \overline{(\underline{K})} \subset \bar{G}_\iota \subset \bar{G}$$

for all $K \in \mathscr{K}_\iota$ so that

$$\bar{G}_\iota = \bar{G}, \qquad \iota = 1, 2, \ldots \tag{2}$$

The interior $G = \operatorname{int}(\bar{G})$ of $\bar{G}$, evidently, satisfies the relation $\underline{K} \subset G$ for all $K \in \mathscr{K}_\iota$ and thus

$$G_\iota \subset G \subset \bar{G}, \qquad \iota = 1, 2, \ldots \tag{3}$$

By (2), this implies that the closure $\operatorname{cl}(G)$ of $G = \operatorname{int}(\bar{G})$ is equal to $\bar{G}$. Consequently $\bar{G}$ is a closed domain and G an open domain,

$$\bar{G} = \operatorname{cl}(\operatorname{int}(\bar{G})), \qquad G = \operatorname{int}(\operatorname{cl}(G)). \tag{4}$$

The boundary ∂K of an element $K \in \mathscr{K}_\iota$ consists of a finite number of uniquely determined $(n-1)$-dimensional convex polyhedra, the *$(n-1)$-faces* F of K. Let $\mathscr{F}_\iota$ be the set of all $(n-1)$-faces of elements $K \in \mathscr{K}_\iota$. Due to property (K2), two distinct elements K, K' have no common interior points. Using (2), one obtains the representation

$$\partial G_\iota = \bar{G} - G_\iota = \bigcup_{F \in \mathscr{F}_\iota} F, \qquad \iota = 1, 2, \ldots, \tag{5}$$

of the boundaries of the open sets G_ι. By virtue of (2) and (3), obviously, $\partial G \subset \partial G_\iota$ for all ι. The boundary ∂G of the domain $G = \operatorname{int}(\bar{G})$ is described in detail in the next theorem. By ri F is meant the *relative interior* of the face F, that is, the interior of F regarded as a subset of the affine hull aff F of F. In view of the alternative stated below, $(n-1)$-faces F of elements $K \in \mathscr{K}_\iota$ are said to be *free* iff F belongs to the boundary ∂G, and *interelement faces* otherwise.

For each $(n-1)$-face $F \in \mathscr{F}_\iota$ the following alternative is true: Either F belongs to ∂G or F is a common face of two distinct elements in $\mathscr{K}_\iota$ and thus an interelement face. Consequently, the boundary ∂G is the union of all free $(n-1)$-faces of elements in $\mathscr{K}_\iota$. (6)

Proof: (i) Let F be any face in $\mathscr{F}_\iota$, then either $F \subset \partial G$ or ri $F \subset G$. For, if F does not belong to ∂G, there exists a point $x \in G \cap \text{ri } F$, an element $K \in \mathscr{K}_\iota$ such that $F \subset \partial K$, and a sequence of points $x_t \in G$, $x_t \notin K$, $t = 1, 2, \ldots$, such that $x_t \to x$ for $t \to \infty$. As $\mathscr{K}_\iota$ is a locally finite covering of G there is an element $K' \in \mathscr{K}_\iota$ and an infinite subsequence $\mathbb{N}' \subset \mathbb{N}$ such that $x_t \in K'$, $t \subset \mathbb{N}'$. K' is closed and $K' \neq K$ so that the limit x of the sequence also belongs to K' and therefore to $K \cap K'$. By virtue of assumption (K2), $K \cap K'$ is a face of both K and K'. Since x is in the relative interior of the $(n-1)$-dimensional face F of K, it follows that $F = K \cap K'$, $F \subset \partial K$ and $F \subset \partial K'$. Hence one immediately obtains the relation

$$\text{ri } F \subset \text{int } (K \cup K') \subset \text{int } (\overline{G}) = G.$$

(ii) The relative interior ri F has the closure cl (ri F) $= F$ so that ri F is dense in F. In view of (5), correspondingly, the set

$$D = \bigcup_{F \in \mathscr{F}_\iota} \text{ri } F,$$

is dense in ∂G_ι and $D \cap \partial G$ dense in the subset ∂G of ∂G_ι. By means of the alternative proved above, we have

$$D \cap \partial G = \bigcup_{F \in \mathscr{F}_\iota} (\text{ri } F \cap \partial G) = \bigcup_{\substack{F \in \mathscr{F}_\iota \\ F \subset \partial G}} \text{ri } F.$$

This gives the representation of the closure ∂G of $D \cap \partial G$:

$$\partial G = \text{cl}(D \cap \partial G) = \bigcup_{\substack{F \in \mathscr{F}_\iota \\ F \subset \partial G}} F. \qquad \square$$

In addition to the topological properties (K1), (K2) of the decompositions $\mathscr{K}_\iota$ of $\overline{G}$, two metric properties (K3), (K4) are needed in the further investigations.

Let $\delta_0(K)$ be the greatest diameter of all balls contained in K, let $\delta_1(K)$ be the greatest diameter of the element K, and let

$$h = \sup_{K \in \mathscr{K}_\iota} \delta_1(K), \qquad \iota = 1, 2, \ldots \tag{7}$$

Notice that h depends on the index ι.

(K3) *There is some constant ζ such that $\delta_1(K) \leqq \zeta\delta_0(K)$ for all $K \in \mathscr{K}_\iota$, $\iota = 1, 2, \ldots$, and $h \to 0$ for $\iota \to \infty$.*

(K4) *There is a positive constant η such that*

$$\eta h \leqq \delta_0(K), \qquad K \subset \mathscr{K}_\iota, \quad \iota = 1, 2, \ldots$$

The condition (K3) is the usual *regularity* property of the decompositions, needed in the derivation of error estimates for finite element approximations, together with the requirement of mesh refinement for $\iota \to \infty$. The condition (K4) guarantees a *quasiuniformity* of the decompositions. Under these conditions, for example, the volume of the elements K can be bounded from below and from above by

$$\omega_n \eta^n h^n \leqq |K| \leqq \omega_n h^n, \tag{8}$$

uniformly for all $K \subset \mathscr{K}_\iota$ and $\iota = 1, 2, \ldots$, where ω_n denotes the volume of balls of diameter 1 in the Euclidean $\mathbb{R}^n$.

The continued refinement of the decompositions $\mathscr{K}_\iota$ for $\iota \to \infty$ casts a closer and closer net of element boundaries ∂G_ι over $\overline{G}$. Each point $x \in \overline{G}$ lies, by (K1), in some element $K \in \mathscr{K}_\iota$ and so at most within a distance h from the boundary ∂K and thus from the boundaries ∂G_ι. As $\partial G_\iota \subset \overline{G}$, the Hausdorff distances between these sets satisfy

$$d(\overline{G}, \partial G_\iota) = \sup_{x \in G} |x, \partial G_\iota| \leqq h \to 0 \qquad (\iota \to \infty). \tag{9}$$

Consequently, the sequence of boundaries ∂G_ι converges in this sense to $\overline{G}$ for $\iota \to \infty$.

1.2. Special properties of decompositions

For arbitrary subsets $S \subset \mathbb{R}^n$ we denote by $\mathscr{K}_\iota(S)$ the set of all elements $K \in \mathscr{K}_\iota$ having a nonvoid intersection with S. In particular

$$\mathscr{K}_\iota(x) = \{K \in \mathscr{K}_\iota \mid x \in K\}, \tag{1}$$

for all $x \in \overline{G}$. When x belongs to the interior $\underline{K}$ of some element $K \in \mathscr{K}_\iota$, $\mathscr{K}_\iota(x)$

contains only this one element K. In case x lies on interelement boundaries, $\mathcal{K}_\iota$ contains more than one element. The *order* $o(\mathcal{K}_\iota)$ of the decomposition or covering $\mathcal{K}_\iota$ of $\bar{G}$ is the maximal number of elements in the sets $\mathcal{K}_\iota(x)$ for all x in $\bar{G}$.

The orders of the decompositions $\mathcal{K}_\iota$ of $\bar{G}$ are bounded by $o(\mathcal{K}_\iota) \leqq (2/\eta)^n$ uniformly for all $\iota = 1, 2, \ldots$ (2)

Proof: Let x be any point in $\bar{G}$. The elements $K \in \mathcal{K}_\iota(x)$ contain the point x and have at most the diameter h. Thus all elements $K \in \mathcal{K}_\iota(x)$ are contained in the ball with midpoint x and radius h. By virtue of assumption (K2), pairwise distinct elements in $\mathcal{K}_\iota$ have no common interior points. Using inequality 1.1.(8), the total volume of all elements in $\mathcal{K}_\iota(x)$ is bounded by

$$\nu\omega_n(\eta h)^n \leqq \sum_{K \in \mathcal{K}_\iota(x)} |K| = \Big| \bigcup_{K \in \mathcal{K}_\iota(x)} K \Big| \leqq \omega_n(2h)^n,$$

where ν is the number of elements in $\mathcal{K}_\iota(x)$. This immediately gives the above appraisal of the orders $o(\mathcal{K}_\iota)$. $\square$

The set $\mathcal{K}_\iota(x)$ is said to be *strongly connected* if every two elements K, $K' \in \mathcal{K}_\iota(x)$ can be connected by a *strong chain* of pairwise distinct elements K_0, $K_1, \ldots, K_l \in \mathcal{K}_\iota(x)$ such that $K_0 = K$, $K_l = K'$, and every two consecutive elements have an $(n-1)$-face in common.

For each point $x \in G$ the associated set $\mathcal{K}_\iota(x)$ is strongly connected. (3)

Proof: Let K_0 be an arbitrary element in $\mathcal{K}_\iota(x)$. Let $\mathcal{L}(K_0)$ be the set of all elements $K \in \mathcal{K}_\iota(x)$ that can be connected by a strong chain with K_0. Then $\mathcal{L}(K_0) = \mathcal{K}_\iota(x)$. If this is not true, there exists an element $K' \in \mathcal{K}_\iota(x)$ not belonging to $\mathcal{L}(K_0)$. Assumption (K2) then implies that the intersection of K' and the set $M = \bigcup_{K \in \mathcal{L}(K_0)} K$ is a subset of the boundary of K'. As K' is convex and possesses interior points, in each neighbourhood of x there are points not belonging to M so that x is in the boundary of M. Applying theorem 1.1.(6) to the polyhedral domain M it is seen that x lies on a free $(n-1)$-face $F \subset \partial M$ of an element $K_1 \in \mathcal{L}(K_0)$. The face F cannot belong to the boundary ∂G because x is in G. Theorem 1.1.(6), applied to $\bar{G}$, next yields the existence of an element $K_2 \in \mathcal{K}_\iota$, $K_2 \neq K_1$, such that $K_1 \cap K_2 = F$. As $x \in F$, also $x \in K_2$ so that $K_2 \in \mathcal{K}_\iota(x)$. Finally $K_1 \in \mathcal{L}(K_0)$ entails $K_2 \in \mathcal{L}(K_0)$ in contradiction to F being a free face of K_1 in M. $\square$

Let a, b be any two points such that the line segment $\overline{ab}$ is in G. The decompositions $\mathscr{K}_\iota$ of $\overline{G}$ induce decompositions $\mathscr{I}_\iota$ of $\overline{ab}$ by the sets of all nonvoid pairwise distinct subsegments

$$I = K \cap \overline{ab}, \qquad K \in \mathscr{K}_\iota, \tag{4}$$

for each $\iota = 1, 2, \ldots$ Since K is convex, also I is convex and thus a closed subinterval of the line segment $\overline{ab}$. Under these conditions, the decompositions $\mathscr{I}_\iota$ of $\overline{ab}$ have properties analogous to (K1), (K2):

The representation

$$\overline{ab} = \bigcup_{I \in \mathscr{I}_\iota} I,$$

is valid and for each pair of distinct elements $I, I' \in \mathscr{I}_\iota$ either $I \cap I'$ is empty or consists of a single point. (5)

Proof: The above representation follows immediately from the property (K1) of the decompositions $\mathscr{K}_\iota$. Let I, I' be any pair of distinct subsegments in $\mathscr{I}_\iota$. The intersection $I \cap I'$ is either empty, or contains one, or at least two points. In case there are at least two points in $I \cap I'$, this set is a subinterval of $\overline{ab}$ having relatively interior points $c \in I \cap I'$. The associated elements K, K' then possess a nonvoid intersection $F = K \cap K'$. Since $I \neq I'$ also $K \neq K'$ such that by (K2) F is a face of both K and K'. The segment I lies in K and the relatively interior point c on the face F of K so that I must be a subset of F (*see* [12, p. 162]). Correspondingly, from $I' \subset K'$ and $c \in F$ one infers $I' \subset F$. Hence $I \subset I \cap I'$, $I' \subset I \cap I'$, that is, $I \subset I'$, $I' \subset I$, contradicting $I \neq I'$. $\square$

1.3. Continuity conditions

Let $C^{m,\infty}(G_\iota)$ be the space of m-times continuously differentiable functions u_ι having uniformly bounded partial derivatives up to the order m over G_ι such that

$$|u_\iota|_{k,\infty,G_\iota} = \sup_{x \in G_\iota} |\nabla^k u_\iota(x)| < \infty, \qquad k = 0, \ldots, m. \tag{1}$$

Functions $u_\iota \in C^{m,\infty}(G_\iota)$ are, in general, not continuous across interelement boundaries. Given any multiindex μ of order $|\mu| \leqq m-1$, $v_\iota = D^\mu u_\iota$ is a member of the space $C^{1,\infty}(G_\iota)$. For every element $K \in \mathscr{K}_\iota$ and every pair of points x, x' in the interior $\underline{K}$ of K, the join $\overline{xx'}$ belongs to $\underline{K}$ because $\underline{K}$ is convex. By the mean value theorem, functions $v_\iota \in C^{1,\infty}(G_\iota)$ satisfy the inequality

$$|v_\iota(x) - v_\iota(x')| \leqq |x - x'| \sup_{\underline{K}} |\nabla v_\iota|, \qquad x, x' \in \underline{K}. \tag{2}$$

From this inequality it is seen that v_ι can be extended continuously to $\overline{(\underline{K})}=K$ and that (2) holds as well for the extension v_ι^K and all $x, x' \in K$. Thus for all points x at interelement boundaries and all elements $K \in \mathscr{K}_\iota(x)$ there exist unique continuous extensions v_ι^K of $v_\iota | \underline{K}$ to K. If not stated otherwise, henceforth, by $v_\iota(x)$ is meant any of the values $v_\iota^K(x)$ for some $K \in \mathscr{K}_\iota(x)$.

In view of compactness theorems for sequences of functions $v_\iota \in C^{1,\infty}(G_\iota)$, restrictions of the discontinuities at interelement boundaries are required. For this purpose the following *weak continuity condition* is introduced:

There exists some constant $\vartheta > 0$ *such that the inequalities*

$$\left| v_\iota^K(x) - v_\iota^{K'}(x) \right| \leqq \vartheta h \sup_{K \in \mathscr{K}_\iota(x)} \sup_K \left| \nabla v_\iota \right|,$$

hold, uniformly for all elements $K, K' \in \mathscr{K}_\iota(x)$, *each point* $x \in G \cap \partial G_\iota$ *and all* $\iota = 1, 2, \ldots$ (3)

Regarding the formulation of this condition, notice that each point $x \in G_\iota$ is interior to some element $K \in \mathscr{K}_\iota$. In this case $\mathscr{K}_\iota(x) = \{K\}$ and the above inequality holds trivially with $K = K'$.

We shall now establish a simple sufficient criterion ascertaining the above weak continuity condition in applications to nonconforming and hybrid finite element spaces.The $(n-1)$-faces $F \in \mathscr{F}_\iota$ of elements $K \in \mathscr{K}_\iota$ are compact subsets of $(n-1)$-dimensional linear manifolds in $\mathbb{R}^n$. By $C(F)$ is meant the Banach space of continuous functions on F endowed with the maximum norm. By theorem 1.1.(6), to each interelement face $F \in \mathscr{F}_\iota$ there are exactly two distinct elements K, K' such that $F = K \cap K'$. The functions $v_\iota \in C^{1,\infty}(G_\iota)$ possess continuous extensions $v_\iota^K, v_\iota^{K'}$ to K, K' the restrictions of which on F belong to $C(F)$. We say, a function $v_\iota \in C^{1,\infty}(G_\iota)$ has the *strong continuity property* if to each interelement face $F \in \mathscr{F}_\iota$ there exists a continuous linear form q_F on $C(F)$ such that

$$q_F(1) = 1, \qquad \| q_F \| \leqq 1, \qquad q_F(v_\iota^K) = q_F(v_\iota^{K'}), \tag{4}$$

where, for convenience, $q_F(v_\iota^K)$ is written instead of $q_F(v_\iota^K | F)$. This condition is true, for example, if to each interelement face F there exists a *point of continuity* c_F of v_ι such that

$$v_\iota^K(c_F) = v_\iota^{K'}(c_F), \qquad c_F \in F = K \cap K'. \tag{5}$$

In this case, q_F is the Dirac functional to the point c_F. Another example is obtained by the requirement of *continuity of mean values* over the faces F,

$$\int_F v_\iota^K \, ds = \int_F v_\iota^{K'} \, ds, \qquad F = K \cap K' \in \mathscr{F}_\iota. \tag{6}$$

The linear form q_F is now specified by

$$q_F(w)=\frac{1}{|F|}\int_F w\,ds, \qquad w\in C(F).$$

A sequence $v_\iota\in C^{1,\infty}(G_\iota)$, $\iota=1, 2, \ldots$, satisfies the weak continuity condition if each function v_ι has the strong continuity property. (7)

Proof: Choose any $\iota=1, 2, \ldots$ and $x\in G\cap\partial G_\iota$. Let K, K' be an arbitrary pair of distinct elements in $\mathscr{K}_\iota(x)$. By lemma 1.2.(3), $\mathscr{K}_\iota(x)$ is strongly connected so that there exists a strong chain of pairwise distinct elements $K_0, \ldots, K_l\in\mathscr{K}_\iota(x)$ joining $K=K_0$ and $K'=K_l$. Each $F_t=K_{t-1}\cap K_t$ is an $(n-1)$-face of both K_{t-1} and K_t. As $x\in G$, F_t does not belong to ∂G. By virtue of the strong continuity property of v_ι, there exist continuous linear forms q_{F_t} having the properties (4) for $F=F_t$. Consequently,

$$q_{F_t}(v_\iota^{K_{t-1}})=q_{F_t}(v_\iota^{K_t}), \qquad t=1, \ldots, l,$$

and

$$v_\iota^K(x)-v_\iota^{K'}(x) = \sum_{t=1}^{l}(v_\iota^{K_{t-1}}(x)-q_{F_t}(v_\iota^{K_{t-1}}))+\sum_{t=1}^{l}(q_{F_t}(v_\iota^{K_t})-v_\iota^{K_t}(x)),$$

the right side of this equation being a telescopic sum. For every $K\in\mathscr{K}_\iota(x)$ and every $(n-1)$-face F of K such that $x\in F$,

$$|q_F(v_\iota^K)-v_\iota^K(x)|=|q_F(v_\iota^K-v_\iota^K(x)1)|\leqq\max_{y\in F}|v_\iota^K(y)-v_\iota^K(x)|\leqq h\max_F|\nabla v_\iota^K|,$$

because $q_F(1)=1$, $\|q_F\|\leqq 1$, and the largest diameter of F is, by 1.1.(7), bounded by h. Majorizing the above sum term by term then yields

$$|v_\iota^K(x)-v_\iota^{K'}(x)|\leqq 2lh\sup_{K\in\mathscr{K}_\iota(x)}|\nabla v_\iota|.$$

By lemma 1.2.(2), the number of elements $K\in\mathscr{K}_\iota(x)$ and so the chain length $l+1$ are bounded by $(2/\eta)^n$. Therefore the weak continuity condition is valid with the constant $\vartheta=2(2/\eta)^n$ uniformly for all $\iota=1, 2, \ldots$ □

An important tool for establishing compactness properties is the following theorem

Let $v_\iota\in C^{1,\infty}(G_\iota)$, $\iota=1, 2, \ldots$, be a sequence of functions satisfying the weak continuity condition. Then, using the constant ϑ in (3), the inequality

$$|v_\iota(a)-v_\iota(b)|\leqq(1+2\vartheta)\,h\sum_{K\in\mathscr{K}_\iota(\overline{ab})}\sup_{\underline{K}}|\nabla v_\iota|$$

holds for all line segments $\overline{ab}$ in G and all $\iota=1, 2, \ldots$ (8)

Proof: (i) The decomposition $\mathcal{K}_\iota$ of $\overline{G}$ induces a decomposition $\mathcal{J}_\iota$ of $\overline{ab}$ having the properties 1.2.(5). Hereby, the segment

$$\overline{ab}=\{x\in\mathbb{R}^n \mid x=a+t(b-a),\ 0\leqq t\leqq 1\}$$

is subdivided into a finite number of subintervals. The coordinates of the end points of these intervals are denoted by

$$0=t_0<t_1<\ldots<t_{l-1}<t_l=1.$$

To each interval $[t_{s-1}, t_s]$ there is exactly one segment $I_s\in\mathcal{J}_\iota$ and accordingly an element $K_s\in\mathcal{K}_\iota$ such that

$$I_s=K_s\cap\overline{ab}=\overline{x_{s-1}\,x_s},\qquad x_s=a+t_s(b-a),\qquad s=1,\ \ldots,\ l.$$

Every function v_ι specifies a function

$$g(t)=v_\iota(a+t(b-a)),\qquad 0\leqq t\leqq 1.$$

This function is continuous in each of the open subintervals (t_{s-1}, t_s). At the points t_s, g has left-and right-sided limits $g(t_s+0)$, $g(t_s-0)$ where we put $g(0-0)=g(0)$, $g(1+0)=g(1)$. This leads to the representation

$$v_\iota(a)-v_\iota(b)=\sum_{s=1}^{l}(g(t_{s-1}+0)-g(t_s-0))\ +\sum_{s=0}^{l}(g(t_s-0)-g(t_s+0)).$$

By means of inequality (2), one obtains the appraisal

$$|g(t_{s-1}+0)-g(t_s-0)|\leqq|x_{s-1}-x_s|\sup_{\underline{K}_s}|\nabla v_\iota|,$$

hence

$$\sum_{s=1}^{l}|g(t_{s-1}+0)-g(t_s-0)|\leqq h\sum_{s=1}^{l}\sup_{\underline{K}_s}|\nabla v_\iota|\leqq h\sum_{K\in\mathcal{K}_\iota(\overline{ab})}\sup_{\underline{K}}|\nabla v_\iota|.$$

where the lengths of the segments $\overline{x_{s-1}\,x_s}$ have been majorized by the largest diameter h of the elements in $\mathcal{K}_\iota$.

(ii) To estimate the second sum, the continuity condition (3) is applied. This results first in the inequality

$$\sum_{s=0}^{l}|g(t_s-0)-g(t_s+0)|\leqq\vartheta h\sum_{s=0}^{l}\sup_{K\in\mathcal{K}_\iota(x_s)}\sup_{\underline{K}}|\nabla v_\iota|.$$

Let $K_s\in\mathcal{K}_\iota(x_s)$ be elements such that

$$\sup_{\underline{K}_s}|\nabla v_\iota|=\sup_{K\in\mathcal{K}_\iota(x_s)}\sup_{\underline{K}}|\nabla v_\iota|,\qquad s=0,\ \ldots,\ l.$$

To each of these elements K_s at most one other element $K_{s'}$ can be equal to K_s. For, otherwise, there are $K_{s_1}=K_{s_2}=K_{s_3}=K$ where $s_1<s_2<s_3$. Then $x_{s_1}, x_{s_2}, x_{s_3}$ lie in K and thus x_{s_2} in the relative interior of $\overline{x_{s_1}x_{s_3}} \subset K$. The point x_{s_2} is the endpoint of an interval $I'=K' \cap \overline{ab}$. The intervals $I=K \cap \overline{ab}$ and I' have a nonvoid intersection neither consisting of a single point nor being the whole interval I what contradicts 1.2.(5). From the above it is then seen that

$$\sum_{s=0}^{l} \left| g(t_s-0)-g(t_s+0) \right| \leqq 2\vartheta h \sum_{s=0}^{l}{}' \sup_{K_s} \left| \nabla v_\iota \right| \leqq 2\vartheta h \sum_{K \in \mathscr{K}_\iota(\overline{ab})} \sup_{K} \left| \nabla v_\iota \right|$$

where the apostrophe in the second sum indicates that it extends only over the subset of pairwise distinct of the elements $K_0, \ldots, K_l$. $\square$

By virtue of the above theorem, the asymptotic equicontinuity of bounded sequences of functions v_ι, $\iota=1, 2, \ldots$, can be established. To see this, some simple properties of the sets $\mathscr{K}_\iota(\overline{ab})$ are needed. The elements $K \in \mathscr{K}_\iota(\overline{ab})$ are contained in the set $S_h(\overline{ab})$ of all points $x \in \mathbb{R}^n$, having the shortest distance $|x, \overline{ab}| \leqq h$ from the segment $\overline{ab}$, because the diameter of K is bounded by h and $K \cap \overline{ab}$ is not empty. Consequently,

$$\bigcup_{K \in \mathscr{K}_\iota(\overline{ab})} K \subset S_h(\overline{ab}). \tag{9}$$

Now

$$|x, \overline{ab}| = \min_{0 \leqq t \leqq 1} |x-(a+t(b-a))| = \min_{0 \leqq t \leqq 1} |x-a-td| = |x-a, \overline{od}|,$$

using the line segment $\overline{od}$ and $d=b-a$. In particular, this shows that

$$S_h(\overline{ab}) = a + S_h(\overline{od}), \qquad a, b \in \mathbb{R}^n. \tag{10}$$

Next it is readily seen that the volume $|S_h(\overline{ab})|$ may be majorized by the volume of a cylinder of height $|d|+2h$ and base $(2h)^{n-1}$, that is,

$$|S_h(\overline{ab})| = |S_h(\overline{od})| \leqq (2h)^{n-1}(|b-a|+2h). \tag{11}$$

Denoting by N the number of elements K in $\mathscr{K}_\iota(\overline{ab})$ and using 1.1.(8), it follows that

$$\omega_n(\eta h)^n N \leqq \sum_{K \in \mathscr{K}_\iota(\overline{ab})} |K| \leqq |S_h(\overline{ab})|, \tag{12}$$

and thus

$$hN \leqq \frac{2^n}{\omega_n \eta^n}(|b-a|+h).$$

Therefore, every sequence of functions $v_\iota \in C^{1,\infty}(G_\iota)$, bounded by $|\nabla v_\iota(x)| \leqq 1$ for $x \in G_\iota$, $\iota = 1, 2, \ldots$, satisfies the inequality

$$|v_\iota(a) - v_\iota(b)| \leqq (1+2\vartheta)\frac{2^n}{\omega_n \eta^n}(|b-a|+h) \tag{13}$$

uniformly for all $\iota = 1, 2, \ldots$ and so is asymptotically equicontinuous.

2. COMPACTNESS PROPERTIES

The purpose of this section is to establish the generalized Rellich theorem for nonconforming and hybrid approximations of the Sobolev spaces $W^{m,p}(G)$. The first step on this way consists in ascertaining the asymptotic equicontinuity in the L^p-mean of functions u_ι and their partial derivatives $D^\mu u_\iota$ up to the $(m-1)$-th order for bounded sequences $u_\iota \in V_\iota \subset W^{m,p}(G_\iota)$, $\iota \in \mathbb{N}$. An essential precondition is that the weak continuity condition and thus theorem 1.3.(8) are valid. The next step, in section 2.2, consists in demonstrating the so-called weak discrete compactness of the sequence of natural embeddings of the subspaces V_ι into $W^{m-1,p}(G_\iota)$. This compactness property already guarantees that to each bounded sequence of functions $u_\iota \in V_\iota$, $\iota \in \mathbb{N}'$, there exist a subsequence $\mathbb{N}'' \subset \mathbb{N}'$ and functions $u^\mu \subset L^p(G)$ such that $D^\mu u_\iota$ tends to u^μ in $L^p(G)$ for $\iota \in \mathbb{N}''$, $\iota \to \infty$, and every multiindex μ in $|\mu| \leqq m-1$. In section 2.3 it is assumed additionally that the sequence of subspaces $V_0, V_1, \ldots$ satisfies the approximability and the closedness condition. These conditions secure that the above limits u^μ are the generalized partial derivatives of a function $u_0 \in V_0$. Thus the desired compactness property of bounded sequences of functions has been achieved.

2.1. Asymptotic equicontinuity in the mean

The main tool in the derivation of compactness properties of sequences of natural embeddings is the equicontinuity of bounded sequences of functions. A first result of this type has been obtained in 1.3.(8), (13). In finite element methods, the approximating spaces V_ι consist of piecewise polynomial functions defined over regular decompositions of the domain G. For such functions the results of section 1.3 lead to analogous statements concerning the asymptotic equicontinuity in the L^p-mean of bounded sequences of functions $u_\iota \in V_\iota$ and their partial derivatives up to the order $m-1$.

Let again a sequence of decompositions $\mathscr{K}_\iota$, $\iota = 1, 2, \ldots$, of the polyhedral domain $\overline{G}$ in $\mathbb{R}^n$ be given having the properties (K1), ..., (K4) collected in section 1.1. By $\mathscr{P}_r(\underline{K})$ is denoted the vector space of all polynomials of at most r-th degree over the open element $\underline{K}$ and by $\mathscr{P}_r(G_\iota)$ the vector space of all piecewise polynomial functions u_ι over G_ι such that $u_\iota | \underline{K} \in \mathscr{P}_r(\underline{K})$ for all $K \in \mathscr{K}_\iota$. In the following, p is assumed to be an arbitrary but fixed real number in $1 < p < \infty$.

Starting point of our further investigations is the lemma (*see* Ciarlet [4, theorem 3.2.6]):

There exists a constant α *such that the inequalities*

$$\sup_{x \in \underline{K}} |\nabla u(x)| \leqq \alpha \left(\frac{1}{|K|} \int_{\underline{K}} |\nabla u(x)|^p \, dx \right)^{1/p},$$

are valid uniformly for all polynomials $u \in \mathscr{P}_r(\underline{K})$, *all elements* $K \in \mathscr{K}_\iota$ *and* $\iota = 1, 2, \ldots$ (1)

Henceforth in section 2, only such sequences of subspaces V_ι, $\iota = 1, 2, \ldots$, are considered that possess the property

(Vo) $V_\iota \subset \mathscr{P}_r(G_\iota) \cap W^{m,p}(G_\iota)$ *for all* $\iota = 1, 2, \ldots$ *and there exists a constant* ϑ *such that all functions in* V_ι *and their partial derivatives up to the order* $m-1$ *satisfy the weak continuity condition* 1.3.(3) *where* $v_\iota = D^\mu u_\iota$ *for* $u_\iota \in V_\iota$ *and* $|\mu| \leqq m-1$.

Under this condition the asymptotic equicontinuity of bounded sequences of functions will be shown now. The first step to this result is the lemma

There is some constant β *such that the inequalities*

$$|v_\iota(a) - v_\iota(b)| \leqq \beta(|a-b| + h) \left\{ \frac{1}{|S_h(\overline{ab})|} \int_{G_\iota \cap S_h(\overline{ab})} |\nabla v_\iota|^p \, dx \right\}^{1/p},$$

are true for every line segment $\overline{ab}$ *in* G, *for all* $v_\iota = D^\mu u_\iota$, $u_\iota \in V_\iota$, $|\mu| \leqq m-1$ *and* $\iota = 1, 2, \ldots$

Proof: The inequality 1.1.(8) yields the appraisal $\omega_n \eta^n h^n \leqq |K|$ of the volume of the elements $K \in \mathscr{K}_\iota$, $\iota = 1, 2, \ldots$ For brevity, set $S = \overline{ab}$, $S_k = S_h(\overline{ab})$.

By theorem 1.3.(8), we first have

$$|v_\iota(a)-v_\iota(b)| \leqq \frac{1+2\vartheta}{\omega_n \eta^n} h^{-n+1} \sum_{K\in\mathscr{K}_\iota(S)} |K| \sup_{\underline{K}} ; v_\iota |.$$

Applying Hölder's inequality for $q=p/(p-1)$, lemma (1) and the relations 1.3.(9), (12), one gains the estimate

$$\sum_{K\in\mathscr{K}_\iota(S)} |K| \sup_{\underline{K}} |\nabla v_\iota| \leqq \Big(\sum_{K\in\mathscr{K}_\iota(S)} |K|\Big)^{1/q} \Big(\sum_{K\in\mathscr{K}_\iota(S)} |K| (\sup_{K} |\nabla v_\iota|)^p\Big)^{1/p}$$

$$\leqq \alpha |S_h| \left(\frac{1}{|S_h|} \int_{G_\iota \cap S_h} |\nabla v_\iota|^p dx \right)^{1/p}.$$

Using the appraisal 1.3.(11) of the volume $|S_h|$, the asserted inequality follows where

$$\beta = \frac{\alpha}{\omega_n} \left(\frac{2}{\eta} \right)^n (1+2\vartheta). \quad \square$$

Having made these preparations we are now in the position to state a theorem ascertaining the following asymptotic equicontinuity in the L^p-mean:

Using the constant β *of lemma* (2),

$$\left(\int_\Gamma |v_\iota(x+d)-v_\iota(x)|^p dx \right)^{1/p} \leqq \beta(|d|+h) \left(\int_{G_\iota} |\nabla v_\iota|^p dx \right)^{1/p}$$

uniformly for every compact subset $\Gamma \subset G$, *for every vector* $d \in \mathbb{R}^n$ *such that* $|d| < |\Gamma, \complement G|$ *and all* $v_\iota = D^\mu u_\iota$, $u_\iota \in V_\iota$, $|\mu| \leqq m-1$ *and* $\iota = 1, 2, \ldots$ (3)

Proof: For the sake of simplicity, the gradients ∇v_ι may be extended trivially to all of $\mathbb{R}^n$ by $\nabla v_\iota = 0$ on $\mathbb{R}^n - G_\iota$. When $a = x \in \Gamma$ and $b = x+d$, the line segment $\overline{ab}$ belongs to G under the above assumptions. Lemma (2) then gives

$$\int_\Gamma |v_\iota(x+d)-v_\iota(x)|^p dx \leqq \beta^p(|a-b|+h)^p \frac{1}{|S_h|} \int_\Gamma \left(\int_{x+S_h} |\nabla v_\iota|^p dy \right) dx,$$

where $S_h = S_h(\overline{od})$. Evidently,

$$\int_{x+S_h} |\nabla v_\iota(y)|^p \, dy = \int_{S_h} |\nabla v_\iota(x+z)|^p \, dz, \qquad x \in \mathbb{R}^n.$$

Hence the double integral in the above inequality can be majorized by

$$\int_{\mathbb{R}^n} \left(\int_{S_h} |\nabla v_\iota(x+z)|^p \, dz \right) dx = \int_{S_h} \left(\int_{\mathbb{R}^n} |\nabla v_\iota(x+z)|^p \, dx \right) dz$$

$$= |S_h| \int_{G_\iota} |\nabla v_\iota|^p \, dx,$$

whereby the asserted inequality is proved. □

2.2. Weak discrete compactness of sequences of natural embeddings

The asymptotic equicontinuity of bounded sequences of functions already permits to establish a first compactness property, namely the weak discrete compactness of the sequence of natural embeddings of V_ι into $W^{m-1,p}(G_\iota)$. Let the same assumptions be valid as in the preceding section, in particular the property (Vo) of the subspaces V_ι is required. Moreover, from now on assume the domain G to be *bounded*. Functions $u_\iota \in W^{m,p}(G_\iota)$ and their partial derivatives $D^\mu u_\iota$, $|\mu| \leqq m$, are defined first only over the subsets $G_\iota \subset G$. Being elements of $L^p(G_\iota)$ and G_ι, G differing only by sets of n-dimensional measure zero, these functions may be viewed as well as elements of $L^p(G)$, what we shall do in the sequel. In the same way, the gradients ∇v_ι of $v_\iota = D^\mu u_\iota$, $|\mu| \leqq m-1$, are regarded as functions in $L^p(G)^n$.

As one readily sees, every finite element K of the decompositions $\mathscr{K}_\iota$ of $\overline{G}$ has the *segment property:* There exists an open covering $O_1, \ldots, O_r$ of the boundary ∂K and an associated system of vectors $a_1, \ldots, a_r$ such that

$$K \cap \overline{O}_k + \varepsilon a_k \subset \underline{K}, \quad 0 < \varepsilon < 1, \quad k = 1, \ldots, r. \tag{1}$$

Hereby an important family of inequalities can be obtained.

There is some constant γ such that for every ε in $0 < \varepsilon < 1$ there exists a compact subset $\Gamma_\varepsilon \subset G$ having the property

$$\| v_\iota \|_{0,p,G_\iota} \leqq \gamma \left(\| v_\iota \|_{0,p,\Gamma_\varepsilon} + (\varepsilon + h) \| \nabla v_\iota \|_{0,p,G_\iota} \right),$$

whenever $v_\iota = D^\mu u_\iota$, $u_\iota \in V_\iota$, $|\mu| \leqq m-1$ *and* $\iota = 1, 2 \ldots$ (2)

Proof: (i) By assumption (K1), the domain $\overline{G}$ is the union of all elements K of the decomposition $\mathscr{K}_\iota$ for each ι and so, in particular, of $\mathscr{K}_1$. Choose any $K \in \mathscr{K}_1$, let $\{O_k\}$ be the open covering of the boundary ∂K, named in the segment property, and let $\{a_k\}$ be the associated system of vectors. Hereby we define the sets

$$K_0 = K - \bigcup_{k=1}^{r} O_k - \bigcup_{k=1}^{r} K \cap O_k,$$

$$K_\varepsilon = K_0 \cup \bigcup_{k=1}^{r} (K \cap \overline{O}_k + \varepsilon a_k), \qquad 0 < \varepsilon < 1.$$

The sets K_0, K_ε are compact and, by virtue of the segment property, located in $\underline{K}$ for all ε in $0 < \varepsilon < 1$. For any $k = 1, \ldots, r$ and $x \in \underline{K} \cap O_k$ the segment property guarantees further that the line segment $\overline{ab}$, where $a = x$, $b = x + \varepsilon a_k$, belongs to $\underline{K} \subset G$. Using lemma 2.1.(2), one obtains the estimate

$$|v_\iota(x)| \leqq |v_\iota(x + \varepsilon a_k)| + \beta(\varepsilon |a_k| + h) \left\{ \frac{1}{|S_h|} \int_{x+S_h} |\nabla v_\iota|^p \, dx \right\}^{1/p},$$

for all functions $v_\iota = D^\mu u_\iota$, $u_\iota \in V_\iota$, and for all $|\mu| \leqq m - 1$, $\iota = 1, 2, \ldots$, where $\nabla v_\iota = 0$ in $\complement G_\iota$, $S_h = S_h(\overline{od})$ and $d = \varepsilon a_k$. Minkowski's inequality yields

$$\begin{aligned}\left(\int_{\underline{K} \cap O_k} |v_\iota|^p \, dx \right)^{1/p} &\leqq \left(\int_{K \cap O_k + \varepsilon a_k} |v_\iota|^p \, dx \right)^{1/p} + \beta(\varepsilon |a_k| + h) \left(\int_{G_\iota} |\nabla v_\iota|^p \, dx \right)^{1/p} \\ &\leqq \| v_\iota \|_{0,p,K_\varepsilon} + \beta(\varepsilon \max |a_k| + h) \| \nabla v_\iota \|_{0,p,G_\iota}\end{aligned}$$

because the double integral on the right side may be majorized by

$$\int_{\mathbb{R}^n} \int_{S_h} |\nabla v_\iota(x+z)|^p \, dz \, dx \leqq |S_h| \int_{G_\iota} |\nabla v_\iota|^p \, dx.$$

Note that

$$K \subset K_0 \cup \bigcup_{k=r}^{r} K \cap O_k, \qquad K_0 \subset K_\varepsilon,$$

and thus

$$\begin{aligned}\| v_\iota \|_{0,p,\underline{K}} &\leqq \left(\int_{K_0} |v_\iota|^p \, dx + \sum_{k=1}^{r} \int_{\underline{K} \cap O_k} |v_\iota|^p \, dx \right)^{1/p} \\ &\leqq \gamma_K \{ \| v_\iota \|_{0,p,K_\varepsilon} + (\varepsilon + h) \| \nabla v_\iota \|_{0,p,G_\iota} \},\end{aligned}$$

where

$$\gamma_K = (r+1)^{1/p}(1+\beta(\max|a_k|+1)).$$

(ii) Under the assumptions (K1), ..., (K4) the number N_1 of elements in $\mathscr{K}_1$ is finite. The above inequalities over elements $K \in \mathscr{K}_1$ of the decomposition of $\overline{G}$ then entail, using Minkowski's inequality,

$$\|v_\iota\|_{0,p,G_\iota} = \Big(\sum_{K \in \mathscr{K}_1} \|v_\iota\|^p_{0,p,\underline{K}}\Big)^{1/p}$$
$$\leqq \max \gamma_K \Big\{\Big(\sum_{K \in \mathscr{K}_1} \|v_\iota\|^p_{0,p,K_\varepsilon}\Big)^{1/p} + (\varepsilon+h) N_1^{1/p} \|\nabla v_\iota\|_{0,p,G_\iota}\Big\},$$

that is, the asserted inequality with

$$\gamma = (1+N_1^{1/p}) \max_{K \in \mathscr{K}_1} \gamma_K, \qquad \Gamma_\varepsilon = \bigcup_{K \in \mathscr{K}_1} K_\varepsilon, \qquad 0 < \varepsilon < 1.$$

The sets K_ε are compact and subsets of $\underline{K}$ so that 1.1.(1), (3) imply the relation $\Gamma_\varepsilon \subset G_1 \subset G$. □

The next theorem now establishes the *weak discrete compactness* of the sequence of natural embeddings of the spaces V_ι into $W^{m-1,p}(G_\iota)$.

For every weakly convergent null sequence of functions $z_\iota \in V_\iota$, $\iota \in \mathbb{N}$, *that means,*

(i) $$D^\mu z_\iota \rightharpoonup 0 \text{ in } L^p(G) \quad (\iota \to \infty), \qquad |\mu| \leqq m,$$

the strong convergence statement

(ii) $$\lim_{\iota \to \infty} \|z_\iota\|_{m-1,p,G_\iota} = 0$$

is true. (3)

Proof: (i) Let μ be any multiindex of order $|\mu| \leqq m-1$ and put $v_\iota = D^\mu z_\iota$, $\iota \in \mathbb{N}$. Then v_ι belongs to $W^{1,p}(G_\iota)$ and the sequences $(D^\lambda v_\iota)$ converge weakly to zero in $L^p(G)$ for every multiindex λ in $|\lambda| \leqq 1$. In particular, this sequence is bounded such that

$$\sup_{\iota \in \mathbb{N}} \|v_\iota\|_{0,p,G_\iota} + \sup_{\iota \in \mathbb{N}} \|\nabla v_\iota\|_{0,p,G_\iota} \leqq \sigma < \infty.$$

Let χ be an arbitrary test function in $C_0^\infty(G)$. This function has compact support $\Gamma = \operatorname{supp} \chi \subset G$, having a positive distance $|\Gamma, \complement G|$ from $\complement G$. On setting $w_\iota = \chi v_\iota$, $\iota \in \mathbb{N}$, and choosing $d \in \mathbb{R}^n$ such that $|d| < |\Gamma, \complement G|$,

$$w_\iota(x+d) - w_\iota(x) = (\chi(x+d) - \chi(x)) v_\iota(x+d) + \chi(x)(v_\iota(x+d) - v_\iota(x)),$$

and, consequently,

$$\left(\int_G |w_\iota(x+d)-w_\iota(x)|^p\,dx\right)^{1/p} \leqq |d| \sup_G |\nabla\chi| \left(\int_{G_\iota} |v_\iota|^p\,dx\right)^{1/p} + \sup_G |\chi| \left(\int_\Gamma |v_\iota(x+d)-v_\iota(x)|^p\,dx\right)^{1/p}.$$

By theorem 2.1.(3), for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|d| \sup_G |\nabla\chi| \sigma \leqq \frac{\varepsilon}{2},$$

$$\sup_G |\chi| \left(\int_\Gamma |v_\iota(x+d)-v_\iota(x)|^p\,dx\right)^{1/p} \leqq \sup_G |\chi| \beta(|d|+h)\sigma \leqq \frac{\varepsilon}{2},$$

thus

$$\left(\int_G |w_\iota(x+d)-w_\iota(x)|^p\,dx\right)^{1/p} \leqq \varepsilon$$

for all $d \in \mathbb{R}^n$, $|d| < \min(\delta, |\Gamma, \complement G|)$ and all $h < \delta$. The sequence $h = h_\iota$, $\iota = 1, 2, \ldots$, tends to zero for $\iota \to \infty$ so that there is an index ν such that $h_\iota < \delta$ whenever $\iota > \nu$. Each of the finite number of functions $w_\iota \in L^P(G)$, $\iota = 1, \ldots, \nu$, is continuous in the L^P-mean. From the above it is thus seen that the sequence (w_ι) is equicontinuous and bounded in $L^P(G)$. Therefore, by a well-known theorem, the sequence is compact in $L^P(G)$. The convergence of $v_\iota \rightharpoonup 0$ in $L^P(G)$ implies $w_\iota \rightharpoonup 0$ in $L^P(G)$ so that the compactness of the sequence (w_ι) leads to the strong convergence $w_\iota \to 0$ in $L^P(G)$ for $\iota \to \infty$.

(ii) By theorem (2), for every $\varepsilon > 0$ there exists a compact subset $\Gamma_\varepsilon \subset G$ such that the sequence (v_ι) satisfies the inequality

$$\|v_\iota\|_{0,p,G_\iota} \leqq \gamma(\|v_\iota\|_{0,p,\Gamma_\varepsilon} + (\varepsilon+h)\sigma), \qquad \iota \in \mathbb{N}.$$

Next there exists a test function $\chi \in C_0^\infty(G)$ such that $\chi = 1$ over Γ_ε and $0 \leqq \chi \leqq 1$. Obviously, $w_\iota = \chi v_\iota$ gives $w_\iota | \Gamma_\varepsilon = v_\iota | \Gamma_\varepsilon$. As it is seen from part (i) of this proof,

$$\|v_\iota\|_{0,p,\Gamma_\varepsilon} = \|w_\iota\|_{0,p,\Gamma_\varepsilon} \leqq \|w_\iota\|_{0,p} \to 0 \qquad (\iota \to \infty).$$

Hence, for every $\varepsilon > 0$,

$$\limsup_{\iota\to\infty} \|v_\iota\|_{0,p,G_\iota} \leqq \varepsilon\gamma\sigma.$$

Therefore,

$$\lim_{\iota\to\infty} \| D^\mu z_\iota \|_{0,p,G_\iota} = \lim_{\iota\to\infty} \| v_\iota \|_{0,p,G_\iota} = 0.$$

This convergence condition is true for all $|\mu| \leqq m-1$ and so proves the assertion of this theorem. □

2.3. Discrete compactness of sequences of natural embeddings and uniform Ehrling inequalities

In the method of nonconforming and hybrid finite elements the partial derivatives $D^\mu u_\iota$ of functions $u_\iota \in W^{m,p}(G_\iota)$ may be viewed as functions in $L^p(G)$ as we have explained already in the preceding section. In this sense a sequence of functions $u_\iota \in W^{m,p}(G_\iota)$, $\iota \in \mathbb{N}' \subset \mathbb{N}$, is said to be *strongly* (*weakly*) convergent iff the sequence $D^\mu u_\iota$ is strongly (weakly) convergent in $L^p(G)$ for $\iota \to \infty$ and every multiindex $|\mu| \leqq m$. Evidently, the strong convergence of the sequence (u_ι) to a function $u \in W^{m,p}(G)$ is then equivalent to

$$\| u_\iota - u \|_{m,p,G_\iota} = \Big(\sum_{|\mu| \leqq m} \sum_{K \in \mathscr{K}_\iota} \int_{\underline{K}} | D^\mu u_\iota - D^\mu u |^p \, dx \Big)^{1/p} \to 0 \qquad (\iota \to \infty). \quad (1)$$

Let $L^{m,p}(G)$ be the space of all vector-valued functions $\mathbf{u} = (u^\mu)$ having components $u^\mu \in L^p(G)$, $|\mu| \leqq m$. This is a Banach space with the norm

$$\| \mathbf{u} \|_{m,p} = \Big(\sum_{|\mu| \leqq m} \int_G | u^\mu |^p \, dx \Big)^{1/p}, \qquad \mathbf{u} \in L^{m,p}(G). \quad (2)$$

The *natural embedding*

$$\mathbf{u}_\iota = (D^\mu u_\iota)_{|\mu| \leqq m}, \qquad u_\iota \in W^{m,p}(G_\iota), \quad \iota = 0, 1, 2, \ldots, \quad (3)$$

assigns to each function u_ι a function $\mathbf{u}_\iota \in L^{m,p}(G)$ where $G_0 = G$. This embedding is an isomorphic and isometric mapping of the Sobolev spaces $W^{m,p}(G_\iota)$ onto closed subspaces ${}_E W^{m,p}(G_\iota)$ in $L^{m,p}(G)$. The above defined strong and weak convergence of sequences of functions $u_\iota \in W^{m,p}(G_\iota)$, $\iota \in \mathbb{N}' \subset \mathbb{N}$, are thus equivalent to the strong and weak convergence of the embedded functions $\mathbf{u}_\iota \in {}_E W^{m,p}(G_\iota)$ in $L^{m,p}(G)$.

We consider sequences of closed linear subspaces $V_\iota \subset W^{m,p}(G_\iota)$ and the associated embeddings $E_\iota =_E V_\iota$ in $L^{m,p}(G)$ for $\iota = 0, 1, 2, \ldots$ The sequence (V_ι) *approximates* the subspace V_0 iff the approximability condition

$$\text{(V1)} \qquad \forall v_0 \in V_0, \quad \inf_{\varphi_\iota \in V_\iota} \| v_0 - \varphi_\iota \|_{m,p,G_\iota} \to 0 \qquad (\iota \to \infty)$$

holds. The sequence $V_0, (V_\iota)$ is *closed* iff the following *closedness condition* is true:

(V2) *The limits of all weakly convergent sequences of functions* $v_\iota \in V_\iota$, $\iota \in \mathbb{N}' \subset \mathbb{N}$ *belong to the subspace* V_0.

In the next section we shall give examples of approximations by nonconforming and hybrid finite elements possessing these properties.

Using the above concepts, we are now able to state the fundamental theorem ascertaining the discrete compactness of the sequence of natural embeddings of V_ι into $W^{m-1,p}(G_\iota)$. This theorem generalizes the well-known Rellich theorem, concerning the compactness of the natural embeddings of $W^{m,p}(G)$ into $W^{m-1,p}(G)$, to approximations of Sobolev spaces by methods of nonconforming and hybrid finite elements. Incidentally, from a theorem [16, p. 30] of another paper it is seen that under the following assumptions the weak discrete compactness, according to theorem 2.2.(3), and the *discrete compactness* of the sequence of natural embeddings, in the sense stated below, are equivalent.

Let a bounded polyhedral domain $\overline{G}$ *and a sequence* $(\mathcal{K}_\iota)$ *of decompositions be given satisfying the assumptions* (K1), ..., (K4). *Let* V_0 *be a subspace of* $W^{m,p}(G)$ *and let* (V_ι) *be a sequence of subspaces consisting of piecewise polynomial functions having the properties* (V0), (V1), (V2). *Then the following statement is true:* (V3) *For every bounded sequence of functions* $v_\iota \in V_\iota$, $\iota \in \mathbb{N}' \subset \mathbb{N}$, *there exists a subsequence* $\mathbb{N}'' \subset \mathbb{N}'$ *and a function* $v_0 \in V_0$ *such that the subsequence* $(v_\iota)_{\iota \in \mathbb{N}''}$ *converges weakly to* v_0 *and, moreover, the strong convergence relation*

$$\| v_\iota - v_0 \|_{m-1,p,G_\iota} \to 0 \qquad (\iota \in \mathbb{N}'', \ \iota \to \infty)$$

is valid. (4)

Proof: By the embedding (3) of the sequence, (v_ι) becomes a bounded sequence $(\mathbf{v}_\iota)$ in $L^{m,p}(G)$. This sequence is weakly compact, that is, there exist a subsequence $\mathbb{N}'' \subset \mathbb{N}'$ and a function $\mathbf{v}_0 \in L^{m,p}(G)$ such that

$$\mathbf{v}_\iota \rightharpoonup \mathbf{v}_0 \quad \text{in } L^{m,p}(G) \qquad (\iota \in \mathbb{N}'', \ \iota \to \infty).$$

Due to assumption (V2), the limit v_0 belongs to ${}_E V_0$. The approximability condition (V1) yields an associated sequence of functions $\varphi_\iota \in V_\iota$ such that

$$\| \varphi_\iota - v_0 \|_{m,p,G_\iota} \to 0 \qquad (\iota \to \infty).$$

The functions

$$z_\iota = \varphi_\iota - v_\iota, \quad \iota \in \mathbb{N}''; \qquad z_\iota = 0, \quad \iota \in \mathbb{N} - \mathbb{N}'';$$

constitute a weakly convergent null sequence. By virtue of theorem 2.2.(3), then

$$\| v_\iota - v_0 \|_{m-1,p,G_\iota} \leqq \| \varphi_\iota - v_0 \|_{m-1,p,G_\iota} + \| z_\iota \|_{m-1,p,G_\iota} \to 0$$

for $\iota \in \mathbb{N}''$, $\iota \to \infty$. □

As a first application of the above compactness theorem, we will establish uniform *Ehrling inequalities* for the sequence of subspaces $V_\iota \subset W^{m,p}(G_\iota)$. First, for every $\varepsilon > 0$ and every ι there exists some constant $\varkappa_\iota(\varepsilon)$ such that

$$\| u_\iota \|_{m-1,p,G_\iota} \leqq \varepsilon \| u_\iota \|_{m,p,G_\iota} + \varkappa_\iota(\varepsilon) \| u_\iota \|_{0,p,G_\iota}, \qquad u_\iota \in W^{m,p}(G_\iota). \tag{5}$$

For, every domain G_ι is the union of the open elements $\underline{K}$ for $K \in \mathscr{K}_\iota$ and for each of the finitely many elements an Ehrling inequality holds (*see* Agmon [1, p. 25], Nečas [9, p. 108]). The inequality for the domain $G = G_0$ is obtained correspondingly. Under the assumptions of the above compactness theorem, the constants $\varkappa_\iota(\varepsilon)$ can be chosen independently of ι. This follows from a general functional analytic theorem (*see* Stummel [14-I, p. 68]). For the sake of completeness, however, the proof will be given here.

For every $\varepsilon > 0$ *there exists a positive constant* $\varkappa(\varepsilon)$ *such that the inequalities*

$$\| v_\iota \|_{m-1,p,G_\iota} \leqq \varepsilon \| v_\iota \|_{m,p,G_\iota} + \varkappa(\varepsilon) \| v_\iota \|_{0,p,G_\iota}, \qquad v_\iota \in V_\iota,$$

hold uniformly for all $\iota = 0, 1, 2, \ldots$ (6)

Proof: As explained above, the inequalities (5) are valid for every ι. If the uniform inequality is not true there exist a positive constant ε_0, a subsequence $\mathbb{N}' \subset \mathbb{N}$ of indices ι, a sequence of numbers $\varkappa_\iota \to \infty$ and associated functions $v_\iota \in V_\iota$ having the properties

$$\| v_\iota \|_{m,p,G_\iota} = 1, \qquad \| v_\iota \|_{m-1,p,G_\iota} > \varepsilon_0 + \varkappa_\iota \| v_\iota \|_{0,p,G_\iota}, \qquad \iota \in \mathbb{N}'.$$

By theorem (4), there is a subsequence $\mathbb{N}'' \subset \mathbb{N}'$ and a function $v \in V_0$ such that $\| v_\iota - v \|_{m-1,p,G_\iota} \to 0$ for $\iota \in \mathbb{N}''$, $\iota \to \infty$. Consequently,

$$\| v_\iota \|_{m-1,p,G_\iota} \to \| v \|_{m-1,p}, \qquad \| v_\iota \|_{0,p,G_\iota} \to \| v \|_{0,p} \quad (\iota \in \mathbb{N}'', \iota \to \infty).$$

The above inequalities thus lead to the contradiction $\| v \|_{m-1,p} \geqq \varepsilon_0 > 0$ and $v=0$. □

3. APPLICATIONS

With regard to applications of the general theorems, in section 3.1 the validity of the strong continuity condition for a series of well-known nonconforming or hybrid finite elements is verified. These elements additionally satisfy the approximability and the closedness condition so that by theorem 2.3.(4) the fundamental compactness statement (V3) is valid for the associated approximations. Compactness properties of sequences of natural embeddings play an important role in functional analytic theorems concerning the convergence of solutions of elliptic variational problems. In section 3.2, using compactness arguments, a fundamental theorem is proved ascertaining the solvability and stability of the sequence of approximating problems as well as the convergence of the approximation solutions for a large class of nonconforming and hybrid approximations of inhomogeneous elliptic variational equations with variable not necessarily smooth coefficients. Section 3.3 describes the corresponding class of generalized elliptic eigenvalue problems and assumptions ensuring the convergence of the associated approximations. In this context, a basic theorem is proved demonstrating the weak collective compactness of a sequence of sesquilinear forms. Finally the main statements are briefly collected concerning the convergence of spectra and resolvent sets, of eigenvalues and eigenspaces of the approximations.

3.1. Examples of nonconforming and hybrid finite elements

The compactness theorem 2.3.(4) requires the validity of the conditions (V0), (V1), (V2). Proofs of the approximability condition (V1) are found frequently in the corresponding literature. The paper Stummel [20] has established a generalized patch test in order to verify the closedness condition. By this test it is shown there that a series of special elements pass the generalized patch test and thus yield closed sequences of subspaces $V_0, V_1, V_2, \ldots$, where V_0 may be $H_0^m(G)$ or $H^m(G)$. We shall now explain that all these elements, additionally, satisfy the continuity condition (V0). In all cases, the functions $v_\iota \in V_\iota$ and their partial derivatives up to the order $m-1$ possess the strong continuity property. As one readily sees this is also true for approximations of $H^2(G)$ by Zienkiewicz triangles. It is well-known, however, that these approximations not necessarily converge for arbitrary decompositions of the domain. Thus one has an example satisfying the conditions (V0), (V1) but not necessarily also the closedness condition (V2).

In the following examples $\overline{G}$ denotes a bounded polyhedral domain in the plane $\mathbb{R}^2$ and the decompositions $\mathscr{K}_\iota$ are assumed to have the properties (K1), ..., (K4).

1. *Wilson's element.*

The elements $K \in \mathscr{K}_\iota$ are rectangles, the subspaces V_ι are contained in $\mathscr{P}_2(G_\iota)$, $\iota \in \mathbb{N}$, and the functions $v_\iota \in V_\iota$ are continuous at the vertices of the rectangles K. Hence to each interelement side of the rectangles there are two points of continuity of v_ι so that these functions possess the strong continuity property and condition (V0) is fulfilled with $r=2$, $m=1$.

2. *Adini's element.*

The elements $K \in \mathscr{K}_\iota$ are also rectangles, the subspaces V_ι belong to $\mathscr{P}_4(G_\iota)$, functions in V_ι and their partial derivatives of first order are continuous at vertices of the rectangles K. Thus the functions $D^\mu v_\iota$ possess the strong continuity property for $|\mu| \leqq 1$ such that the continuity condition (V0) is true for $r=4$, $m=2$.

3. *The elements of Crouzeit-Raviart* are hybrid finite elements, weak continuity at interelement boundaries of the triangulations is achieved by orthogonality to all polynomials up to a certain degree. In this way, the piecewise polynomial functions v_ι are also continuous at the associated Gaussian points at interelement sides of the triangles such that these elements may as well be regarded as nonconforming finite elements In the simplest case, V_ι consists of piecewise linear functions being continuous at midside nodes of the triangles $K \in \mathscr{K}_\iota$. The functions $v_\iota \in V_\iota$ have the strong continuity property and condition (V0) is fulfilled for $m=1$.

4. *Morley's element* is specified over triangulations $\mathscr{K}_\iota$ of $\overline{G}$ by subspaces $V_\iota \subset \mathscr{P}_2(G_\iota)$. Function values of $v_\iota \in V_\iota$ at the vertices of the triangles and the first derivatives in normal direction at midside nodes are continuous. From the continuity of the function values at vertices one immediately concludes that the mean values of the first derivatives in tangential direction over interelement sides of the triangles are continuous. The midpoint rule is exact in this case and yields that at midside nodes of interelement sides also the derivatives in tangential direction and thus of the gradients of v_ι are continuous. Therefore, the functions $D^\mu v_\iota$, $|\mu| \leqq 1$, have the strong continuity property such that the condition (V0) for $r=2$, $m=2$ is valid.

5. *De Veubecke's element* is defined by subspaces $V_\iota \subset \mathscr{P}_3(G_\iota)$ and triangulations $\mathscr{K}_\iota$ of $\overline{G}$. Function values at vertices of the triangles and the first derivatives in normal direction at Gaussian points of second order at

interelement sides are continuous. This immediately implies the continuity of the mean values of the first derivatives in normal direction over interelement sides. As for Morley's element one further obtains that the mean values of the first derivatives in tangential direction and consequently the mean values of the gradients of functions $v_\iota \in V_\iota$ are continuous at interelement sides. By theorem 1.3.(7) the functions $D^\mu v_\iota$ then possess the weak continuity property for $|\mu| \leqq 1$ and condition (V0) is true where $r=3$, $m=2$.

3.2. The stability and convergence theorem, generalized Poincaré and Friedrichs inequalities

The approximations of Sobolev spaces, generated by nonconforming and hybrid finite elements of the above examples, have the approximability, closedness and compactness properties (V1), (V2), (V3). On this basis, a fundamental stability theorem is proved ascertaining the convergence of approximate solutions for a general class of variational equations. Note that by theorem (10) the validity of the assumption (V) in Stummel [20, section 1.2] is valid for the sequence of variational equations (4) for all $\iota \geqq \nu$ and a suitable index ν.

The large class of approximations, studied here, encompasses also conforming finite elements of piecewise polynomial subspaces $V_\iota \subset V_0 \subset H^m(G)$. In this case the closedness condition and the strong continuity condition are trivially valid. Rellich's theorem is true for bounded polyhedral domains $\overline{G}$, having the properties (K1), (K2). This immediately yields the discrete compactness of the sequence of natural embeddings of the conforming subspaces V_ι into $H^{m-1}(G)$. Thus, of the conditions (V1), (V2), (V3) only the approximability condition (V1) has to be verified in applications of the stability and convergence theorem (10) to conforming approximations.

We consider a class of generalized boundary value problems and associated approximations specified by a bounded sesquilinear form a on the space $L^{m,2}(G)$,

$$|a(\mathbf{v}, \mathbf{w})| \leqq \alpha_1 \|\mathbf{v}\|_m \|\mathbf{w}\|_m, \qquad \mathbf{v}, \mathbf{w} \in L^{m,2}(G), \tag{1}$$

and a sequence of closed subspaces $V_\iota \subset H^m(G_\iota) = W^{m,2}(G_\iota)$ for $\iota = 1, 2, \ldots$, where for convenience $G_0 = G$. By the natural embeddings 2.3.(3) of the spaces V_ι one obtains the closed subspaces

$$E_\iota = {}_E V_\iota \subset {}_E H^m(G_\iota), \qquad \iota = 0, 1, 2, \ldots, \tag{2}$$

in $L^{m,2}(G)$. Hereby the variational equation

$$\mathbf{u}_0 \in E_0; \qquad a(\varphi, \mathbf{u}_0) = l(\varphi), \quad \varphi \in E_0; \tag{3}$$

and the associated sequence of approximations

$$\mathbf{u}_\iota \in E_\iota; \qquad a(\varphi_\iota, \mathbf{u}_\iota) = l(\varphi_\iota), \quad \varphi_\iota \in E_\iota; \quad \iota = 1, 2, \ldots, \tag{4}$$

are defined where l denotes any continuous linear form on $L^{m,2}(G)$.

A typical example is given by the sesquilinear form

$$\left.\begin{aligned} a(\mathbf{v}, \mathbf{w}) &= \sum_{|\sigma|, |\tau| \leq m} \int_G a_{\sigma\tau}\, v^\sigma\, \bar{w}^\tau\, dx, \\ \mathbf{v} &= (v^\sigma), \qquad \mathbf{w} = (w^\sigma) \in L^{m,2}(G), \end{aligned}\right\} \tag{5}$$

with coefficients $a_{\sigma\tau} \in L^\infty(G)$ for $|\sigma|, |\tau| \leq m$. Every continuous linear form on $L^{m,2}(G)$ may be written in the form

$$l(\mathbf{v}) = \sum_{|\sigma| \leq m} \int_G v^\sigma\, \bar{f}_\sigma\, dx, \qquad \mathbf{v} = (v^\sigma) \in L^{m,2}(G), \tag{6}$$

where $f_\sigma \in L^2(G)$, $|\sigma| \leq m$. In this example, (3) is equivalent to the generalized boundary value problem

$$\left.\begin{aligned} & u_0 \in V_0; \\ \sum_{|\sigma|, |\tau| \leq m} \int_G a_{\sigma\tau} D^\sigma \varphi\, \overline{D^\tau u_0}\, dx &= \sum_{|\sigma| \leq m} \int_G D^\sigma \varphi\, \bar{f}_\sigma\, dx, \\ & \varphi \in V_0; \end{aligned}\right\} \tag{7}$$

and the approximating equations (4) are equivalent to

$$\left.\begin{aligned} & u_\iota \in V_\iota; \\ \sum_{|\sigma|, |\tau| \leq m} \sum_{K \in \mathscr{K}_\iota} \int_{\underline{K}} a_{\sigma\tau} D^\sigma \varphi_\iota\, \overline{D^\tau u_\iota}\, dx &= \sum_{|\sigma| \leq m} \sum_{K \in \mathscr{K}_\iota} \int_{\underline{K}} D^\sigma \varphi_\iota\, \bar{f}_\sigma\, dx, \\ & \varphi_\iota \in V_\iota; \end{aligned}\right\} \tag{8}$$

for $\iota = 1, 2, \ldots$

The given problem (3) is said to be *properly posed* if it is uniquely and continuously solvable for every inhomogeneous term of the form (6). The sequence of approximating problems (4) is said to be *uniformly coercive* if there exist constants $\gamma_0 > 0$, $\gamma_1 \geq 0$ such that the inequalities

$$\operatorname{Re} a(\mathbf{v}_\iota) \geq \gamma_0 \|\mathbf{v}_\iota\|_m^2 - \gamma_1 \|\mathbf{v}_\iota\|_{m-1}^2, \qquad \mathbf{v}_\iota \in_E V_\iota, \tag{9}$$

hold uniformly for all $\iota = 1, 2, \ldots$ A set $\mathbb{N}_1$ contains *almost all* natural numbers if it is, save a finite number of elements, equal to $\mathbb{N} = \{1, 2, 3, \ldots\}$. As a *general* *assumption* in the following we require, that the domain $\overline{G}$ is bounded, the sequence $(\mathscr{K}_\iota)$ of decompositions of $\overline{G}$ has the properties (K1), (K2) and the subspaces V_0, V_ι, $\iota \in \mathbb{N}$, satisfy the approximability condition (V1), the closedness condition (V2) and the compactness condition (V3). Under these conditions the stability and convergence theorem (10) below is true.

Note that in [20, section 1.2] an associated two-sided discretization error estimate is given. In applications to the method of hybrid finite elements and the model Dirichlet problem $-\Delta u = f$, the corresponding discretization error equation [20, 1.2.(16)] is identical to the one of Raviart-Thomas [11, (6.6)]. In contrast to the results of Ciarlet [3], Lascaux-Lesaint [8], for nonconforming approximations and Raviart-Thomas [11], Thomas [21] for hybrid approximations, theorem (10) admits general sesquilinear forms with variable nonsmooth coefficients, not necessarily being symmetric or V_0-elliptic in the sense of Lions. Moreover, convergence is established in Sobolev norms according to 2.3.(1), for $p=2$, and not only in the energy norm defined by the sesquilinear form a.

Let the given variational equation (3) *be properly posed and the sequence of approximating equations* (4) *be uniformly coercive. Then there exist positive constants* α_0, α_1 *such that for almost all* ι *the approximating equations are uniquely solvable for all inhomogeneous terms* (6) *and the bistability inequalities*

$$\text{(i)} \qquad \alpha_0 \| \mathbf{v}_\iota \|_m \leqq \sup_{0 \neq \varphi_\iota \in_E V_\iota} \frac{|a(\varphi_\iota, \mathbf{v}_\iota)|}{\| \varphi_\iota \|_m} \leqq \alpha_1 \| \mathbf{v}_\iota \|_m, \qquad \mathbf{v}_\iota \in_E V_\iota,$$

hold. The approximate solutions $\mathbf{u}_\iota$ *of* (4) *converge to the solution* $\mathbf{u}_0$ *of* (3) *according to*

$$\text{(ii)} \qquad \| \mathbf{u}_\iota - \mathbf{u}_0 \|_m = \| u_\iota - u_0 \|_{m, G_\iota} \to 0 \qquad (\iota \to \infty)$$

for all right-hand sides of the form (6). (10)

Proof: (i) The right side of the bistability inequalities follows immediately from (1). Let us assume that the left side of the inequalities is not true. Then there exist a subsequence $\mathbb{N}' \subset \mathbb{N}$ and elements $\mathbf{v}_\iota \in_E V_\iota$ such that

$$\left. \begin{aligned} & \| \mathbf{v}_\iota \|_m = 1, \\ & \sup_{0 \neq \varphi_\iota \in_E V_\iota} \frac{|a(\varphi_\iota, \mathbf{v}_\iota)|}{\| \varphi_\iota \|_m} \to 0 \qquad (\iota \in \mathbb{N}', \ \iota \to \infty). \end{aligned} \right\} \tag{11}$$

From the compactness condition (V3) it is seen that there exist a subsequence $\mathbb{N}'' \subset \mathbb{N}'$ and a function $v_0 \in V_0$ such that $\mathbf{v}_\iota \rightharpoonup \mathbf{v}_0$ in $L^{m,2}(G)$ and $\| \mathbf{v}_\iota - \mathbf{v}_0 \|_{m-1} \to 0$ for $\iota \in \mathbb{N}''$, $\iota \to \infty$. Since the subspace V_0 is approximated by the sequence (V_ι), for every $\varphi \in V_0$ there is a sequence $\varphi_\iota \in V_\iota$, $\iota \in \mathbb{N}$, having the property $\|\varphi_\iota - \varphi\|_m \to 0$ for $\iota \to \infty$. Using (11), we then have

$$a(\varphi, \mathbf{v}_0) = \lim_{\substack{\iota \in \mathbb{N}'' \\ \iota \to \infty}} a(\varphi_\iota, \mathbf{v}_\iota) = 0$$

for each $\varphi \in V_0$. Therefore, the function $\mathbf{v}_0$ is a solution of the homogeneous equation associated with (3) and thus by assumption $\mathbf{v}_0 = 0$. Using this fact, now $\|\mathbf{v}_\iota\|_{m-1} \to 0$ for $\iota \in \mathbb{N}''$, $\iota \to \infty$. The coerciveness inequality (9) then leads to the contradiction

$$0 < \gamma_0 = \gamma_0 \|\mathbf{v}_\iota\|_m \leqq \sup_{0 \neq \varphi_\iota \in_E V_\iota} \frac{|a(\varphi_\iota, \mathbf{v}_\iota)|}{\|\varphi_\iota\|_m} + \gamma_1 \|\mathbf{v}_\iota\|_{m-1} \to 0 \quad (\iota \in \mathbb{N}'', \iota \to \infty).$$

(ii) By assumption, the given problem (3) is uniquely and continuously solvable for each right-hand side. Thus the adjoint sesquilinear form

$$a^*(\mathbf{v}, \mathbf{w}) = \overline{a(\mathbf{w}, \mathbf{v})}, \qquad \mathbf{v}, \mathbf{w} \in L^{m,2}(G),$$

specifies a homogeneous equation $a^*(\varphi, \mathbf{w}) = 0$, $\varphi \in_E V_0$, having only the trivial solution $\mathbf{w} = 0$. By an analogous conclusion as in part (i) of this proof, one infers the existence of a positive constant α_0^* such that the inequalities

$$\alpha_0^* \|\mathbf{v}_\iota\|_m \leqq \sup_{0 \neq \varphi_\iota \in_E V_\iota} \frac{|a^*(\varphi_\iota, \mathbf{v}_\iota)|}{\|\varphi_\iota\|_m}, \qquad \mathbf{v}_\iota \in_E V_\iota, \tag{12}$$

hold uniformly for almost all ι. The inequalities (10 i), (12) together demonstrate that the mappings, defined by a from ${}_E V_\iota$ to the space of continuous linear forms on ${}_E V_\iota$, are bijective and bicontinuous. Consequently, the approximating equations are uniquely solvable for almost all ι.

(iii) In part (i), (ii) of the proof we have shown that the assumption (V) of [20, theorem 1.2.(12)] is fulfilled. By the approximability and closedness condition (V1), (V2) the condition [20, 1.2.(12 iii)] is satisfied. The cited theorem thus ascertains the convergence statement (10 ii). □

In applications, frequently the sesquilinear forms are *nonnegative*, that is, symmetric and $a(\mathbf{v}_\iota) \geqq 0$ for all $\mathbf{v}_\iota \in_E V_\iota$, $\iota = 1, 2, \ldots$. In this case the following corollary is true. It shows that the sesquilinear form a specifies scalar products for the subspaces ${}_E V_\iota$ and that the associated a-norms are uniformly equivalent to the $\|.\|_m$-norm on ${}_E V_\iota$ for almost all ι.

Let the assumptions of theorem (10) *be valid and let the sesquilinear form a on* ${}_E V_\iota$ *be nonnegative for every* $\iota = 1, 2, \ldots$ *Then, using the constants* α_0, α_1 *of theorem* (10), *the inequalities*

$$\text{(i)} \qquad \alpha_0 \| \mathbf{v}_\iota \|_m^2 \leqq a(\mathbf{v}_\iota) \leqq \alpha_1 \| \mathbf{v}_\iota \|_m^2, \qquad \mathbf{v}_\iota \in {}_E V_\iota,$$

hold uniformly for almost all ι. (13)

Proof: The sesquilinear form a on E_ι has the representation

$$a(\mathbf{v}_\iota, \mathbf{w}_\iota) = (\mathbf{v}_\iota, A_\iota \mathbf{w}_\iota), \qquad \mathbf{v}_\iota, \mathbf{w}_\iota \in V_\iota,$$

where A_ι denotes bounded symmetric operators in ${}_E V_\iota$. By assumption on a, the operators A_ι are nonnegative and, accordingly, possess nonnegative square roots $A_\iota^{1/2}$. From theorem (10) we conclude that the inverse operators A_ι^{-1} exist and are bounded by $\| A_\iota^{-1} \| \leqq 1/\alpha_0$ for almost all ι. Together with A_ι also A_ι^{-1} is nonnegative. It is well-known that the norm of a symmetric operator is equal to the norm of the associated quadratic form so that

$$(\mathbf{w}_\iota, A_\iota^{-1} \mathbf{w}_\iota)_m \leqq \frac{1}{\alpha_0} \| \mathbf{w}_\iota \|_m^2, \qquad \mathbf{w}_\iota \in {}_E V_\iota.$$

On setting $\mathbf{w}_\iota = A_\iota^{1/2} \mathbf{v}_\iota$, it follows that

$$\alpha_0 \| \mathbf{v}_\iota \|_m^2 \leqq (\mathbf{v}_\iota, A_\iota \mathbf{v}_\iota)_m = a(\mathbf{v}_\iota), \qquad \mathbf{v}_\iota \in {}_E V_\iota,$$

whereby the first inequality in (13 i) is proved. The second inequality is an immediate consequence of (10 i). □

The general theorems will now be applied in deriving two important sequences of inequalities. For brevity we use the notation

$$[v_\iota, w_\iota]_{m, G_\iota} = \sum_{|\sigma| \leqq m} \int_{G_\iota} D^\sigma v_\iota \overline{D}^\sigma w_\iota \, dx = \sum_{|\sigma| \leqq m} \sum_{K \in \mathscr{K}_\iota} \int_{\underline{K}} D^\sigma v_\iota \overline{D}^\sigma w_\iota \, dx \qquad (14)$$

and

$$| v_\iota |_{m, G_\iota} = [v_\iota, v_\iota]_{m, G_\iota}^{1/2}$$

for $v_\iota, w_\iota \in H^m(G_\iota)$. The nonconforming and hybrid approximations of $H_0^m(G)$ studied in the papers of Ciarlet [3], Lascaux-Lesaint [8], Raviart-Thomas [11], Stummel [20], Thomas [21], have the property that $| . |_{m, G_\iota}$ define norms for the subspaces V_ι. On finite dimensional spaces every two norms are equivalent. Hence the inequalities stated below are valid for all $\iota = 1, 2, \ldots,$ in those approximations. Note that the generalized Poincaré-Friedrichs inequality is also proved by Thomas [21, p. V-38] for $m=1$ in the context of hybrid finite element methods and for "uniformly regular" triangulations.

There exist positive constants α_0, ν *such that the generalized inequality of Poincaré-Friedrichs*

$$\alpha_0 \| v_\iota \|^2_{m,G_\iota} \leqq | v_\iota |^2_{m,G_\iota}, \qquad v_\iota \in V_\iota,$$

is true uniformly for all $\iota \geqq \nu$ *provided that* $V_0^{\circ} = H_0^m(G)$. (15)

Proof: The sesquilinear form

$$a(\mathbf{v}, \mathbf{w}) = [\mathbf{v}, \mathbf{w}]_m = \sum_{|\sigma| \leqq m} \int_G v^\sigma \bar{w}^\sigma \, dx, \qquad \mathbf{v}, \mathbf{w} \in L^{m,2}(G),$$

is bounded, symmetric and nonnegative. The well-known Poincaré-Friedrichs inequality holds for the space $H_0^m(G)$ and, consequently,

$$\alpha_0 \| v \|^2_m \leqq | v |^2_m, \qquad v \in H_0^m(G),$$

(*see* Agmon (1, p. 73], Nečas [9, p. 20]). Thus ${}_E H_0^m(G)$ with the scalar product $a = [.,.]_m$ is a Hilbert space and the variational equation (3), defined by a, is properly posed. Evidently,

$$\| \mathbf{v} \|^2_m - \| \mathbf{v} \|^2_{m-1} = | \mathbf{v} |^2_m = a(\mathbf{v}), \qquad \mathbf{v} \in L^{m,2}(G),$$

Hence the sequence of approximating problems (4) is uniformly coercive so that corollary (13) yields the asserted inequality. □

Analogously as above, the following inequalities hold for all $\iota = 1, 2, \ldots$, if the quadratic forms on the right sides of the inequalities are positive definite. As it is readily seen, this is the case, for example, when the domain $\bar{G}$ is connected and the functions $v_\iota \in V_\iota$, $\iota \in \mathbb{N}$, together with their partial derivatives up to the order $m-1$ have the strong continuity property.

There exist positive constants α_0, ν *such that the generalized Poincaré inequalities*

$$\alpha_0 \| v_\iota \|^2_{m,G_\iota} \leqq | v_\iota |^2_{m,G_\iota} + \sum_{|\sigma| < m} \left| \int_{G_\iota} D^\sigma v_\iota \, dx \right|^2, \qquad v_\iota \in V_\iota,$$

hold uniformly for all $\iota \geqq \nu$. (16)

Proof: The sesquilinear form a is now specified by

$$a(\mathbf{v}, \mathbf{w}) = \sum_{|\sigma| \leqq m} \int_G v^\sigma \bar{w}^\sigma \, dx + \sum_{|\sigma| < m} \int_G v^\sigma \, dx \int_G \bar{w}^\sigma \, dx,$$
$$\mathbf{v}, \mathbf{w} \in L^{m,2}(G).$$

The generalized Poincaré inequality for $H^m(G)$ reads

$$\alpha_0 \| v \|_m^2 \leqq | v |_m^2 + \sum_{|\sigma| < m} \left| \int_G D^\sigma v \, dx \right|^2, \qquad v \in H^m(G),$$

(*see* Nečas [9, p. 18]). Accordingly, ${}_E H^m(G)$ endowed with the bounded symmetric nonnegative sesquilinear form a as scalar product is a Hilbert space and the associated variational equation (3) is properly posed. Further we have

$$\| \mathbf{v} \|_m^2 - \| \mathbf{v} \|_{m-1}^2 = | \mathbf{v} |_m^2 \leqq a(\mathbf{v}), \qquad \mathbf{v} \in L^{m,2}(G).$$

Consequently, a on ${}_E V_\iota$, $\iota \in \mathbb{N}$, is uniformly coercive and corollary (13) implies the assertion. □

3.3. Approximation of eigenvalue problems

The fundamental compactness theorems of the present paper allow as well to establish very general statements concerning the convergence of spectra and resolvent sets, of eigenvalues and eigenspaces of approximations of generalized elliptic boundary value problems by methods of nonconforming and hybrid finite elements. The class of eigenvalue problems, considered here, is given by

$$0 \neq \mathbf{w}_0 \in E_0; \qquad a(\boldsymbol{\varphi}, \mathbf{w}_0) = \overline{\lambda}_0 \, b(\boldsymbol{\varphi}, \mathbf{w}_0), \qquad \boldsymbol{\varphi} \in E_0; \tag{1}$$

and the associated sequence of approximating eigenvalue problems reads

$$0 \neq \mathbf{w}_\iota \in E_\iota; \qquad a(\boldsymbol{\varphi}_\iota, \mathbf{w}_\iota) = \overline{\lambda}_\iota \, b(\boldsymbol{\varphi}_\iota, \mathbf{w}_\iota), \qquad \boldsymbol{\varphi}_\iota \in E_\iota; \quad \iota = 1, 2, \ldots, \tag{2}$$

a, b being bounded sesquilinear forms on $L^{m,2}(G)$ and the spaces $E_\iota = {}_E V_\iota \subset L^{m,2}(G)$ being embedded subspaces $V_\iota \subset H^m(G_\iota)$, $\iota = 0, 1, 2, \ldots$, where $G_0 = G$.

A typical example is obtained again by sesquilinear forms of the form 3.2.(5) with coefficients $a_{\sigma\tau}$, $b_{\sigma\tau} \in L^\infty(G)$. Problem (1) then becomes

$$\left.\begin{array}{c} 0 \neq w_0 \in V_0; \\ \displaystyle\sum_{|\sigma|, |\tau| \leqq m} \int_G a_{\sigma\tau} D^\sigma \varphi \, \overline{D^\tau w_0} \, dx = \overline{\lambda}_0 \sum_{|\sigma|, |\tau| \leqq m} \int_G b_{\sigma\tau} D^\sigma \varphi \, \overline{D^\tau w_0} \, dx, \\ \varphi \in V_0; \end{array}\right\} \tag{3}$$

and the approximations (2) take on the form

$$\left.\begin{array}{c} 0 \neq w_\iota \in V_\iota; \\ \displaystyle\sum_{|\sigma|,|\tau| \leqq m} \int_{G_\iota} a_{\sigma\tau} D^\sigma \varphi_\iota \overline{D}^\tau w_\iota \, dx = \overline{\lambda}_\iota \sum_{|\sigma|,|\tau| \leqq m} \int_{G_\iota} b_{\sigma\tau} D^\sigma \varphi_\iota \overline{D}^\tau w_\iota \, dx, \\ \varphi_\iota \in V_\iota; \quad \iota = 1, 2, \ldots \end{array}\right\} \tag{4}$$

We shall assume as in the preceding section that the domain $\overline{G}$ is bounded, the decompositions $(\mathscr{K}_\iota)$ satisfy the conditions (K1), (K2) and the spaces V_0, V_ι, $\iota \in \mathbb{N}$, the conditions (V1), (V2), (V3). Let P_ι be the orthogonal projections of $L^{m,2}(G)$ onto the subspaces E_ι for $\iota = 0, 1, 2, \ldots$ The validity of the approximability and closedness condition (V1), (V2) guarantees the convergence of the orthogonal projections

$$P_\iota \to P_0 \qquad (\iota \to \infty) \tag{5}$$

(*see* Stummel [16, theorem 1.2.(7)]). The sesquilinear form a is bounded on $L^{m,2}(G)$ such that the inequality 3.2.(1) holds. Let the sesquilinear form b be bounded by

$$|b(\mathbf{v}, \mathbf{w})| \leqq \beta(\|\mathbf{v}\|_m \|\mathbf{w}\|_{m-1} + \|\mathbf{v}\|_{m-1} \|\mathbf{w}\|_m), \qquad \mathbf{v}, \mathbf{w} \in L^{m,2}(G). \tag{6}$$

This is true in example (3), (4) when $b_{\sigma\tau} = 0$ for $|\sigma| = |\tau| = m$. The above eigenvalue problems are defined by the restrictions a_{E_ι}, b_{E_ι} of the sesquilinear forms a, b to the subspaces E_ι. Let the sequence a_{E_ι}, $\iota = 1, 2, \ldots$, be uniformly coercive so that the inequalities 3.2.(9) hold. Using the approximability condition (V1) it follows that also the corresponding coerciveness condition over the subspace E_0 is valid,

$$\operatorname{Re} a(\mathbf{v}) \geqq \gamma_0 \|\mathbf{v}\|_m^2 - \gamma_1 \|\mathbf{v}\|_{m-1}^2, \qquad \mathbf{v} \in E_0. \tag{7}$$

Finally assume that the pair a, b on the subspaces E_ι is *strongly definite*, that is, there exist real constants α_ι, β_ι such that

$$\alpha_\iota \operatorname{Re} a(\varphi_\iota) + \beta_\iota \operatorname{Re} b(\varphi_\iota) > 0, \qquad 0 \neq \varphi_\iota \in E_\iota, \quad \iota = 0, 1, 2, \ldots \tag{8}$$

With regard to applying general functional analytic theorems, one finally still needs the following property:

Under the above assumptions on b, the sequence b_{E_ι}, $\iota = 0, 1, 2, \ldots$, is weakly collectively compact, that is, b_{E_ι} is compact for each ι and the convergence statement

$$\text{(i)} \qquad \mathbf{z}_\iota \rightharpoonup 0 \;\Rightarrow\; \sup_{0 \neq \varphi_\iota \in E_\iota} \frac{|b(\mathbf{z}_\iota, \varphi_\iota)|}{\|\varphi_\iota\|_m} \to 0 \qquad (\iota \to \infty)$$

is true for every weakly convergent null sequence $\mathbf{z}_\iota \in E_\iota$, $\iota \in \mathbb{N}$. (9)

Proof: (i) The domain G is bounded, $\overline{G}$ admits the decompositions (K1), (K2) and every element $K \in \mathcal{K}_\iota$ has the segment property. Hence Rellich's theorem is valid ensuring the compactness of the natural embeddings of $H^m(G_\iota)$ into $H^{m-1}(G_\iota)$ for each $\iota = 0, 1, 2, \ldots$ From property (6) we then infer the compactness of b on ${}_E V_\iota \subset {}_E H^m(G_\iota)$ (*see* [13, p. 34]).

(ii) Let $(\mathbf{z}_\iota)$ be any weakly convergent null sequence in $L^{m,2}(G)$ and (ε_ι) be a null sequence of positive numbers. Then there exists a sequence of functions $\psi_\iota \in E_\iota$, $\|\psi_\iota\|_m = 1$, such that

$$\sup_{0 \neq \varphi_\iota \in E_\iota} \frac{|b(\mathbf{z}_\iota, \varphi_\iota)|}{\|\varphi_\iota\|_m} \leqq |b(\mathbf{z}_\iota, \psi_\iota)| + \varepsilon_\iota, \qquad \iota = 1, 2, \ldots$$

Let $\mathbb{N}'$ be a subsequence of $\mathbb{N} = \{1, 2, \ldots\}$ specified by

$$\limsup_{\iota \to \infty} |b(\mathbf{z}_\iota, \psi_\iota) = \lim_{\substack{\iota \in \mathbb{N}' \\ \iota \to \infty}} |b(\mathbf{z}_\iota, \psi_\iota)|.$$

Using the compactness theorem 2.3.(4), one obtains a subsequence $\mathbb{N}'' \subset \mathbb{N}'$ and an element $\psi_0 \in V_0$ such that

$$\psi_\iota \rightharpoonup \psi_0, \qquad \|\psi_\iota - \psi_0\|_{m-1} \to 0 \qquad (\iota \in \mathbb{N}'', \iota \to \infty).$$

For brevity, set $y_\iota = \psi_\iota - \psi_0$ for $\iota \in \mathbb{N}''$, $y_\iota = 0$ for $\iota \in \mathbb{N} - \mathbb{N}''$. Then

$$b(\mathbf{z}_\iota, \psi_\iota) = b(\mathbf{z}_\iota, \psi_0) + b(\mathbf{z}_\iota, \mathbf{y}_\iota).$$

The first term on the right side tends to zero due to $\mathbf{z}_\iota \rightharpoonup 0$ for $\iota \to \infty$. Inequality (6) and the compactness theorem 2.2.(3) then imply the convergence of $b(\mathbf{z}_\iota, \mathbf{y}_\iota) \to 0$ for $\iota \to \infty$. Consequently, $b(\mathbf{z}_\iota, \psi_\iota) \to 0$ for $\iota \in \mathbb{N}''$ and thus necessarily for all $\iota \in \mathbb{N}$, $\iota \to \infty$. By the above estimate of the supremum, this entails the convergence relation (9 i). □

Now the proconditions for applying our perturbation theory [15, 18] for elliptic sesquilinear forms in Hilbert spaces are given such that the general theorems hold for the class of approximations considered here: the eigenvalue problems (1), (2) have discrete spectra of sequences of eigenvalues of finite multiplicities having no finite accumulation point. Spectra and resolvent sets of the approximating problems (2) converge to those of the given problem (1). To each eigenvalue λ_0 of (1), having the algebraic multiplicity m, there exist exactly m eigenvalues $\lambda_\iota^{(1)}, \ldots, \lambda_\iota^{(m)}$ of (2) converging to λ_0 for $\iota \to \infty$. The associated sums of algebraic eigenspaces of (2) converge to the algebraic eigenspace of (1).

Under these conditions the assumption $(a\,F)$ of the paper Grigorieff [6] is valid. There one finds further convergence results, in particular, statements concerning the order of convergence, error estimates and asymptotic developments.

In the case that a, b are symmetric sesquilinear forms and a is positive definite on the subspaces $E_\iota =_E V_\iota$, our perturbation theory [18] is applicable to the eigenvalue problems (1), (2). By corollary 3.2.(13), a on E_τ specifies scalar products $(.\,,\,.)_{E_\tau}$. The definiteness condition (8) is trivially satisfied. In [18] one finds, together with general convergence theorems, associated error estimates for eigenvalue and eigenvector approximations. In particular, the specific form of discretization errors is explained and the quadratic convergence behaviour, compared to the discretization errors, of eigenvalue approximations is established there.

REFERENCES

1. Sh. AGMON, *Lectures on Elliptic Boundary Value Problems*, van Nostrand, New York, 1965.
2. P. ALEXANDROFF and H. HOPF, *Topologie*, Springer-Verlag, Berlin, 1935; Chelsea, New York, 1965.
3. P. C. CIARLET, *Conforming and Nonconforming Finite Element Methods for Solving the Plate Problem*, Proc. Conf. Numer. Solution of Differential Equations, Dundee, 1973, G. A. WATSON, Ed., Lecture Notes in Math., Vol. 363, 1974, pp. 150-168.
4. P. C. CIARLET, *The Finite Element Method for Elliptic Problems*, North Holland, Amsterdam, 1978.
5. M. CROUZEIX and P. A. RAVIART, *Conforming and Nonconforming Finite Element Methods for Solving the Stationary Stokes Equations I*, R.A.I.R.O., Série Rouge, Anal. Numér., Vol. 3, 1973, pp. 33-76.
6. R. D. GRIGORIEFF, *Diskrete Approximation von Eigenwertproblemen*. I. *Qualitative Konvergenz*, Numer. Math., Vol. 24, 1974, pp. 355-374; II. *Konvergenzordnung*, Numer. Math., Vol. 24, 1974, pp. 415-433; III. *Asymptotische Entwicklungen*, Numer. Math., Vol. 25, 1975, pp. 79-97.
7. B. GRÜNBAUM, *Convex Polytopes*, Wiley-Interscience, London-New York, 1967.
8. P. LASCAUX and P. LESAINT, *Some Nonconforming Finite Elements for the Plate Bending Problem*, R.A.I.R.O., Série Rouge, Anal. Numér., Vol. 9, 1975, pp. 9-53.
9. J. NEČAS, *Les méthodes directes en théorie des équations elliptiques*, Masson, Paris, 1967.
10. R. RANNACHER, *Nonconforming Finite Element Methods for Eigenvalue Problems in Linear Plate Theory*, Preprint, No. 191, S.F.B. 72, Universität Bonn, 1978.
11. P. A. RAVIART and J. M. THOMAS, *Primal Hybrid Finite Element Methods for 2nd Order Elliptic Equations*, Math. Comp., Vol. 31, 1977, No. 138, pp. 391-413.
12. R. T. ROCKAFELLAR, *Convex Analysis*, University Press, Princeton, 1970.
13. F. STUMMEL, *Rand- und Eigenwertaufgaben in Sobolewschen Rämen*, Lecture Notes in Mathematics, No. 102, Springer-Verlag, Berlin-Heidelberg-New York, 1969.

14 F STUMMEL, *Diskrete Konvergenz linearer Operatoren* I, Math Ann, Vol 190 1970, pp 45-92, II, Math Z, Vol 120, 1971, pp 231-264, III, Proc Conf Linear Operators and Approximation, Oberwolfach, 1971, P L BUTZER, J-P KAHANE and B Sz-NAGY, Eds, Birkhauser Basel, I S N M Vol 20 1972 pp 196-216

15 F STUMMEL, *Singular Perturbations of Elliptic Sesquilinear Forms*, Proc Conf Differential Equations, Dundee, 1972, W N EVERITT and B D SLEEMAN, Eds, Lecture Notes in Mathematics, No 280, 1972, pp 155-180

16 F STUMMEL, *Perturbation Theory for Sobolev Spaces*, Proc Royal Soc Edinburgh, Vol 73A, 1974/1975, pp 1-49

17 F STUMMEL *Perturbation of Domains in Elliptic Boundary Value Problems*, Proc Conf Application of Methods of Functional Analysis to Problems of Mechanics, Marseille, 1975, P GERMAIN and B NAYROLES, Eds, Lecture Notes in Mathematics, No 503, 1976, pp 110-136

18 F STUMMEL, *Approximation Methods for Eigenvalue Problems in Elliptic Differential Equations*, Proc Conf Numerik und Anwendungen von Eigenwertaufgaben und Verzweigungsproblemen, Oberwolfach 1976 E BOHL L COLLATZ and K P HADELER Eds Birkhauser Basel I S N M Vol 38 1977 pp 133-165

19 F STUMMEL *The Limitations of the Patch Test* Int J Numer Meth Eng

20 F STUMMEL *The Generalized Patch Test* S I A M J Numer Anal Vol 16 1979 pp 449-471

21 J M THOMAS, *Sur l'analyse numerique des methodes d element finis hybrides et mixtes*, These Universite P-et-M-Curie, Paris, 1977