ROLF RANNACHER

On nonconforming an mixed finite element methods for plate bending problems. The linear case

RAIRO. Analyse numérique, tome 13, nº 4 (1979), p. 369-387
<http://www.numdam.org/item?id=M2AN_1979__13_4_369_0>

R A I R O Analyse numerique/Numerical Analysis
(vol 13 n° 4 1979 p 369 a 387)

ON NONCONFORMING AND MIXED FINITE ELEMENT METHODS FOR PLATE BENDING PROBLEMS. THE LINEAR CASE (*)

by Rolf RANNACHER ([1])

Communique par P G CIARLET

Abstract — *This paper deals with the approximate solution of linear 4th-order elliptic boundary-value problems by the finite element method For a displacement method using the nonconforming plate element of Morley and for a mixed method known as Herrmann-Miyoshi-scheme quasi optimal* L^2- *and* L^∞-*error estimates are derived The proof essentially uses* L^1-*estimates for regularized Green's functions and their discrete analogues*

Resume — *Cet article traite de l approximation par elements finis de problemes aux limites elliptiques lineaires du* 4^e *ordre On obtient des majorations de l'erreur quasi optimales dans les normes* L^2 *et* L^∞ *pour une methode "deplacement" non conforme utilisant l'element de Morley et pour une methode mixte connue sous le nom de schema d'Herrmann-Miyoshi La demonstration utilise de façon essentielle des estimations en norme* L^1 *pour les fonctions de Green regularisees et pour leurs analogues discrets*

1. INTRODUCTION

Let Ω be a bounded region in Euclidean space $\mathbb{R}^2$ with piecewise smooth boundary $\partial\Omega$ We consider the standard model problem in linear plate theory

$$\Delta^2 u=f \quad \text{in } \Omega, \qquad u=\partial_n u=0 \quad \text{on } \partial\Omega \tag{1 1}$$

The corresponding "primal" variational function is

(P) *Find a function* $u \in H_0^{2,2}(\Omega)$ *such that*

$$a(u, \varphi)=(f, \varphi), \qquad \forall \varphi \in H_0^{2,2}(\Omega),$$

with the bilinear forms

$$a(u, v)=\int_\Omega \left\{ \Delta u \Delta v-(1-\nu)(\partial_1^2 u\, \partial_2^2 v+\partial_2^2 u\, \partial_1^2 v-2\, \partial_1 \partial_2 u\, \partial_1 \partial_2 v) \right\} dx,$$

$$(u, v)=\int_\Omega uv\, dx$$

(*) Manuscrit reçu le 28 août 1978 revision reçue le 5 janvier 1979

([1]) Institut fur Angewandte Mathematik der Universitat Bonn (Fed Rep of Germany)

R A I R O Analyse numerique/Numerical Analysis 0399-0516/1979/369/$ 4 00

and the Poisson ratio $0<\nu<1$ of the plate.

We shall use the standard notation $L^p(\Omega)$ and $H^{m,p}(\Omega)$, $m\in\mathbb{N}$, $1\leqq p\leqq\infty$, for the Lebesgue and Sobolev spaces, respectively, and for their norms

$$\|u\|_p=\|u\|_{p;\Omega}=\left(\int_\Omega |u|^p\,dx\right)^{1/p},\qquad \|u\|_{m,p}=\left(\sum_{k=0}^{m}\|\nabla^k u\|_p^p\right)^{1/p},$$

with the usual modifications for $p=\infty$.

$H_0^{m,p}(\Omega)$ is the closure in $H^{m,p}(\Omega)$ of the space $C_0^\infty(\Omega)$ of test functions on Ω. Further we write $\partial_i u=\partial u/\partial x_i$, $i=1, 2$, $\partial_0 u=u$ for the (generalized) derivatives and $\nabla^k u$ for the fields of all k-th derivatives. "c" denotes a positive generic constant which may vary with the context, but which is independent of all parameters and functions that appear.

The standard finite element *displacement method* based on the variational formulation (P) yields approximations to the displacement function u, from which one can compute also approximations, in general discontinuous, to the most interesting bending moments $M=\nabla^2 u$. For the *conforming methods*, which use proper subspaces $S_0^h\subset H_0^{2,2}(\Omega)$ of piecewise polynomial shape functions, optimal convergence results in $L^2(\Omega)$ as well as in $L^\infty(\Omega)$ are well known (*see* e. g. [8] and the survey article [7]). The proofs proceed analogously to those for the second order case. Hence we shall only consider here *nonconforming* elements violating this compatibility condition: $S_0^h\not\subset H_0^{2,2}(\Omega)$. The simplest but rather typical representative of this class, the quadratic element of Morley (*see* e. g. [4]), will be analysed in detail. Another type of finite element method is based on mixed variational formulations of problem (1.1). For this purpose we introduce the notation

$$H^{1,2}(\Omega)^3=\{\Psi=(\Psi^{ij})_{i,\,j=1,\,2},\ \Psi^{ij}=\Psi^{ji}\in H^{1,2}(\Omega)\}$$

for the space of $H^{1,2}$-fields and correspondingly the forms

$$A(\Phi,\Psi)=\int_\Omega\{(\Phi^{11}+\Phi^{22})(\Psi^{11}+\Psi^{22}) -(1-\nu)(\Phi^{11}\Psi^{22}+\Phi^{22}\Psi^{11}-2\Phi^{12}\Psi^{12})\}\,dx,$$

$$D(\Phi,\Psi)=\int_\Omega\{\partial_1\Phi^{11}\partial_1\Psi^{11}+\nu\partial_1\Phi^{22}\partial_1\Psi^{22}+\nu\partial_2\Phi^{11}\partial_2\Psi^{11}+\partial_2\Phi^{22}\partial_2\Psi^{22} +\nabla\Phi^{12}\nabla\Psi^{12}+(1-\nu)(\partial_1\Phi^{12}\partial_2\Psi^{22}+\partial_2\Phi^{12}\partial_1\Psi^{11})\}\,dx.$$

The usual embedding of $H^{1,2}(\Omega)$ into $H^{1,2}(\Omega)^3$ is always indicated by the use of capital letters as follows

$$\varphi\in H^{1,2}(\Omega)\to\Phi\in H^{1,2}(\Omega)^3:\qquad \Phi^{11}=\Phi^{22}:=\varphi,\ \Phi^{12}:=0.$$

Using this notation, one easily verifies that any "weak" solution $u \in \overset{\circ}{H}{}^{2,2}(\Omega)$ of problem (P) generates by $(u, M)=(u, \nabla^2 u)$ a solution of the following mixed formulation

($\tilde{\mathrm{P}}$) *Find a pair* $(u, M) \in \overset{\circ}{H}{}^{1,2}(\Omega) \times H^{1,2}(\Omega)^3$ *such that*

$$D(M, \Phi)+(f, \varphi)=0. \qquad \forall \varphi \in \overset{\circ}{H}{}^{1,2}(\Omega),$$
$$D(\Psi, U)+A(\Psi, M)=0, \qquad \forall \Psi \in H^{1,2}(\Omega)^3,$$

where U and Φ denote the fields obtained by embedding u and φ, respectively, into $H^{1,2}(\Omega)^3$. Using this Ansatz, the trial functions only need to be in $C^0(\overline{\Omega})$, and one simultaneously gets continuous approximations to the displacement u as well as to the field $M=\nabla^2 u$ of bending moments. The discrete analogues of problem (P) are equivalent to the well known Herrmann-Miyoshi-scheme (*see* [5 and 1]). In detail we shall analyse its version using piecewise quadratic shape functions.

The present study for linear problems leads also to a corresponding analysis for nonlinear plate models which will be carried out in a forthcoming paper. For this purpose we are mainly interested in pointwise error estimates for the bending moments which need to be controlled during some linearization process. We note that the results, stated below, even hold for more general linear 4th-order systems

$$\sum_{i=1}^{2}\left\{\sum_{\alpha, \beta, \gamma, \lambda=1}^{2} \partial_\alpha \partial_\beta (a^{ik}_{\alpha\beta\gamma\lambda} \partial_\gamma \partial_\lambda u^i)+\sum_{\alpha, \beta=0}^{2} \partial_\alpha (a^{ik} \partial_\beta u^i)\right\}=f^k \text{in } \Omega,$$
$$u^i=\partial_n u^i=0 \quad \text{on } \partial\Omega_c, \qquad u^i=0 \quad \text{on } \partial\Omega_s, \qquad i, k=1, 2,$$

where the unknown $u=(u^1, u^2)$ is a vector function, and the coefficients $a^{ik}_{\alpha\beta\gamma\lambda}$, $a^{ik}_{\alpha\beta}$ are such that the usual ellipticity and regularity conditions are satisfied (*see* e.g. [10] for the general scalar case).

2. FINITE ELEMENT APPROXIMATION

For the following we assume that Ω is a *convex polygonal* domain. This does not restrict our results essentially, but it simplifies the presentation and proofs. Otherwise, in the case of a curved boundary, one has to deal with some type of boundary approximation (e.g. polygonal approximation, isoparametric elements, etc.). The presence of "nonconvex" angular points of the boundary would cause additional difficulties because of singular behaviour of the solution u of (1.1).

For a discretization parameter $0 \leqq h \leqq h_0 < 1$, tending to zero, let $\Pi_h = \{T\}$ be finite triangulations of $\bar{\Omega}$ such that the usual regularity condition is satisfied:

(T) *Any two triangles in Π_h may meet at most in whole common sides or in common vertices. Each triangle contains a circle with radius $c_0 h$ and is contained in a circle with radius $c_0^{-1} h$.*

The finite element spaces of Morley are defined by (*see* [4]):

$S_0^h = \{ v_h \in L^\infty(\Omega) \mid 1.$ On each triangle the restriction $v_{h|T}$ is a polynomial of degree less or equal two. 2. v_h and the normal derivatives $\partial_n v_h$ are continuous at vertices and midpoints of sides, respectively, and vanish at the nodal points on the boundary $\partial\Omega \}$.

Obviously we even have $S_0^h \not\subset C(\bar{C})$, so that the spaces S_0^h are highly nonconforming. The finite dimensional analogues of the formulation (P) are

(P_h) *Find a function $u_h \in S_0^h$ such that*

$$a_h(u_h, \varphi_h) = (f, \varphi_h), \qquad \forall \varphi \in S_0^h,$$

where the modified bilinear forms are used

$$a_h(\varphi, \psi) = \sum_{T \in \Pi_h} \int_T \{ \Delta\varphi \, \Delta\psi - (1-\nu)(\partial_1^2 \varphi \, \partial_2^2 \psi + \partial_2^2 \varphi \, \partial_1^2 \psi - 2 \, \partial_1 \partial_2 \varphi \, \partial_1 \partial_2 \psi) \} \, dx.$$

Further we introduce the notation

$$\| u \|_{k,p;h} = \Big(\sum_{T \in \Pi_h} \| u \|_{k,p;T}^p \Big)^{1/p}, \qquad \| u \|_{p;h} = \Big(\sum_{T \in \Pi_h} \| u \|_{p;T}^p \Big)^{1/p}$$

with the usual modification for $p = \infty$.

For the nonconforming method (P_h) Lascaux and Lesaint [4] proved the existence of unique approximation solutions $u_h \in S_0^h$ and the L^2-error estimate

$$\| u - u_h \|_{2;h} + h \| \nabla^2 (u - u_h) \|_{2;h} \leqq ch^2 (\| u \|_{3,2} + h \| u \|_{4,2}). \tag{2.1}$$

The L^∞-error estimate

$$\| u - u_h \|_\infty \leqq ch^2 |\ln h| . \| u \|_{4,2} \tag{2.2}$$

was given by the author in [8]. These results are in some sense unsatisfactory. The estimate (2.1) requires the $H^{4,2}$-regularity of the solution u which cannot be expected in general even in convex polygonal domains. The L^∞-estimate (2.2) is not sufficient for an analysis of quasi-linear 4th-order problems as mentioned in paragraph 1. These problems are solved by the following theorem.

THEOREM 1: *For the displacement method* (P_h) *using the nonconforming finite element of Morley the asymptotic error estimates hold:*

$$\| u-u_h \|_{1,2;h}+h \| \nabla^2 (u-u_h) \|_{2;h} \leqq ch^2 \| u \|_{3,2},$$
$$\| u-u_h \|_{1,\infty;h}+h \| \nabla^2 (u-u_h) \|_{\infty;h} \leqq ch^2 |\ln h|^{3/2} \| u \|_{3}.$$

These results are optimal with respect to the power of h as well as to the regularity requirement.

In order to describe the mixed method based on the formulation $(\tilde{P})$ of problem (1.1) we introduce the following spaces

$V^h = \{ v_h \in H^{1,2}(\Omega) \mid$ On each triangle the restriction $v_{h|T}$ is a polynomial of degree less or equal two $\}$,

$$V_0^h = V^h \cap H_0^{1,2}(\Omega), \qquad W^h = (V^h)^3 \subset H^{1,2}(\Omega)^3.$$

Obviously the product spaces $V_0^h \times W^h$ are conforming for the formulation $(\tilde{P})$. Then the Herrmann-Miyoshi-scheme may be defined by

$(\tilde{P}_h)$ *Find a pair* $(u_h, M_h) \in V_0^h \times W^h$ *such that*

$$D(M_h, \Phi_h)+(f, \varphi_h)=0, \qquad \forall \varphi_h \in V_0^h,$$
$$D(\Psi_h, U_h)+A(\Psi_h, M_h)=0, \qquad \forall \Psi_h \in W^h.$$

These finite dimensional problems are uniquely solvable since any solution of the corresponding homogeneous problem necessarily equals zero. For this mixed method Brezzi, Ciarlet and Raviart [1, 3] proved the error estimate

$$\| u-u_h \|_{1,2}+\| M-M_h \|_2 \leqq ch \| u \|_{3,2}, \tag{2.3}$$

and Scholz [11] derived the improved L^2-estimate

$$\| u-u_h \|_2 \leqq ch^2 \| u \|_{3,2}. \tag{2.4}$$

Corresponding results also hold for higher order finite elements. The case of piecewise linear elements was treated originally by Miyoshi [5] on locally uniform meshes and recently by Scholz [12] for the special case $\nu=1$ even on general meshes. The L^∞-estimate for the piecewise quadratic case

$$\| u-u_h \|_\infty \leqq ch^2 |\ln h|^{1/2} \| u \|_{4,2}, \tag{2.5}$$

was proved by the author in [8].

Here we present the following improved result:

THEOREM 2: *For the Herrmann-Miyoshi-scheme* ($\tilde{\mathrm{P}}_h$) *using piecewise quadratic finite elements the asymptotic error estimates hold:*

$$\|u-u_h\|_{1,2}+h\|M-M_h\|_2\leqq ch^2\|u\|_{3,2},$$
$$\|u-u_h\|_{1,\infty}+h\|M-M_h\|_\infty\leqq ch^2|\ln h|^{3/2}\{\|u\|_{3,\infty}+\|u\|_{4,2}\}.$$

We note that this result even holds for the limit case $\nu=1$ which corresponds to problems in fluid mechanics. Then the scheme ($\tilde{\mathrm{P}}_h$) reduces to:

($\tilde{\mathrm{P}}'_h$) *Find a pair* $(u_h, \omega_h)\in S_0^h\times S^h$ *such that*

$$(\nabla\omega_h, \nabla\varphi_h)+(f, \varphi_h)=0, \qquad \forall\varphi_h\in S_0^h,$$
$$(\nabla\psi_h, \nabla u_h)+(\psi_h, \omega_h)=0, \qquad \forall\psi_h\in S^h,$$

and yields approximations $\omega_h\in S^h$ to the quantity $\omega=\Delta u$. Obviously it is again uniquely solvable and the techniques of proof, presented below, can be directly carried over. Also the results of theorem 1 for the Morley-triangle remain valid for the case $\nu=1$ if one uses the discrete bilinear forms

$$a_h(v, w)=\sum_T\int_T\{\partial_1^2 v\,\partial_1^2 w+\partial_2^2 v\,\partial_2^2 w+2\,\partial_1\partial_2 v\,\partial_1\partial_2 w\}\,dx.$$

3. PROOF OF THEOREM 1

Our approach to the above L^∞-error estimates is closely related to those used by J. A. Nitsche [14], R. Scott [15] and by J. Frehse and the author in [13]. Unfortunately the rather elegant technique of [14] seems to yield only a reduced order $O(h^{3/2})$, which depends on the fact that the duality between L^2 and H^4 cannot be full utilized with quadratic finite elements. The method in [15] needs explicit pointwise estimates for the derivatives of Green's functions and is less convenient for treating the additional technical difficulties arising for nonconforming or mixed methods. Hence our proofs are essentially based on a modification of the method described in [13] which uses so-called regularized Green's functions and weighted *a priori*-estimates.

The proof will be given for a more general version of problem (P). For this purpose we introduce the spaces

$$\hat{S}_0^h=S_h^0\oplus H_0^{2,2}(\Omega)$$

which are provided with the norms $\|.\|_{2,2,h}$. For some arbitrarily fixed $h>0$ let

$F(.)$ be a bounded linear functional on $\hat{S}_0^h$ and let $v \in H_0^{2,2}(\Omega)$ and $v_h \in S_0^h$ be defined by

$$\left.\begin{aligned} a(v, \varphi) &= F(\varphi), \qquad \forall \varphi \in H_0^{2,2}(\Omega), \\ a_h(v_h, \varphi_h) &= F(\varphi_h), \qquad \forall \varphi_h \in S_0^h. \end{aligned}\right\} \tag{3.1}$$

We note that on a convex polygonal domain the biharmonic operator is a homeomorphism from $H_0^{2,2}(\Omega) \cap H^{3,2}(\Omega)$ onto $H^{-1,2}(\Omega)$ and satisfies the *a priori*-estimate

$$\| u \|_{3,2} \leq c \| \Delta^2 u \|_{-1,2}, \qquad u \in H_0^{2,2}(\Omega) \cap H^{3,2}(\Omega).$$

This result may be obtained by considering $u \in H_0^{2,2}(\Omega) \cap H^{3,2}(\Omega)$ as the stream function of a viscous incompressible flow with velocity field $\underline{v} := \underline{curl}\; u$ satisfying the usual Stokes equation. Then the regularity results of Kellogg and Osborn [16] applied to v lead to the desired result for the biharmonic operator.

So we may assume that the functional $F(.)$ is in $H^{-1,2}(\Omega)$ and hence satisfies

$$\| v \|_{3,2} \leq c \sup \left\{ \frac{|F(\varphi)|}{\| \varphi \|_{1,2}}, \varphi \in H_0^{1,2}(\Omega) - \{0\} \right\}. \tag{3.2}$$

In the following the generic constant $c > 0$ will always be independent of $F(.)$ and $h > 0$. For abbreviation we set $e := v - v_h$.

(A) Here we prove the L^2-error estimates.

We recall the following standard estimate for nonconforming methods (*see* e.g. [2] or [4]):

$$\| \nabla^2 e \|_{2,h} \leq c \inf_{\varphi \in S_0^h} \| \nabla^2 (v - \varphi) \|_{2,h} + c \sup_{\varphi \in S_0^h - \{0\}} \frac{|N_h(v, \varphi)|}{\| \nabla^2 \varphi \|_{2,h}}, \tag{3.3}$$

where

$$N_h(v, \varphi) = a_h(e, \varphi) = a_h(v, \varphi) - F(\varphi).$$

Green's formula yields for any function $\varphi \in \hat{S}_0^h$ that

$$N_h(v, \varphi) = -(\nabla \Delta v, \nabla \varphi)_h - F(\varphi) + B_h(v, \varphi) \tag{3.4}$$

where (using standard notation):

$$B_h(v, \varphi) = \sum_{T \in \Pi_h} \int_{\partial T} \{ \Delta v \, \partial_n \varphi + (1 - \nu)(\partial_n \partial_s v \, \partial_s \varphi - \partial_s^2 v \, \partial_n \varphi) \} \, ds$$

and

$$(w, \varphi)_h = \sum_{T \in \Pi_h} \int_T w \varphi \, dx.$$

Corresponding to the triangulations Π_h we introduce for any $\varphi \in \hat{S}_0^h$ the natural piecewise linear interpolant φ^L which satisfies $\varphi^L \in H_0^{1,2}(\Omega)$ and

$$\| \varphi - \varphi^L \|_{k, p; h} \leqq c h^{2-k} \| \nabla^2 \varphi \|_{p; h}, \qquad k=0,1. \tag{3.5}$$

Using this notation we conclude by Green's formula

$$\begin{aligned} |(\nabla \Delta v, \nabla \varphi)_h - F(\varphi)| &= |(\nabla \Delta v, \nabla(\varphi - \varphi^L))_h - F(\varphi - \varphi^L)| \\ &\leqq c h \| \nabla^2 \varphi \|_{p;h} \| \nabla^3 v \|_q, \qquad 1/p + 1/q = 1. \end{aligned} \tag{3.6}$$

In order to estimate the boundary term $B_h(v, \varphi)$ we denote by Γ the edges of the triangles $T \in \Pi_h$ and by $[\varphi]$ the jump of φ along Γ. Then

$$B_h(v, \varphi) = \sum_\Gamma \int_\Gamma \{ \Delta u [\partial_n \varphi] + (1-\nu)(\partial_n \partial_s v [\partial_s \varphi] - \partial_s^2 v [\partial_n \varphi]) \} \, ds.$$

The continuity properties of $\varphi \in \hat{S}_0^h$ imply that $[\partial_n \varphi] = 0$ at midpoints and $[\varphi] = 0$ at endpoints of all Γ. Consequently

$$\int_\Gamma [\partial_n \varphi] \, ds = \int_\Gamma [\partial_s \varphi] \, ds = 0.$$

This allows us to insert appropriate meanvalues as follows

$$\int_\Gamma \Delta v [\partial_n \varphi] \, ds = \int_\Gamma (\Delta v - \omega) [\partial_n \varphi] \, ds$$

and analogously for the other terms in $B_h(v, \varphi)$. From this one obtains by a Poincaré-type argument the local estimate

$$\left| \int_\Gamma \Delta v [\partial_n \varphi] \, ds \right| \leqq c h \| \nabla^3 v \|_{q; T} (\| \nabla^2 \varphi \|_{p; T} + \| \nabla^2 \varphi \|_{p; T'}), \tag{3.7}$$

where T and T' are the triangles which meet at Γ ($T' = \emptyset$ along the boundary $\partial\Omega$). Assembling these estimates for all edges gives

$$| B_h(v, \varphi) | \leqq c h \| \nabla^3 v \|_q \| \nabla^2 \varphi \|_{p;h}, \tag{3.8}$$

and consequently

$$| N_h(v, \varphi) | \leqq c h \| \nabla^3 v \|_q \| \nabla^2 \varphi \|_{p;h}, \qquad 1/p + 1/q = 1. \tag{3.9}$$

Now let $I_h v \in S_0^h$ denote the usual interpolant of v satisfying

$$\| v - I_h v \|_{k, p; h} \leqq c h^{3-k} \| \nabla^3 v \|_{p;h}, \qquad k = 0, 1, 2, \quad T \in \Pi_h. \tag{3.10}$$

This, combined with (3.3) and (3.9), proves the energy-norm estimate

$$\|\nabla^2 e\|_{2;h} \leqq ch \|\nabla^3 v\|_2 \leqq ch \|F\|_{-1,2}. \tag{3.11}$$

In order to estimate $\|e\|_{1,2;h}$ *we apply a standard duality argument (see* e. g. *[6]* or [4]). Let a linear functional L(.) as introduced above be chosen as $L(\varphi) = (\nabla I_h e, \nabla \varphi)_h$ so that for the corresponding solutions $w \in H_0^{2,2}(\Omega) \cap H^{3,2}(\Omega)$ and $w_h \in S_0^h$ of (3.1) the following relation holds

$$\|w\|_{3,2} \leqq c \|\nabla I_h e\|_{2;h} \tag{3.12}$$

Further

$$\begin{aligned} L(I_h e) &= a_h(w - w_h, v - I_h v) + a_h(w_h, e) - a_h(w, v - I_h v) \\ &= a_h(w - w_h, v - I_h v) + N_h(v, w_h - w) - B_h(w, v - I_h v) \\ &\qquad + (\nabla \Delta w, \nabla(v - I_h v))_h \end{aligned} \tag{3.13}$$

where $N_h(.,.)$, $B_h(.,.)$ are defined as above and the relations are used: $N_h(v, w) = B_h(v, w) = 0$. Hence we obtain using (3.8) and (3.9) for v and w, respectively,

$$\begin{aligned} \|\nabla I_h e\|_{2;h} \leqq c \|\nabla^2(w - w_h)\|_{2;h} \{ \|\nabla^2(v - I_h v)\|_{2;h} + h\|v\|_{3,2} \} \\ + c\{ h\|\nabla^2(v - I_h v)\|_{2;h} + \|\nabla(v - I_h v)\|_{2;h} \} \|w\|_{3,2}. \end{aligned}$$

and consequently, using (3.10), (3.12) and the result (3.11) for the error $w - w_h$,

$$\|\nabla I_h e\|_{2;h} \leqq ch^2 \|v\|_{3,2}.$$

This obviously proves the desired estimate

$$\|e\|_{1,2;h} \leqq ch^2 \|v\|_{3,2}. \tag{3.14}$$

(B) To prove the L^∞-error estimates we use the technique of regularized Green's functions already known from the second order case: For any $h > 0$ let $\hat{T} \in \Pi_h$ be an arbitrary fixed triangle. Then by the local properties of polynomials

$$\|\nabla I_h e\|_{\infty;\hat{T}} \leqq ch^{-2} \int_{\hat{T}} |\nabla I_h e| \, dx.$$

With the functions $\delta_i = h^{-2} \operatorname{sgn}(\partial_i I_h e) \chi_{\hat{T}}$, $\chi_{\hat{T}}$ = characteristic function of $\hat{T}$, we define the linear bounded functionals on $\hat{S}_0^h$:

$$L(\varphi) = \sum_{i=1}^{2} (\delta_i, \partial_i \varphi)_h, \qquad \varphi \in \hat{S}_0^h.$$

Then by (3.10) with $p=\infty$:

$$\|\nabla e\|_{\infty, T} \leqq c\, L(I_h e) + ch^2 \|v\|_{3,\infty}.$$

Let the corresponding solutions of the auxiliary problems (3.1) be denoted by $g \in H_0^{2,2}(\Omega) \cap H^{3,2}(\Omega)$ and $g_h \in S_0^h$, respectively. From (3.13) we obtain, using the L^1- and L^∞-analogues of the estimates (3.8) and (3.9), respectively,

$$\begin{aligned} |L(I_h e)| = |a_h(g-g_h, v-I_h v) + N_h(v, g_h-g) \\ - B_h(g, v-I_h v) + (\nabla \Delta g, \nabla(v-I_h v))_h| \\ \leqq c \|\nabla^2(g-g_h)\|_{1;h} \{\|\nabla^2(v-I_h v)\|_{\infty;h} + h\|v\|_{3;\infty}\} \\ + c\|\nabla^3 g\|_1 \{\|\nabla(v-I_h v)\|_{\infty;h} + h\|\nabla^2(v-I_h v)\|_{\infty;h}\} \end{aligned}$$

and consequently by (3.10):

$$|L(I_h e)| \leqq ch\|v\|_{3,\infty} \{\|\nabla^2(g-g_h)\|_{1,h} + h\|\nabla^3 g\|_1\}.$$

In the next step we shall prove the following estimate

$$\|\nabla^2(g-g_h)\|_{1;h} + h\|\nabla^3 g\|_1 \leqq ch|\ln h|^{3/2}, \tag{3.15}$$

by which all the desired L^∞-error estimates can easily be obtained.

(C) The error $E := g - g_h$ will be estimated by a technique which was already used by J. Frehse and the author in [13] for second order problems. For this purpose we introduce the weight-function

$$\sigma(x) = (|x-\hat{x}|^2 + \varkappa^2 h^2)^{1/2}, \qquad \varkappa \geqq 1,$$

where $\hat{x}$ is the center of the fixed triangle $\hat{T} \in \Pi_h$, and the corresponding weighted L^2-norms

$$\|w\|_{(a)} = \left(\sum_{T \in \Pi_h} \int_T \sigma^a |w|^2\, dx\right)^{1/2}, \qquad a \in \mathbb{R}.$$

The following relations will be frequently used

$$|\nabla^k \sigma| \leqq c_k \sigma^{1-k}, \qquad \sigma^{-1} \leqq (\varkappa h)^{-1}.$$

For $\varkappa \geqq \varkappa_0$, sufficiently large, the relation

$$\max\{\max_{x \in T} \sigma(x)/\min_{x \in T} \sigma(x),\ T \in \Pi_h\} \leqq c$$

holds uniformly for $h>0$. This allows us to carry the interpolation estimates (3.5), (3.10) and even the estimates (3.8), (3.9) over to the weighted norms $\|\cdot\|_{(a)}$, $a \in \mathbb{R}$.

Using this notation, we have

$$\|\nabla^2 E\|_{1;h} \leqq c|\ln h|^{1/2}\|\nabla^2 E\|_{(2)}. \tag{3.16}$$

Then a simple calculation leads to

$$\|\nabla^2 E\|_{(2)}^2 \leqq c a_h(E, \sigma^2 E) + c\|\nabla^2 E\|_{(2)}\{\|\nabla E\|_{2;h} + \|E\|_{(-2)}\}$$

and with the interpolant $\chi_h = I_h(\sigma^2 E) \in S_0^h$:

$$a_h(E, \sigma^2 E) = a_h(E, \sigma^2 E - \chi_h) + N_h(g, \chi_h).$$

Using the estimates (3.9) and (3.10) for weighted norms, we find

$$|a_h(E, \sigma^2 E)| \leqq c\|\nabla^2 E\|_{(2)}\, h\|\nabla^3(\sigma^2 E)\|_{(-2)} + ch\|\nabla^3 g\|_{(2)}\|\nabla^2 \chi_h\|_{(-2)}$$

and further

$$\|\nabla^2 \chi_h\|_{(-2)} \leqq ch\|\nabla^3(\sigma^2 E)\|_{(-2)} + c\|\nabla^2(\sigma^2 E)\|_{(-2)}.$$

Carrying out the differentiation in the above norms one obtains the estimate

$$|a_h(E, \sigma^2 E)| \leqq c\varkappa^{-1}\|\nabla^2 E\|_{(2)}^2 + c\|\nabla E\|_{2;h}^2 + c\|E\|_{(-2)}^2 + ch^2\|\nabla^3 g\|_{(2)}^2.$$

Hence, for $\varkappa = \varkappa_1$, sufficiently large,

$$\|\nabla^2 E\|_{(2)} \leqq c\{\|\nabla E\|_{2\cdot h} + \|E\|_{(-2)} + h\|\nabla^3 g\|_{(2)}\}. \tag{3.17}$$

To estimate the three terms on the right we note that the results (3.11) and (3.14), applied for $e = E$, yield

$$\|E\|_{1,2;h} + h\|\nabla^2 E\|_{2;h} \leqq ch^2\|g\|_{3,2}. \tag{3.18}$$

Further one easily verifies for the piecewise linear interpolant $E^L \in H_0^{1,2}(\Omega)$ of E that

$$\|E^L\|_{(-2)} \leqq c|\ln h|\,\|E^L\|_{1,2},$$

and consequently, using (3.5) and (3.18),

$$\|E\|_{(-2)} \leqq ch|\ln h|\,\|\nabla^2 E\|_{2;h} \leqq ch^2|\ln h|\,\|g\|_{3,2}. \tag{3.19}$$

Now we obtain from (3.16)-(3.19) :

$$\|\nabla^2 E\|_{1;h} \leqq ch^2|\ln h|^{3/2}\|g\|_{3,2} + ch|\ln h|^{1/2}\|\nabla^3 g\|_{(2)}. \tag{3.20}$$

Thus the proof of the crucial estimate (3.15) is completed by the following lemma:

LEMMA: *The regularized Green's functions g are bounded by*

$$h\|g\|_{3,2} + |\ln h|^{-1/2}\|\nabla^3 g\|_1 + \|\nabla^3 g\|_{(2)} \leqq c|\ln h|^{1/2},$$

where the constant c is independent of h and of the triangle $\hat{T} \in \Pi_h$.

Proof: The usual L^2- *a priori* estimate (3.2) immediately yields

$$\|g\|_{3,2} \leqq c\|\delta\|_2 \leqq ch^{-1}.$$

Applying the same estimate for the functions $(.-\hat{x})_i g \in H_0^{2,2}(\Omega)$, we obtain

$$\|\nabla^3 g\|_{(2)}^2 = \sum_{i=1}^{2} \|(.-\hat{x})_i \nabla^3 g\|_2^2 + \varkappa^2 h^2 \|\nabla^3 g\|_2^2 \leqq c(1+\|\nabla^2 g\|_2^2).$$

Further we have

$$\|\nabla^2 g\|_2^2 \leqq ca(g, g) = c \sum_{i=1}^{2} (\delta_i, \partial_i g)_h. \tag{3.21}$$

Now we define by

$$-\Delta G_i = \delta_i \quad \text{in } \Omega, \qquad G_i = 0 \quad \text{on } \partial\Omega,$$

regularized Green's functions of second order which satisfy by [13; th. B4]:

$$\|G_i\|_{1,2} \leqq c(1+|\ln h|)^{1/2}.$$

Using this one obtains from (3.21):

$$\|\nabla^2 g\|_2^2 = \sum_{i=1}^{2} (\nabla G_i, \nabla\partial_i g) \leqq c|\ln h|^{1/2} \|\nabla^2 g\|_2.$$

Q.E.D.

4. PROOF OF THEOREM 2

In the following we continue using some of the notation introduced in paragraph 3. We start with the proof of the L^2-error estimates assuming again that the right hand side f of problems (1.1) is a functional $f \in H^{-1,2}(\Omega)$.

(A) By combining the equations of problems ($\tilde{\mathrm{P}}$) and ($\tilde{\mathrm{P}}_h$) one obtains the orthogonality relations

$$\left.\begin{aligned} D(M-M_h, \Phi_h) = 0, \qquad \forall \varphi_h \in V_0^h, \\ D(\Psi_h, U-U_h) + A(\Psi_h, M-M_h) = 0, \qquad \forall \Psi_h \in W^h. \end{aligned}\right\} \tag{4.1}$$

For any continuous function v we shall use the notation $I_h v$ for some approximate of v in V^h or V_0^h, respectively, which satisfies the usual local estimate

$$\|v - I_h v\|_{k,p,T} \leqq ch^{r-k} \|\nabla^r v\|_{p,T}, \qquad 0 \leqq k \leqq r \leqq 3, \quad T \in \Pi_h, \tag{4.2}$$

where T' may be some appropriate neighborhood of the triangle T of size h. The field generated by embedding of $I_h v$ into W^h will also be denoted by $I_h v$.

With this notation one finds by the relations (4.1) that

$$\|M-M_h\|_2 \leqq c\{A(M-I_h M, M-M_h) \\ -D(I_h M-M, U-U_h)-D(M-M_h, U-I_h U)\},$$

and consequently, using (4.2) and the well known inverse relation

$$\|M-M_h\|_{1,2} \leqq ch^{-1}\|M-M_h\|_2+c\|M\|_{1,2},$$

that with arbitrary $\varepsilon \in (0, 1)$:

$$\|M-M_h\|_2 \leqq c\frac{h}{\varepsilon}\|M\|_{1,2}+\frac{\varepsilon}{h}\|u-u_h\|_{1,2}. \tag{4.3}$$

To estimate the error $u-u_h$ we again use a duality argument. Analogously to the procedure in paragraph 3(B) we define by

$$L(\varphi)=(\nabla(u-u_h), \nabla\varphi)$$

a bounded linear functional on $H_0^{1,2}(\Omega)$, so that the corresponding solution $v \in H_0^{2,2}(\Omega)$ of the auxiliary problem

$$a(v, \varphi)=L(\varphi), \qquad \forall \varphi \in H_0^{2,2}(\Omega),$$

is in $H^{3,2}(\Omega)$ and satisfies the *a* priori estimate

$$\|v\|_{3,2} \leqq c\|\nabla(u-u_h)\|_2.$$

Then, applying Green's formulas, one finds

$$L(\varphi)=-D(\nabla^2 v, \varphi), \qquad \forall \varphi \in H_0^{1,2}(\Omega). \tag{4.4}$$

Now, for technical reasons, we introduce a Ritz-projection $R_h : H^{1,2}(\Omega)^3 \to W^h$ corresponding to the coercive bilinear form $(D+A)(.,.)$ on $H^{1,2}(\Omega)^3$ by

$$(D+A)(W-R_h W, \Psi_h)=0, \qquad \forall \Psi_h \in W^h. \tag{4.5}$$

For this the well known error estimate holds

$$\|W-R_h W\|_{k,2} \leqq ch^{r-k}\|W\|_{r,2}, \qquad k=0, 1, \quad r=1, 2, 3. \tag{4.6}$$

Now, using (4.5), we find

$$L(u-u_h)=-D(\nabla^2 v, U-U_h)=(D+A)(R_h\nabla^2 v-\nabla^2 v, U-U_h) \\ -A(R_k\nabla^2 v-\nabla^2 v, U-U_h)-D(R_h\nabla^2 v, U-U_h) \\ =(D+A)(R_h\nabla^2 v-\nabla^2 v, U-I_h u) \\ -A(R_h\nabla^2 v-\nabla^2 v, U-U_h)-D(R_h\nabla^2 v, U-U_h)$$

and, observing that $A(\Psi, \nabla^2 v) = -D(\Psi, V)$, by (4.1) :

$$D(R_h \nabla^2 v, U - U_h) = A(\nabla^2 v - R_h \nabla^2 v, M - M_h) - A(M - M_h, \nabla^2 v)$$
$$= A(\nabla^2 v - R_h \nabla^2 v, M - M_h) + D(M - M_h, V - I_h V).$$

Then, by the estimates (4.2) and (4.6):

$$|L(u-u_h)| \leqq c \| v \|_{3,2} \{ h^2 \| u \|_{3,2} + h \| u - u_h \|_2 + h \| M - M_h \|_2 \}$$

and for sufficiently small $h > 0$:

$$\| u - u_h \|_{1,2} \leqq c \, | L(u-u_h) |^{1/2} \leqq ch^2 \| u \|_{3,2} + ch \| M - M_h \|_2. \tag{4.7}$$

Finally, combining (4.3) and (4.7), we obtain the desired L^2-error estimates if we choose ε sufficiently small

$$\| u - u_h \|_{1,2} + h \| M - M_h \|_2 \leqq ch^2 \| u \|_{3,2}. \tag{4.8}$$

(B) The L^∞-error estimates will be proved in a similar way as is done in paragraph 3 (B), (C) for the Morley-triangle. We start with a pointwise estimate for the moments

$$\| M - M_h \|_\infty \leqq ch^{-1} \| \nabla (u - u_h) \|_\infty + ch \| M \|_{1,\infty}. \tag{4.9}$$

To prove this we estimate, using again the notation introduced in paragraph 3(C), with some fixed $a \in \mathbb{R}$:

$$\| M - M_h \|_{\infty, T} \leqq c \varkappa^{a/2} h^{a/2-1} \| M - M_h \|_{(-a)} + ch \| M \|_{1,\infty}. \tag{4.10}$$

The weighted norm $\| M - M_h \|_{(-a)}$ will be handled following the line which led to (4.3). With the approximate $\Xi_h = I_h(\sigma^{-a} I_h(M - M_h)) \in W^h$ according to (4.2), we find using (4.1) that

$$\| M - M_h \|^2_{(-a)} \leqq c \{ A(\sigma^{-a}(M - I_h M), M - M_h)$$
$$+ A(\sigma^{-a} I_h(M - M_h) - \Xi_h, M - M_h)$$
$$- D(\Xi_h - \sigma^{-a} I_h(M - M_h), U - U_h) - D(\sigma^{-a}(M - M_h), U - U_h) \}$$
$$\leqq c \| M - M_h \|_{(-a)} \{ \| M - I_h M \|_{(-a)} + \| \sigma^{-a} I_h(M - M_h) - \Xi_h \|_{(a)} \}$$
$$+ c \| \nabla (u - u_h) \|_{(-a)} \{ \| \nabla (\Xi_h - \sigma^{-a} I_h(M - M_h)) \|_{(a)}$$
$$+ \| \nabla (\sigma^{-a}(M - M_h)) \|_{(a)} \}.$$

The weighted norm analogues [*see* § 3 (C)] of the estimate (4.2) and of the usual inverse relation for finite elements yield

$$\|\sigma^{-a} I_h(M-M_h)-\Xi_h\|_{(a)} \leqq ch^3 \|\nabla^3(\sigma^{-a} I_h(M-M_h)\|_{(a)}$$
$$\leqq c \sum_{j=0}^{2} h^{3-j}(\varkappa h)^{j-3} \|I_h(M-M_h)\|_{(-a)}$$
$$\leqq c\|M-I_h M\|_{(-a)}+\frac{c}{\varkappa}\|M-M_h\|_{(-a)}$$

and analogously,

$$\|\nabla(\sigma^{-a} I_h(M-M_h)-\Xi_h)\|_{(a+2)} \leqq c\|M-I_h M\|_{(-a)}+c\|M-M_h\|_{(-a)}.$$

Hence we obtain by a straightforward calculation

$$\|M-M_h\|_{(-a)} \leqq c\|M-I_h M\|_{(-a)}+\frac{c}{\varkappa}\|M-M_h\|_{(-a)}+c\|\nabla(u-u_h)\|_{(-a)}$$

and consequently for $\varkappa=\varkappa_1$, sufficiently large,

$$\|M-M_h\|_{(-a)} \leqq ch\|\nabla M\|_{(-a)}+\frac{c}{h}\|\nabla(u-u_h)\|_{(-a)}. \tag{4.11}$$

Now we choose $a=4$ and find

$$\|M-M_h\|_{(-4)} \leqq c\|M\|_{1,\infty}+ch^{-2}\|u-u_h\|_{1,\sigma},$$

which obviously proves *via* (4.10) the desired estimate (4.9).

(C) In order to estimate $\|\nabla(u-u_h)\|_\infty$ we state a technical lemma which will be proved below in section (E).

LEMMA: *For each $h>0$ there is a smooth function $g \in C_0^\infty(\Omega)$ with the properties*

$$\|\nabla^k g\|_{(2)}+h|\ln h|^{1/2}\|g\|_{k,2} \leqq ch^{3-k}|\ln h|^{1/2}, \qquad k=3,4,5, \tag{4.12}$$

such that

$$\|\nabla(u-u_h)\|_\infty \leqq ch^2\|u\|_{3,\infty}+c|(\nabla\Delta g, \nabla(u-u_h)))|, \tag{4.13}$$

where the constants c do not depend on h.

Using this notation we set $\Theta=\nabla^2 g \in H^{1,2}(\Omega)^3$ and find

$$-(\nabla\Delta g, \nabla(u-u_h))=D(\Theta, U-U_h). \tag{4.14}$$

Now let $(g_h, \Theta_h) \in V_0^h \times W^h$ be the (unique) solution of the corresponding discrete problem

$$\left.\begin{aligned} D(\Theta-\Theta_h, \Phi_h)=0, \qquad & \forall \Phi_h \in V_0^h, \\ D(\Psi_h, G-G_h)+A(\Psi_h, \Theta-\Theta_h)=0, \qquad & \forall \Psi_h \in W^h, \end{aligned}\right\} \tag{4.15}$$

where G and G_h again denote the fields obtained by embedding g and g_h, respectively, into $H^{1,2}(\Omega)^3$. Using (4.1) and the relation

$$A(\Theta, M-M_h) = -D(M-M_h, G)$$

we conclude that

$$D(\Theta, U-U_h) = D(\Theta-\Theta_h, U-U_h) + A(\Theta-\Theta_h, M-M_h) + D(M-M_h, G-G_h)$$

and further using (4.15):

$$\begin{aligned} D(\Theta, U-U_h) = & D(\Theta-\Theta_h, U-I_h U) + A(\Theta-\Theta_h, M-M_h) \\ & + D(M-I_h M, G-G_h) - A(I_h M-M_h, \Theta-\Theta_h) \\ = & D(\Theta-\Theta_h, U-I_h U) + D(M-I_h M, G-G_h) + A(\Theta-\Theta_h, M-I_h M). \end{aligned}$$

This leads us to

$$|D(\Theta, U-U_h)| \leqq ch|\ln h|^{1/2}\{\|\Theta-\Theta_h\|_{(2)} + \|\nabla(g-g_h)\|_2 + h^3\|\nabla^5 g\|_{(2)}\} B$$

where

$$B = \|u\|_{3,\infty} + \|u\|_{4,2},$$

and the well known inverse relation was used

$$\|\nabla(\Theta-\Theta_h)\|_{(2)} \leqq ch^{-1}\|\Theta-\Theta_h\|_{(2)} + ch^2\|\nabla^3\Theta\|_{(2)}. \tag{4.16}$$

The above lemma states that

$$\|\nabla^3\Theta\|_{(2)} \leqq \|\nabla^5 g\|_{(2)} \leqq ch^{-2}|\ln h|^{1/2},$$

and in the next section (D) we shall show that

$$\|\Theta-\Theta_h\|_{(2)} + \|\nabla(g-g_h)\|_2 \leqq ch|\ln h|. \tag{4.17}$$

Thus we obtain

$$|D(\Theta, U-U_h)| \leqq ch^2|\ln h|^{3/2} B$$

and *via* (4.14) and (4.13):

$$\|\nabla(u-u_h)\|_\infty \leqq ch^2|\ln h|^{3/2} B.$$

This, combined with (4.9), also yields

$$\|M-M_h\|_\infty \leqq ch|\ln h|^{3/2} B,$$

which proves the assertion of theorem 2.

(D) Now we shall prove the estimate (4.17). To this end we introduce the approximates

$$\Xi_h = I_h(\sigma^2(\Theta-\Theta_h)) \in W^h, \qquad \chi_h = I_h(\sigma^2(g-g_h)) \in V_0^h,$$

and find by a straightforward calculation, again using the relations (4.16) and (4.12),

$$\|\sigma^2(\Theta-\Theta_h)-\Xi_h\|_{(-2)} \leqq ch^3 \|\nabla^3(\sigma^2(\Theta-\Theta_h))\|_{(-2)} \leqq c\varkappa^{-1}\|\Theta-\Theta_h\|_{(2)} + ch|\ln h|^{1/2},$$

$$\|\nabla(\sigma^2(\Theta-\Theta_h)-\Xi_h)\|_2 \leqq ch^2 \|\nabla^3(\sigma^2(\Theta-\Theta_h))\|_{(0)} \leqq c\|\Theta-\Theta_h\|_{(2)} + ch|\ln h|^{1/2},$$

$$\|\nabla(\sigma^2(g-g_h)-\chi_h)\|_{(-2)} \leqq ch^2 \|\nabla^3(\sigma^2(g-g_h))\|_{(-2)} \leqq ch\|g-g_h\|_{(-2)} + ch^2|\ln h|^{1/2}.$$

The relations (4.15) lead us to

$$c\|\Theta-\Theta_h\|_{(2)}^2 \leqq A(\sigma^2(\Theta-\Theta_h)-\Xi_h,\ \Theta-\Theta_h) - D(\Xi_h-\sigma^2(\Theta-\Theta_h),\ G-G_h) - D(\sigma^2(\Theta-\Theta_h),\ G-G_h)$$

and

$$D(\sigma^2(\Theta-\Theta_h),\ G-G_h) = D(\Theta-\Theta_h,\ \sigma^2(G-G_h)-\chi_h) + A$$

where

$$A \leqq c\|\Theta-\Theta_h\|_{(2)}\{\|\nabla(g-g_h)\|_2 + \|g-g_h\|_{(-2)}\}.$$

Then by the above approximation estimates and again by (4.16) we obtain

$$\|\Theta-\Theta_h\|_{(2)} \leqq c\varkappa^{-1}\|\Theta-\Theta_h\|_{(2)} + c\|\nabla(g-g_h)\|_2 + c\|g-g_h\|_{(-2)} + ch|\ln h|^{1/2}$$

and choosing again $\varkappa=\varkappa_2$, sufficiently large,

$$\|\Theta-\Theta_h\|_{(2)} \leqq c\|\nabla(g-g_h)\|_2 + c\|g-g_h\|_{(-2)} + ch|\ln h|^{1/2}. \qquad (4.18)$$

Analogously as done in (3.19) of paragraph 3 we conclude that

$$\|g-g_h\|_{(-2)} \leqq c|\ln h|\,\|\nabla(g-g_h)\|_2 + ch^2|\ln h|.\ \|g\|_{3,2}.$$

Further the L^2-result (4.8) applied for $g-g_h$ states that

$$\|\nabla(g-g_h)\|_2 \leqq ch^2\|g\|_{3,2}.$$

Thus we find, using (4.12),

$$\|g-g_h\|_{(-2)} + \|\nabla(g-g_h)\|_2 \leqq ch|\ln h|. \qquad (4.19)$$

This combined with (4.18) proves the desired estimate (4.17).

(E) *Proof of the lemma:* We only give a sketch of the highly technical but standard argument. All constants c appearing below are independent of h. Let the error $|\nabla(u-u_h)|$ attain its maximum value at some triangle $T\in\Pi_h$ which, by assumption (T), contains a circle $B(x_0;\, c_0 h)$ with center x_0 and radius $c_0 h$. Then we certainly can find some point $x_1\in B(x_0;\, c_0 h/2)$ such that

$$\|\nabla(I_h u-u_h)\|_{\infty;\,T}\leqq c\,|\partial_i(I_h u-u_h)(x_1)|$$

and some smoothed δ-function such that

$$\delta\in C_0^\infty(T),\quad |\nabla^k\delta|\leqq ch^{-2-k},\qquad k=0,1,$$

and

$$\partial_i(I_h u-u_h)(x_1)=\int_\Omega \delta\,\partial_i(I_h u-u_h)\,dx. \tag{4.20}$$

Introducing the linear functional $L(\varphi)=(\underline{\delta},\,\nabla\varphi)$, $\underline{\delta}=(\delta\delta_{1i},\,\delta\delta_{2i})$, we conclude that

$$\|\nabla(u-u_h)\|_\infty\leqq ch^2\|u\|_{3,\infty}+c_1\,|L(u-u_h)|. \tag{4.21}$$

Now let $\tilde g\in H_0^{2,2}(\Omega)\cap H^{3,2}(\Omega)$ be the solution of the problem

$$a(\tilde g,\,\varphi)=L(\varphi),\qquad \forall\,\varphi\in H_0^{2,2}(\Omega), \tag{4.22}$$

which, by the lemma in paragraph 3 (C), satisfies the *a priori*-estimate

$$h\|\tilde g\|_{3,2}+\|\nabla^3\tilde g\|_{(2)}\leqq c\,|\ln h|^{1/2}. \tag{4.23}$$

Furthermore, by a standard smoothing process one obtains regularizations $g\in C_0^\infty(\Omega)$ of $\tilde g$ with the following properties

$$\left.\begin{aligned}\|\nabla^k g\|_2&\leqq ch^{3-k}\|\nabla^3\tilde g\|_2\leqq ch^{2-k},\qquad k=3,4,5,\\ \|\nabla^k g\|_{(2)}&\leqq ch^{3-k}\|\nabla^3\tilde g\|_{(2)}\leqq ch^{3-k}|\ln h|^{1/2},\end{aligned}\right\} \tag{4.24}$$

and

$$\|\underline{\delta}-\nabla\Delta g\|_1\leqq\frac{1}{2c_1},$$

where c_1 is the constant appearing in (4.21). The technical details of this construction are omitted. Then we conclude from the estimate (4.21) that

$$\|\nabla(u-u_h)\|_\infty\leqq ch^2\|u\|_{3,\infty}+c\,|(\nabla\Delta g,\,\nabla(u-u_h))|.$$

REFERENCES

1. F. BREZZI and P. A. RAVIART, *Mixed Finite Element Methods for 4th Order Elliptic Equations* In *Topics in Numerical Analysis*, Vol. III, J. J. H. MILLER, Ed., Academic Press, 1978.
2. P. G. CIARLET, *Conforming and Nonconforming Finite Element Methods for Solving the Plate Problem* In *Numerical Solution of Differential Equations*, G. A. WATSON, Ed., Springer, 1974.
3. P. G. CIARLET and P. A. RAVIART, *A mixed Finite Element Method for the Biharmonic Equation* In *Mathematical Aspects of Finite Elements in Partial Differential Equations*, C. DE BOOR, Ed., Academic Press, 1974.
4. P. LASCAUX and P. LESAINT, *Some Nonconforming Finite Elements for the Plate Bending Problem*, R.A.I.R.O., Anal. Numér., Vol. 1, 1975, pp. 9-53.
5. T. MIYOSHI, *A Finite Element Method for the Solution of 4-th Order Partial Differential Equations*, Kumamoto J. Sc. Math., Vol. 9, 1973, pp. 87-116.
6. J. A. NITSCHE, *Convergence of Nonconforming Methods* In *Numerical Solution of Differential Equations*, G. A. WATSON, Ed., Springer, 1974.
7. J. A. NITSCHE, *On Projection Methods for the Plate Problem* In *Numerical Analysis*, J. DESCLOUX and J. MARTI, Ed., Birkhäuser, 1977.
8. R. RANNACHER, *Punktweise Konvergenz der Methode der finiten Elemente beim Plattenproblem*, Manuscripta math., Vol. 19, 1976, pp. 401-416.
9. R. RANNACHER, *Finite Element Approximation of Simply Supported Plates and the Babuska Paradox*, Z. Angew. Math. Mech., Vol. 59, 1979, pp. T 73-T 76.
10. R RANNACHER, *Nonconforming Finite Element Methods for Eigenvalue Problems in Linear Plate Theory, Numer. Math.*, Vol. 32, 1979 (to appear).
11. R. SCHOLZ, *Approximation von Sattelpunkten mit finiten Elementen* In *Finite Elemente, Tagungsband, Bonn*, Math. Schr., Vol. 89, 1976, pp. 53-66.
12. R. SCHOLZ, *A Mixed Method for 4th Order Problems using Linear Finite Elements*, R.A.I.R.O., Anal. Numer., Vol. 12, 1978, pp. 85-90.
13. J. FREHSE and R, RANNACHER, *Eine L^1-Fehlerabschätzung für diskrete Grundlösungen in der Methode der finiten Elemente* In *Finite Elemente, Tagungsband, Bonn*, Math. Schr., Vol. 89, 1976, pp. 92-114.
14. J. A. NITSCHE, *L^∞-Convergence of Finite Element Approximation*. Second Conference on Finite Elements, Rennes, 1975.
15. R. SCOTT *Optimal L^∞-Estimates for the Finite Element Method on Irregular Meshes*, Math. Comp., Vol. 30, 1976, pp. 681-697.
16. R. B. KELLOGG and J. E. OSBORN, *A Regularity Result for the Stokes Problem in a Convex Polygon*, J. Funct. Anal., Vol. 21, 1976, pp. 397-431.