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SUPERCONVERGENCE OF THE GRADIENT OF FINITE ELEMENT SOLUTIONS (*)

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Abstract — *Superconvergence of the gradient of approximate solutions to second order elliptic equations is analysed and justified for a large class of curved isoparametric quadrilateral elements*

Résumé — *On analyse et on justifie la superconvergence du gradient des solutions approchées obtenues lors de la résolution d'équations elliptiques du second ordre à l'aide d'éléments isoparamétriques courbes de type quadrilatéral, de plusieurs types courants*

1. INTRODUCTION

Superconvergence of the gradient of finite element solutions was observed by engineers when curved isoparametric linear and quadratic elements of the Serendipity family were applied for stress computation at the so called Gaussian points (*see* references introduced in [10]). In [9] and [10] the second of the authors gave a justification of this phenomenon for some cases. [10] contains a complete analysis for quadratic elements of the Serendipity family. In this paper we construct a large class of curved isoparametric quadrilateral elements of an arbitrary degree n in each variable. We take a Dirichlet problem to a second order elliptic equation as a model problem and we prove superconvergence of the gradient at Gauss-Legendre points (called Gaussian points in the above references). A relatively highest improvement of the convergence rate is achieved when linear elements are used. The average convergence rate of the gradient is $O(h)$ whereas at Gauss-Legendre points (in case of linear elements these are centroids of the quadrilaterals) the rate is $O(h^2)$. The best numerical results were won when computation of the element stiffness matrices and of the right-hand sides was carried out by the Gauss-Legendre product 1×1 formula even if superconvergence is true for other formulas, too (*see* theorem 4.1).

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There is a limitation of our results We need that the finite element partitions be n-strongly regular, in particular that (2 6) be true A sufficient condition for (2 6) (even if not necessary, *see* remark 2 in [10]) is that the elements are close to parallelograms Numerical results indicate convincingly in case of linear elements the same what indicated numerical results won by quadratic elements (*see* [10], section 6) superconvergence does not set in if the condition (2 6) is not satisfied Nevertheless we think that superconvergence of the gradient has a considerable practical importance, especially when linear elements are used inside the given domain and quadratic elements are applied along the boundary if necessary The inner elements can often be chosen to be almost parallelograms whereas along the boundary a better approximation by quadratic elements guarantees the convergence rate $O(h^2)$ even if the elements are arbitrary convex quadrilaterals Computing the gradient at Gauss-Legendre points and interpolating it to internal nodes (in a similar way as proposed in [10], section 6) we can expect the convergence rate $O(h^2)$ at all nodes The same situation is expected in three dimensions

2. PRELIMINARIES

Let Ω be a bounded domain in R^2 with a sufficiently smooth boundary Γ We consider the Dirichlet problem

$$\left.\begin{aligned} &Au=f(x), \qquad \forall x\in\Omega, \quad u|_\Gamma=0, \\ &Au=-\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left[a_{ij}(x)\frac{\partial u}{\partial x_j}\right], \end{aligned}\right\} \tag{2 1}$$

here $x=(x_1, x_2)$ Let us remark at this point that we could add a term $a_0 u$ with $a_0 \geqq 0$ in the definition (2 1) of the operator Au All that follows applies equally well to this case, with a straightforward supplementary analysis To (2 1) there is associated the bilinear functional

$$a(u, v)=\int_\Omega \sum_{i,j=1}^{2} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx \tag{2 2}$$

We assume that the coefficients are Lipschitz continuous on $\bar{\Omega}$ and that

$$\left.\begin{aligned} &a_{ij}(x)=a_{ji}(x) \qquad \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \geqq \alpha_0 \sum_{i=1}^{2}\xi_i^2, \\ &\forall x\in\Omega, \quad \alpha_0=\text{const}>0 \end{aligned}\right\} \tag{2 3}$$

Hence $a(u, v)$ is $H_0^1(\Omega)$-elliptic

The weak solution of the problem (2.1) is a function $u \in H_0^1(\Omega)$ which satisfies

$$a(u, v) = (f, v)_{0\,\Omega}, \qquad \forall v \in H_0^1(\Omega). \tag{2.4}$$

We are using the usual notation for the Sobolev spaces

$$H^m(\Omega) = \{ u \in L^2(\Omega), D^\alpha u \in L^2(\Omega), \forall |\alpha| \leqq m \}, \qquad m = 0, 1, \ldots$$
$$H_0^1(\Omega) = \{ u \in H^1(\Omega), u|_\Gamma = 0 \}.$$

The norm in $H^m(\Omega)$ is denoted by $\|.\|_{m\,\Omega}$ and defined by

$$\|u\|_{m\,\Omega} = \{ \sum_{|\alpha| \leqq m} \|D^\alpha u\|^2_{L^2(\Omega)} \}^{1/2},$$

the inner product in $H^m(\Omega)$ is denoted by $(.,.)_{m\,\Omega}$. Often we shall use the seminorms

$$|u|_{m\,\Omega} = \{ \sum_{|\alpha| = m} \|D^\alpha n\|^2_{L^2(\Omega)} \}^{1/2},$$

$$[u]_{m\,\Omega} = \left\{ \left\| \frac{\partial^m u}{\partial x_1^m} \right\|^2_{0\,\Omega} + \left\| \frac{\partial^m u}{\partial x_2^m} \right\|^2_{0\,\Omega} \right\}^{1/2}.$$

To construct the finite element space V_h in which the approximate solution will lie let us cover Ω by curved isoparametric quadrilateral elements defined as follows: We consider the points $\{ s_k, s_l \}_{k,l=0}^n$ where $s_0 = -1$, $s_n = 1$ and $s_k (k = 1, \ldots, n-1)$ are zeros of $P_n'(s)$ (by P_n we denote the Legendre polynomial of degree n). The numbers $s_k (k = 0, \ldots, n)$ are points of Lobatto formulas (*see* [4]) and they belong to the interval $[-1, 1]$. We call the points $\{ s_k, s_l \}_{k,l=0}^n$ nodes of the square $\hat{K} : -1 \leqq \xi_i \leqq 1$, $i = 1, 2$. We also use the notation $\hat{a}_j$ for the nodes so that $\{ \hat{a}_j \}_{j=1}^{(n+1)^2}$ is the set of all nodes. We denote by $\hat{P}(n)$ the class of polynomials of degree $\leqq n$ in the variables ξ_1, ξ_2 and by $\hat{Q}(n)$ the class of polynomials of degree $\leqq n$ in each variable ξ_1 and ξ_2. Now any polynomial $\hat{v}$ from $\hat{Q}(n)$ is uniquely determined by its values v_j at $\hat{a}_j$. Let namely $\hat{v}(\hat{a}_j) = 0$, $j = 1, \ldots, (n+1)^2$. The function $\hat{v}(\xi_1, s_l)$ is a polynomial of degree not greater than n and it vanishes for $\xi_1 = s_k$, $k = 0, \ldots, n$. Therefore $\hat{v}(\xi_1, s_l) \equiv 0$. Similarly we find $\hat{v}(s_k, \xi_2) \equiv 0$. Therefore $\hat{v}(\xi_1, \xi_2)$ is divisible by $\prod_{i=0}^{n} (\xi_1 - s_i)(\xi_2 - s_i)$. This is a polynomial of degree $n+1$ in each variable, hence $\hat{v}$ must vanish identically which proves the unisolvability of the Lagrange interpolation problem $\hat{v}(\hat{a}_j) = v_j$, $j = 1, \ldots, (n+1)^2$. Let $N_j(\xi_1, \xi_2) \in \hat{Q}(n)$ be basic functions, i.e. $N_j(\hat{a}_i) = \delta_j^i$. Consider $(n+1)^2$ points $a_j = (x_1^j, x_2^j)$ in the x_1, x_2-plane lying in Ω or on Γ and the mapping $F_K : \hat{K} \to R^2$ defined by

$$x_i = x_i^u(\xi_1, \xi_2) \equiv \sum_{j=1}^{(n+1)^2} x_i^j N_j(\xi_1, \xi_2), \qquad i = 1, 2. \tag{2.5}$$

If (2.5) maps the square $\hat{K}$ one-to-one on a closed domain K lying in the x_1, x_2-plane we call K a curved quadrilateral element. The points a_j are nodes of this element.

We "cover Ω" by such elements and we suppose that every partition of Ω by these elements is a n-strongly regular partition. By a k-strongly regular partition we understand a partition with the following properties:

(a) for every element the mapping (2.5) is a C^{k+1}-diffeomorphism [in particular, (2.5) is invertible];

(b) to every element K there is associated a positive parameter h_K and the mapping (2.5) is such that on K:

$$|D^\alpha x_i^K(\xi_1, \xi_2)| \leqq c_1 h_K^{|\alpha|}, \qquad \forall |\alpha| \leqq k+1, \quad i=1, 2, \tag{2.6}$$

$$c_2^{-1} h_K^2 \leqq |\mathscr{J}_K(\xi_1, \xi_2)| \leqq c_2 h_K^2. \tag{2.7}$$

Here $\mathscr{J}_K(\xi_1, \xi_2)$ is the Jacobian of (2.6) and c_1, c_2 are positive constants independent on h_K as well as on the chosen partition (they depend on n which we do not denote). If h is defined by

$$h = \max_K h_K,$$

then the constants c_1, c_2 are independent of h, too.

k-strongly regular partitions were introduced in [10] and we refer the reader to remarks 1, 2, 3 in [10]. In particular, we may assume that for every element K:

$$\mathscr{J}_K(\xi_1, \xi_2) > 0, \qquad \forall \xi \in \hat{K}. \tag{2.8}$$

We will consider a family of n-strongly regular partitions of Ω such that $h \to 0$. We denote by Ω_h the interior of the union of all elements of the given partition (in general, $\Omega_h \neq \Omega$); Γ_h is its boundary.

The functions v from the finite element space V_h are defined piecewise

$$v(x_1, x_2) = \hat{v}[\xi_1^K(x_1, x_2), \xi_2^K(x_1, x_2)], \qquad \hat{v}(\xi_1, \xi_2) = \sum_{j=1}^{(n+1)^2} v_j N_j(\xi_1, \xi_2). \tag{2.9}$$

Here $\xi_i = \xi_i^K(x_1, x_2)$, $i=1, 2$, is the inverse mapping to (2.5) and the values v_j of v at nodes lying on Γ are equal zero, hence it is easy to see that $v|_{\Gamma_h} = 0$. Evidently,

$$V_h \subset C(\overline{\Omega}_h), \qquad V_h \subset H_0^1(\Omega_h).$$

Let us notice the special cases of V_h corresponding to $n=1, 2, 3$. If $n=1$ Ω_h consists of straight quadrilaterals. The nodes are vertices of these quadrilaterals and the functions $\hat{v}$ are bilinear polynomials. If $n=2$ the square $\hat{K}$ has 9 nodes. These are vertices, midpoints of sides and the center of $\hat{K}$. The polynomials $\hat{v}$ are biquadratic polynomials with 9° of freedom (in [10] we considered an element with 8° of freedom). If $n=3$ the element has 16° of freedom. The nodes are points $\{s_k, s_l\}_{k,l=0}^{3}$ with $s_1=-\sqrt{5}/5$, $s_2=\sqrt{5}/5$. The polynomials $\hat{v}$ are bicubic polynomials.

To define the approximate solution of the problem (2.4) we proceed in a similar way as in [3]. We extend the solution u and the coefficients a_{ij} according to Calderon's extension theorem (*see* Necas [7], p. 80) to R^2 and denote this extensions by $\tilde{u}$ and $\tilde{a}_{ij}$, respectively. We also extend f as follows:

$$\tilde{f}=-\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(\tilde{a}_{ij}\frac{\partial \tilde{u}}{\partial x_j}\right).$$

Denote by $\tilde{a}(w, v)$ the bilinear functional

$$\tilde{a}(w, v)=\int_{\Omega_h}\sum_{i,j=1}^{2}\tilde{a}_{ij}\frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_j}dx.$$

Due to $v|_{\Gamma_h}=0$ we get for any $v\in V_h$ by Green's theorem $\tilde{a}(\tilde{u}, v)=(\tilde{f}, v)_{0\,\Omega_h}$. For simplicity of writing we will leave out the sign $\sim$ and write

$$\left.\begin{aligned} &a(u, v)=(f, v)_{0,\Omega_h}, \qquad \forall v\in V_h,\\ &a(n, v)=\int_{\Omega_h}\sum_{i,j=1}^{2}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx.\end{aligned}\right\} \tag{2.10}$$

This will not cause any confusion in the estimates carried out later. All constants will depend on $\|\tilde{u}\|_{n+3\,\Omega_h}$. This norm is bounded, according to Calderon's theorem, by $\|u\|_{n+3\,\Omega}$. If the extensions of the coefficients are continuous the matrix $\{a_{ij}(x)\}_{i,j=1}^{2}$ is positive definite also in a greater domain $\Omega^0\supset\bar{\Omega}$. As $\Omega_h\subset\Omega^0$ for h sufficiently small it holds under these conditions

$$\sum_{i,j=1}^{2}a_{ij}(x)\xi_i\xi_j\geqq\alpha_1\sum_{i=1}^{2}\xi_i^2, \qquad \forall x\in\Omega_h, \tag{2.11}$$

where α_1 is a positive constant independent on h.

In general, numerical integration is the usual and only possible way how to compute the bilinear functional $a(u, v)$. To this end let us consider quadrature formulas $\hat{I}(\hat{\varphi})$ for the square $\hat{K}$ of the form

$$\hat{I}(\hat{\varphi})=\sum_r \hat{A}_r\hat{\varphi}(\hat{Q}_r). \tag{2.12}$$

We make the assumption that the points $\hat{Q}_r$ of the formula belong to the interior of $\hat{K}$ or are nodes of $\hat{K}$ and that the coefficients $\hat{A}_r$ are positive (the last assumption is not necessary but it yields simpler proofs) Any such formula induces a quadrature formula $\hat{I}_K(\varphi)$ for the element K of the form

$$I_K(\varphi)=\sum_r A_r\,\varphi(Q_r), \qquad A_r=\hat{A}_r\mathscr{J}_K(\hat{Q}_r), \qquad Q_r=F_K(\hat{Q}_r)$$

and

$$I_K(\varphi)=\hat{I}(\mathscr{J}_K\hat{\varphi}) \tag{2 13}$$

Here we use the following notation [in agreement with the notation in (2 9)] for any function g defined on $\overline{\Omega}_h$ $\hat{g}(\xi_1, \xi_2)=g[x_1^K(\xi_1, \xi_2)\ x_2^K(\xi_1, \xi_2)]$ on every K

Expressing $a(w, v)$ and $(f, v)_{0\ \Omega_h}$ as sums of integrals over the elements K we get the approximate values $a_h(w, v)$ and $(f, v)_h$ of $a(w, v)$ and $(f, v)_{0\ \Omega_h}$, respectively

$$\left.\begin{aligned} a_h(w, v)&=\sum_K I_K\left(\sum_{i\ j=1}^{2} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\right)=\sum_K \hat{I}\left(\mathscr{J}_K\sum_{i\ j=1}^{2}\hat{a}_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\right), \\ (f, v)_h&=\sum_K I_K(fv)=\sum_K \hat{I}(\mathscr{J}_K\hat{f}\hat{v}) \end{aligned}\right\} \tag{2 14}$$

Our assumption concerning the points $\hat{Q}_r$ guarantees that, at least for h sufficiently small, we do not need for the computation of $a_h(w, v)$ and $(f, v)_h$ values of data at other points than at points from $\overline{\Omega}$ Now the approximate solution $u_h\in V_h$ is defined formally by

$$a_h(u_h, v)=(f, v)_h, \qquad \forall v\in V_h \tag{2 15}$$

All quadrature formulas considered in the sequel are such that $a_h(v, v)$ is a positive definite quadratic form on V_h This implies existence and unicity of u_h

3. SOME LEMMAS

In what follows we denote by C a generic positive constant not necessarily the same in any two places which does not depend on h_K, h and on some functions (it depends on n) It will be clear from the context of which functions the constant is independent

LEMMA 3 1 *We have for any $\hat{v}\in\hat{Q}(n)$*

$$|\hat{v}|_{j\ \hat{K}}\leqq C|\hat{v}|_{i\ \hat{K}}, \qquad 0\leqq i<j \tag{3 1}$$

$$\max_{\hat{K}}|D^\alpha\hat{v}|\leqq C|\hat{v}|_{|\alpha|\ \hat{K}}, \qquad |\alpha|\geqq 0 \tag{3 2}$$

LEMMA 3 2 *We have for any* $g \in H^{i}(\Omega_h)$

$$|\hat{g}|_{i\,K} \leqq C h_K^{i-1} \|g\|_{i\,K}, \qquad 0 \leqq i \leqq n+1 \tag{3 3}$$

LEMMA 3 3 (special case of Bramble-Hilbert lemma on linear functionals, *see* [1]) *Let* $\mathscr{A}$ *be any subset of the set of multi-indices of length* $k+1$ *which contains the indices of the form* $(k+1,\,0)$, $(0,\,k+1)$ *The set of polynomials such that* $D^{\alpha}p=0$ *for all* $\alpha \in \mathscr{A}$ *will be denoted by* $P_{\mathscr{A}}$ *Let* f *be a continuous linear functional on* $H^{k+1}(\Omega)$ *satisfying* $f(p)=0$, $\forall p \in P_{\mathscr{A}}$ *Then there is a constant* $c=c(k,\,\Omega)$ *such that*

$$|f(v)| \leqq C \|f\|^{*}_{k+1\,\Omega} \sum_{\alpha \in \mathscr{A}} \|D^{\alpha} v\|_{0\,\Omega}, \qquad \forall v \in H^{k+1}(\Omega) \tag{3 4}$$

Two extreme cases of $\mathscr{A}$ *are the set of all multiindices of length* $k+1$ *and the set* $(k+1,\,0)$ $(0\;k+1)$ *Then* $P_{\mathscr{A}}=P(k)$ *and* $P_{\mathscr{A}}=Q(k)$, *respectively and* (3 4) *has the form*

$$\left.\begin{aligned} &|f(v)| \leqq c\,\|f\|^{*}_{k+1\,\Omega}\,|v|_{k+1\,\Omega} \qquad (3\;5)\\ &|f(v)| \leqq c\,\|f\|^{*}_{k+1\,\Omega}\,[v]_{k+1\,\Omega} \qquad (3\;6)\end{aligned}\right\}\ \forall v \in H^{k+1}(\Omega)$$

The Bramble-Hilbert lemma allows to estimate the interpolation error for a given function The interpolate $\hat{\varphi}_I$ of a function $\hat{\varphi}$ defined on $\hat{K}$ is the polynomial $\sum_{j=1}^{(n+1)^2} \hat{\varphi}_j N_j(\xi_1,\,\xi_2)$ where $\hat{\varphi}_j$ are values of $\hat{\varphi}$ at the nodes $\hat{a}_j$ of $\hat{K}$ The interpolate g_I of a function g defined on $\overline{\Omega}_h$ is the function which is on each element $K \subset \overline{\Omega}_h$ of the form (2 9) with $\hat{v}$ interpolating $\hat{g}$

LEMMA 3 4 *If* $\hat{\varphi} \in H^{n+1}(\hat{K})$ *then*

$$\|\hat{\varphi}-\hat{\varphi}_I\|_{j\,K} \leqq C[\hat{\varphi}]_{n+1\,K}, \qquad 0 \leqq j \leqq n+1, \tag{3 7}$$

Also

$$\left.\begin{aligned} &\|\hat{\varphi}-\hat{\varphi}_I\|_{W^1_{\infty}(\hat{K})} \leqq C[\hat{\varphi}]_{n+1\,K} \qquad \text{if} \quad n>1,\\ &\|\hat{\varphi}-\hat{\varphi}_I\|_{W^1_{\infty}(\hat{K})} \leqq C\{[\hat{\varphi}]_{2\,K}+[\hat{\varphi}]_{3\,K}\} \qquad \text{if} \quad n=1\end{aligned}\right\} \tag{3 7'}$$

The proofs of lemmas 3 1, 3 2 and 3 4 differ little from proofs of the correspond lemmas of [10] with one difference To prove the second part of (3 7′) one must use lemma 3 7 introduced later

We shall need estimates of the error functional $E(\hat{\varphi})=\int_{\hat{K}} \hat{\varphi}\,d\xi - \hat{I}(\hat{\varphi})$ Such estimates follow immediately from (3 6)

LEMMA 3 5 *Let* $\hat{I}(\hat{\varphi})$ *be a formula which integrates exactly all polynomials from* $\hat{Q}(k)$ *If* $\hat{\varphi} \in H^{k+1}(\hat{K})$ *then*

$$|E(\hat{\varphi})| \leqq C[\hat{\varphi}]_{k+1\ \hat{K}} \tag{3 8}$$

LEMMA 3 6 *Let the finite element partitions be 0-strongly regular and the formula* $I(\hat{\varphi})$ *be either the Lobatto product* $n+1 \times n+1$ *formula or a formula integrating exactly the class* $\hat{Q}(2n)$ *Then* $\{a_h(v, v)\}^{1/2}$ *is a norm on* V_h *equivalent uniformly with respect to h to the norm* $|v|_{1\ \Omega_h}$, i e *there is a constant* c_4 *independent of h such that*

$$c_4^{-1}|v|_{1\ \Omega_h}^2 \leqq a_h(v, v) \leqq c_4|v|_{1\ \Omega_h}^2, \qquad \forall v \in V_h, \tag{3 9}$$

for h suffiently small

Proof From positivity of the coefficients $\hat{A}_r$ (Lobatto formulas have positive coefficients) and from (2 11) we easily get (*see* part *b*) of the proof of lemma 3 6 in [10])

$$a_h(v, v) \geqq C \sum_K \hat{I}(\hat{\psi}), \qquad \hat{\psi} = \left(\frac{\partial \hat{v}}{\partial \xi_1}\right)^2 + \left(\frac{\partial \hat{v}}{\partial \xi_2}\right)^2 \tag{3 10}$$

(3 9) follows if we prove

$$\hat{I}(\hat{\psi}) \geqq C|\hat{v}|_{1\ K}^2 \tag{3 11}$$

If $\hat{I}$ integrates exactly the class $\hat{Q}(2n)$ then $\hat{I}(\hat{\psi}) = |\hat{v}|_{1\ \hat{K}}^2$ So let $\hat{I}$ be the Lobatto formula The term $|\hat{v}|_{1\ K}$ is a norm over the finite dimensional factor space $\hat{Q}(2n)/\hat{P}(0)$ When we show that from $\hat{I}(\hat{\psi}) = 0$ it follows $\hat{v} = \text{const}$, $\{\hat{I}(\hat{\psi})]^{1/2}$ is also such a norm and (3 9) is true

From $\hat{I}(\hat{\psi}) = 0$ it follows

$$\frac{\partial \hat{v}(\xi_1, s_l)}{\partial \xi_1} = 0 \qquad \text{for} \quad \xi_1 = s_k, \qquad k = 0, \qquad , n$$

As $\partial \hat{v}/\partial \xi_1$ is a polynomial of ξ_1 of degree not greater than $n-1$ it follows

$$\frac{\partial \hat{v}(\xi_1, s_l)}{\partial \xi_1} = 0 \qquad \text{for} \quad \xi_1 \in [-1, 1]$$

Further, $\partial \hat{v}(\xi_1, \xi_2)/\partial \xi_1$ is a polynomial of ξ_2 of degree not greater than n As it vanishes for $\xi_2 = s_l$, $l = 0, \quad , n$ it follows $\partial \hat{v}/\partial \xi_1 = 0$ on $\hat{K}$ Similarly, $\partial \hat{v}/\partial \xi_2 = 0$ on $\hat{K}$, thus $\hat{v} = \text{const}$

REMARK 1 The Gauss-Legendre product $n \times n$ formula is exact for all polynomials from $\hat{Q}(2n-1)$ as is the Lobatto product $n+1 \times n-1$ formula

Gauss-Legendre formula has less points, namely n^2, and this is why $\{a_h^*(v, v)\}^{1/2}$ where $a_h^*(v, v)$ is the approximate value of $a(v, v)$ computed by means of Gauss-Legendre $n \times n$ formula is not equivalent uniformly with respect to h to the norm $|v|_{1, \Omega_h}$. It was noticed by Girault [5] for $n=1$. Nevertheless, it is a norm on V_h, too. To prove it we remark first that (3.10) where Gauss-Legendre $n \times n$ formula $\hat{I}^*(\hat{\varphi})$ stands for $\hat{I}(\hat{\varphi})$ is again valid. Hence it is sufficient to prove that from $\sum_K \hat{I}^*(\hat{\psi})=0$ it follows $v=0$ on $\overline{\Omega}_h$. Denote by $t_k (k=1, \ldots, n)$ the zeros of P_n. The points of the formula $\hat{I}^*$ are $\{(t_k, t_l)\}_{k, l=1}^n$. As $\partial\hat{v}(\xi_1, t_l)/\partial\xi_1$ vanishes for $\xi_1=t_k$, $k=1, \ldots, n$ and it is a polynomial of degree not greater than $n-1$ it vanishes identically. As $\partial\hat{v}(\xi_1, \xi_2)/\partial\xi_1$ is a polynomial of the variable ξ_2 of degree not greater than n vanishing for $\xi_2=t_l (l=1, \ldots, n)$ it must be of the form $\alpha(\xi_1)\prod_{i=1}^n(\xi_2-t_i)$, hence $\hat{v}=\alpha^*(\xi_1)\prod_{i=1}^n(\xi_2-t_i)$. Similarly, $\hat{v}=\beta^*(\xi_2)\prod_{i=1}^n(\xi_1-t_i)$, thus

$$\frac{\alpha^*(\xi_1)}{\prod_{i=1}^n(\xi_1-t_i)}=\frac{\beta^*(\xi_2)}{\prod_{i=1}^n(\xi_2-t_i)}$$

which can be true only if these ratios are constant. It follows $\hat{v}=c\prod_{i=1}^n(\xi_1-t_i)(\xi_2-t_i)$, $c=\text{const}$. Take first a boundary element. $\hat{v}$ vanishes on a part of the boundary of $\hat{K}$, therefore $c=0$ and $\hat{v}=0$. We can repeat the reasoning for neighbors of boundary elements and prove successively that $v=0$ on $\overline{\Omega}_h$.

LEMMA 3.7. *Let f be a continuous linear functional on $H^{k+r+1}(\Omega)$ satisfying $f(p)=0$, $\forall p \in P(k)$ and $\forall p \in Q(k)$, respectively. Then there is a constant $c=c(k, r, \Omega)$ such that*

$$|f(v)| \leqq c\|f\|_{k+r+1, \Omega}^* \sum_{s=k+1}^{k+r+1}|v|_{s, \Omega} \tag{3.12}$$

and $\forall v \in H^{k+r+1}(\Omega)$,

$$|f(v)| \leqq c\|f\|_{k+r+1, \Omega}^* \sum_{s=k+1}^{k+r+1}[v]_{s, \Omega} \tag{3.13}$$

respectively.

The proof is given in [6] (lemma 3, p. 8), nevertheless we shall repeat it. We shall need the following.

Tartar's lemma Let E be a Banach space and E_1, E_2 be two normed spaces We consider two linear continuous operators $A_i \in \mathscr{L}(E, E_i)$, $i=1, 2$ such that

(i) $v \to \| A_1 v \|_{E_1} + \| A_2 v \|_{E_2}$ is a norm on E equivalent to $\| v \|_E$,

(ii) A_1 is compact

Let P be the kernel of the operator A_2 Then the mapping $v \to \| A_2 v \|_{E_2}$ is a norm on the quotient space E/P equivalent to the usual quotient norm $\inf_{p \in P} \| v+p \|_E$

Tartar's lemma (private communication) was not published A different proof of a slightly more general lemma can be found in Brezzi, Marini [2] (p 25, lemma 4 1)

Proof of lemma 3 7 We apply Tartar's lemma with $E = H^{k+r+1}(\Omega)$, $E_1 = H^k(\Omega)$, $E_2 = (L^2(\Omega))^N$ where N denotes the number of all derivatives of order s where $k+1 \leqq s \leqq k+r+1$ A_1 is the identity operator and the operator A_2 is defined as follows for each function $v \in H^{k+r+1}(\Omega)$ $A_2 v$ denotes the set of all derivatives of v of order s, $k+1 \leqq s \leqq k+r+1$ A_1 is a compact operator from $H^{k+r+1}(\Omega)$ into $H^k(\Omega)$ The kernel of A_2 is the set $P(k)$ The norm on E is equivalent to $\| A_1 v \|_{E_1} + \| A_2 v \|_{E_2}$ By Tartar's lemma the seminorm $\sum_{r=k+1}^{k+r+1} | v |_{s, \Omega}$ is a norm on the quotient space $H^{k+r+1}(\Omega)/P(k)$ equivalent to the usual quotient norm $\inf_{p \in P(k)} \| v+p \|_{k+r+1, \Omega}$

Now let $f \in (H^{k+r+1}(\Omega))^*$ be such that $f(p)=0$, $\forall p \in P(k)$ We have $f(v) = f(v+p)$, $\forall p \in P(k)$, hence

$$| f(v) | \leqq \| f \|^*_{k+r+1, \Omega} \inf_{p \in P(k)} \| v+p \|_{k+r+1, \Omega} \leqq c \| f \|^*_{k+r+1, \Omega} \sum_{s=k+1}^{k+r+1} | v |_{s, \Omega}$$

The proof of (3 13) is quite similar

4. LOBATTO AND MORE ACCURATE FORMULAS

We introduce the norm

$$| v |_h = \{ d_h^*(v, v) \}^{1/2}$$

$$\mathscr{I} = \left\{ \sum_K \hat{I}^* \left(\mathscr{I}_K \left[\left(\widehat{\frac{\partial v}{\partial x_1}} \right)^2 + \left(\widehat{\frac{\partial v}{\partial x_2}} \right)^2 \right] \right) \right\}^{1/2}$$

$$= \left\{ \sum_K \sum_{r=1}^{n^2} \hat{A}_r^* \mathscr{I}_K(\hat{Q}_r^*) \left[\left(\widehat{\frac{\partial v}{\partial x_1}}(\hat{Q}_r^*) \right)^2 + \left(\widehat{\frac{\partial v}{\partial x_2}}(\hat{Q}_r^*) \right)^2 \right] \right\}^{1/2} \quad (4\ 1)$$

where $d(v, v)$ is the quadratic functional associated to the Laplace operator

$$\left(d(v, v)=\int_{\Omega}\left[\left(\frac{\partial v}{\partial x_1}\right)^2+\left(\frac{\partial v}{\partial x_2}\right)^2\right]dx\right)$$

and $d_h^*(v, v)$ is its approximate value computed by means of Gauss-Legendre $n \times n$ formula $\hat{I}^*(\hat{\varphi})$ According to remark 1 $|v|_h$ is a norm on V_h Notice that the sum appearing on the right-hand side of (4 1) is a sum over Gauss-Legendre points, i e over all points which are maps of the points $\{(t_k, t_l)\}_{k\ l=1}^n$ We will denote the set of all Gauss-Legendre points by G

The space $\hat{Q}(n)$ contains the space $\hat{P}(n)$, however it does not contain $\hat{P}(n+1)$ Therefore (*see* Ciarlet and Raviart [3]) the best estimate of the error $u-u_h$ in the H^1-norm is

$$\|u-u_h\|_{1\ \Omega\cap\Omega_i} \leqq C h^n \tag{4 2}$$

We shall prove that $|u-u_h|_h \leqq C h^{n+1}$ and this is the reason that we speak about superconvergence In fact, let us denote by N_G the number of all Gauss-Legendre points and by $e(P)$ the error of the gradient

$$e(P)=\left[\left(\frac{\partial(n-u_h)(P)}{\partial x_1}\right)^2+\left(\frac{\partial(u-u_h)(P)}{\partial x_2}\right)^2\right]^{1/2}$$

We have

$$\operatorname{mes}\Omega_h=\sum_K\int_K dx=\sum_K\int_K \mathscr{I}_K\, d\xi \leqq C h^2 N_G,$$

therefore $N_G \geqq C \overline{h}^{-2}$ By the Cauchy inequality we prove easily under the additional assumption

$$\frac{\underline{h}}{\overline{h}} \geqq \vartheta, \qquad \vartheta=\text{const}>0, \qquad \underline{h}=\min_K h_K, \tag{4 3}$$

that

$$N_G^{-1}\sum_{P\in G} e(P) \leqq C\,|u-u_h|_h \tag{4 4}$$

Hence it follows that the arithmetic mean of errors of the gradient at Gauss-Legendre points is $O(h^{n+1})$

In this section we prove superconvergence in case that the evaluation of $a(w, v)$ and $(f, v)_{0\ \Omega_i}$ is done either exactly or there is used an integration formula $\hat{I}$ which integrates exactly the class $\hat{Q}(2n)$ or $\hat{I}$ is the Lobatto product $n+1 \times n+1$ formula [this integrates exactly $\hat{Q}(2n-1)$ but not $\hat{Q}(2n)$] We assume that the finite element partitions are n-strongly regular Numerical results indicate convincingly (*see* also [10], section 6) that superconvergence does not set in if the condition (2 6) is not satisfied Condition (2 6) with $k=n$ is just characteristic for n-strong regularity

We shall need one more property of the finite element partitions, namely that for any two adjacent elements K, K' we have

$$\left| \mathscr{J}_K^{-1} \frac{\partial x_i^K}{\partial \xi_1} \frac{\partial x_j^K}{\partial \xi_2} - \mathscr{J}_{K'}^{-1} \frac{\partial x_i^{K'}}{\partial \xi_1} \frac{\partial x_j^{K'}}{\partial \xi_2} \right| \leqq C h, \qquad i, j = 1, 2 \tag{4 5}$$

This condition is satisfied if e g the meshes consist of elements which differ little from parallelograms having sides nearly parallel to sides of its neighbors We refer the reader to remark 6 in [10]

THEOREM 4 1 *Let the finite element partitions be n-strongly regular and satisfy* (4 5) *Let the functional* $a(w, v)$ *and* $(f, v)_{0\,\Omega_h}$ *be evaluated either exactly or by means of a formula which integrates exactly the class* $\hat{Q}(2n)$ *or by means of the Lobatto product* $n+1 \times n+1$ *formula Finally, let the solution* u *belong to* $H^{n+3}(\Omega)$ *and, in case of numerical integration, let the coefficients* a_{ij} *belong to* $C^{n+2}(\overline{\Omega})$ *Then we have*

$$|u-u_h|_h \leqq C h^{n+1} \| u \|_{n+3\,\Omega} \tag{4 6}$$

Proof Subtracting $a_h(u_I, v)$ (u_I is the interpolate of u) from both sides of (2 15) we have

$$a_h(u_h-u_I, v) = (f, v)_h - a_h(u_I, v) = (f, v)_h - a_h(u, v) + a_h(u-u_I, v)$$

Hence

$$a_h(u_I-u_h, v) = a_h(u, v) - (A u, v)_h + a_h(u_I-u, v), \qquad \forall v \in V_h \tag{4 7}$$

(4 7) is true also in case of exact evaluation if instead of $a_h(u, v)$ and $(A u\ v)_h$ we set $a(u, v)$ and $(A u, v)_{0\,\Omega_h}$ Suppose that we prove

$$\left.\begin{aligned} |a_h(u, v) - (A u, v)_h| &\leqq C h^{n+1} \| u \|_{n+3\,\Omega} |v|_{1\,\Omega_h} \quad (4\ 8)\\ |a_h(u-u_I, v)| &\leqq C h^{n+1} \| u \|_{n+2\,\Omega} |v|_{1\,\Omega_h} \quad (4\ 9) \end{aligned}\right\} \forall v \in V_h$$

Putting $v = u_I - u_h \in V_h$ in (4 7) and using (3 9) we get

$$|u_I - u_h|_{1\,\Omega_h} \leqq C h^{n+1} \| u \|_{n+3\,\Omega} \tag{4 10}$$

The quadratic functional $d_h^*(v, v) = |v|_h^2$ satisfies also an inequality of the form (3 9), i e

$$c_4^{-1} |v|_{1\,\Omega_h}^2 \leqq d_h^*(v, v) \leqq c_4 |v|_{1\,\Omega_h}^2, \qquad \forall v \in V_h$$

Therefore by (4 10)

$$|u_I - u_h|_h \leqq C h^{n+1} \| u \|_{n+3\,\Omega} \tag{4 11}$$

It is sufficient to prove

$$|u-u_I|_h \leqq C h^{n+1} \|u\|_{n+2,\Omega} \tag{4.12}$$

and (4.6) follows by the triangle inequality.

Proof of (4.8): If $a(u, v)$ and $(A u, v)_{0\,\Omega_h}=(f, v)_{0,\Omega_h}$ are evaluated exactly we have nothing to prove. So let at this moment $\hat{I}$ denote any quadrature formula of the form (2.12) and let I_K be the induced formula (2.13). Using the symmetry $a_{ij}=a_{ji}$ we have

$$a_h(u, v)-(A u, v)_h=\sum_K I_K\left(\sum_{i,j=1}^{2}\left[a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}+\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right)v\right]\right). \tag{4.13}$$

We estimate only the sum of terms with $i=j=1$. The other three sums can be estimated in the same way. Setting $\sigma=a_K(\partial u/\partial x_1)$ we have to estimate $\sum_K I_K((\partial/\partial x_1)[\sigma v])$. Using the transformation (2.9) we get

$$\begin{aligned}\sum_K I_K\left(\frac{\partial}{\partial x_1}[\sigma v]\right) &=\sum_K\left\{\hat{I}\left(\frac{\partial}{\partial \xi_1}[\hat{\sigma}\hat{v}]\frac{\partial x_2^K}{\partial \xi_2}\right)-\hat{I}\left(\frac{\partial}{\partial \xi_2}[\hat{\sigma}\hat{v}]\frac{\partial x_2^K}{\partial \xi_1}\right)\right\}\\ &=\sum_K \hat{I}\left(\frac{\partial}{\partial \xi_1}\left[\frac{\partial x_2^K}{\partial \xi_2}\hat{\sigma}\hat{v}\right]\right)-\sum_K \hat{I}\left(\frac{\partial}{\partial \xi_2}\left[\frac{\partial x_2^K}{\partial \xi_1}\hat{\sigma}\hat{v}\right]\right).\end{aligned} \tag{4.14}$$

Again we restrict ourselves to estimation of the first sum which appears on the right-hand side of (4.14), i. e. :

$$S=\sum_K \hat{I}\left(\frac{\partial}{\partial \xi_1}\left[\frac{\partial x_2^K}{\partial \xi_2}\hat{\sigma}\hat{v}\right]\right). \tag{4.15}$$

First let $\hat{I}$ be a formula which integrates exactly the class $\hat{Q}(2n)$. Setting

$$\hat{z}^K=\frac{\partial x_2^K}{\partial \xi_2}\hat{\sigma}=\frac{\partial x_2^K}{\partial \xi_2}\hat{a}_{11}\frac{\partial \hat{u}}{\partial x_1}, \tag{4.16}$$

we have

$$S=\sum_K \hat{I}\left(\frac{\partial}{\partial \xi_1}[\hat{z}^K\hat{v}]\right)=\sum_K\left\{\hat{I}\left(\frac{\partial}{\partial \xi_1}[\hat{z}^K\hat{v}]\right)\right.\\ \left.-\int_{-1}^{1}[(\hat{z}^K\hat{v})(1, \xi_2)-(\hat{z}^K\hat{v})(-1, \xi_2)]\,d\xi_2\right\}.$$

We could subtract the sum $\sum_K\int_{-1}^{1}[(\hat{z}^K\hat{v})\,(1, \xi_2)-(\hat{z}^K\hat{v})\,(-1, \xi_2)]\,d\xi_2$ from S because it is equal to zero. In fact, in this sum they appear either integrals over element sides which lie on Γ_h and as $v|_{\Gamma_h}=0$ these are equal to zero Or they appear couples of integrals over a common side of two adjacent elements taken in opposite directions with integrands which are the same The functions σ as well as v assume namely the same values on such side (they are continuous on $\overline{\Omega}_h$), also x_2^K and hence $\partial x_2^K/\partial \xi_2$ and consequently the function z defined on each K by $z_K=z^K$ assume the same values on such side

Set

$$\hat{F}(\hat{z}, \hat{v})=\hat{I}\left(\frac{\partial}{\partial \xi_1}[\hat{z}\hat{v}]\right)-\int_{-1}^{1}[(\hat{z}\hat{v})(1, \xi_2)-(\hat{z}\hat{v})(-1, \xi_2)]\, d\xi_2, \tag{4 17}$$

so that

$$S=\sum_K \hat{F}(\hat{z}^K, \hat{v}) \tag{4 18}$$

LEMMA 4 1 *We have for* $\hat{z}\in H^{n+2}(\hat{K})$

$$|\hat{F}(\hat{z}, \hat{v})|\leqq C\{|\hat{v}|_{1\ K}[\hat{z}]_{n+1\ K}+\|\hat{v}\|_{0\ K}[\hat{z}]_{n+2\ K}\} \tag{4 19}$$

Proof We express $\hat{F}(\hat{z}, \hat{v})$ as follows

$$\hat{F}(\hat{z}, \hat{v})=\hat{F}(\hat{z}, \hat{v}-\hat{v}^0)+\hat{F}(\hat{z}, \hat{v}^0), \qquad \hat{v}^0=\hat{v}(0, 0) \tag{4 20}$$

Consider first $n>1$ $\hat{z}\to\hat{F}(\hat{z}, \hat{v}-\hat{v}^0)$ is a continuous linear functional on $H^{n+1}(\hat{K})$ bounded by

$$|\hat{F}(\hat{z}, \hat{v}-\hat{v}^0)|\leqq C\|\hat{v}-\hat{v}^0\|_{1\ K}\|\hat{z}\|_{n+1\ K}\leqq C|\hat{v}|_{1\ K}\|\hat{z}\|_{n+1\ K}$$

If $\hat{z}\in\hat{Q}(n)$ then $\hat{z}\hat{v}\in\hat{Q}(2n)$ and

$$\hat{F}(\hat{z}, \hat{v}-\hat{v}^0)=\int_{\hat{K}}\frac{\partial}{\partial \xi_1}[\hat{z}(\hat{v}-\hat{v}^0)]\, d\xi$$

$$-\int_{-1}^{1}[(\hat{z}(\hat{v}-\hat{v}^0))(1\ \ \xi_2)-(\hat{z}(\hat{v}-\hat{v}^0))(-1, \xi_2)]\, d\xi_2=0$$

According to (3 6)

$$|\hat{F}(\hat{z}, \hat{v}-\hat{v}_0)|\leqq C|\hat{v}|_{1\ K}[\hat{z}]_{n+1\ \hat{K}}, \qquad n>1 \tag{4 21}$$

Further, $\hat{z}\to\hat{F}(\hat{z}, \hat{v}^0)$ is a continuous linear functional on $H^{n+2}(\hat{K})$ bounded by $C\|\hat{v}\|_{0\ K}\|\hat{z}\|_{n+2\ K}$ if $n\geqq 1$ By the same argument it follows $\hat{F}(\hat{z}, \hat{v}^0)=0$, $\forall\, \hat{z}\in\hat{Q}(n+1)$ Therefore by (3 6)

$$|\hat{F}(\hat{z}, \hat{v}^0)|\leqq C\|\hat{v}\|_{0\ K}[\hat{z}]_{n+2\ K}, \qquad n\geqq 1 \tag{4 22}$$

For $n>1$ (4 19) follows from (4 20), (4 21) and (4 22) If $n=1$ then $\hat{F}(\hat{z}, \hat{v}-\hat{v}^0)$ is a continuous linear functional on $H^3(\hat{K})$ bounded by $C|\hat{v}|_{1\ K}\|\hat{z}\|_{3\ \hat{K}}$ and vanishing for $\hat{z} \in \hat{Q}(1)$. We use (3.13) and get

$$|\hat{F}(\hat{z}, \hat{v}-\hat{v}^0)| \leqq C |\hat{v}|_{1,\hat{K}} \{[\hat{z}]_{2,\hat{K}}+[\hat{z}]_{3,\hat{K}}\}. \tag{4.23}$$

For $n=1$ (4.19) follows from (4.20), (4.22), (4.23) and (3.1).

Now let the integration formula be the Lobatto product $n+1 \times n+1$ formula denoted by $\hat{I}^0$. As before we must estimate

$$S=\sum_K \hat{I}^0\left(\frac{\partial}{\partial \xi_1}[\hat{z}^K \hat{v}]\right)$$

$$=\sum_K \left\{\hat{I}^0\left(\frac{\partial}{\partial \xi_1}[\hat{z}^K \hat{v}]\right)-\hat{J}^0\left(\sum(\hat{z}^K \hat{v})(1, \xi_2)-(\hat{z}^K \hat{v})(-1, \xi_2)]\right\}.$$

Here $\hat{J}^0$ is the Lobatto $n+1$ formula over the interval $[-1, 1]$. We could subtract the sum $\sum_K \hat{J}^0([(\hat{z}^K \hat{v})(1, \xi_2)-(\hat{z}^K \hat{v})(-1, \xi_2)])$ because it is equal to zero from the same reason as above. Set

$$\hat{F}(\hat{z}, \hat{v})=\hat{I}^0\left(\frac{\partial}{\partial \xi_1}[\hat{z}\hat{v}]\right)-\hat{J}^0((\hat{z}\hat{v})(1, \xi_2)-(\hat{z}\hat{v})(-1, \xi_2)). \tag{4.24}$$

Again (4.18) holds.

LEMMA 4.2: (4.19) *is true also for* $\hat{F}$ *defined by* (4.24).

Proof: The arguments are the same or similar to those above. Let us only show that

$$\hat{F}(\hat{z}, \hat{v}-\hat{v}^0)=0, \qquad \forall \hat{z} \in \hat{Q}(n). \tag{4.25}$$

$\hat{J}^0$ is of the form $\hat{J}^0(\hat{\varphi})=\sum_{k=0}^{n} \hat{\mu}_k \hat{\varphi}(s_k)$. Then $\hat{I}^0(\hat{\varphi})=\sum_{k,l=0}^{n} \hat{\mu}_k \hat{\mu}_l \hat{\varphi}(s_k, s_l)$. If $\hat{z} \in \hat{Q}(n)$ the derivative $(\partial/\partial\xi_1)(\hat{z}\hat{v})$ is a polynomial of degree $\leqq 2n-1$ of the variable ξ_1. As $\hat{J}^0$ integrates exactly such polynomials we have

$$\hat{I}^0\left(\frac{\partial}{\partial \xi_1}[\hat{z}\hat{v}]\right)=\sum_{l=0}^{n} \hat{\mu}_l \sum_{k=0}^{n} \hat{\mu}_k\left(\frac{\partial}{\partial \xi_1}[\hat{z}\hat{v}]\right)(s_k, s_l)$$

$$=\sum_{l=0}^{n} \hat{\mu}_l \hat{J}^0\left(\left(\frac{\partial}{\partial \xi_1}[\hat{z}\hat{v}]\right)(\xi_1, s_l)\right)$$

$$=\sum_{l=0}^{n} \hat{\mu}_l \int_{-1}^{1}\left(\frac{\partial}{\partial \xi_1}[\hat{z}\hat{v}]\right)(\xi_1, s_l)\, d\xi_1$$

$$=\sum_{l=0}^{n} \hat{\mu}_l [(\hat{z}\hat{v})(1, s_l)-(\hat{z}\hat{v})(-1, s_l)] =\hat{J}^0([(\hat{z}\hat{v})(1, \xi_2)-(\hat{z}\hat{v})(-1, \xi_2)]),$$

which proves (4.25).

To finish the proof of (4.8) we return to (4.18) where $\hat{F}$ is defined either by (4.17) or by (4.24). In both cases (4.19) is true. Thus by (3.3):

$$|S| \leqq C \sum_K \{ |v|_{1\ K} [\hat{z}^K]_{n+1\ \hat{K}} + h_K^{-1} \|v\|_{0\ K} [\hat{z}^K]_{n+2\ \hat{K}} \}. \tag{4.26}$$

Denote

$$\hat{\alpha} = \frac{\partial x_2^K}{\partial \xi_2} \hat{a}_{11}, \qquad \hat{w} = \frac{\partial \hat{u}}{\partial x_1};$$

then $\hat{z}^K = \hat{\alpha}\, \hat{w}$. Using Leibnitz formula we obtain

$$[\hat{z}^K]_{n+1\ \hat{K}} \leqq C \sum_{\iota=0}^{n+1} \sum_{j=1}^{2} \left\| \frac{\partial^\iota \hat{\alpha}}{\partial \xi_j^\iota} \right\|_{L^\infty(\hat{K})} [\hat{w}]_{n+1-\iota\ \hat{K}}. \tag{4.27}$$

Using again Leibnitz formula we get

$$\left\| \frac{\partial^\iota \hat{\alpha}}{\partial \xi_j^\iota} \right\|_{L^\infty(\hat{K})} \leqq C \sum_{r=0}^{\iota} \left\| \frac{\partial^{r+1} x_2^K}{\partial \xi_2\, \partial \xi_j^r} \right\|_{L^\infty(\hat{K})} \left\| \frac{\partial^{\iota-r} \hat{a}_{11}}{\partial \xi_j^{\iota-r}} \right\|_{L^\infty(\hat{K})}$$

$$\leqq C \sum_{r=0}^{\iota} h_K^{r+1} h_K^{\iota-r} \|a_{11}\|_{W^{\iota,\infty}} \leqq C h_K^{\iota+1} \|a_{11}\|_{W^{\iota,\infty}} = O(h_K^{\iota+1}).$$

In the last inequality we used the fact that

$$\left| \frac{\partial^{r+1} x_2^K}{\partial \xi_2\, \partial \xi_j^r} \right| \leqq C h_K^{r+1},$$

if $r \leqq n$ and

$$D^\alpha x_\iota^K = 0 \qquad \text{if} \quad \alpha_1 \geqq n+1 \quad \text{or} \quad \alpha_2 \geqq n+1. \tag{4.28}$$

From (4.27) and (3.3) it follows

$$[\hat{z}^K]_{n+1\ \hat{K}} \leqq C h_K^{n+1} \|w\|_{n+1\ K} \leqq C h_K^{n+1} \|u\|_{n+3\ K}. \tag{4.29}$$

In the same way we prove

$$[\hat{z}^K]_{n+2\ \hat{K}} \leqq C h_K^{n+2} \|u\|_{n+3\ K}. \tag{4.30}$$

As a matter of fact, in addition to (3.3) we must use the estimate

$$\left\| \frac{\partial^{n+2} \hat{u}}{\partial \xi_1^{\alpha_1} \partial \xi_2^{\alpha_2}} \right\|_{0\ \hat{K}} \leqq C h_K^{n+1} \|u\|_{n+2\ K} \qquad \text{if} \quad \alpha_2 = 0,\ 1 \quad \text{or}\ \alpha_1 = 0,\ 1 \tag{4.31}$$

which follows from (3.3) and (4.28).

From (4.26), (4.29) and (4.30) we have

$$|S| \leqq C \sum_K h_K^{n+1} \|v\|_{1\ K} \|u\|_{n+3\ K} \leqq C h^{n+1} \|v\|_{1\ \Omega_h} \|u\|_{n+3,\ \Omega_h}.$$

u is, in fact, the extension $\hat{u}$, and by Calderon's theorem

$$\|\hat{u}\|_{n+3\ \Omega_k} \leqq C \|u\|_{n+3\ \Omega}.$$

Further $v|_{\Gamma_k}=0$, therefore from Friedrich's inequality (applied to a fixed domain $\Omega_0 > \overline{\Omega}_k$ so that the constant of the inequality does not depend on h) it follows $\|v\|_{1\ \Omega_k} \leqq C |v|_{1\ \Omega_k}$, hence

$$|S| \leqq C h^{n+1} \|u\|_{n+3\ \Omega} |v|_{1\ \Omega_k}$$

which proves (4.8).

Proof of (4.9): 1) Set $\omega = u - u_I$. Let $\hat{I}(\hat{\varphi})$ denote either the integration formula (i.e. a formula which integrates exactly $\hat{Q}(2n)$ or the Lobatto product $n+1 \times n+1$ formula) or let $\hat{I}(\hat{\varphi}) = \int_{\hat{K}} \hat{\varphi}\, d\xi$. Using (2.5) we get

$$a_k(\omega, v) = \sum_K I_K \left(\sum_{i,j=1}^{2} a_{ij} \frac{\partial \omega}{\partial x_i} \frac{\partial v}{\partial x_j} \right) = \sum_K \hat{I} \left(\sum_{i,j=1}^{2} b_{ij} \frac{\partial \hat{\omega}}{\partial \xi_i} \frac{\partial \hat{v}}{\partial \xi_j} \right) \tag{4.32}$$

$\left(a_h(\omega, v) = a(\omega, v) \text{ in case } \hat{I}(\hat{\varphi}) = \int_{\hat{K}} \hat{\varphi}\, d\xi \right)$. The coefficients b_{ij} are easy to calculate by means of the coefficients a_{ij} and the functions $x_i^K(\xi_1, \xi_2)$. The explicit formulas are given in [10], equation (3.15). Denote by b_{ij}^0 the values $b_{ij}(0, 0)$. Then

$$a_k(\omega, v) = \sum_{i,j=1}^{2} \sum_K b_{ij}^0 \hat{I} \left(\frac{\partial \hat{\omega}}{\partial \xi_i} \frac{\partial \hat{v}}{\partial \xi_j} \right) + \sum_{i,j=1}^{2} \sum_K \hat{I} \left([b_{ij} - b_{ij}^0] \frac{\partial \hat{\omega}}{\partial \xi_i} \frac{\partial \hat{v}}{\partial \xi_i} \right). \tag{4.33}$$

As a_{ij} are Lipschitz continuous and x_i^K satisfy (2.6) for $|\alpha| \leqq 2$ we easily estimate that $b_{ij} - b_{ij}^0 = O(h_K)$ on each element K. By (3.7') $\|\hat{\omega}\|_{W^{1\,\infty}(\hat{K})} \leqq C [\hat{u}]_{n+1\ \hat{K}}$ if $n > 1$. As $\hat{I}$ is of the form (2.12) or $\hat{I}(\hat{\varphi}) = \int_{\hat{K}} \hat{\varphi}\, d\xi$ we get

$$\begin{aligned} \left| \sum_{i,j=1}^{2} \sum_K \hat{I} \left([b_{ij} - b_{ij}^0] \frac{\partial \hat{\omega}}{\partial \xi_i} \frac{\partial \hat{v}}{\partial \xi_j} \right) \right| &\leqq C \sum_K h_K [\hat{u}]_{n+1\ \Omega} |\hat{v}|_{1, \hat{K}} \\ &\leqq C \sum_K h_K^{n+1} \|u\|_{n+1\ K} |v|_{1\ K} \leqq C h^{n+1} \|u\|_{n+1\ \Omega} |v|_{1, \Omega_k}. \end{aligned} \tag{4.34}$$

For $n=1$ we get

$$\left| \sum_{i,j=1}^{2} \sum_K \hat{I} ([b_{ij} - b_{ij}^0]) \frac{\partial \hat{\omega}}{\partial \xi_i} \frac{\partial \hat{v}}{\partial \xi_j} \right| \leqq C h^2 \|u\|_{3\ \Omega} |v|_{1\ \Omega_k}. \tag{4.34'}$$

2) It remains to estimate the first sum in (4.33). We must investigate separately the case $i=j$ and $i \neq j$. Consider the first case and take $i=j=1$. The

functional $f(\hat{u}) = I\,((\partial\hat{\omega}/\partial\xi_1)\,(\partial\hat{v}/\partial\xi_1))$ is linear and bounded on $H^{n+2}(\hat{K})$ by $C|\hat{v}|_{1\ \hat{K}}\|\hat{u}\|_{n+2\ \hat{K}}$ [it follows from (3 7') and (3 2)] It vanishes for $\hat{u} \in \hat{Q}(n)$ because $\hat{\omega} \equiv 0$ It also vanishes for $\hat{u} = \xi_2^{n+1}$ because $\partial\hat{\omega}/\partial\xi_1 \equiv 0$ If $\hat{u} = \xi_1^{n+1}$ then by inspection we find

$$\hat{u}_I = \xi_1^{n+1} - \frac{1}{nc_n}(\xi_1^2 - 1)\,P_n'(\xi_1)$$

where c_n is the coefficient at ξ_1^n of the Legendre polynomial $P_n(\xi_1)$ Hence

$$\frac{\partial\hat{\omega}}{\partial\xi_1} = \frac{1}{nc_n}\,\frac{d}{d\xi_1}\,[(\xi_1^2 - 1)\,P_n'(\xi_1)] = \frac{n+1}{c_n}P_n(\xi_1) \tag{4 35}$$

Evidently, $(\partial\hat{\omega}/\partial\xi_1)(\partial\hat{v}/\partial\xi_1) \in \hat{Q}(2n-1)$, therefore

$$f(\hat{u}) = \int_{\hat{K}} \frac{\partial\hat{\omega}}{\partial\xi_1}\,\frac{\partial\hat{v}}{\partial\xi_1}\,d\xi$$

As $\partial\hat{v}/\partial\xi_1$ is a polynomial of degree $\leqq n-1$ in ξ_1 and integration with respect to ξ_1 is done over the interval $[-1, 1]$, $f(\hat{u})$ vanishes, too So $f(\hat{u})$ vanishes for

$$\hat{u} \in \hat{Q}(n) + \{\xi_1^{n+1}, \xi_2^{n+1}\} = \hat{Q}(n+1) - \{p = \xi_1^{n+1}\xi_2^{\alpha_2},\ 1 \leqq \alpha_2 \leqq n+1\}$$
$$- \{p = \xi_1^{\alpha_1}\xi_2^{n+1},\ 1 \leqq \alpha_1 \leqq n+1\}$$

By (3 4)

$$|f(\hat{u})| \leqq C\left\{\left\|\frac{\partial^{n+2}\hat{u}}{\partial\xi_1^{n+2}}\right\|_{0\ K} + \left\|\frac{\partial^{n+2}\hat{u}}{\partial\xi_1^{n+1}\,\partial\xi_2}\right\|_{0\ \hat{K}} + \left\|\frac{\partial^{n+2}\hat{u}}{\partial\xi_1\,\partial\xi_2^{n+1}}\right\|_{0\ K} + \left\|\frac{\partial^{n+2}\hat{u}}{\partial\xi_2^{n+2}}\right\|_{0\ \hat{K}}\right\}|\hat{v}|_{1\ \hat{K}} \tag{4 36}$$

and by (4 31), (3 3)

$$|f(\hat{u})| \leqq C\,h_K^{n+1}\|u\|_{n+2\ K}|v|_{1\ K} \tag{4 37}$$

The same bound is true for $i = j = 2$ Hence

$$\left|\sum_{i=1}^{2}\sum_K b_{ii}^0\,\hat{I}\left(\frac{\partial\hat{\omega}}{\partial\xi_i}\,\frac{\partial\hat{v}}{\partial\xi_i}\right)\right|$$
$$\leqq C\sum_K h_K^{n+1}\|u\|_{n+2\ K}|v|_{1\ K} \leqq C\,h^{n+1}\|u\|_{n+2\ \Omega}|v|_{1\ \Omega_h} \tag{4 38}$$

3) Consider the case $i = 1$, $j = 2$ and first let $\hat{I}$ be the formula which integrates exactly $\hat{Q}(2n)$ or let $\hat{I}(\hat{\varphi}) = \int_{\hat{K}} \hat{\varphi}\,d\xi$ Denote by $L(\hat{u})$ the func-

tional $\hat{I}((\partial\hat{\omega}/\partial\xi_1)(\partial\hat{v}/\partial\xi_2))$ We have to estimate $S=\sum_K b_{12}^0 L(\hat{u})$ Express S as follows

$$\left.\begin{aligned} S&=\sum_K b_{12}^0\{L(\hat{u})-H(\hat{u})\}+\sum_K b_{12}^0 H(\hat{u}),\\ H(\hat{u})&=\int_{-1}^{1}\frac{\partial\hat{\omega}(\xi_1,1)}{\partial\xi_1}[\hat{v}(\xi_1,1)-\hat{v}(0,1)]\,d\xi_1\\ &\quad-\int_{-1}^{1}\frac{\partial\hat{\omega}(\xi_1-1)}{\partial\xi_1}[\hat{v}(\xi_1-1)-\hat{v}(0,-1)]\,d\xi_1 \end{aligned}\right\}\qquad(4\ 39)$$

We begin with estimation of $\sum_K b_{12}^0 H(\hat{u})$ In this sum they appear either integrals over element sides which lie on Γ_k and these integrals vanish Or they appear couples of integrals over a common side of two adjacent elements taken in opposite directions with integrands which are the same The factors b_{12}^0 need not be the same, however their difference is $O(h)$ on basis of (4 5) (*see* remark 6 in [10]) Therefore using the inequality

$$\int_{-1}^{1}\hat{\varphi}^2\,d\xi_1\leqq C\|\hat{\varphi}\|_{1\ \hat{K}}^2,\qquad\forall\hat{\varphi}\in H^1(\hat{K})$$

we easily get by (3 2), (3 7′), (3 3)

$$\begin{aligned}\Big|\sum_K b_{12}^0 H(\hat{u})\Big|&\leqq C\,h\sum_K\left\|\frac{\partial\hat{\omega}}{\partial\xi_1}[\hat{v}-\hat{v}(0,\pm1)]\right\|_{1\ \hat{K}}\\ &\leqq C\,h\sum_K\|\hat{\omega}\|_{2\ \hat{K}}|\hat{v}|_{1\ \hat{K}}\leqq C\,h\sum_K[\hat{u}]_{n+1\ \hat{K}}|\hat{v}|_{1\ \hat{K}}\\ &\leqq C\,h\sum_K h_K^n\|u\|_{n+1\ K}|v|_{1\ K}\leqq C\,h^{n+1}\|u\|_{n+1\ \Omega}|v|_{1\ \Omega_k}\end{aligned}\qquad(4\ 40)$$

To estimate the sum $\sum_K b_{12}^0\{L(\hat{u})-H(\hat{u})\}$ consider the functional $f(\hat{u})=L(\hat{u})-H(\hat{u})$ It is a continuous linear functional on $H^{n+2}(\hat{K})$ bounded by $C|\hat{v}|_{1\ K}\|\hat{u}\|_{n+2\ \hat{K}}$ Evidently, it vanishes for $\hat{u}\in\hat{Q}(n)$ and $\hat{u}=\xi_2^{n+1}$ If $\hat{u}=\xi_1^{n+1}$ then [*see* (4 35)]

$$\frac{\partial\hat{\omega}}{\partial\xi_1}=\frac{n+1}{c_n}P_n(\xi_1)$$

and

$$f(\hat{u})=\frac{n+1}{c_n}\int_{-1}^{1}P_u(\xi_1)\int_{-1}^{1}\frac{\partial\hat{v}}{\partial\xi_2}\,d\xi_2\,d\xi_1$$
$$-\frac{n+1}{c_n}\int_{-1}^{1}P_u(\xi_1)[\hat{v}(\xi_1,1)-\hat{v}(\xi_1,-1)]\,d\xi_1=0$$

Exactly as before we prove (4 37) Consequently

$$|S| \leqq C h^{n+1} \|u\|_{n+2\ \Omega} |v|_{1\ \Omega_h} \tag{4 41}$$

Let the integration formula be the Lobatto product $n+1 \times n+1$ formula denoted before by $\hat{I}^0$ Now we choose

$$L(\hat{u}) = \hat{I}^0 \left(\frac{\partial \hat{\omega}}{\partial \xi_1} \frac{\partial \hat{v}}{\partial \xi_2} \right),$$

$$H(\hat{u}) = \hat{J}^0 \left(\frac{\partial \hat{\omega}(\xi_1, 1)}{\partial \xi_1} [\hat{v}(\xi_1, 1) - \hat{v}(0, 1)] \right)$$

$$- \hat{J}^0 \left(\frac{\partial \hat{\omega}(\xi_1, -1)}{\partial \xi_1} [\hat{v}(\xi_1 -1) - \hat{v}(0, -1)] \right)$$

and we make use of the argument which we used to prove (4 25) Proceeding as before we get again the estimate (4 41), (4 33), (4 34), (4 34'), (4 38) and (4 41) imply (4 9)

Proof of (4 12) We set again $\omega = u - u_I$ We first estimate $f(\hat{u}) = \partial \hat{\omega} (\hat{Q}_r^*)/\partial \xi_0'$ Let, say, $j = 1$ $f(\hat{u})$ is a continuous linear functional on $H^{n+2}(\hat{K})$ bounded by $C \|\hat{u}\|_{n+2\ \hat{K}}$ It vanishes for $\hat{u} \in \hat{Q}(n)$ and for $\hat{u} = \xi_2^{n+1}$ By (4 35) it also vanishes for $\hat{u} = \xi_1^{n+1}$ because the coordinates of $\hat{Q}_r^*$ are zeros of P_n As before [*see* the estimation of $f(\hat{u}) = \hat{I}((\partial \hat{\omega}/\partial \xi_1)(\partial \hat{v}/\partial \xi_1))$] it follows

$$|f(\hat{u})| \leqq C h_K^{n+1} \|u\|_{n+2\ K} \tag{4 42}$$

From (4 1) we easily find out using (2 6) and (2 7) that

$$|\omega|_h \leqq C \{ \sum_K h_K^{2(n+1)} \|u\|_{n+2, K}^2 \}^{1/2} \leqq C h^{n+1} \|u\|_{n+2\ \Omega}$$

5. GAUSS-LEGENDRE INTEGRATION

In this section we consider the case that the evaluation of $a(w, v)$ and $(f, v)_{0\ \Omega_h}$ is done by Gauss-Legendre product $n \times n$ formula which has the smallest number of points among formulas integrating exactly the class $\hat{Q}(2n-1)$ The functional $a_h^*(v, v)$ is not bounded from below by $C|v|_{1\ \Omega_h}^2$ uniformly with respect to h (*see* remark 1, section 3), nevertheless we prove that superconvergence of the gradient at Gauss-Legendre points sets in, too In fact, numerical experiments show that we can expect results better than those won by Lobatto or by more accurate formulas

Concerning the finite element partitions we do not need condition (4 5) We needed this assumption to prove (4 9), but we did not need it to prove (4 12) and therefore we shall not need it to prove (5.9). However, we assume that the partitions are topologically equivalent to rectangular meshes in the following sense: If a_k is a corner node (i.e. a node which is map of a corner of K) we call neighbors of this node all corner nodes a_l such that a_k and a_l are endpoints of an element side. A finite element partition will be called topologically equivalent to a rectangular partition if its corner nodes can be numbered by two indices $i, j\,(i=j=0, 1, \ldots)$ in such a way that all neighbors of a corner node a_{ij} belong to the set $\{a_{i+1j}, a_{i-1j}, a_{ij+1}, a_{ij-1}\}$. Let the numbering be such that for a given j we have $0 \leqq m_j \leqq i \leqq M_j$ and for a given i we have $0 \leqq n_i \leqq j \leqq N_i$. Let

$$M = \max_j M_j, \qquad N = \max_i N_i, \qquad \Delta x = M^{-1}, \qquad \Delta y = N^{-1}.$$

In the sequel we assume that all finite element partitions, besides being topologically equivalent to rectangular partitions, are such that

$$h^2 \leqq c_5\, \Delta x\, \Delta y, \qquad \frac{\min(\Delta x, \Delta y)}{\max(\Delta x, \Delta y)} \geqq c_5 > 0, \tag{5.1}$$

where the constant c_5 does not depend on h.

THEOREM 5.1: *Let the finite element partitions be n-strongly regular, topologically equivalent to rectangular partitions and satisfy the condition* (5.1). *Let $u \in H^{n+3}(\Omega)$, $a_{ij} \in C^{n+2}(\overline{\Omega})$. Finally, let the evaluation of $a(w, v)$ and $(f, v)_{0\,\Omega_h}$ be carried out by means of Gauss-Legendre product $n \times n$ formula. Then*

$$|u - u_h|_h \leqq C h^{n+1} \|u\|_{n+3,\,\Omega}. \tag{5.2}$$

Proof: (4.7) is true if instead of a_h and $(f, v)_h$ we set a_h^* and $(f, v)_h^*$, respectively. Hence

$$a_h^*(u_I - u_h, v) = a_h^*(u, v) - (Au, v)_h^* + a_h^*(u_I - u, v), \qquad \forall v \in V_h. \tag{5.3}$$

We prove later that

$$|a_h^*(u, v) - (Au, v)_h^*| \leqq C h^{n+1} \|u\|_{n+3\,\Omega} |v|_h, \qquad \forall v \in V_h. \tag{5.4}$$

From positivity of the coefficients of Gauss-Legendre formulas, from ellipticity of the operator Au and from boundedness of its coefficients it follows

$$C^* |z|_h^2 \leqq a_h^*(z, z) \leqq C |z|_h^2 \tag{5.5}$$

for any function z such that $\partial z / \partial x_i$, $i = 1, 2$, exist at all Gauss-Legendre points. Therefore by (4.12) and (5.5):

$$|a_h^*(u - u_I, v)| \leqq C h^{n+1} \|u\|_{n+2\,\Omega} |v|_h, \qquad \forall v \in V_h. \tag{5.6}$$

Setting $v=u_I-u_h\in V_h$ in (5.3) we get by (5.4), (5.6) and by (5.5):

$$|u_I-u_h|_h\leqq Ch^{n+1}\|u\|_{n+3\ \Omega}.$$

(5.2) follows by the triangle inequality.

Proof of (5.4): Proceeding as in the proof of (4.8) we find out that we have to estimate certain sums a prototype of which is

$$S=\sum_K \hat{I}^*\left(\frac{\partial}{\partial\xi_1}[\hat{z}^K\hat{v}]\right),\qquad \hat{z}^K=\frac{\partial x_2^K}{\partial\xi_2}\hat{a}_{11}\widehat{\frac{\partial u}{\partial x_1}}.$$

Denote by $\hat{v}_k$ ($k=1,\ \ldots,\ n$) the coefficients of the one-dimensional Gauss-Legendre formula. From the same reason as above

$$\sum_K\sum_{l=1}^n \hat{v}_l[(\hat{z}\hat{v})(1,\ t_l)-(\hat{z}\hat{v})(-1,\ t_l)]=0.$$

Therefore the sum S can be written in the form

$$S=\sum_K \hat{F}(\hat{z}^K,\ \hat{v}),$$

where

$$\hat{F}(\hat{z},\ \hat{v})=\sum_{l=1}^n \hat{v}_l\sum_{k=1}^n \hat{v}_k\frac{\partial}{\partial\xi_1}(\hat{z}\hat{v})(t_h,\ t_l)-\sum_{l=1}^n \hat{v}_l[(\hat{z}\hat{v})(1,\ t_l)-(\hat{z}\hat{v})(-1,\ t_l)].$$

LEMMA 5.1: *We have for* $\hat{z}\in H^{n+2}(\hat{K})$.

$$|\hat{F}(\hat{z},\ \hat{v})|\leqq C\left\{\left[\hat{I}^*\left(\mathscr{J}_K\left[\left(\widehat{\frac{\partial v}{\partial x_1}}\right)^2+\left(\widehat{\frac{\partial v}{\partial x_2}}\right)^2\right]\right)\right]^{1/2}[\hat{z}]_{n+1\ \hat{K}}+[\hat{I}^*(\hat{v}^2)]^{1/2}[\hat{z}]_{n+2\ \hat{K}}\right\}.\quad(5.7)$$

Proof: Let $\hat{\pi}_{n-1}\hat{v}$ be the interpolate of $\hat{v}$ in $\hat{Q}(n-1)$ determined uniquely by values of $\hat{v}$ at the points $(t_k,\ t_l)$, $k,\ l=1,\ \ldots,\ n$. We write

$$\hat{F}(\hat{z},\ \hat{v})=\hat{F}(\hat{z},\ \hat{v}-\hat{\pi}_{n-1}\hat{v})+\hat{F}(\hat{z},\ \hat{\pi}_{n-1}\hat{v})$$

and estimate $\hat{f}(\hat{z})=\hat{F}(\hat{z},\hat{w})$, $\hat{w}=\hat{v}-\hat{\pi}_{n-1}\hat{v}$. We consider $f(\hat{z})$ as a linear functional on $H^{n+1}(\hat{K})$. It is a bounded functional because we easily get

$$|\hat{f}(\hat{z})|\leqq C\left\{\sum_{l=1}^n \hat{v}_l\left[\sum_{k=1}^n \hat{v}_k\left(\frac{\partial\hat{w}(t_k,\ t_l)}{\partial\xi_1}\right)^2+\hat{w}^2(1,\ t_l)+\hat{w}^2(-1,\ t_l)\right]\right\}^{1/2}\|\hat{z}\|_{n+1\ \hat{K}}.$$

Now

$$\sum_{k=1}^n \hat{v}_k\left(\frac{\partial\hat{w}(t_h,\ t_l)}{\partial\xi_1}\right)^2+\hat{w}^2(1,\ t_l)+\hat{w}^2(-1,\ t_l)\leqq C\sum_{k=1}^n \hat{v}_k\left(\frac{\partial\hat{v}(t_h,\ t_l)}{\partial\xi_1}\right)^2.$$

This is true because if the right-hand side vanishes then $\hat{v}(\xi_1, t_l) = \text{const}$ and the left-hand side also vanishes Hence

$$|f(\hat{z})| \leqq C\left\{\sum_{l=1}^{n} \hat{v}_l \sum_{h=1}^{n} \hat{v}_k \left(\frac{\partial \hat{v}(t_k, t_l)}{\partial \xi_1}\right)^2\right\}^{1/2} \|\hat{z}\|_{n+1\ \hat{K}}$$

$$= C\left\{\hat{I}^*\left(\left[\frac{\partial \hat{v}}{\partial \xi_1}\right]^2\right)\right\}^{1/2} \|\hat{z}\|_{n+1\ \hat{K}}$$

$$\leqq C\left\{\hat{I}^*\left(\mathscr{I}_K\left[\left(\widehat{\frac{\partial v}{\partial x_1}}\right)^2 + \left(\widehat{\frac{\partial v}{\partial x_2}}\right)^2\right]\right)\right\}^{1/2} \|\hat{z}\|_{n+1\ \hat{K}} \quad (5\ 8)$$

If $\hat{z} \in \hat{Q}(n)$ then $(\partial/\partial \xi_1)(\hat{z}\ \hat{w})$ is a polynomial of degree $\leqq 2n-1$ of the variable ξ_1 Therefore

$$\hat{F}(\hat{z}, \hat{w}) = \sum_{l=1}^{n} \hat{v}_l \int_{-1}^{1} \frac{\partial}{\partial \xi_1}(\hat{z}\ \hat{w})(\xi_1, t_l)\, d\xi_1 - \sum_{l=1}^{n} \hat{v}_l [(\hat{r}\ \hat{w})(1, t_l) - (\hat{z}\ \hat{w})(-1, t_l)] = 0$$

From (5 8) and the Bramble-Hilbert lemma it follows

$$|\hat{F}(\hat{z}, \hat{v} - \hat{\pi}_{n-1}\hat{v})| \leqq C\left\{\hat{I}^*\left(\mathscr{I}_K\left[\left(\widehat{\frac{\partial v}{\partial x_1}}\right)^2 + \left(\widehat{\frac{\partial v}{\partial x_2}}\right)^2\right]\right)\right\}^{1/2} [\hat{z}]_{n+1\ \hat{K}} \quad (5\ 9)$$

We estimate the other term, i e $\hat{F}(\hat{z}, \hat{\pi}_{n-1}\hat{v})$ We have

$$|\hat{F}(\hat{z}, \hat{\pi}_{n-1}\hat{v})| \leqq C \|\hat{\pi}_{n-1}\hat{v}\|_{1\ \hat{K}} \|\hat{z}\|_{n+2\ \hat{K}} \leqq C \|\hat{\pi}_{n-1}\hat{v}\|_{0\ K} \|\hat{z}\|_{n+2\ \hat{K}}$$

As

$$\|\hat{\pi}_{n-1}\hat{v}\|_{0\ \hat{K}}^2 = \sum_{k\ l=1}^{n} \hat{v}_h \hat{v}_l [(\hat{\pi}_{n-1}\hat{v})(t_k, t_l)]^2 = \sum_{k, l=1}^{n} \hat{v}_k \hat{v}_2 \hat{v}^2(t_h, t_l) = \hat{I}^*(\hat{v}^2)$$

we get

$$|\hat{F}(\hat{z}, \hat{\pi}_{n-1}\hat{v})| \leqq C\{\hat{I}^*(\hat{v}^2)\}^{1/2} \|\hat{z}\|_{n+2\ K}$$

We prove in the same way as above that $\hat{F}(\hat{z}, \hat{\pi}_{n-1}\hat{v}) = 0$ for $\hat{z} \in \hat{Q}(n+1)$ The Bramble-Hilbert lemma gives

$$|\hat{F}(\hat{z}, \hat{\pi}_{n-1}\hat{v})| \leqq C\{\hat{I}^*(\hat{v}^2)\}^{1/2} [\hat{z}]_{n+2\ \hat{K}},$$

which together with (5 9) proves (5 7)

We continue in the proof of (5 4) We introduce the norm $\|\ \|_h$ on V_h defined by

$$\|v\|_h = \{\sum_K \hat{I}^*(\mathscr{I}_K \hat{v}^2)\}^{1/2}$$

As $J_K \geqq c_2^1 h_K^2$ we get by (5 7), (4 29) and (4 30)

$$|S| \leqq C \sum_K h_K^{n+1} \left\{ \left[\hat{I}^* \left(\mathscr{J}_K \left[\left(\widehat{\frac{\partial v}{\partial x_1}} \right)^2 + \left(\widehat{\frac{\partial v}{\partial x_2}} \right)^2 \right] \right) \right]^{1/2} \|u\|_{n+2\ K} \right.$$
$$\left. + [\hat{I}^* (\mathscr{J}_K \hat{v}^2)]^{1/2} \|u\|_{n+3\ K} \right\}$$
$$\leqq C h^{n+1} \left\{ |v|_h \|u\|_{n+2\ \Omega} + \|v\|_h \|u\|_{n+3\ \Omega} \right\},$$

(5 4) follows from the following discrete analog of Friedrich's inequality

LEMMA 5 2 *Let the finite element partitions be O-strongly regular and topologically equivalent to rectangular meshes in such a way that* (5 1) *is satisfied Then there is a constant* $c = c(\Omega)$ *such that*

$$\|v\|_h \leqq c |v|_h, \qquad \forall v \in V_h \tag{5 10}$$

Proof We consider the unit square S^1 $0 < x_1 < 1$, $0 < x_2 < 1$ and the mesh $\{(i\Delta x,\ j\Delta y)\}_{i=0\ \ M}^{j=0\ \ N}$ We denote by W_h the space of trial functions defined on this mesh (of the same form as the functions $v \in V_h$, of course, (2 5) is the (linear) mapping corresponding to rectangular elements of the mesh $\{(i\Delta x, j\Delta y)\}$) and vanishing on ∂S^1 To every $v \in V_h$ we associate a $w \in W_h$ in the following way if K is an element of a given partition of Ω then the numbering of corner nodes by two indices associates a unique rectangular element R of S The function w assumes at all nodes of R (not only at corner nodes) the same values as the function v at the corresponding nodes of K At all remaining nodes of S^1 w is equal to zero We remark that either $\hat{w} = \hat{v}$ or $\hat{w} = 0$ and w vanishes on all elements $R \subset S^1$ to which no $K \subset \Omega_h$ is associated We have $\mathscr{J}_R = (1/4)\Delta x \Delta y$ Therefore

$$\|v\|_h^2 = \sum_K \hat{I}^* (\mathscr{J}_K \hat{v}^2) \leqq Ch^2 \sum_K \hat{I}^* (\hat{v}^2) = Ch^2 \sum_R \hat{I}^* (\hat{w}^2)$$
$$\leqq C \Delta x \Delta y \sum_R \hat{I}^* (\hat{w}^2) = 4 C \sum_R \hat{I}^* (\mathscr{J}_R \hat{w}^2)$$

Denote

$$\|w\|_h = \{ \sum_R \hat{I}^* (\mathscr{J}_R \hat{w}^2) \}^{1/2},$$

$$|w|_h = \left\{ \sum_R \hat{I}^* \left(\mathscr{J}_R \left[\left(\widehat{\frac{\partial w}{\partial x_1}} \right)^2 + \left(\widehat{\frac{\partial w}{\partial x_2}} \right)^2 \right] \right) \right\}^{1/2}$$

We have just proved

$$\|v\|_h \leqq C \|w\|_h \tag{5 11}$$

Suppose that we prove

$$\| w \|_h \leqq C \, | w |_h, \tag{5.12}$$

i. e. that we prove (5.10) for a uniform rectangular mesh of the unit square S^1. Then

$$| w |_h^2 = \sum_R \hat{I}^* \left(\frac{\Delta y}{\Delta x} \left(\frac{\partial \hat{w}}{\partial \xi_1} \right)^2 + \frac{\Delta x}{\Delta y} \left(\frac{\partial \hat{w}}{\partial \xi_2} \right)^2 \right)$$

$$\leqq c_5^{-1} \sum_R \hat{I}^* \left(\left(\frac{\partial \hat{w}}{\partial \xi_1} \right)^2 + \left(\frac{\partial \hat{w}}{\partial \xi_2} \right)^2 \right)$$

$$= C \sum_K \hat{I}^* \left(\left(\frac{\partial \hat{v}}{\partial \xi_1} \right)^2 + \left(\frac{\partial \hat{v}}{\partial \xi_2} \right)^2 \right) \leqq C \sum_K \hat{I}^* \left(\mathscr{J}_K \left[\left(\widehat{\frac{\partial v}{\partial x_1}} \right)^2 + \left(\widehat{\frac{\partial v}{\partial x_2}} \right)^2 \right] \right),$$

hence

$$| w |_h \leqq C \, | v |_h. \tag{5.13}$$

(5.11) and (5.13) gives (5.10).

Proof of (5.12): As $\hat{w}$ is a polynomial of degree $\leqq n$ of the variable ξ_1 it holds

$$\max_{1 \leqq \xi_1 \leqq 1} | \hat{w}(\xi_1, t_l) | \leqq C \left\{ \int_{-1}^{1} \hat{w}^2 (\xi_1, t_l) \, d\xi_1 \right\}^{1/2}.$$

Therefore

$$\| w \|_h^2 = \frac{1}{4} \Delta x \, \Delta y \sum_R \sum_{l=1}^{n} \hat{\nu}_l \sum_{k=1}^{n} \hat{\nu}_k \hat{w}^2 (t_k, t_l) \leqq C \, \Delta x \, \Delta y \sum_R \sum_{l=1}^{n} \hat{\nu}_l \int_{-1}^{1} \hat{w}^2 (\xi_1, t_l) \, d\xi_1.$$

Denote by R_{ij} the element with corners $(i \Delta x, j \Delta y)$, $((i+1) \Delta x, j \Delta y)$, $((i+1) \Delta x, (j+1) \Delta y)$, $(i \Delta x, (j+1) \Delta y)$. The mapping (2.5) has for R_{ij} the form

$$x_1 = \frac{1}{2} \Delta x (2i + 1 + \xi_1), \qquad x_2 = \frac{1}{2} \Delta y (2j + 1 + \xi_2).$$

Let (g_k^i, g_l^j) be the map of (t_k, t_l) by this mapping. Then

$$\| w \|_h^2 \leqq C \, \Delta y \sum_{j=0}^{N-1} \sum_{i=0}^{M-1} \sum_{l=1}^{n} \hat{\nu}_l \int_{i \Delta x}^{(i+1) \Delta x} w^2 (x_1, g_l^j) \, dx_1$$

$$= C \, \Delta y \sum_{j=0}^{N-1} \sum_{l=1}^{n} \hat{\nu}_l \int_0^1 w^2 (x_1, g_l^j) \, dx_1.$$

Applying the one-dimensional Friedrich's inequality we get

$$\|w\|_h^2 \leqq C\,\Delta y \sum_{j=0}^{N-1} \sum_{l=1}^{n} \hat{v}_l \int_0^1 \left(\frac{\partial w(x_1, g_l^j)}{\partial x_1}\right)^2 dx_1$$

$$= C\,\Delta y \sum_{j=0}^{N-1} \sum_{l=1}^{n} \hat{v}_l \sum_{i=0}^{M-1} \int_{i\Delta x}^{(i+1)\Delta x} \left(\frac{\partial w(x_1, g_l^j)}{\partial x_1}\right)^2 dx_1$$

$$= 2\,C \frac{\Delta y}{\Delta x} \sum_{i,j} \sum_{k,l=1}^{n} \hat{v}_k \hat{v}_l \left(\frac{\partial \hat{w}(t_h, t_l)}{\partial \xi_1}\right)^2$$

$$= C \sum_R \hat{I}^*\left(\left(\frac{\Delta y}{\Delta x}\right)\left[\frac{\partial \hat{w}}{\partial \xi_1}\right]^2\right)$$

$$\leqq C \sum_R \hat{I}^*\left(\mathscr{J}_R\left[\left(\widehat{\frac{\partial w}{\partial x_1}}\right)^2 + \left(\widehat{\frac{\partial w}{\partial x_2}}\right)^2\right]\right) = C\,|w|_h^2 .$$

This proves (5.12).

6. NUMERICAL RESULTS. SERENDIPITY FAMILY

1) The following problem was solved [3]:

$$-\Delta u = -12x - 2y + 16x^2 + 54xy + 16y^2 - 4x^3 - 42x^2y - 12xy^2 - 14y^3 \quad \text{in } \Omega,$$

$$u|_\Gamma = 0, \qquad \Omega:\ 0<x<1,\ 0<y<1.$$

The exact solution is $u(x, y) = x(1-x)\,y(1-y)(1+2x+7y)$. We used bilinear polynomials ($n=1$) and partitions consisting of square elements with vertices $\{(ih, jh)\}^M_{i,j=0}$, $M = h^{-1}$, $h = 1/4,\ 1/6,\ 1/8,\ 1/10$. There were applied Gauss-Legendre product 1×1 formula, Gauss-Legendre product 2×2 formula (substituting exact integration) and Lobatto product 2×2 formula (product trapezoidal rule). The norm $|u - u_h|_h$ is denoted by E_G and is equal in this case to

$$E_G = \left\{ N_G^{-1} \sum_{p \in G} \left[\left(\frac{\partial(u-u_h)(p)}{\partial x}\right)^2 + \left(\frac{\partial(u-u_h)(p)}{\partial y}\right)^2\right]\right\}^{1/2}.$$

Here $N_G = 4h^{-2}$ is the number of Gauss-Legendre points. Also the gradient at vertices of square elements was computed (the unique values of the gradient were

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[3] The authors are indebted to M. Kovaříková who carried out all computations on the computer DATASAAB D21

won by averaging) and as a measure of the error the number

$$E_V=\left\{N_V^{-1}\sum_{p\in V}\left[\left(\frac{\partial(u-u_h)(p)}{\partial x}\right)^2+\left(\frac{\partial(u-u_h)(p)}{\partial y}\right)^2\right]\right\}^{1/2}$$

is taken The set V consists of all vertices of square elements with exception of the vertices of Ω In table I Gauss-Legendre product 1×1 formula was used The table shows on one hand the superconvergence and the big difference between the magnitudes of E_G and E_V, on the other hand it shows that E_V goes to zero just fast as h

TABLE I

h	E_G	$h^{-2}E_G$	E_V	$h^{-1}E_V$
1/4	0 055	0 87	0 27	1 07
1/6	0 025	0 90	0 18	1 07
1/8	0 014	0 90	0 13	1 07
1/10	0 0091	0 91	0 11	1 07

Table II compares the values $h^{-2}E_G$ when Gauss-Legendre product 1×1 and 2×2 formula and Lobatto product 2×2 formula were used

TABLE II

h	$h^{-2}E_G$		
	$G-L\,1\times 1$	$G-L\,2\times 2$	Lob 2×2
1/4	0 874	0 980	1 462
1/6	0 897	0 994	1 504
1/8	0 906	0 998	1 519
1/10	0 910	1 001	1 526

Evidently, Gauss-Legendre 1×1 formula gives the best values

2) In engineering applications the curved isoparametric elements of the Serendipity family (*see* Zienkiewicz [8]) are mostly used The linear elements of this family are the simplest elements defined in this paper ($n=1$) The quadratic and cubic elements are different from elements introduced here for $n=2$ and $n=3$ Instead of complete biquadratic and bicubic polynomials, respectively, there are used incomplete polynomials formed from these classes In the first case

there is missing the term $\xi_1^2 \xi_2^2$ (as nodes we take eight nodes of the class $\hat{Q}(2)$ lying on the boundary of $\hat{K}$), in the second there are missing the terms $\xi_1^2 \xi_2^2$, $\xi_1^3 \xi_2^2$, $\xi_1^2 \xi_2^3$, $\xi_1^3 \xi_2^3$ (as nodes we take twelve nodes of the class $\hat{Q}(3)$ lying on the boundary of $\hat{K}$) Superconvergence of the gradient at Gauss-Legendre points can be proved by the same technique which we used for polynomials from $\hat{Q}(n)$ The proof is simpler because the functional $a_h(v, v)$ is bounded from below by $C|v|^2_{1\,\Omega_h}$ uniformly with respect to h even for Gauss-Legendre formulas

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