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## A MIXED METHOD FOR 4TH ORDER PROBLEMS USING LINEAR FINITE ELEMENTS (\*)

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Communiqué par J. A. Nitsche

Abstract. — *The solution of the fourth order problem  $\Delta^2 u = f$  in  $\Omega$ ,  $u = \partial u/\partial \nu = 0$  on  $\partial\Omega$ ,  $\Omega$  bounded in  $\mathbf{R}^2$ , and its Laplacian are approximated by linear finite elements.  $L_2$ - and  $L_\infty$ -error estimates are given.*

### 1

Let  $\Omega \subseteq \mathbf{R}^2$  be a bounded domain with sufficiently smooth boundary. We consider the fourth order boundary value problem

$$\left. \begin{aligned} \Delta^2 u &= f && \text{in } \Omega, \\ u &= \partial u/\partial \nu = 0 && \text{on } \partial\Omega \end{aligned} \right\} \quad (1)$$

with  $f \in L_2$ .

The basic idea of the mixed method considered here—due to Ciarlet-Raviart [3]—is to write the equation (1) as a system

$$\left. \begin{aligned} -\Delta u_2 &= f \\ -\Delta u_1 &= u_2 \end{aligned} \right\} \text{ in } \Omega, \quad \left. \begin{aligned} u_1 &= \partial u_1/\partial \nu = 0 && \text{on } \partial\Omega \end{aligned} \right\} \quad (2)$$

and to approximate  $u_1$  and  $u_2$  simultaneously by suitably chosen subspaces. (Another mixed method can be used if one is interested to approximate  $u$  and all second derivatives of  $u$ . In this context we refer to Brezzi-Raviart [1] and the references given there.)

Using finite element spaces of piecewise polynomials of degree  $r \geq 2$  as approximating subspaces the first  $L_2$ -error estimates were given by Ciarlet-Raviart [3]. In [9] improved  $L_2$ -estimates have been obtained, and in the case of quadratic finite elements Rannacher [8] proved an  $L_\infty$ -estimate.

In this note we show that also in the case of linear finite elements the mixed method approximations are convergent, and we derive an error estimate in the  $L_2$ - as well as in the  $L_\infty$ - norm.

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For  $h > 0$  let  $\Gamma_h$  be a  $\kappa$ -regular partition of  $\Omega$  in generalized triangles, that means:

(i)  $\Delta \in \Gamma_h$  is a triangle if  $\Delta$  and  $\partial\Omega$  have at most one common point, otherwise one of the sides of  $\Delta$  may be curved;

(ii) there is a fixed  $\kappa > 0$  such that for each  $\Delta \in \Gamma_h$  two circles  $\underline{K}$  and  $\bar{K}$  exists with radii  $\kappa^{-1}h$  and  $\kappa h$  and  $\underline{K} \subseteq \Delta \subseteq \bar{K}$ .

Let  $S_h = S_h(\Gamma_h)$  be the space of continuous functions which are linear in each triangle of  $\Gamma_h$  with the usual modification for the curved elements (see Ciarlet-Raviart [2], Zlamal [10]). The space  $\hat{S}_h$  is the intersection of  $S_h$  and  $\hat{W}_2^1$ . The approximation properties of the spaces  $S_h$  and  $\hat{S}_h$  are well-known; confer Ciarlet-Raviart [2] for example.

Further we denote with  $(\cdot, \cdot)$  and  $D(\cdot, \cdot)$  the inner product in  $L_2$  and the Dirichlet integral. Finally the Ritz operators  $R_h : W_2^1 \rightarrow S_h$  and  $\hat{R}_h : \hat{W}_2^1 \rightarrow \hat{S}_h$  are defined by

$$D(v - R_h v, \eta) = 0 \quad \text{for all } \eta \in S_h,$$

$$\int_{\Omega} (v - R_h v) ds = 0$$

and

$$D(u - \hat{R}_h u, \xi) = 0 \quad \text{for all } \xi \in \hat{S}_h$$

respectively.

The following lemma is fundamental for the derivation of our estimates. The proof rests on  $L_{\infty}$ -error estimates for the Ritz approximation of second order problems (see Nitsche [7], confer also Fehse-Rannacher [4], Nitsche [6]).

LEMMA: For all  $u \in \hat{W}_2^1 \cap W_{\infty}^2$  and for all  $\eta \in S_h$  we have

$$|D(u - \hat{R}_h u, \eta)| \leq C h^{1/2} |\ln h| \|\Delta u\|_{L_{\infty}} \|\eta\|_{L_2}, \quad (3)$$

with  $C$  independent of  $u$ ,  $\eta$  and  $h$ .

*Proof:* Let  $\Omega_h$  be the union of all triangles  $\Delta$  with  $\Delta \cap \partial\Omega \neq \emptyset$ . With  $\xi \in \hat{S}_h$  we denote the function which interpolates  $\eta$  at the interior grid-points of the triangulation  $\Gamma_h$ . For  $\varphi := \eta - \xi$  it follows

$$\varphi(s) = 0, \quad s \in \Omega - \Omega_h.$$

Setting  $\varepsilon := u - \hat{R}_h u$  and using the inverse estimate

$$\|\varphi\|_{W_{\infty}^1(\Delta)} \leq C h^{-1} \|\varphi\|_{L_{\infty}(\Delta)}$$

we therefore get

$$\begin{aligned}
 |D(\varepsilon, \eta)| &= |D(\varepsilon, \varphi)| \\
 &= \left| \sum_{\Delta} \int_{\Delta} \text{grad } \varepsilon \cdot \text{grad } \varphi \, ds \right| \\
 &\leq C h^2 \|\varepsilon\|_{W^1} \sum \|\varphi\|_{W^1_{\infty}(\Delta)} \\
 &\leq C h \|\varepsilon\|_{W^1_{\infty}} \sum \|\varphi\|_{L_{\infty}(\Delta)}, \tag{4}
 \end{aligned}$$

where the sum has to be taken over all triangles  $\Delta \in \Omega_h$ . Now it is easy to see that also in the curved triangles

$$\|\varphi\|_{L_{\infty}(\Delta)} \leq C \max_{k=1,2,3} |\varphi(s_k)|,$$

where  $s_k$  denotes the vertices of  $\Delta$ . Therefore we find

$$\|\varphi\|_{L_{\infty}(\Delta)} \leq C \|\eta\|_{L_{\infty}(\Delta)},$$

and with the inverse estimate

$$\|\eta\|_{L_{\infty}(\Delta)} \leq C h^{-1} \|\eta\|_{L_2(\Delta)}$$

we get from (4)

$$\begin{aligned}
 |D(\varepsilon, \eta)| &\leq C \|\varepsilon\|_{W^1_{\infty}} \sum \|\eta\|_{L_2(\Delta)} \\
 &\leq C h^{-1/2} \|\varepsilon\|_{W^1_{\infty}} \|\eta\|_{L_2(\Omega_h)},
 \end{aligned}$$

since the number of triangles in  $\Omega_h$  is of order  $h^{-1}$ .

With the  $L_{\infty}$ -estimate for the Ritz-approximation (Nitsche [7])

$$\|\varepsilon\|_{W^1_{\infty}} = \|u - R_h u\|_{W^1_{\infty}} \leq C h |\ln h| \|\Delta u\|_{L_{\infty}}$$

the lemma is proven.

3

The mixed finite element approximation  $(u_1^h, u_2^h) \in \dot{S}_h \times S_h$  for the solution  $(u_1, u_2)$  of the problem (2) is given by

$$\left. \begin{aligned}
 D(u_2^h, \xi) &= (f, \xi) \quad \text{for all } \xi \in \dot{S}_h \\
 D(u_1^h, \eta) &= (u_2^h, \eta) \quad \text{for all } \eta \in S_h
 \end{aligned} \right\} \tag{5}$$

(see Ciarlet-Raviart [3], Scholz [9].)

Since  $\dot{S}_h \subseteq S_h$  holds, the system (5) is uniquely solvable, and with  $e_i := u_i - u_i^h$ ,  $i = 1, 2$ , we can rewrite (5) in the following form

$$\left. \begin{aligned}
 D(e_2, \xi) &= 0 \quad \text{for all } \xi \in \dot{S}_h \\
 D(e_1, \eta) &= (e_2, \eta) \quad \text{for all } \eta \in S_h.
 \end{aligned} \right\} \tag{5'}$$

In the  $L_2$ -norm we get the following estimates.

**THEOREM 1:** *The differences  $e_i = u_i - u_i^h$ ,  $i = 1, 2$ , between the exact solution of the problem (2) and the mixed finite element approximation can be estimated by*

$$\|e_1\|_{L_2} + h^{1/2} |\ln h| \|e_2\|_{L_2} \leq Ch |\ln h|^2 \|u_1\|_{W_2^2}, \quad (6)$$

where  $C$  is independent of  $(u_1, u_2)$  and  $h$ .

**COROLLARY:** *As a consequence of (6) combined with the second part of (5') we get for  $e_1$  in the  $W_2^1$ -norm*

$$\|e_1\|_{W_2^1} \leq Ch^{3/4} |\ln h|^{3/2} \|u_1\|_{W_2^2}. \quad (7)$$

*Proof:* Let  $\varphi_1 \in \mathring{S}_h$  and  $\varphi_2 \in S_h$  be the Ritz approximations of  $u_1$  respective  $u_2$  as defined above. Using the equations (5') we find

$$\begin{aligned} \|u_2^h - \varphi_2\|_{L_2}^2 &= (u_2^h - \varphi_2, u_2^h - \varphi_2) - D(u_1^h - \varphi_1, u_2^h - \varphi_2) \\ &\quad + D(u_1^h - \varphi_1, u_2^h - \varphi_2) \\ &= (u_2 - \varphi_2, u_2^h - \varphi_2) - D(u_1 - \varphi_1, u_2^h - \varphi_2) \\ &\quad + D(u_1^h - \varphi_1, u_2 - \varphi_2). \end{aligned} \quad (8)$$

With the standard error estimates for the Ritz approximations in the  $L_2$ -norm the first term on the right-hand side can be estimated by

$$|(u_2 - \varphi_2, u_2^h - \varphi_2)| \leq Ch^2 \|u_2\|_{W_2^2} \|u_2^h - \varphi_2\|_{L_2}.$$

For the second term we get with the help of the lemma

$$|D(u_1 - \varphi_1, u_2^h - \varphi_2)| \leq Ch^{1/2} |\ln h| \|\Delta u_1\|_{L_\infty} \|u_2^h - \varphi_2\|_{L_2},$$

and the third term is equal to zero by definition of  $\varphi_2$ . Combining these inequalities with (8) the estimate for  $e_2$  follows.

The other part of (6) is proven by a duality argument. Let  $w \in \mathring{W}_2^2 \cap W_2^4$  be the solution of

$$\begin{aligned} \Delta^2 w &= e_1 \quad \text{in } \Omega, \\ w &= \partial w / \partial \nu = 0 \quad \text{on } \partial \Omega. \end{aligned}$$

Observing (5') we get

$$\|e_1\|_{L_2}^2 = -D(e_1, \Delta w - R_h \Delta w) + (e_2, \Delta w - R_h \Delta w) + D(e_2, w - \mathring{R}_h w). \quad (9)$$

Since  $u_1^h$  and  $\varphi_1$  are elements of  $\mathring{S}_h$ , we obtain for the first term

$$\begin{aligned} |D(e_1, \Delta w - R_h \Delta w)| &= |D(u_1 - \varphi_1, \Delta w - R_h \Delta w)| \\ &\leq Ch^2 \|u_1\|_{W_2^2} \|w\|_{W_2^4}, \end{aligned}$$

and the second term can be estimated by

$$|(e_2, \Delta w - R_h \Delta w)| \leq Ch^2 \|e_2\|_{L_2} \|w\|_{W_2^2}.$$

Finally we get—again with the help of (3)—

$$|D(e_2, w - \dot{R}_h w)| \leq Ch^2 \|u_2\|_{W_2^2} (\|w\|_{W_2^2} + \|\Delta w\|_{L_\infty}) + Ch^{1/2} |\ln h| \|e_2\|_{L_2} \|\Delta w\|_{L_\infty}.$$

Using Sobolev's embedding theorem we obtain

$$\|\Delta w\|_{L_\infty} \leq C \|w\|_{W_2^2}.$$

Together with the estimate for  $e_2$  we find

$$|D(e_2, w - \dot{R}_h w)| \leq Ch |\ln h|^2 \|u_2\|_{W_2^2} \|w\|_{W_2^2}.$$

Combining these inequalities with (9) and observing

$$\|w\|_{W_2^2} \leq C \|e_1\|_{L_2},$$

the theorem is proven.

The second result is an  $L_\infty$ -estimate for  $e_1$ .

**THEOREM 2:** *The error  $u_1 - u_1^h$  in the  $L_\infty$ -norm can be estimated by*

$$\|e_1\|_{L_\infty} \leq Ch |\ln h|^3 \|u_1\|_{W_2^2}, \tag{10}$$

where  $C$  is independent of  $(u_1, u_2)$  and  $h$ .

*Proof:* We choose  $\Delta \in \Gamma_h$  such that

$$\|e_1\|_{L_\infty} = \|e_1\|_{L_\infty(\Delta)}.$$

With standard arguments we get

$$\|e_1\|_{L_\infty(\Delta)} \leq C(h^{-1} \|e_1\|_{L_2(\Delta)} + h^2 \|u_1\|_{W_\infty^-}) \tag{11}$$

and it suffices to estimate  $\|e_1\|_{L_2(\Delta)}$ . This again is done by a duality technique. Let  $w \in \dot{W}_2^2 \cap W_2^4$  be the solution of

$$\begin{aligned} \Delta^2 w &= e_1 \chi_\Delta \quad \text{in } \Omega, \\ w &= \partial w / \partial \nu = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where  $\chi_\Delta$  is the characteristic function of the triangle  $\Delta$ . With the same arguments as above we obtain

$$\|e_1\|_{L_2(\Delta)}^2 \leq C(h^2 \|w\|_{W_2^2} + h |\ln h|^2 \|\Delta w\|_{L_\infty}) \|u_1\|_{W_2^2}, \tag{12}$$

and from *a priori*-estimates the inequality

$$\|w\|_{W_2^2} \leq Ch \|e_1\|_{L_\infty} \tag{13}$$

immediately follows. Further with the help of Sobolev's integral identity (see for instance Mikhlin [5], pp. 60-66) we find for all real  $\varepsilon > 0$ :

$$\|\Delta w\|_{L_\infty} \leq C(\|\Delta w\|_{L_2} + \varepsilon \|\Delta w\|_{W_2^1} + |\log \varepsilon|^{1/2} \|\Delta w\|_{W_2^1}) \tag{14}$$

with  $C$  independent of  $\varepsilon$ . From Sobolev's imbedding theorem it follows for all  $p > 2$ :

$$\begin{aligned} \|\Delta w\|_{W_\infty^1} &\leq C_p \|w\|_{W_2^2} \\ &\leq C_p h^{2/p} \|e_1\|_{L_\infty}, \end{aligned} \quad (15)$$

$C_p$  independent of  $h$ , and from Frehse-Rannacher [4], Theorem 4.B we derive

$$\|\Delta w\|_{W_2^1} \leq C h^2 |\ln h|^{1/2} \|e_1\|_{L_\infty}. \quad (16)$$

Fixing  $p > 2$  and choosing  $\varepsilon := h^{2-2/p}$ , we get from (13), (14), (15), and (16):

$$\|e_1\|_{L_2(\Delta)}^2 \leq C h^3 |\ln h|^3 \|e_1\|_{L_\infty} \|u_1\|_{W_2^1}.$$

Together with (11) the last inequality gives the desired result.

*Remark:* An analogue of the estimate (3) can be shown for finite element spaces of higher degree. Therefore the error estimates in this case can be improved too, especially in the case of quadratic finite elements, provided that the solution is sufficiently smooth.

#### REFERENCES

1. F. BREZZI and P. A. RAVIART, *Mixed Finite Element Methods for 4th Order Elliptic Equations* (to appear).
2. P. G. CIARLET and P. A. RAVIART, *Interpolation Theory Over Curved Elements, with Applications to Finite Element Methods*, Computer Meth. Appl. Mech. Engrg., 1, 1972, pp. 217-249.
3. P. G. CIARLET and P. A. RAVIART, *A Mixed Finite Element Method for the Biharmonic Equation in Mathematical Aspects of Finite Elements in Partial Differential Equations*, C. DE BOOR, Ed., Proc. Symp. Math. Res. Center, Univ. Wisconsin, April 1-3, 1974, Academic Press, New York - San Francisco - London, 1974, pp. 125-145.
4. J. FREHSE and R. RANNACHER, *Eine  $L_1$ -Fehlerabschätzung für diskrete Grundlösungen in der Methode der finiten Elemente*, Bonner Math. Schriften, 89, 1976, pp. 92-114.
5. S. G. MIKHLIN, *The Problem of the Minimum of a Quadratic Functional*, Holden-Day, Inc., San Francisco - London - Amsterdam, 1965.
6. J. NITSCHKE,  *$L_\infty$ -Convergence of Finite Element Approximations*, Rom, 1975 (to appear).
7. J. NITSCHKE,  *$L_\infty$ -Convergence of the Ritz-Method with Linear Finite Elements for Second Order Elliptic Boundary Value Problems* (to appear in the anniversary volume dedicated to academician I. N. Vekua).
8. R. RANNACHER, *Punktweise Konvergenz der Methode der finiten Elemente beim Plattenproblem*, Manuscripta Math., 19, 1976, pp. 401-416.
9. R. SCHOLZ, *Approximation von Sattelpunkten mit finiten Elementen*, Bonner Math. Schriften, 89, 1976, pp. 53-66.
10. M. ZLAMAL, *Curved Elements in the Finite Element Method*, I. S.I.A.M. J. Numer. Anal., 10, 1973, pp. 229-240; II. S.I.A.M. J. Numer. Anal., 11, 1974, pp. 347-362.