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## AN ANALYSIS OF THE CONVERGENCE OF MIXED FINITE ELEMENT METHODS (\*) (1)

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Communiqué par P.-A. RAVIART

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*Abstract. — This paper deals with convergence proofs in mixed finite element methods. After recalling abstract conditions of Brezzi, one shows that these conditions are, in some cases, equivalent to the possibility of building an uniformly continuous operator  $\Pi_h$  from  $V$  into  $V_h$ . Moreover some properties of discrete operators involved in the approximation are characterized. Two examples show that building the operator  $\Pi_h$  can be done through an interpolation operator. A third example presents a case which is still out of reach of present techniques.*

### 1. INTRODUCTION

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The aim of this paper, is to study, in a rather general setting, the convergence properties of approximations, by finite elements, of saddle-point problems related to the minimization of convex functionals under a linear constraint. Applications are, of course, mixed finite elements methods and hybrid methods, but the results given here are mainly adapted to the case of mixed methods.

The problem we consider has already been treated in Brezzi [1] and Brezzi-Raviart [2], among others. The case we consider is slightly more general in a sense to be precised later. However the main result will be to give sufficient (and in some cases necessary) conditions to verify the abstract "stability" condition of [1]. These new conditions can, in many cases, be quite easily verified, thus simplifying, in a considerable way, convergence proofs. Although it would be too long to present a full account of the previous works on the subject, the reader may refer, apart from the above cited papers, to Oden [6-7], and Johnson [5] for a more complete view of the problem.

The exposition will proceed as follows. In No. 2, we study the abstract continuous problem and give an existence and uniqueness theorem. In No. 3 we recall the general abstract condition of [1] for the convergence of approximations. In No. 4, we present a few lemmas characterizing the Kernels and Images of some operators appearing in the problem and we use these results to give practical convergence conditions. Finally in No. 5, we give some examples of application of these results.

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## 2. THE GENERAL PROBLEM

Let  $V$  and  $W$  be two real Hilbert spaces whose norms and scalar products are respectively denoted  $|\cdot|_V$ ,  $(\cdot, \cdot)_V$ ,  $|\cdot|_W$  and  $(\cdot, \cdot)_W$ .

We give on  $V \times V$  a continuous, symmetrical, bilinear form  $a(u, v)$  and on  $V \times W$  a continuous bilinear form  $b(v, \varphi)$ . Continuity of  $b$  implies that there exist a constant, denoted  $\|b\|$ , such that,

$$|b(v, \varphi)| \leq \|b\| |v|_V |\varphi|_W, \quad \forall v \in V, \quad \forall \varphi \in W. \quad (2.1)$$

In the same way, the norm of  $a$  as a bilinear form on  $V \times V$  will be denoted  $\|a\|_V$ . Let  $f \in V'$  and  $g \in W'$  be given. The brackets  $\langle \cdot, \cdot \rangle$  will denote duality between both  $V'$  and  $V$  and  $W'$  and  $W$ , no ambiguity being possible. We consider the functional, on  $V \times W$ ,

$$L(v, \varphi) = \frac{1}{2} a(v, v) - \langle f, v \rangle + b(v, \varphi) - \langle g, \varphi \rangle, \quad (2.2)$$

and we want to find a pair  $(u, \lambda) \in V \times W$ , saddle-point of  $L(v, \varphi)$  and  $V \times W$ , that is,

$$L(u, \varphi) \leq L(u, \lambda) \leq L(v, \lambda), \quad \forall v \in V, \quad \forall \varphi \in W. \quad (2.3)$$

This is, of course, equivalent to solving, the linearly constrained, quadratic problem,

$$\frac{1}{2} a(u, u) - \langle f, u \rangle \leq \frac{1}{2} a(v, v) - \langle f, v \rangle, \quad \forall v \in Z(g), \quad u \in Z(g); \quad (2.4)$$

$$Z(g) = \{v \mid v \in V, b(v, \varphi) = \langle g, \varphi \rangle, \forall \varphi \in W\}. \quad (2.5)$$

The saddle-point  $(u, \lambda)$  is then also solution of the system,

$$a(u, v) + b(v, \lambda) = \langle f, v \rangle, \quad \forall v \in V, \quad (2.6)$$

$$b(u, \varphi) = \langle g, \varphi \rangle, \quad \forall \varphi \in W, \quad (2.7)$$

$$u \in V, \quad \lambda \in W. \quad (2.8)$$

We remind that under some hypotheses, this saddle-point problem has a solution, eventually a unique solution. We first recall a few classical notations. First let us remind that the continuous bilinear form  $b(v, \varphi)$  defines a continuous linear operator  $B$  from  $V$  into  $W'$ , precisely,

$$\langle Bv, \varphi \rangle = b(v, \varphi), \quad \forall \varphi \in W. \quad (2.9)$$

In the same way, the transpose  $B^*$  of  $B$ , from  $W$  into  $V'$  is defined by

$$\langle v, B^* \varphi \rangle = b(v, \varphi), \quad \forall v \in V. \quad (2.10)$$

Condition (2.7) is thus clearly equivalent to

$$Bu = g \tag{2.11}$$

and we also have, according to (2.5):

$$Z(g) = \{ v \mid v \in V, Bv = g \}. \tag{2.12}$$

A necessary condition for the existence of a solution  $u$  to (2.11) is, of course,  $g \in \text{Im } B$ . We shall assume that this is fulfilled. Let then  $v_g$  be any element of  $Z(g)$ . Our problem may then be written, writing  $u = u_0 + v_g$  in the equivalent form,

$$a(u_0, v_0) = \langle f, v_0 \rangle - a(v_g, v_0), \quad \forall v_0 \in \text{Ker } B; \tag{2.13}$$

$$u_0 \in \text{Ker } B. \tag{2.14}$$

According to the Lax-Milgram theorem, we have for the existence of  $u_0$  (and then of  $u = u_0 + v_g$ ) the classical coerciveness condition:

$$a(v_0, v_0) \geq \alpha \|v_0\|_v^2, \quad \forall v_0 \in \text{Ker } B. \tag{2.15}$$

This condition implies the existence of a unique solution  $u$  to (2.4)-(2.5) i. e. to the *primal problem*.

To prove the existence of a saddle-point, we must show the existence of a Lagrange multiplier for the linear constraint (2.11).

Before doing so, we recall, some facts about the properties of  $B$  and  $B^*$ .

LEMMA 2.1: *The following statements are equivalent:*

$$\text{The range Im } B \text{ is closed in } W', \tag{2.16}$$

$$\sup_{v \in V} \frac{|b(v, \varphi)|}{\|v\|_V} \geq k \inf_{\varphi_0 \in \text{Ker } B^*} |\varphi + \varphi_0|_W \tag{2.17}$$

$$\|B^* \varphi\|_V \geq k \|\varphi\|_{W/\text{Ker } B^*}, \tag{2.18}$$

$$\|Bu\|_{W'} \geq k \|u\|_{V/\text{Ker } B}, \tag{2.19}$$

$$B \text{ admits a continuous lifting from } W' \text{ into } V. \tag{2.20}$$

*Proof:* This a restatement of the closed range theorem (cf. e. g. Yosida [9]). ■ We then have the following result:

PROPOSITION 2.1: *Let Im } B be closed in } W' and let (2.15) be satisfied. Then the saddle-point problem (2.3) has a unique solution (u, λ) in V × W/Ker } B^\*. The Lagrange multiplier λ is thus unique up to the addition of any element of Ker } B^\*.*

*Proof:* See Brezzi [1]. ■

### 3. ABSTRACT CONVERGENCE RESULTS

We approximate here the saddle-point problem (2.3) by internal approximation which in practice will be realized by finite elements. We consider two finite dimensional spaces,

$$V_h \subset V; \quad W_h \subset W \quad (3.1)$$

with the topology induced by  $V$  and  $W$  respectively.

We now consider a discrete saddle-point problem,

$$L(u_h, \varphi_h) \leq L(u_h, \lambda_h) \leq L(v_h, \lambda_h), \quad \forall v_h \in V_h, \quad \forall \varphi_h \in W_h, \quad (3.2)$$

which is characterized by the optimality conditions,

$$a(u_h, v_h) + b(v_h, \lambda_h) = \langle f, v_h \rangle, \quad \forall v_h \in V_h, \quad (3.3)$$

$$b(u_h, \varphi_h) = \langle g, \varphi_h \rangle, \quad \forall \varphi_h \in W_h, \quad (3.4)$$

$$u_h \in V_h, \quad \lambda_h \in W_h. \quad (3.5)$$

The continuous bilinear form  $b(\cdot, \cdot)$  still defines here a continuous operator,  $B_h : V_h \rightarrow W'_h$ , and its transpose  $B_h^* : W_h \rightarrow V'_h$ . In general one cannot identify  $B_h$  as the restriction of  $B$  to  $V_h$  but one has

$$B_h v_h = P_{W'_h}(B v_h), \quad (3.6)$$

where  $P_{W'_h}$  is the projection operator from  $W'$  to  $W'_h$ . Let  $\tilde{g}_h = P_{W'_h}(g)$  (that is  $\langle \tilde{g}_h, \varphi_h \rangle = \langle g, \varphi_h \rangle, \forall \varphi_h \in W_h$ ).

Then (3.4) can be written as

$$B_h u_h = \tilde{g}_h \quad (3.7)$$

and a necessary condition for existence is of course:

$$\tilde{g}_h \in \text{Im } B_h. \quad (3.8)$$

Under a proper coerciveness condition, for instance:

$$a(v_h, v_h) \geq \alpha |v_h|_{V_h/\text{Ker } B_h}^2, \quad (3.9)$$

Proposition 2.1, implies the existence of a discrete saddle-point  $(u_h, \lambda_h)$ , for in this finite dimensional case,  $\text{Im } B_h$  is always closed. We would then want to know if  $(u_h, \lambda_h)$  is an approximation to  $(u, \lambda)$ . In order to solve this problem, we first present abstracts results, extending to the case where  $B$  is not surjective, the results of Brezzi [1].

It is clear Lemma 2.1 is still valid, and even trivial in finite dimensional spaces. However, the various constants  $C$ , and  $k$  generally depend on  $h$ . Convergence proofs will rely heavily on the following definition.

DEFINITION 3.1: We say that  $B_h$  satisfies the *uniformly continuous lifting property (UCLP)* if the following equivalent conditions hold with  $k$ , and  $c$  independent of  $h$

$$\sup_{v_h \in V_h} \frac{|b(v_h, \varphi_h)|}{|v_h|_{V_h}} \geq k \inf_{\varphi_{0h} \in \text{Ker } B_h^*} |\varphi_h + \varphi_{0h}|_{W_h}, \tag{3.10}$$

$$|B_h^* \varphi_h|_{V_h'} \geq k |\varphi_h|_{W_h/\text{Ker } B_h^*}, \tag{3.11}$$

$$|B_h u_h|_{W_h'} \geq k |u_h|_{V_h/\text{Ker } B_h}. \tag{3.12}$$

For any  $g_h \in \text{Im } B_h$ , there exists  $u_h \in V_h$  such that

$$B_h u_h = g_h, \quad |u_h|_{V_h} < c |g_h|_{W_h}. \quad \blacksquare \tag{3.13}$$

We now define

$$Z_h(g) = \{v_h \in V_h \mid b(v_h, \varphi_h) = \langle g, \varphi_h \rangle, \forall \varphi_h \in W_h\}, \tag{3.14}$$

or equivalently

$$Z_h(g) = \{v_h \in V_h \mid B_h v_h = \tilde{g}_h\}. \tag{3.15}$$

We now recall the following classical results of Brezzi [1] and Brezzi-Raviart [2].

PROPOSITION 3.1: Let (3.8) hold and let  $a(\cdot, \cdot)$  be  $V$ -coercive, that is

$$a(v, v) \geq \alpha |v|_V^2. \tag{3.16}$$

Then there exists a constant  $C$  independent of  $h$  such that

$$|u - u_h|_V \leq C \left( \inf_{v_h \in Z_h(g)} |u - v_h|_V + \inf_{\varphi_h \in W_h} |\lambda - \varphi_h|_W \right). \tag{3.17}$$

Moreover, if  $B_h$  satisfies the UCLP condition and if we denote  $\lambda$  and  $\lambda_h$  the minimal norm Lagrange multipliers (that is with zero component in  $\text{Ker } B^*$  and  $\text{Ker } B_h^*$  respectively), then there exists a constant  $C$  independent of  $h$  such that:

$$|u - u_h|_V + |\lambda - \lambda_h|_W \leq C \left\{ \inf_{w_h \in V_h} |u - w_h|_V + \inf_{\varphi_h \in \text{Im } B_h} |\lambda - \varphi_h|_W \right\}. \quad \blacksquare \tag{3.18}$$

Another special case is of special interest: Let us suppose that we have  $V \hookrightarrow H$ , where  $H$  is a Hilbert space, and that  $a$  is  $H$ -coercive, that is

$$a(v, v) \geq \alpha |v|_H^2, \tag{3.19}$$

but not  $V$ -coercive [i. e. (3.16)].

The following result can then be proved:

PROPOSITION 3.2: Let (3.8)-(3.19) and the UCLP condition hold. Then if one has

$$\text{Ker } B_h \subset \text{Ker } B, \tag{3.20}$$

There exists a constant  $C$  independent of  $h$  such that

$$|u - u_h|_H + |\lambda - \lambda_h|_W \leq C \left\{ \inf_{w_h \in V_h} |u - w_h|_V + \inf_{\varphi_h \in \text{Im } B_h} |\lambda - \varphi_h|_W \right\}. \quad \blacksquare \quad (3.21)$$

It must be noted that we have not stated the results of [2] in their most general form. In order to use them in practice the following questions must be answered:

*Q 3.1:* When is, in general, (3.8) satisfied?

*Q 3.2:* When is the UCLP condition satisfied?

*Q 3.3:* When is (3.20) satisfied?

*Q 3.4:* Can we replace in (3.21),  $\inf_{\varphi_h \in \text{Im } B_h} |\lambda - \varphi_h|_W$  by an infimum over all  $\varphi_h \in W_h$ ?

We shall try in the following section to give equivalent or sufficient conditions for the answers to be positive.

#### 4. EQUIVALENT FORMS FOR CONVERGENCE CONDITIONS

When trying to apply the abstract results of No. 3 to a precise case, the main problem lies in the verification of the continuous lifting property or the condition on Kernels (3.16). We shall first give some algebraic lemmas that will clarify the relations between the ranges and kernels of  $B$  and  $B_h$ . As may be expected, the continuous lifting property will then be splitted in a consistency and a stability condition and we shall give sufficient conditions for the stability to hold. We restrict ourselves, to simplify the proofs to the case where  $W$  and  $W_h$  are identified to their dual spaces. We first have.

**LEMMA 4.1:** *The following statements are equivalent:*

$$\left. \begin{array}{l} \text{For any } u \in V, \text{ there exists } \hat{u}_h = \Pi_h u \in V_h, \text{ such that,} \\ b(u - \hat{u}_h, \varphi_h) = 0, \forall \varphi_h \in W_h, \text{ or equivalently } \hat{u}_h \in Z_h(Bu); \end{array} \right\} \quad (4.1)$$

$$\text{Im } B_h = P_{W_h}(\text{Im } B), \quad (4.2)$$

$P_{W_h}$  being the projection operator of  $W$  on  $W_h$ .

$$\text{Ker } B_h^* = \text{Ker } B^* \cap W_h \subset \text{Ker } B^*. \quad (4.3)$$

*Proof:* The equivalence of (4.1) and (4.2) is trivial: by definition, one always has

$$\text{Im } B_h = P_{W_h}(B V_h) \subset P_{W_h}(\text{Im } B);$$

it is therefore sufficient to consider the reverse inclusion which is nothing that another statement of (4.1). To show the equivalence of (4.1) or (4.2) with (4.3), let us suppose that (4.1) is satisfied and let  $\bar{\varphi}_h$  be given in  $\text{Ker } B_h^*$ , i. e.,

$$b(v_h, \bar{\varphi}_h) = 0, \quad \forall v_h \in V_h.$$

We have to show that  $b(v, \bar{\varphi}_h) = 0$ , for any  $v \in V$ , which implies  $\bar{\varphi}_h \in \text{Ker } B^*$ . But for  $v \in V$ , there exists by (4.1),  $\hat{v}_h \in V_h$  such that

$$b(v, \varphi_h) = b(\hat{v}_h, \varphi_h), \quad \varphi_h \in W_h.$$

In particular this is true for  $\bar{\varphi}_h$  so  $b(v, \bar{\varphi}_h) = b(\hat{v}_h, \bar{\varphi}_h) = 0$ .

Conversely, let  $u \in V$  and consider  $P_{W_h}(Bu)$ .

We then have, by definition of the projection operator,

$$(\varphi_h, Bu - P_{W_h}(Bu))_W = 0, \quad \varphi_h \in W_h. \tag{4.4}$$

We want to show that  $P_{W_h}(Bu) \subset \text{Im } B_h$ , or equivalently

$$P_{W_h}(Bu) \in (\text{Ker } B_h^*)^\perp. \tag{4.5}$$

Let then  $\bar{\varphi}_h \in \text{Ker } B_h^* \subset \text{Ker } B^*$  be given, and take  $\varphi_h = \bar{\varphi}_h$  in (4.4), we obtain,

$$(\bar{\varphi}_h, P_{W_h}(Bu)) = (\bar{\varphi}_h, Bu) = (B^* \bar{\varphi}_h, u) = 0. \quad \blacksquare \tag{4.6}$$

*Remark 4.1:* The previous proof shows, in fact, that (4.1) is equivalent to,

$$\text{Ker } B_h^* \subset \text{Ker } B^* \cap W_h, \tag{4.7}$$

the reverse inclusion always being true. Moreover the identification of  $W$  to  $W'$  is not essential for the proof, it is sufficient to restrict the analysis to  $W'_h \cap W' = \tilde{W}'_h$ . By definition of  $B_h$ , one has immediately  $B_h v_h \in \tilde{W}'_h$ ,  $v_h \in V_h$  and the previous proof can be extended, with a few technical subtilities.  $\blacksquare$

Merely exchanging the roles of  $V$  and  $W$ , and taking into account the preceding remark, we have thus shown:

**LEMMA 4.2:** *The following statements are equivalent:*

$$\begin{aligned} \text{For any } \varphi \in W, \text{ there exists } \hat{\varphi}_h \in W_h \text{ such that,} \\ b(v_h, \varphi - \hat{\varphi}_h) = 0, \quad \forall v_h \in V_h, \end{aligned} \tag{4.8}$$

$$\text{Im } B_h^* = P_{V'_h \cap V'}(\text{Im } B^*). \tag{4.9}$$

$$\text{Ker } B_h = \text{Ker } B \cap V_h \subset \text{Ker } B. \quad \blacksquare \tag{4.10}$$

We have thus obtained a characterization of condition (3.20)  $\text{Ker } B_h \subset \text{Ker } B$  and given a partial answer to Q 3.3.  $\blacksquare$

Finally to conclude this analysis, we prove.

**LEMMA 4.3:** *The following statements are equivalent:*

$$\text{Im } B_h \subset (\text{Im } B) \cap W_h, \tag{4.11}$$

$$P_{W_h}(\text{Ker } B^*) \subset \text{Ker } B_h^*. \tag{4.12}$$



*Proof:* Let (4.11) hold and  $\varphi \in \text{Ker } B^*$ ; we want to show that

$$\bar{\varphi}_h = P_{W_h}(\varphi) \in \text{Ker } B_h^*.$$

But, by definition, we have

$$(\varphi, \lambda_h) = (\varphi_h, \lambda_h), \quad \forall \lambda_h \in W. \tag{4.13}$$

We have to show that  $(\varphi_h, \lambda_h) = 0$ , if  $\lambda_h \in \text{Im } B_h$ . However in this case we have  $\lambda_h \in \text{Im } B$  and  $(\varphi, \lambda_h) = 0$ , for  $\varphi \in \text{Ker } B^*$ .

Conversely, let (4.12) hold and  $\lambda_h \in \text{Im } B_h$ ; we want to show that  $\lambda_h \in \text{Im } B$ , that is  $\lambda_h$  is orthogonal to  $\text{Ker } B^*$ . But for any  $\varphi \in \text{Ker } B^*$ , one has

$$(\varphi, \lambda_h) = (\varphi_h, \lambda_h) = 0, \text{ for } \varphi_h = P_{W_h} \varphi \in \text{Ker } B_h^*. \quad \blacksquare$$

*Remark 4.1:* (4.12) is of course satisfied if  $\text{Ker } B_h^* = \text{Ker } B^*$ , in particular if  $B$  and  $B_h$  are surjective. Moreover if  $\text{Ker } B_h^* \subset \text{Ker } B^*$  inclusions in (4.11) and (4.12) may be replaced by equalities.  $\blacksquare$

*Remark 4.2:* In proposition 3.1, we supposed that  $\tilde{g}_h = P_{W_h}(g)$  belonged to  $\text{Im } B_h$ . From Lemma 4.1, we deduce that this will be the case in general, if and only if  $\text{Ker } B_h^* \subset \text{Ker } B^*$ . An important case is however  $g = \tilde{g}_h = 0$  where this last condition needs not be satisfied. This answers in part Q.3.1.  $\blacksquare$

*Remark 4.3:* In the same way, let  $\text{Ker } B_h^* \subset \text{Ker } B^*$ . Then by (4.2), we have as  $\lambda \in \text{Im } B$

$$\inf_{\varphi_h \in \text{Im } B_h} |\bar{\lambda} - \varphi_h| = \inf_{\varphi_h \in W_h} |\bar{\lambda} - \varphi_h|. \tag{4.14}$$

This answers Q. 3.4.  $\blacksquare$

The main problem that remains is to characterize the UCLP condition. This will be done in two steps. We first prove.

**PROPOSITION 4.1:** *Let  $\text{Im } B$  be closed, and let (4.1)-(4.3) be satisfied, the linear operator  $\Pi_h : V \rightarrow V_h$  being uniformly continuous, that is, there exists a constant  $c$ , independant of  $h$  such that*

$$|\hat{u}_h|_{V_h} = |\Pi_h u|_{V_h} \leq c |u|_V, \tag{4.15}$$

*Then there exists a constant  $k$ , independant of  $h$ , such that*

$$|B_h^* \varphi_h|_{V_h'} \geq k \inf_{\varphi_0 \in \text{Ker } B^*} |\varphi_0 + \varphi_h|_W. \tag{4.16}$$

*Proof:* We clearly have

$$|B_h^* \varphi_h|_{V_h'} = \sup_{v_h} \frac{|b(v_h, \varphi_h)|}{|v_h|_{V_h}} \geq \sup_{v \in V} \frac{|b(\Pi_h v, \varphi_h)|}{|\Pi_h v|_{V_h}}. \tag{4.17}$$

By (4.1) and (4.15) we have

$$\sup_{v \in V} \frac{|b(\Pi_h v, \varphi_h)|}{|\Pi_h(v)|_{V_h}} \geq \sup_{v \in V} \frac{1}{c} \frac{|b(v, \varphi_h)|}{|v|_V}, \tag{4.18}$$

and  $\text{Im } B$ , being closed,

$$\sup_{v \in V} \frac{|b(v, \varphi_h)|}{|v|_V} \geq k \inf_{\varphi_0 \in \text{Ker } B^*} |\varphi_0 + \varphi_h|. \quad \blacksquare \tag{4.19}$$

*Remark 4.4:* We have thus shown, that if  $\text{Ker } B_h^* \subset \text{Ker } B^*$  then building a uniformly continuous operator  $\Pi_h$  implies (4.16) which is *almost* the UCLP condition. Moreover from Remark 4.2, we see that  $\text{Ker } B_h^* \subset \text{Ker } B^*$  is also, in general, a condition for the existence of a discrete solution. We shall then, try to see what must be added to get the uniform lifting which is necessary for convergence proofs.  $\blacksquare$

We now prove

**PROPOSITION 4.2:** *If (4.11)-(4.12) hold, that is  $\text{Im } B_h \subset \text{Im } B$ , then (4.16) implies the UCLP condition. If moreover  $\text{Ker } B_h^* \subset \text{Ker } B^*$  [cf. (4.1)-(4.3)] then (4.16) and the UCLP condition are equivalent.*

*Proof:* Let (4.16) be satisfied, and consider  $\varphi_0 \in \text{Ker } B^*$ . If  $\text{Im } B_h \subset \text{Im } B \cap W_h$ , then by Lemma 4.3,  $P_{W_h} \varphi_0 \in \text{Ker } B_h^*$ . Moreover, one has

$$P_{W_h}(\varphi_h + \varphi_0) = \varphi_h + P_{W_h} \varphi_0,$$

and

$$|\varphi_h + P_{W_h} \varphi_0|_W = |P_{W_h}(\varphi_h + \varphi_0)|_W \leq |\varphi_h + \varphi_0|_W. \tag{4.20}$$

Thus

$$\inf_{\varphi_{0h} \in \text{Ker } B_h^*} |\varphi_h + \varphi_{0h}| \leq \inf_{\varphi_0 \in \text{Ker } B^*} |\varphi_h + \varphi_0| \leq 1/k |B_h^* \varphi_h|_{V_h}. \tag{4.30}$$

*Conversely*, if  $\text{Ker } B_h^* \subset \text{Ker } B^*$ , one has

$$\inf_{\varphi_{0h} \in \text{Ker } B_h^*} |\varphi_h + \varphi_{0h}| \leq \inf_{\varphi_0 \in \text{Ker } B^*} |\varphi_h + \varphi_0|, \tag{4.31}$$

so that (3.25) implies (3.14).  $\blacksquare$

*Remark 4.5:* Let us suppose that (4.3) and (4.11) are both satisfied. That is

$$\text{Ker } B_h^* = (\text{Ker } B^*) \cap W_h \quad \text{and} \quad \text{Im } B_h = (\text{Im } B) \cap W_h.$$

The decomposition,

$$W_h = (\text{Im } B_h) \oplus (\text{Ker } B_h^*), \tag{4.29}$$

is then merely the restriction to  $W_h$  of the decomposition  $W = (\text{Im } B) \oplus (\text{Ker } B^*)$ , if  $\varphi_h \in W_h$  is written as  $\varphi_h = \varphi_h^I + \varphi_h^k$  with  $\varphi_h^I \in \text{Im } B_h$  and  $\varphi_h^k \in \text{Ker } B_h^*$ , then  $\varphi_h^I \in \text{Im } B$  and  $\varphi_h^k \in \text{Ker } B^*$ .

The preceding result shows that this (hard to realize) case has a special importance. ■

*Remark 4.6:* The hypotheses of Proposition 4.1 and 4.2 are readily satisfied if one has  $\text{Ker } B_h^* = \text{Ker } B^*$ , in particular if  $B$  and  $B_h$  are both surjective. This situation will hold in most practical cases, so that we have reduced the verification of UCLP to building an uniformly continuous operator  $\Pi_h$ . We shall give examples in the next section showing how our results may be applied. ■

5. SOME EXAMPLES OF APPLICATIONS

The main result of this paper is that the abstract convergence condition of Brezzi [1], may be checked through the construction of an operator  $\Pi_h$  satisfying (4.15). We want to give rapidly here two examples where this operator may be explicitly built. Let us also refer to Brezzi-Raviart [2], where to our suggestion, Proposition 4.1 has been used to prove the convergence of the Hermann-Johnson's scheme for the biharmonic problem. Our first example treats of the approximation of Stokes' creeping flow problem in fluid mechanics and the second one to the approximation of Dirichlet's problem by mixed finite elements. Finally we give an example of a case where  $\text{Ker } B_h^* \not\subset \text{Ker } B^*$ .

*Example 5.1:* We consider in a domain  $\Omega \subset \mathbf{R}^2$ , with polygonal boundary, the Stokes problem. Let  $\vec{u} = (u_1, u_2)$  the velocity of the fluid,  $p$  the pressure, we have to solve:

$$-\Delta \vec{u} + \text{grad } p = \vec{f}, \tag{5.1}$$

$$\text{div } \vec{u} = 0, \tag{5.2}$$

$$\vec{u} \in (H_0^1(\Omega))^2, p \in L^2(\Omega). \tag{5.3}$$

We so have,

$$V = (H_0^1(\Omega))^2, \quad W = L^2(\Omega), \quad B = \text{div}.$$

Let us note that  $B$  is not surjective and that  $\text{Ker } B^*$  is formed by constants. Let us consider an approximation of  $H_0^1(\Omega)$  by quadratic conforming finite elements. The domain is triangulated and on each triangle, a function of  $V_h$  is defined by twelve degrees of freedom, which are the values of  $u_1$  and  $u_2$  at the vertices and at the midpoint of the sides. (Fig. 5.1).

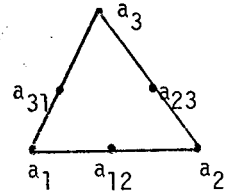


Figure 5.1.

These nodes are numbered as on the Figure.

We now consider, as in Fortin [4], and Crouzeix-Raviart [3], an approximation  $W_h$  of  $L^2(\Omega)$  by functions which are piecewise constants on the triangles. The operator  $B_h$  then associates to  $\vec{v}_h \in V_h$  its average divergence on each triangle.

Given  $\vec{u} \in V$ , it would be natural to define  $\hat{u}_h = \Pi_h u$ , taking,

$$\hat{u}_{kh}(a_i) = u_k(a_i), \quad i = 1, 2, 3; \quad k = 1, 2, \tag{5.4}$$

$$u_{kh}(a_{ij}) = \frac{1}{|a_i a_j|} \int_{a_i}^{a_j} u_k d\sigma, \quad i, j = 1, 2, 3, \quad k = 1, 2. \tag{5.5}$$

It is then easy to check that

$$b(u - u_h, \varphi_h) = \int_{\Omega} \operatorname{div}(u - u_h) \varphi_h dx = 0, \quad \varphi_h \in W_h. \tag{5.6}$$

However this definition is not possible as the functions of  $V$  are not smooth enough to define a point value  $u(a_i)$ .

Crouzeix and Raviart have shown that it is possible to build  $\Pi_h$  in an indirect way, first taking,

$$\tilde{u}_h = P_{V_h}(\vec{u}), \tag{5.7}$$

and then

$$\hat{u}_{kh}(a_i) = \tilde{u}_{kh}(a_i), \quad i = 1, 2, 3; \quad k = 1, 2. \tag{5.8}$$

$$\hat{u}_{kh}(a_{ij}) = \frac{1}{|a_i - a_j|} \int_{a_i}^{a_j} \tilde{u}_{kh} d\sigma, \quad i, j = 1, 2, 3; \quad k = 1, 2. \tag{5.9}$$

UCLP condition is then proved in [3], for a slightly more general case. By Lemma 4.1, this proves that  $\operatorname{Ker} B_h^* = \operatorname{Ker} B^*$  and is therefore formed of constant functions. ■

*Example 5.2:* Raviart and Thomas [8], introduce a mixed approximation for Dirichlet's problem in  $\mathbb{R}^2$ , using the following functions spaces

$$V = H(\operatorname{div}; \Omega) = \{ p \mid p = (p_1, p_2) \in (L^2(\Omega))^2, \operatorname{div} p \in L^2(\Omega) \}. \tag{5.10}$$

$$W = \{ v \mid v \in L^2(\Omega) \}. \tag{5.11}$$

They then solve for  $f \in L^2(\Omega)$ ,

$$(p, q)_{(L^2(\Omega))^2} + (\operatorname{div} q, u) = 0, \quad \forall q \in V, \tag{5.12}$$

$$(\operatorname{div} p, v) = (f, v), \quad \forall v \in W. \tag{5.13}$$

We thus have for  $q \in V, v \in W$ :

$$b(q, v) = (\operatorname{div} q, v), \tag{5.14}$$

that is

$$Bq = \operatorname{div} q. \tag{5.15}$$

Let  $u$  be the unique solution of

$$\left. \begin{aligned} -\Delta u &= f, \\ u|_{\Gamma} &= 0. \end{aligned} \right\} \tag{5.16}$$

Then  $(-\text{grad } u, u)$  is the unique solution to (5.12)-(5.13).

Following [8], we define  $V_h \subset V$ , using piecewise-polynomials of degree  $k + 1$  on a triangulation  $\mathcal{T}_h$  of  $\Omega$ . It is required that on any triangle boundary, the normal trace  $q_h \cdot \nu$  of  $q_h \in V$ , be a polynomial of degree  $k$  and that this normal trace be continuous from one triangle to another. We define  $W_h$  using piecewise polynomials of degree  $\leq k$  on each triangle, without any continuity condition. With respect to  $V_h$ , Raviart and Thomas show that such a space can be built and that the degrees of freedom on each triangle  $K$  can be chosen as

$$\text{the moments of order } \leq k \text{ of } q_h \cdot \nu \text{ on } \partial K, \tag{5.17}$$

$$\text{the moments of order } \leq k - 1 \text{ of } q_h \text{ on } K. \tag{5.18}$$

The degrees of freedom (5.17) indeed insure the continuity of  $q_h \cdot \nu$  on interfaces. We now show that we can use the results of section 4 to prove the convergence of this approximation. In order to do so, we have to build an uniformly continuous linear operator  $\Pi_h$  from  $V$  into  $V_h$  such that  $b(q - \Pi_h q, v_h) = 0, \forall v_h \in W$ , or more precisely

$$\int_{\Omega} (\text{div } q - \text{div } \Pi_h q) v_h \, dx = 0, \quad \forall v_h \in W_h. \tag{5.19}$$

Integrating by parts on each triangle  $K$ , this becomes,

$$-\sum_K \int_K \text{grad } v_h \cdot (q - \Pi_h q) \, dx + \int_{\partial K} v_h (q - \Pi_h q) \cdot \nu \, d\sigma = 0. \tag{5.20}$$

Let us define tentatively  $\Pi_h$  as the interpolation operator on the degrees of freedom of  $V_h$ , that is on each triangle  $K$ , and for any side  $K'$  of  $K$ .

$$\int_{K'} (q - \Pi_h q) \cdot \nu \varphi \, d\sigma = 0, \quad \forall \varphi \in P_k(K'), \tag{5.21}$$

$$\int_K (q - \Pi_h q) \varphi \, dx = 0, \quad \forall \varphi \in P_{k-1}(K). \tag{5.22}$$

Then, as  $\text{grad } v_h|_K \in P_{k-1}(K)$  and  $v_h|_{K'} \in P_k(K')$ , for any  $v_h \in W_h$ , condition (5.19) is evidently satisfied.

A problem however arises, as for  $q \in V$ , the moments on the sides may not be defined due to a lack of regularity. If however we can take  $q \in (H^1(\Omega))^2$ , we can use (5.21) and (5.22) and moreover we have.

$$|\tilde{q} - \Pi_h \tilde{q}|_{H(\text{div}, \Omega)} \leq Ch |\tilde{q}|_{1, \Omega}, \tag{5.23}$$

which is indeed stronger than the uniform continuity requirement.

We now show that we can deduce the result for  $q \in H(\text{div}; \Omega)$  from the result for  $q \in (H^1(\Omega))^2$ . To do so, we build for any  $q \in H(\text{div}, \Omega)$ , a  $\tilde{q} \in (H^1(\Omega))^2$

such that

$$\left. \begin{aligned} \operatorname{div} q &= \operatorname{div} \tilde{q}, \\ |\tilde{q}|_{(H^1(\Omega))^2} &\leq C |q|_{H(\operatorname{div}, \Omega)}. \end{aligned} \right\} \quad (5.24)$$

This is easily done, if the boundary  $\Gamma$  of  $\Omega$  is smooth enough, by solving

$$\left. \begin{aligned} -\Delta \varphi &= \operatorname{div} q \\ \varphi|_{\Gamma} &= 0. \end{aligned} \right\} \quad (5.25)$$

Then  $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$  and setting

$$\tilde{q} = \operatorname{grad} \varphi, \quad (5.26)$$

solves (5.24).

We now define:

$$\Pi_h q = \Pi_h \tilde{q}. \quad (5.27)$$

We then have

$$\int_{\Omega} (\operatorname{div} q - \operatorname{div} \Pi_h q) v_h \, dx = \int_{\Omega} (\operatorname{div} \tilde{q} - \operatorname{div} \Pi_h \tilde{q}) v_h \, dx = 0 \quad (5.28)$$

and the uniform continuity of  $\Pi_h$  follows from (5.24) and (5.23).

This proves from Lemma 4.1 that  $\operatorname{Ker} B_h^* \subset \operatorname{Ker} B^*$  and therefore that  $B_h^*$  is surjective. Proposition 4.2 then implies UCLP condition.

It is also a trivial task to prove  $\operatorname{Ker} B_h \subset \operatorname{Ker} B$ .

Indeed for  $q_h \in V_h$ ,  $\operatorname{div} q_h|_K \in P_k(K)$ . Let then  $v_h$  be the  $L^2(\Omega)$  projection of  $v \in W$  on  $W_h$ . Then:

$$\int_{\Omega} \operatorname{div} q_h (v - v_h) \, dx = 0, \quad q_h \in V_h, \quad (5.29)$$

hence the result by Lemma 4.2. ■

*Example 5.3:* We go back to the problem of Example 5.1 but we now consider bilinear finite elements on a rectangular mesh, the degrees of freedom are the values of  $u_1$  and  $u_2$  at the vertices.

It is then classical to get in this way an approximation of  $(H_0^1(\Omega))^2$  (cf. [4]).

We then consider  $W_h$  formed by piecewise constants on the rectangle. The operator  $B_h$  still associates to  $u_h$  its average divergence on each rectangle. It is an easy task to verify that in this case, the Kernel of the discrete gradient is generated by two piecewise constant functions,

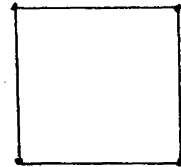


Figure 5.2.

taking two different values on a chess-board like pattern (Fig. 5.3).

We have a very simple case where  $\text{Ker } B_h^* \not\subset \text{Ker } B^*$ . The convergence of this approximation has been checked experimentally, but the proof seems to be an open question. The same is true of some approximations used by engineers using conforming elements for both the velocity and the pressure. ■

$c_1$	$c_2$	$c_1$
$c_2$	$c_1$	$c_2$
$c_1$	$c_2$	$c_1$

Figure 5.3.

## 6. CONCLUSION

We have shown that in some circumstances, convergence proofs for the approximations of saddle-point problem may be obtained through building an operator  $\Pi_h$  which in many practical cases turns out to be an "interpolation" operator, in a more or less generalized sense. This fact has been useful to get proofs in the biharmonic problem [2] and for the second-order elliptic problems [8]. The result developed here was in fact implicit in [3] and [4] where convergence proof for the approximation of Stokes problem were studied. Example 5.3 shows that very simple cases are still out of reach of the present theory. The author is thankful to P.A. Raviart and F. Brezzi for helpful discussions and suggestions.

## REFERENCES

1. F. BREZZI, *On the existence, uniqueness and approximation of saddle-point problems arising from langrangian multipliers*, R.A.I.R.O., vol. 8, août 1974, 2, pp. 129-151.
2. F. BREZZI and P. A. RAVIART, *Mixed finite element methods for fourth order elliptic equations* (to appear). Rapport interne No. 9, École polytechnique, Centre de Mathématiques Appliquées.
3. M. CROUZEIX and P. A. RAVIART, *Conforming and non conforming finite elements methods for solving the stationary Stoke equations*, R.A.I.R.O., 3, 1974, pp. 33-76.
4. M. FORTIN, *Utilisation de la méthode des éléments finis en mécanique des fluides*, CALCOLO, vol. XII, fasc. IV, pp. 405-441 and Vol. XII, fasc. 1, pp. 1-20.
5. C. JOHNSON, *On the convergence of a mixed finite element method for plate bending problems*, Num. Math., vol. 21, 1973, pp. 43-62.
6. J. T. ODEN, *Some contributions to the mathematical theory of mixed finite element approximations*, Tokyo Seminar on Finite Elements, Tokyo, 1973.
7. J. T. ODEN et J. N. REDDY, *On mixed finite element approximations*, Texas Institute for Computational Mechanics, The University of Texas at Austin, 1974.
8. P. A. RAVIART and J. M. THOMAS, *Amixed finite element method for 2nd order elliptic problems* (to appear).
9. K. YOSIDA, *Functional analysis*, Springer Verlag 1965.