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SOME NONCONFORMING FINITE ELEMENTS FOR THE PLATE BENDING PROBLEM

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Abstract : *We consider various non-conforming finite element methods to solve the plate bending problem. We show that all the elements pass the "Patch test", a pratical condition for convergence. Using general results, we derive in each case the error bounds for both strains and displacements.*

1 – INTRODUCTION

We shall study the convergence and accuracy of the approximate solutions of the plate bending problem obtained with finite element methods using some nonconforming elements. These elements are the quadratic triangular element of Morley [15], two cubic triangular elements recently introduced by Fraeijs de Veubeke [9], the rectangular element of Adini [1] and the triangular element of Zienkiewicz [3].

To obtain the corresponding error estimates, the keystone is the patch test of Irons [10]. The first three elements pass the patch test for polynomials of degree less than or equal to 2 whatever the mesh geometry. Zienkiewicz's triangle passes it only if the mesh is generated by three sets of parallel lines. Adini's rectangle passes it for polynomials with degree less than or equal to 2, whatever are the dimensions of the rectangles and it passes a "super" patch test – so called by Strang [19] – when all rectangles are equal.

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The main results are the following : Let u be the exact solution, let u_h be the approximate solution that one gets on a mesh of order h (the supremum of the element side length), then :

$$\| u - u_h \|_2 \leqslant c\, h \quad (| u |_3 + h\, | u |_4)$$

$$\| u - u_h \|_0 \leqslant c\, h^2 \; (| u |_3 + h\, | u |_4)$$

for the first three triangular elements, and :

$$\| u - u_h \|_2 \leqslant c\, h \quad | u |_3$$

$$\| u - u_h \|_1 \leqslant c\, h^2 \quad | u |_3$$

for Zienkiewicz's triangular element and Adini's element, and finally :

$$\| u - u_h \|_2 \leqslant c\, h^2 \quad | u |_4$$

for Adini's element, when all rectangles are equal.

In these inequalities, c is a constant independent of h and $\| \cdot \|_k$ (resp. $| \cdot |_k$) denotes the discrete equivalent of the Sobolev H^k norm (resp. semi norm).

We have assumed that the exact and approximated domains are similar, which is an important restriction if one wants to use Zienkiewicz' or Adini's element (see the last section). The proofs for the error bounds are first given in the case of the clamped plate, and we show (in the last section) that they are still valid with some other boundary conditions, such as, for example, in the case of a simply supported plate.

An outline of the paper is as follows :

In section II, we introduce some notations and the plate equations, and we derive some general estimates which are used subsequently.

In section III, we study the elements of Morley and Fraeijs de Veubeke, which seem to have similar properties. Section IV is devoted to the element of Adini, and section V to the element of Zienkiewicz. In each of these sections, we first describe the elements, we check that the discrete energy is positive definite (for $\sigma < 1$), we show that the patch test is satisfied, we give an interior error estimate, we derive an error estimate for the terms arising at the interfaces between the elements, and finally, using techniques similar to these developped by Ciarlet et Raviart [6], we prove the general error estimates.

In the last section, we consider the remarks already done above, and we also show that if one sets the Poisson ratio σ equal to the (irrealistic) value one in order to simplify the energy formula, which is valid in the continuous case for the clamped plate, the first three elements should no longer be used since the discrete energy is no longer a positive definite form. For the last two elements, the energy is still a positive definite form. However, it is not known whether this positive definitiveness is uniform with respect to h.

II – EQUATIONS OF THE PLATE BENDING PROBLEM AND GENERAL ESTIMATES

The problem of the plate bending can be written as follows [13]. Let Ω be a bounded domain of the plane (x,y), with boundary Γ. We shall denote by s a curvilinear abscissa along Γ and $\frac{\partial}{\partial n}$ the derivative along the outer normal on Γ and $\frac{\partial}{\partial s}$ the tangential derivative along Γ. We define the inner product (f,g) by $(f,g) = \int_\Omega f(x,y)\, g(x,y)\, dx\, dy$ and the corresponding norm by :

$$\|f\|_{0,\Omega} = (f,f)^{1/2}$$

Given an integer $m \geqslant 0$, we consider the usual Sobolev space :

$$H^m(\Omega) = \left\{ v,\ v \in L^2(\Omega)\ ,\ \partial^\alpha v \in L^2(\Omega)\ ,\ |\alpha| \leqslant m \right\} \quad ,$$

with the norm and semi norm $\|\ \ \|_{m,\Omega}$ and $|\ \ |_{m,\Omega}$ defined by :

$$\|v\|_{m,\Omega} = \Big(\sum_{|\alpha| \leqslant m} \|\partial^\alpha v\|^2_{0,\Omega}\Big)^{1/2},$$

$$|v|_{m,\Omega} = \Big(\sum_{|\alpha| = m} \|\partial^\alpha v\|^2_{0,\Omega}\Big)^{1/2},$$

where α is a multiindex such that $\alpha = (\alpha_1, \alpha_2)$, $\alpha_i \geqslant 0$,

$$|\alpha| = \alpha_1 + \alpha_2 \text{ and } \partial^\alpha = \Big(\frac{\partial}{\partial x}\Big)^{\alpha_1} \Big(\frac{\partial}{\partial y}\Big)^{\alpha_2} .$$

The following space will be of particular importance :

$$H_0^2(\Omega) = \left\{ v \;;\; v \in H^2(\Omega) \;,\; v = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma \right\}.$$

Over the Sobolev space $H_0^2(\Omega)$, the semi norm $|\cdot|_{2,\Omega}$ is a norm, which is equivalent to the norm $\|\cdot\|_{2,\Omega}$. We let :

$$(2.1) \quad \langle u,v \rangle (x,y) = \Delta u \cdot \Delta v + (1-\sigma)\left(2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2}\right),$$

where the constant σ is the Poisson's coefficient and satisfies $0 < \sigma \leqslant \frac{1}{2}$.

The bilinear form a (. , .) is defined by :

$$(2.2) \quad a(u,v) = \int_\Omega \langle u,v \rangle (x,y)\; dx\; dy.$$

The problem of the clamped plate can be formulated as follows [13] :

$$(2.3) \quad \begin{cases} \textbf{To find } u \in H_0^2(\Omega) & \textbf{such that} \\ a(u,v) = (f,v) & \textbf{for all } v \in H_0^2(\Omega) \end{cases}$$

The bilinear form $a(.,.)$ is H_0^2 – elliptic , since

$$(2.4) \quad a(v,v) = \sigma\, |\Delta v|_{0,\Omega}^2 + (1-\sigma)\, |v|_{2,\Omega}^2 \quad \text{for all } v \in H_0^2(\Omega).$$

Since it is also continuous over the space H_0^2, the problem (2.3) has a unique solution $u \in H_0^2$, and it is known that $u \in H^3(\Omega) \cap H_0^2(\Omega)$ if Ω is a convex polygon ([11], [12]).

If we use the **Green's formula,** we get [13] :

$$(2.5) \quad a(u,v) = \int_\Omega \Delta^2 u \cdot v\; dx\, dy + \int_\Gamma \left(\Delta u - (1-\sigma)\frac{\partial^2 u}{\partial s^2}\right)\frac{\partial v}{\partial n}\, ds - $$
$$- \int_\Gamma \frac{\partial}{\partial n}(\Delta u)\cdot v\, ds + (1-\sigma)\int_\Gamma \frac{\partial^2 u}{\partial n \partial s}\frac{\partial v}{\partial s}\, ds.$$

If the solution u of problem (2.3) is smooth enough, then it is also the solution of the problem :

(2.6) $$\Delta^2 u = f \qquad \text{in } \Omega \ ,$$

(2.7) $$u = \frac{\partial u}{\partial n} = \qquad \text{on } \Gamma$$

Hence, the solution u seems to be independent of the Poisson's coefficient σ (see section VI).

We shall now define a finite element approximation of problem (2.3).

Definition 2.1 : We let $P(\ell)$ denote the space of all polynomials in x and y, of degree less than or equal to ℓ.

Consider a triangulation $\mathcal{T}_h$ of Ω with elements K with boundary ∂K (the elements will be either triangles or rectangles). To each element K, we associate a set of degrees of freedom and a finite dimensional space P_K of shape functions uniquely defined by their values at the degrees of freedom. We assume that the functions of P_K are at least *twice continuously differentiable* and that P_K *contains the space* $P(2)$ so that a first heuristical criterion of convergence ([21]) is satisfied.

We define the following geometrical parameters :

$$h(K) = \text{diameter of } K \ , \quad h = \sup_{K \in \mathcal{T}_h} \{ h(K) \} \ ,$$

$$p(K) = \sup \{ \text{diameter of the circles inscribed in } K \} \ .$$

In what follows, we shall always assume that we have $\frac{h(K)}{p(K)} \leqslant \lambda$, where λ is a constant > 0 independent of h.

Now let V_h be a finite dimensional space of real valued functions defined on $\bar{\Omega}$, continuous at the degrees of freedom of the elements K belonging to $\mathcal{T}_h$ and whose restrictions to each elements K belong to P_K. A second heuristical criterion of convergence ([21]) would imply that the functions of V_h should be continuously differentiable on $\bar{\Omega}$. On the contrary, as is often done for practical calculations we shall consider a space V_h of functions which are not continuously differentiable on $\bar{\Omega}$; the elements which are used are then *nonconforming or incompatible* ([10], [19], [20], [21]). Consider the following bilinear form :

(2.8) $$a_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} \int_K < u_h, v_h > (x, y) \, dx \, dy \ , \text{ for } u_h, v_h \in V_h.$$

The finite element approximation of problem (2.3) *will be defined as follows:*

$$(2.9)\qquad \begin{cases} \textit{To find } u_h \in V_h & \textit{such that} \\ a_h(u_h, v_h) = (f, v_h) & \textit{for all } v_h \in V_h \ . \end{cases}$$

REMARK 2.1. : This way of defining the problem is quite natural since, for pratical purposes, the stiffness matrix of the problem is assembled element by element.

We let :

$$\| v_h \|_h = \left(\sum_{K \in \mathcal{T}_h} \int_K \left(\left(\frac{\partial^2 v_h}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 v_h}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 v_h}{\partial y^2}\right)^2 \right) \mathrm{d}x\,\mathrm{d}y \right)^{1/2}$$

for $v_h \in V_h$.

We then have :

$$(2.10)\qquad a_h(v_h, v_h) \geqslant (1-\sigma)\ \| v_h \|_h^2 \quad \text{for all } v_h \in V_h \,.$$

If the following hypothesis holds :

$$(2.11)\qquad \| \cdot \|_h \ \textit{ is a norm over the space } V_h \ ,$$

then problem (2.9) has a unique solution $u_h \in V_h$.

We shall now derive some general estimates for the errors done on the *strains* (measured by the norm $\| \cdot \|_h$) and on the *displacements* (measured by the norm $\| \ \|_{0,\Omega}$). In the next paragraphs, we shall apply those results to different types of elements : Morley's quadratic triangular element ([15]) called T.Q.M. in what follows, FRAEIJS de VEUBEKE's ([9]) triangular elements (F.V.1 and F.V.2), ARI ADINI's rectangle ([1]) and ZIENKIEWICZ's triangle ([3]).

For the first three elements, the displacements are not continuous at the interfaces between elements and for the last two elements, the displacements are continuous, but the first derivatives of the displacements are not continuous at the interfaces.

We have the following result ([19] [20]).

Theorem 2.1 : *Assume that hypothesis* (2.11) *holds, and let* $u_h \in V_h$ *be the solution of problem* (2.9) *and* u *the solution of problem* (2.3).

Then we have :

(2.12) $$\|u - u_h\|_h \leqslant C \left(\inf_{v \in V_h} \| u - v \|_h + \sup_{w \in V_h} \frac{|E_h(u,w)|}{\| w \|_h} \right)$$

where C is a constant > 0 independent of h, and where :

$$E_h(u,w) = \begin{cases} E_1(u,w) & \text{if } V_h \subset C^0(\bar{\Omega}) \\ E_1(u,w) + E_2(u,w) + E_3(u,w) & \text{if } V_h \not\subset C^0(\bar{\Omega}) \end{cases}, \text{ with}$$

$$E_1(u,w) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \left((1-\sigma) \frac{\partial^2 u}{\partial s^2} - \Delta u \right) \frac{\partial w}{\partial n} \, ds ,$$

$$E_2(u,w) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} -(1-\sigma) \frac{\partial^2 u}{\partial n \partial s} \frac{\partial w}{\partial s} \, ds$$

$$E_3(u,w) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial}{\partial n} (\Delta u) \, w \, ds$$

Proof : We consider the expression X_h defined as follows :

$$X_h = a_h(u_h - v, u_h - v) \quad \text{for any } v \in V_h.$$

We have : $X_h \geqslant (1-\sigma) \| u_h - v \|_h^2$ and

$$X_h = (f, u_h - v) - a_h(v, u_h - v) = a_h(u-v, u_h - v) + (f, u_h - v) - a_h(u, u_h - v).$$

But : $(f,u_h - v) - a_h(u,u - v) = \sum_{K \in \mathcal{T}_h} \int_K (\Delta^2 u \cdot (u_h - v) - <u, u_h - v>) \, dx \, dy.$

Applying Green's formula (2.5) to each element $K \in \mathcal{T}_h$, we get :

$$(f,u_h - v) - a_h(u,u_h - v) = \sum_{K \in h} \int_{\partial K} \left\{ \left((1-\sigma) \frac{\partial^2 u}{\partial s^2} - \Delta u \right) \frac{\partial}{\partial n} \right) (u_h - v)$$

$$+ \frac{\partial}{\partial n} (\Delta u) \cdot (u_h - v) - (1-\sigma) \frac{\partial^2 u}{\partial n \partial s} \frac{\partial}{\partial s} (u_h - v) \Bigg\} ds,$$

which yields inequality (2.12).

REMARK 2.2 : The expressions E_1 (u,w) and E_2 (u,w) can be defined for any $w \in V_h$ and any $u \in H^3$ (Ω). The expression E_3 (u,w) can be defined for any $w \in V_h$ and any $u \in H^4$ (Ω).

Definition 2.2 : For the non-conforming elements, the second heuristical criterion of convergence (continuity requirements) will be replaced by a weaker one called *Patch Test* ([10], [19]). In its global form, *the Patch Test will consist in showing that :*

$$E_h(u,w) \equiv 0 \quad \textit{for any } u \in P(2) \quad \textit{and any } w \in V_h.$$

This definitions can be made local if we replace w by the different basis functions of V_h.

We shall see later on that all the elements previously mentioned pass the patch test, which is *a practical condition for convergence.* For these five examples, the patch test combined with the continuity at the nodes will provide necessary and sufficient conditions for convergence. For this purpose, the following lemma [4] may be useful :

Lemma 2.1 : *Let Ω be an open bounded subset of R^2 with a sufficiently smooth boundary, let k and ℓ be two integers and let W be a space of functions satisfying $P(\ell) \subset W \subset H^{\ell+1}(\Omega)$; the space W is considered as being equipped with the norm $\|\cdot\|_{\ell+1,\Omega}$. Finally, let $A : H^{k+1}(\Omega) \times W \to R$ be a continuous bilinear form which satisfies.*

$$A(u,v) = 0 \quad \textit{for all } u \in P(k),\ v \in W,$$

$$A(u,v) = 0 \quad \textit{for all } u \in H^{k+1}(\Omega),\ v \in P(\ell).$$

Then there exists a constant $C = C(\Omega)$ such that :

$$(2.13) \quad |A(u,v)| \leqslant C\,\|A\|\ |u|_{k+1,\Omega}\ |v|_{\ell+1,\Omega} \quad \textit{for all } u \in H^{k+1}(\Omega),\ \ v \in W.$$

This lemma will be applied in the case where Ω is an element $K \in \mathcal{T}_h$, and where $k = 0$, $\ell = 0$, W being the space of the first order derivative along a given direction of the functions of P_K. We shall now derive some general estimates for the displacements and their first order derivatives. We shall need the following result for the biharmonic problem, which is true whenever Ω has a sufficiently smooth boundary ([14]).

$$(2.14)\quad \begin{cases} \text{For any } g \in H^{-\ell}(\Omega), \text{ for } \ell = 0, 1, \text{ the problem of finding } \varphi \text{ such that} \\ \Delta^2 \varphi = 0 \quad \text{in } \Omega, \\ \varphi = \dfrac{\partial \varphi}{\partial n} = 0 \quad \text{on } \Gamma, \\ \text{has a unique solution in } H^{4-\ell}(\Omega) \cap H_0^2(\Omega) \text{ and we have} \\ \|\varphi\|_{4-\ell,\Omega} \leqslant C \quad \|g\|_{H^{-\ell}(\Omega)} \quad \text{for } \ell = 0, 1. \end{cases}$$

Theorem 2.2 : *Assume that hypothesis* (2.11) *holds and let* $u_h \in V_h$ *be the solution of problem* (2.9). *Then we can write :*

$$(2.15)\quad |u - u_h|_{L^2(\Omega)} \leqslant C \sup_{\varphi \in H^k(\Omega)} \left\{ \inf_{\varphi_h \in V_h} \frac{|E(u, u_h, \varphi\ \varphi_h)|}{\|\varphi\|_{k,\Omega}} \right\}$$

for $k = 3$ *or* 4,

Where :

$$E(u, u_h, \varphi, \varphi_h) = a_h(u - u_h, \varphi - \varphi_h) + E_h(\varphi, u - u_h) + E_h(u, \varphi - \varphi_h).$$

If the inclusion $V_h \subset C^0(\bar{\Omega})$ *holds, then we have :*

$$(2.16)\quad \|u - u_h\|_{1,\Omega} \leqslant C \sup_{\varphi \in H^3(\Omega)} \inf_{\varphi_h \in V_h} \left\{ \frac{|E(u, u_h, \varphi, \varphi_h)|}{\|\varphi\|_{3,\Omega}} \right\}.$$

Proof : We shall use the classical *duality argument* ([2], [17]) :

We can write :

$$(2.17)\quad \|u - u_h\|_{\ell,\Omega} = \sup_{g \in H^{-\ell}(\Omega)} \frac{|(u - u_h, g)|}{\|g\|_{H^{-\ell}(\Omega)}}, \text{ with}$$

$\ell = 0$ or 1 when $V_h \subset C^0(\bar{\Omega})$ and $\ell = 0$ when $V_h \not\subset C^0(\bar{\Omega})$.

Let φ be the solution of the biharmonic problem :

$$\begin{cases} \Delta^2 \varphi = g & \text{in } \Omega, \\ \varphi = \dfrac{\partial \varphi}{\partial n} = 0 & \text{on } \Gamma. \end{cases}$$

Using Green's formula, we may write :

$$(u-u_h, g) = (u-u_h, \Delta^2 \varphi) = a_h (u-u_h, \varphi) + E_h(\varphi, u-u_h).$$

Moreover, for any $\varphi_h \in V_h$, we have :

$$a_h (u-u_h, \varphi_h) = -E_h (u, \varphi_h) = -E_h (u, \varphi_h - \varphi).$$

From the last two equalities, we get :

$$(2.18)\ (u-u_h, g) = a_h (u-u_h, \varphi_h - \varphi_h) + E_h (\varphi, u-u_h) + E_h (u, \varphi_h - \varphi).$$

Equalities (2.17) and (2.18) combined with hypothesis (2.14) lead to inequalties (2.15) and (2.16).

III – MORLEY'S TRIANGLE AND FRAEIJS DE VEUBEKE ELEMENTS

Given a triangle K with vertices A_i with coordinates (x_i, y_i), $1 \leqslant i \leqslant 3$, we let :

λ_i = *area coordinates* relative to the vertices A_i, $1 \leqslant i \leqslant 3$,

$b_1 = y_2 - y_3$, $c_1 = x_3 - x_2$, $d_1 = \frac{1}{2}(c_2 - c_3)\, c_1 + \frac{1}{2}(b_2 - b_3)\, b_1$, b_i, c_i and d_i for $i = 2$ or 3 being defined by cyclic permutation of the indices,

$\ell_i = (b_i^2 + c_i^2)^{1/2}$ = length of the edge $A_j A_k$ opposite to A_i, $1 \leqslant i \leqslant 3$,

$\vec{m}_i = \overrightarrow{A_i A_{jk}}$, where A_{jk} is the mid-point of the edge $A_j A_k$, $1 \leqslant i \leqslant 3$,

A_{jjk} and A_{kkj} = the points dividing the edge $A_j A_k$ in the ratio :

$\dfrac{\overrightarrow{A_j A_{jjk}}}{\overrightarrow{A_j A_k}} = \dfrac{\overrightarrow{A_k A_{kkj}}}{\overrightarrow{A_k A_j}} = \dfrac{3 - \sqrt{3}}{6}$ (= the *Gauss quadrature points* for the edge $A_j A_k$), $1 \leqslant j \leqslant 3$, $k \neq j$,

Δ = area of K and G = centroid of K (figure 1).

Given a function P defined and continuously differentiable on K , we let :

$$P_i = P(A_i),\ \frac{\partial P}{\partial n_i} = \frac{\partial P}{\partial n}(A_{jk}),\ \left[\frac{\partial P}{\partial n}\right]_i = \frac{1}{\ell_i}\int_{A_j}^{A_k} \frac{\partial P}{\partial n}\, ds,\ 1 \leqslant i \leqslant 3, \text{where}$$

$\frac{\partial P}{\partial n}$ denotes the normal derivatives to the edge $A_j\ A_k$,

$$P_{jk} = P(A_{jk}) \quad . \quad P_G = P(G)$$

$$\frac{\partial P}{\partial n_{jjk}} = \frac{\partial P}{\partial n}(A_{jjk}) \quad , \quad 1 \leqslant j \leqslant 3, k \neq j \ .$$

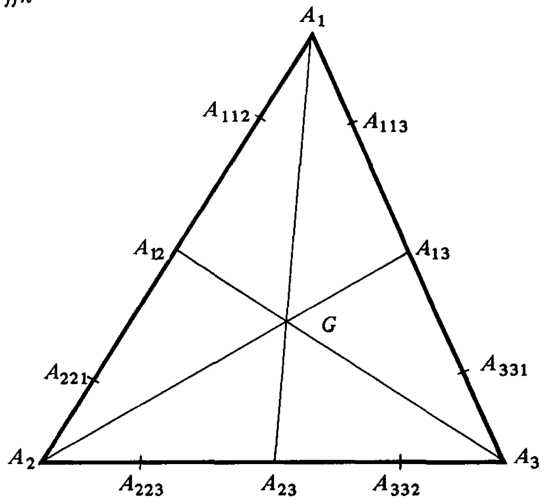


Figure 1.
Some particular points on a triangle

Definition 3.1 : *Morley's element* (T.Q.M.) is defined as follows [15] :

(i) The degrees of freedom are the values of the function at the vertices of the triangle and the values of the first normal derivatives at the mid-point of the edges of the triangle (figure 2).

ii) The space P_K of the shape functions is $P(2)$.

It can be shown that any function of P_K is uniquely determined by its degrees of freedom described above, and we have :

$$(3.1) \qquad P(x,y) = \sum_{i=1}^{3} \varphi_i(x,y)\, P_i + \sum_{i=1}^{3} \psi_i(x,y) \frac{\partial P}{\partial n_i}\ , \text{ where}$$

$$\psi_i(x,y) = -2\Delta \cdot \ell_i^{-1} \cdot \lambda_i(1-\lambda_i) \quad , \quad 1 \leqslant i \leqslant 3 \quad ,$$

$$\varphi_1(x,y) = \lambda_1^2 + \alpha_2 \psi_2 + \alpha_3 \psi_3 \text{ , with } \alpha_i = \frac{b_1 b_i + c_1 c_i}{2\Delta \cdot \ell_i} \text{ ,i = 2 or 3,}$$

φ_2 and φ_3 being defined by cyclic permutation of the indices.

Given a function u defined and continuously differentiable on K, its *interpolate* $r_K u$ will be the function of P_K which is equal to u at the vertices of K and whose first normal derivative at the mid point of each edge are equal to the mean value of the first normal derivative of u along this edge.

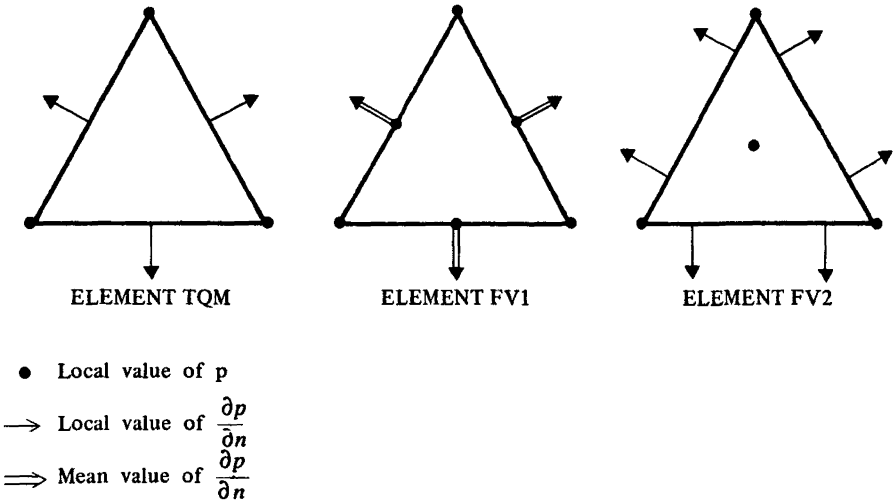


Figure 2.
Elements TQM, FV1 and FV2

Definition 3.2 : The first *Fraeijs de Veubeke's element* (F.V.1) is defined as follows [9].

i) The degrees of freedom are the values of the functions at the vertices, at the centroid and at the mid-points of the edges of the element, and the mean value of the first normal derivative along each edge (figure 2).

ii) The problem of finding a complete cubic assuming any degrees of freedom has generally no solution, unless the following relationship is satisfied :

(3.2) $$P_G = \frac{1}{27} \sum_{i=1}^{3} P_i + \frac{8}{27} \sum_{i \neq j} P_{ij} - \frac{2}{27} \sum_{i=1}^{3} \left[\vec{D}_P \cdot \vec{m}_i \right] \text{, where}$$

$$\vec{D}_P \cdot \vec{m}_1 \equiv \frac{1}{\ell_1} \int_{A_2}^{A_3} \vec{D}_P \cdot \vec{m}_1 \, ds = \frac{2\Delta}{\ell_1^2} \int_{A_2}^{A_3} \frac{\partial P}{\partial n} \, ds + \frac{d_1}{\ell_1^2} (P_3 - P_2),$$

$\left[\vec{D}_P \cdot \vec{m}_i \right]$, $i = 2$ or 3, being defined by cyclic permutation of the indices.

iii) The space P_K of the shape functions will be the space of all polynomials of $P(3)$ which satisfy equality (3.2), and we have :

(3.3) $$P(x,y) = \sum_{i=1}^{3} \Phi_i(x,y) P_i + \sum_{\substack{i \neq j \\ k = 6-i-j}} (4\lambda_k (1-\lambda_k)(1-2\lambda_k) + 4\lambda_i \lambda_j -$$
$$- 12 \lambda_1 \lambda_2 \lambda_3) P_{ij} - \sum_{i=1}^{3} \frac{2\Delta}{l_i} \lambda_i (2\lambda_i - 1)(\lambda_i - 1) \left[\frac{\partial P}{\partial n} \right]_i ,$$

where $\Phi_1(x,y) = \lambda_1(\lambda_1 - \frac{1}{2})(\lambda_1 + 1) + 3\lambda_1 \lambda_2 \lambda_3 - \frac{d_2}{\ell_2^2} \lambda_2 (2\lambda_2 - 1)(\lambda_2 - 1)$

$+ \frac{d_3}{\ell_3^2} \lambda_3 (2\lambda_3 - 1)(\lambda_3 - 1)$, $\Phi_i(x,y)$ for $i = 2$ or 3 being defined by cyclic permutation of the indices.

We have the inclusions $P(2) \subset P_K \subset P(3)$.

Given a function u defined and continuously differentiable on K, its *interpolate* $r_K u$ will be the function of P_K which is equal to u at the vertices and at the mid-points of the edges of K and for which the mean value of the first normal derivative along each edge is equal to the mean value of the first normal derivative of u along this edge.

Definition 3.3 : *Fraeijs de Veubeke's element* F.V.2. is defined as follows.

i) The degrees of freedom are the values of the functions at the vertices and at the centroid of the element, and the values of the first normal derivative at the Gauss points on each edge (figure 2).

ii) The space P_K of the shape functions is $P(3)$.

It can be shown that any function of P_K is uniquely determined by the degrees of freedom previously described.

Given a function u defined and continuously differentiable on K, its interpolate $r_K u$ will be the function of P_K which takes the same values as u at the degrees of freedom.

For the three elements defined above, the space V_h will be the space of functions whose restriction to each triangle $K \in \mathcal{T}_h$ belongs to P_K which are continuous at the degrees of freedom, and equal to zero at the degrees of freedom located on the boundary Γ. Given a function u defined and continuously differentiable on Ω, equal to zero, (along) with its first order derivatives on the boundary Γ, its *interpolate* $r_h u$ will be the function of V_h whose restriction to each $K \in \mathcal{T}_h$ is equal to $r_K u$.

We let $\mathbf{S}_h$ be the set of all the edges S which are not contained in Γ and $\bar{S}_h$ be the set of all the edges $\bar{\mathbf{S}}$ contained in Γ.

Generally, the inclusion $V_h \subset C^0\ (\Omega)$ does not hold, but we have :

Lemma 3.1 : *For the three elements, the mean values along the edges $\not\subset \Gamma$ of the first order derivatives of the functions of V_h are the same on both sides of the edges, and they are equal to zero along the edges $\subset \Gamma$.*

Proof : For the three elements, we have along any edge $A_i\ A_j$:

(3.4) $$\int_{A_i}^{A_j} \overrightarrow{Dw} \cdot \overrightarrow{A_i A_j}\ ds = |\overrightarrow{A_i A_j}|\ (w(A_j) - w(A_i)) \text{ for all w.}$$

Since we have continuity at the vertices, the mean value of the first order derivative directed along the edge is the same on both sides.

For the element (I), we have $P_K = P(2)$, so $\frac{\partial w}{\partial n} \in P(1)$ and

(3.5) $$\int_{A_i}^{A_j} \frac{\partial w}{\partial n} ds = |\overrightarrow{A_i A_j}|\ \frac{\partial w}{\partial n}\ (A_{ij})\ ,\ \text{which implies}$$

that the mean value of the first normal derivative is the same on both sides of the edge.

This property is also true, by definition, for element (II)

For element (III) we have :

$$(3.6) \quad \int_{A_i}^{A_j} \frac{\partial w}{\partial n} ds = \frac{1}{2} \, |\overrightarrow{A_i A_j}| \, \left(\frac{\partial w}{\partial n} (A_{iij}) + \frac{\partial w}{\partial n} (A_{jji}) \right), \text{ for all } w \in V_h ,$$

since $\frac{\partial w}{\partial n}$ is a polynomial with degree $\leqslant 2$ on $A_i A_j$ and the points A_{iij} and A_{jji} are Gauss quadrature points.

Lemma 3.2 : *Problem* (2.9) *has a unique solution* $u_h \in V_h$, *when* V_h *is constructed with any of the three elements defined above.*

Proof : We only need to show that for any $v_h \in V_h$ and satisfying $\| v_h \|_h = 0$, we have $v_h = 0$. If for some $v_h \in V_h$, we have $\| v_h \|_h = 0$, then the first order derivatives of v_h are constant on each element K : since the mean value of these derivatives are continuous at the inter element edges and equal to zero along the edges $\subset \Gamma$, the first order derivatives are equal to zero, Hence v_h is constant on each element ; since v_h is continuous at the vertices of the elements and equal to zero at the vertices $\in \Gamma$, we have $v_h = 0$.

Lemma 3.3 : *Patch test. For any* $u \in P(2)$ *and any* $w \in V_h$, *we have* : $E_h(u,w) = 0$ *for the elements defined above, where* E_h *is defined in Theorem* 2.1.

Proof : Given an edge S belonging to $\mathcal{S}_h$, we let K_s and K'_s denote the two triangles which are adjacent to S, w and w' denote the restrictions of w to K_s and K'_s and $\frac{\partial}{\partial n}$, $\frac{\partial}{\partial n'}$ the first normal derivatives on S directed outward K_s, K'_s (we have $\frac{\partial}{\partial n} = - \frac{\partial}{\partial n'}$). The expression $E_h(u,w)$ can be written as follows :

$$(3.7) \quad E_h(u,w) = \sum_{S \in \mathcal{S}_h} \left\{ \int_S \left(\frac{\partial}{\partial n} (\Delta u) \cdot w + \frac{\partial}{\partial n'} (\Delta u) . w' \right) ds - \right.$$

$$- \int_S (1 - \sigma) \left(\frac{\partial^2 u}{\delta n \delta s} \frac{\partial w}{\partial s} + \frac{\partial^2 u}{\partial n' \partial s} \frac{\partial w'}{\partial s} \right) ds +$$

$$+ \int_S (1 - \sigma) \left(\frac{\partial^2 u}{\partial s^2} - \Delta u \right) \left(\frac{\partial w}{\partial n} + \frac{\partial w'}{\partial n'} \right) ds +$$

$$+ \sum_{S \in \bar{S}_h} \int_S \left(\frac{\partial}{\partial n} (\Delta u) \cdot w - (1 - \sigma) \frac{\partial^2 u}{\partial n \partial s} \frac{\partial w}{\partial s} + \right.$$

$$\left. + \left((1 - \sigma) \frac{\partial^2 u}{\partial s^2} - \Delta u \right) \frac{\partial w}{\partial n} \right) ds .$$

Then $E_h(u,w) = 0$ for all $w \in V_h$ and all $u \in P_2$ if we can show that :

$$(3.8) \qquad \int_S \left(\frac{\partial w}{\partial s} - \frac{\partial w'}{\partial s} \right) ds = 0 \qquad \text{for all } S \in S_h ,$$

$$(3.9) \qquad \int_S \frac{\partial w}{\partial s} \, ds = 0 \qquad \text{for all } S \in \bar{S}_h ,$$

$$(3.10) \qquad \int_S \left(\frac{\partial w}{\partial n} + \frac{\partial w'}{\partial n'} \right) ds = 0 \qquad \text{for all } S \in S_h ,$$

$$(3.11) \qquad \int_S \frac{\partial w}{\partial n} \, ds = 0 \qquad \text{for all } S \in \bar{S}_h ,$$

Lemma 3.1 says that equalities (3.8) to (3.11) are satisfied for all $w \in V_h$, and the Patch Test is then satisfied.

Before being able to give the error bounds, we need some results in approximation theory.

Lemma 3.4 : *For the elements T.Q.M. and F.V.1, we have :*

$$(3.12) \quad | u - r_h u |_{m,K} \leqslant C (h(K))^{3-m} | u |_{3,K} \quad , \text{ for } 0 \leqslant m \leqslant 3 \quad ,$$

for any $u \in H^3(K)$, *where* c *is a constant independent of* h *and* u.

For the element F.V.2, we have :

$$| u - r_h u |_{m,K} \leqslant C (h(K))^{k-m} | u |_{k,K} \quad , \text{ for } 0 \leqslant m \leqslant k \quad ,$$

for any $u \in H^2(\Omega)$, *with* $k = 3$ *or* 4, *where* C *is a constant independent of* h *and* u.

Proof : To be able to apply an affine theory as in [6], we define for each case a function $\tilde{r}_K u$. For the elements T.Q.M. and F.V.1, we show that we have in fact $\tilde{r}_K u = r_K u$. For the element F.V.2, we prove that the difference between $\tilde{r}_K u$ is small.

For the element T.Q.M., we let $\tilde{r}_K\, u$ be the unique function of P_K such that :

$$\tilde{r}_K u = u \quad \text{at the vertices of } K$$

$$\overrightarrow{D\tilde{r}_K\, u} \cdot \overrightarrow{m_i} = \frac{1}{\ell_i} \int_{A_j}^{A_k} \overrightarrow{Du} \cdot \overrightarrow{m_i}\, ds \;, \quad 1 \leqslant i \leqslant 3 \;.$$

For the element F.V.1, we let $r_K\, u$ be the the unique function of P_K such that:

$\tilde{r}_K u = u$ at the vertices and at the mid points of the edges of K.

$$\left[\overrightarrow{D\,\tilde{r}_K\, u} \cdot \overrightarrow{m_i}\right] = \left[\overrightarrow{Du} \cdot \overrightarrow{m_i}\right] \quad , \quad 1 \leqslant i \leqslant 3 \;.$$

If we apply an affine theory as in [6], we have in both cases, since $P(2) \subset P_K$:

$$(3.14) \quad |u - \tilde{r}_K\, u|_{m,K} \leqslant C\, (h(K))^{3-m}\, |u|_{3,K} \quad , \quad 0 \leqslant m \leqslant 3 \quad ,$$

for all $u \in H^3(K)$ and any $K \in \mathcal{C}_h$.

For the element T.Q.M., we can show that :

$$\frac{\partial}{\partial n_i} (\tilde{r}_K\, u) = \left[\frac{\partial u}{\partial n}\right]_i = \frac{\partial}{\partial n_i} (r_K\, u) \quad , \quad 1 \leqslant i \leqslant 3 \quad ,$$

and for the element F.V.1, we can show that :

$$\left[\frac{\partial}{\partial n} \tilde{r}_K\, u\right]_i = \left[\frac{\partial u}{\partial n}\right]_i = \left[\frac{\partial}{\partial n} r_K\, u\right]_i \quad , \quad 1 \leqslant i \leqslant 3 \quad ,$$

Hence in both cases, we have $\tilde{r}_K\, u = r_K\, u$, for all u and for all $K \in \mathcal{C}_h$; inequality (3.14) gives immediately inequality (3.12).

The proof is more complicated for the element F.V.2. We proceed as in [7], [8]. Let $\tilde{r}_K\, u$ be the unique function of P_K such that :

$$\tilde{r}_K\, u = u \quad \text{at the vertices and at the centroid of } K,$$

$\overrightarrow{D\tilde{r}_K\, u} \cdot \overrightarrow{m_i} = \overrightarrow{Du} \cdot \overrightarrow{m_i}$ at the Gauss points of the edge opposite to the vertice

$$A_i \;, \quad 1 \leqslant i \leqslant 3 \;.$$

If we apply the affine theory [6] we get :

(3.15) $|u-\tilde{r}_K u|_{m,K} \leqslant C(h(K))^{k-m} \ |u|_{k,K}$, $0 \leqslant m \leqslant k$,

for all $u \in H^k(\Omega)$, $k = 3$ or 4 and any $K \in \mathcal{T}_h$.

We can write :

(3.16) $\tilde{r}_K u - r_K u = \sum_{i \neq j} \varphi_{iij} \ |\overrightarrow{A_i A_j}| \ (\frac{\partial}{\partial n} \tilde{r}_K u - \frac{\partial u}{\partial n})_{iij}$, where the φ_{iij}'s are the shape functions relative to the normal derivatives at the points A_{iij}, $i \neq j$, $1 \leqslant i, j \leqslant 3$.

If we apply an affine theory [6] , we can get :

(3.17) $\max \left\{ \| D\tilde{r}_k u - Du \| \ ; (x,y) \in K \right\} \leqslant c\,(h(K))^{k-2} \ |u|_{k,K}$,

for all $u \in H^k(K)$, $k = 3$ or 4, and any $K \in \mathcal{T}_h$, where Du denotes the total derivative of u and $\| \cdot \|$ the operator norm induced by the euclidean norm. Moreover, one can show that :

(3.18) $|\varphi iij|_{m,K} \leqslant C(h(K))^{1-m}$, $m \geqslant 0$, $1 \leqslant i, j \leqslant 3$, $i \neq j$.

Combining inequalities (3.17) and (3.18) with equality (3.16), we get :

(3.19) $|\tilde{r}_K u - r_K u|_{m,K} \leqslant C\ (h(K))^{k-m} \ |u|_{k,K}$, $0 \leqslant m \leqslant k$,

for all $u \in H^k(K)$, $k = 3$ or 4 and for any $K \in \mathcal{T}_h$.

Inequalities (3.15) and (3.19) give inequality (3.13).

Lemma 3.5 : *For the three elements, we have the estimates :*

(3.20) $|E_1(u,w-w_h)| + |E_2(u,w-w_h)| \leqslant C \ h|u|_{3,\Omega} \ \|w-w_h\|_h$,

for all $u \in H^3(\Omega)$, $w \in H^2(\Omega)$ *and* $w_h \in V_h$.

(3.21) $|E_3(u,w-w_h)| \leqslant C(h\,|u|_{3,\Omega} + h^2\,|u|_{4,\Omega}) \cdot \|w-w_h\|_h$

for all $u \in H^4(\Omega)$, $w \in H^2(\Omega)$, *and* $w_h \in V_h$.

Proof : For any function g defined on an edge $S \in \mathrm{S}_h \cup \bar{\mathrm{S}}_h$, we define its mean value $\pi_0 g$ by :

$\pi_o g = \dfrac{1}{L(S)} \displaystyle\int_S g \, ds$, where $L(S)$ is the length of S.

According to equalities (3.8), (3.9), 3.10) and (3.11) :

$$E_1(u,w-w_h) = \sum_{K\in\mathcal{T}_h} \int_{\partial K} \left((1-\sigma)\frac{\partial^2 u}{\partial s^2} - \Delta u\right)\left(\frac{\partial}{\partial n}(w-w_h) - \pi_o \frac{\partial}{\partial n}(w-w_h)\right) ds ,$$

$$E_1(u,w-w_h) = \sum_{K\in\mathcal{T}_h} \int_{\partial K} -(1-\sigma)\frac{\partial^2 u}{\partial n\,\partial s}\left(\frac{\partial}{\partial s}(w-w_h) - \pi_o \frac{\partial}{\partial s}(w-w_h)\right) ds .$$

Now consider the following expression :

$$E_1^S(u,w-w_h) = \int_S \left((1-\sigma)\frac{\partial^2 u}{\partial s^2} - \Delta u\right)\left(\frac{\partial}{\partial n}(w-w_h) - \pi_o \frac{\partial}{\partial n}(w-w_h)\right) ds ,$$

where S is a face of a triangle $K \in \mathcal{T}_h$.

Let v be a function defined on K, and equal to the first order derivative of $w-w_h$ in the direction defined by the outer normal on S and let : $g = (1-\sigma)\dfrac{\partial^2 u}{\partial s^2} - \Delta u$. If F_K denotes the affine transformation which maps the reference triangle $\hat{K}$ onto K we let $\hat{q} = q \circ K_K$ for any function q defined on K.

The bilinear form $A(\hat{g},\hat{v}) = \displaystyle\int_{\hat{S}} \hat{g}\,(\hat{v} - \hat{\pi}_o \hat{v})\, d\hat{s}$, where $\hat{\pi}_o \hat{v}$ = mean value of $\hat{v}$ on $\hat{S}$, with $\hat{S} = F_K^{-1}(S)$, satisfies the hypothesis of Lemma 2.1 with $\ell = k = 0$, and we get :

$$|A(\hat{g},\hat{v})| \leqslant C\,|\hat{g}|_{1,\hat{K}}\,|\hat{v}|_{1,\hat{K}}$$

Going back to the element K, we get :

$$|E_1^S(u,w-w_h)| \leqslant C h(K)\,|u|_{3,K}\,|w-w_h|_{2,K}$$

Summing over all the edges $S \in \mathsf{S}_h \cup \bar{\mathsf{S}}_h$, we get :

$| E_1 (u, w-w_h) | \leqslant C h | u |_{3,\Omega} \; \| w-w_h \|_h$ for all $u \in H^3 (\Omega), w \in H^2 (\Omega)$ and $w_h \in V_h$.

The same inequality can also be derived for $| E_2 (u, w-w_h) |$ and we get the estimate (3.20).

For any function q defined and continuous at the nodes, we let $\widetilde{q}$ be the continuous function equal to q at the vertices of all the triangles K and whose restriction to any triangle $K \in \mathcal{T}_h$ belongs to $P(1)$.

We have :

$$E_3 (u, w-w_h) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial}{\partial n} (\Delta u) \cdot (w-w_h - (\widetilde{w}-\widetilde{w}_h)) \, ds .$$

We consider the bilinear form :

$$E_3^S = \int_S P (v-\widetilde{v}) \, ds \quad , \quad \text{where } P = \frac{\partial}{\partial n} (\Delta u)$$

and $v = w-w_h$, and the corresponding bilinear form on $\hat{K}$:

$$\hat{E}_3^S = \int_{\hat{S}} \hat{P} \, (\hat{v}-\widetilde{\hat{v}}) \, ds .$$

One can show that [6], [14] :

$$| \hat{E}_3^S | \leqslant C \, | \hat{P} |_{o,\hat{K}}^{1/2} \; \| \hat{P} \|_{1,\hat{K}}^{1/2} \; | \hat{v} |_{2,\hat{K}} \text{ so that}$$

$$| \hat{E}_3^S | \leqslant C (| \hat{P} |_{0,\hat{K}} + | \hat{P} |_{1,\hat{K}}) \; | \hat{v} |_{2,\hat{K}}$$

Going back to the element K, we easily get :

$$| E_3^S | \leqslant C \left(h(K) \; | u |_{3,K} + h(K)^2 \; | u |_{4,K} \right) | w-w_h |_{2,K}$$

Summing over all the elements K, we get inequality (3.21).

We are now able to show the following estimates :

Therorem 3.1 : *Assume that* $u \in H^4 (\Omega)$ *and let* $u_h \in V_h$ *be the solution of problem* (2.9). *We have, for the three elements :*

(3.22) $$\| u - u_h \|_h \leqslant C \, (h | u |_{3,\Omega} + h^2 \, | u |_{4,\Omega}) ,$$

where C is a constant independent of h.

Proof : We use inequality (2.12).

According to Lemma (3.4) applied with $m = 2$, we have :

$$\inf_{v \in V_h} \| u - v \|_h \leqslant C\, h \, | u |_{3,\Omega}$$

Lemma 3.5 shows that :

$$E_h(u,w) \leqslant C\,(h \, | u |_{3,\Omega} + h^2 \, | u |_{4,\Omega}) \, \| w \|_h \, ,$$

for all $w \in V_h$.

These two inequalities lead to inequality (3.22).

Theorem 3.2 : *Assume that* $u \in H^4(\Omega)$, *that hypothesis* (2.14) holds with $\ell = 0$ *and let* $u_h \in V_h$ *be the solution of problem* (2.9). *We have for the three elements :*

$$\| u - u_h \|_{0,\Omega} \leqslant C\, h^2 (| u |_{3,\Omega} + | u |_{4,\Omega}) \quad (3.23)$$

Proof : We use Theorem 2.2. In expression $E(u, u_h, \varphi, \varphi_h)$, we choose $\varphi_h = r_h \varphi$ as defined above. Applying Lemma 3.4 and Theorem 3.1, we get :

$$| a_h (u - u_h, \varphi - r_h \varphi) | \leqslant C \| u - u_h \|_h \, \| \varphi - r_h \varphi \|_h \, .$$

According to Lemma 3.5, we have :

$$| E_h (\varphi, u - u_h) | \leqslant C\, h \, \| \varphi \|_{4,\Omega} \, \| u - u_h \|_h \, ,$$

$$| E_h (u, \varphi - r_h \varphi) | \leqslant C h \, (| u |_{3,\Omega} + | u |_{4,\Omega}) \, \| \varphi - r_h \varphi \|_h$$

But from Theorem 3.1, we know that :

$$\| u - u_h \|_h \leqslant C h \, (| u |_{3,\Omega} + h \, | u |_{4,\Omega}) \, ,$$

and from Lemma 3.4, we know that :

$$\| \varphi - r_h \varphi \|_h \leqslant C\, h \, | \varphi |_{3,\Omega} \, ,$$

so we get inequality (3.23).

IV – ARI ADINI'S RECTANGULAR ELEMENT

Definition 4.1 : *Ari Adini's element* [1] *is defined as follows :*

i) The geometrical shape is a rectangle (or a parallelogram),

ii) The degrees of freedom are the values of the function and of its first order derivatives at the vertices of the rectangle (figure 3),

iii) The space P_K of the shape functions is defined by :

$$P_K = \left\{ P = \hat{P} \circ F_K^{-1} \ ; \ \forall \hat{p} \in \hat{P} \right\} , \text{ with}$$

$$\hat{P} = \left\{ P(3) \cup \left\{ \xi^3 \eta \right\} \cup \left\{ \xi \eta^3 \right\} \right\} .$$

The problem of finding a function $\in P_K$ assuming any given degrees of freedom has a unique solution and the shape functions can be expressed easily in local coordinates ξ, η if we use the isoparametric mapping F_K defined for $-1 \leqslant \xi , \eta \leqslant 1$ by :

$$(4.1) \quad F_K : \begin{cases} x = \dfrac{(1+\xi)(1+\eta)}{4} x_1 + \dfrac{(1-\xi)(1+\eta)}{4} x_2 + \dfrac{(1-\xi)(1-\eta)}{4} x_3 + \dfrac{(1+\xi)(1-\eta)}{4} x_4 , \\ y = \dfrac{(1+\xi)(1+\eta)}{4} y_1 + \dfrac{(1-\xi)(1+\eta)}{4} y_2 + \dfrac{(1-\xi)(1-\eta)}{4} y_3 + \dfrac{(1+\xi)(1-\eta)}{4} y_4 , \end{cases}$$

where (x_i, y_i) are the coordinates of the vertices A_i, $1 \leqslant i \leqslant 4$.

Let P_i denote the values of $P \in P_K$ at the vertices A_i, $1 \leqslant i \leqslant 4$, we have :

$$(4.2) \quad P(x,y) = \sum_{i=1}^{4} P_i \ \varphi_i(x,y) + \sum_{(i,j) \in I} DP(A_i) . \overrightarrow{A_i A_j} \ \varphi_{i,j}(x,y) \quad , \quad \text{where}$$

$$I = \left\{ (i,j) \ ; \ i,j \leqslant 4 \quad , \quad i \neq j, \quad |i-j| = 1 \text{ or } 3 \right\} \quad \text{and where}$$

$$\varphi_i(x,y) = \hat{\varphi}_i \circ F_K^{-1} , 1 \leqslant i \leqslant 4, \ \varphi_{i,j}(x,y) = \hat{\varphi}_{ij} \circ F_K^{-1}, (i,j) \in I,$$

with :

$$(4.3) \qquad \hat{\varphi}_1(\xi \eta) = \frac{(1+\xi)(1+\eta)}{4} \left(1 + \frac{\xi+\eta}{2} - \frac{\xi^2+\eta^2}{2}\right) \quad ,$$

$$(4.4) \qquad \hat{\varphi}_{1,2}(\xi \eta) = \frac{(1+\eta)(1+\xi)^2(1-\xi)}{16}$$

(4.5) $$\varphi_{1,4}(\xi,\eta) = \frac{(1+\xi)(1+\eta)^2(1-\eta)}{16}$$

the other shape functions being derived by changing ξ in $-\xi$ or (and) η in $-\eta$.

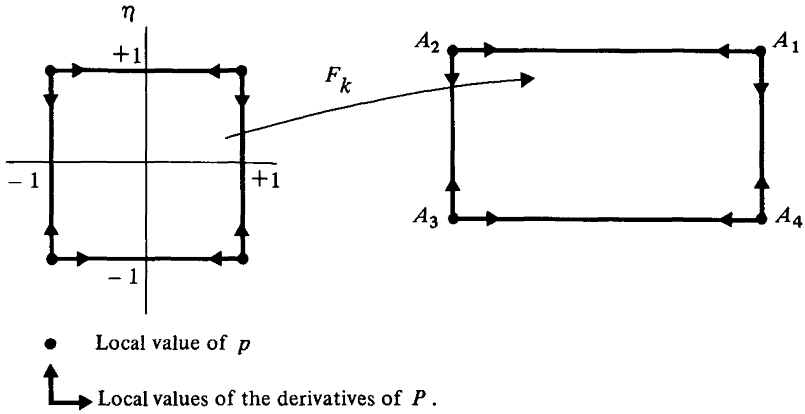


Figure 3.
Ari Adini's element.

The space V_h will be the space of functions whose restriction to each quadrilateral K belongs to P_K , and which are continuous along with their first order derivatives at the vertices of the rectangles. At the vertices belonging to Γ ,the degrees of freedom are set equal to zero.

The functions of V_h are continuous over $\bar{\Omega}$, equal to zero on Γ, but generally they are not continuously differentiable.

Lemma 4.1 : *Problem* (2.9) *has a unique solution* $u_h \in V_h$, *when* V_h *is constructed with the element described above.*

Proof : Let $v_h \in V_h$ be such that $\| v_h \|_h = 0$. Then the first order derivatives of v_h are constant on each element ; since they are continuous at the vertices of the elements and equal to zero at the vertices $\in \Gamma$, they are equal to zero on $\bar{\Omega}$. Then v_h is constant on each element ; since v_h is continuous on $\bar{\Omega}$ and equal to zero on Γ, v_h is equal to zero on $\bar{\Omega}$.

Lemma 4.2 : *Patch Test. For any* $u \in P(2)$ *and any* $w \in V_h$, *we have :* $E_h(u,w) = 0$, *where* E_h *is defined in Theorem* 2.1.

Proof : Since we have the inclusion $V_h \subset C^0(\bar{\Omega})$, we can write :

$$(4.6) \qquad E_h(u,w) \equiv E_1(u,w) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \left((1-\sigma)\frac{\partial^2 u}{\partial s^2} - \Delta u\right)\frac{\partial w}{\partial n}\, ds .$$

We shall consider separately the edges parallel to the x and y axes.

We write : $E_1 = E_x + E_y$, *with*

$$(4.7) \qquad E_x(u,w) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \left((1-\sigma)\frac{\partial^2 u}{\partial s^2} - \Delta u\right)\frac{\partial w}{\partial x}\, n_x \, ds ,$$

$$(4.8) \quad E_y(u,w) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \left((1-\sigma)\frac{\partial^2 u}{\partial s^2} - \Delta u\right)\frac{\partial w}{\partial y}\, n_y \, ds , \quad \text{for}$$

$$u \in H^3(\Omega), \; w \in V_h .$$

Let $\dfrac{\partial \tilde{w}}{\partial x}$ be the continuous function whose restriction to each element K belongs to $Q(1)$, the space of all polynomials q of the form $q = a + bx + cy + dxy$, and which takes the same values as $\dfrac{\delta w}{\delta x}$ at the vertices of the elements.

$$(4.9) \qquad z_h = \frac{\partial w}{\partial x} - \frac{\partial \tilde{w}}{\partial x} \quad . \text{ We have :}$$

$$E_x(u,w) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \left((1-\sigma)\frac{\partial^2 u}{\partial s^2} - \Delta u\right) z_h \, n_x \, ds .$$

The Patch test consists in showing that :

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} z_h \, n_x \, ds = 0 \quad \text{for all } w \in V_h .$$

We shall show a stronger and more local result, i.e. :

$$\int_{\partial K} z_h \, n_x \, ds = 0 \qquad \text{for all } w \in V_h \text{ and all } K \in \mathcal{T}_h .$$

The calculations will be done on the reference element $\hat{K}$, in coordinates ξ , η . On the element $\hat{K}$, any function $w \in V_h$ can be written as :

$$\hat{w} = P_0(\xi) + \eta P_1(\xi) + \eta^2 P_2(\xi) + \eta^3 P_3(\xi) \quad \text{, where}$$

P_0 and P_1 (resp. P_2 and P_3) are polynomials of degree $\leqslant 3$ (resp. $\leqslant 1$).

It is easy to see that :

$$\hat{z}_h(1,\eta) = \hat{z}_h(-1,\eta) = \frac{2}{\Delta x}\left(\frac{\partial P_2}{\partial \xi} + \eta \frac{\partial P_3}{\partial \xi}\right)(\eta^2 - 1).$$

Hence we have :

$$(4.11) \qquad \int_{\partial K} z_h\, n_x\, ds = \frac{\Delta y}{2} \int_{-1}^{+1} (\hat{z}_h(1,\eta) - \hat{z}_h(-1,\eta))\, d\eta = 0 \quad ,$$

where Δy is the length of the edge parallel to the y-axis.

Remark 4.1 : We shall show later on that if *the elements are equal rectangles,* then we have :

$$(4.12) \qquad E_h(u,w) = 0 \qquad \textit{for all } u \in P(3) \textit{ and all } w \in V_h .$$

Error bounds : We define the interpolate $r_h u \in V_h$ of any continuously differentiable function u as the unique function of V_h which is equal to u and whose first order derivatives are equal to those of u at the vertices of the elements. If we apply the result of [6] on Hermite interpolation, we get :

Lemma 4.1 : *We have the estimate :*

$$(4.13) \qquad |v - r_h v|_{m,K} \leqslant C(h(K))^{k-m} |v|_{k,K} \quad \textit{for } 0 \leqslant m \leqslant k \ ,$$

for any $v \in H^k(K)$, $k = 3$ *or* 4 *and for any* $K \in \mathcal{T}_h$, *where* C *is a constant* > 0 *independent of* K *and* v.

We shall need the following technical result whose proof is straightforward :

Lemma 4.2 : *For any* $w \in V_h$, *we have :*

$$(4.14) \qquad |w|_{3,K} \leqslant C(h(K))^{-1} |w|_{2,K} \quad , \textit{for any } K \in \mathcal{T}_h .$$

Lemma 4.3 : *We have the following estimates :*

$$(4.15) \qquad E_1(u,w) \leqslant Ch^2 |u|_{3,\Omega} \Big(\sum_{K \in \mathcal{T}_h} |w|^2_{3,K}\Big)^{1/2} \quad ,$$

(4.16) $E_1(u,w) \leqslant Ch \; |u|_{3,\Omega} \; \|w\|_h$, *for all* $u \in H^3(\Omega)$ *and* $w \in V_h$.

Proof : Consider the expression $E_x\,(u,w)$ defined by equality (4.10) and define :

$$E_x^K(u,w) = \int_{\partial K} \left((1-\sigma) \frac{\partial^2 u}{\partial s^2} - \Delta u \right) z_h \, n_x \, ds \; , \text{ with}$$

$$z_h = \frac{\partial w}{\partial x} - \frac{\widetilde{\partial w}}{\partial x} \; , \; w \in V_h \; .$$

Let : $v = \dfrac{\partial w}{\partial x}$, $g = (1-\sigma) \dfrac{\partial^2 u}{\partial y^2} - \Delta u$, $\hat{v} = v_o \, F_K$ and $\hat{g} = g_o \; F_K$.

We have : $$E_x^K(u,w) = \frac{\Delta y}{2} \, E_x\,(\hat{g},\hat{v}) \; , \text{ with}$$

$$\hat{E}_x\,(\hat{g},\hat{v}) = \int_{-1}^{+1} (\hat{g}\,.\,(\hat{v}-\widetilde{\hat{v}})\,(1,\eta) - \hat{g}\,.(\hat{v}-\widetilde{\hat{v}})\,(-1\,,\eta)\,)\; d\eta \; .$$

We can apply Lemma 2.1 with $\ell = 1$, $k = 0$, $\Omega = \hat{K}$ and

$W = \left\{ \hat{v} \,;\, \hat{v} = \dfrac{\partial w}{\partial x} \, o \, F_K \right\}$, since according to (4.9) and (4.11) we have :

$$\hat{E}_x\,(\hat{g},\hat{v}) = 0 \quad \text{for all } \hat{g} \in P(0) \; , \; \hat{v} \in W \; ,$$
$$\hat{E}_x\,(\hat{g},\hat{v}) = 0 \quad \text{for all } \hat{g} \in H^1(K), \; \hat{v} \in P(1).$$

Hence we have the estimate :

$$|\hat{E}_x\,(\hat{g},\hat{v})| \leqslant C \; |\hat{g}|_{1,\hat{K}} \; |\hat{v}|_{2,\hat{K}} \; .$$

Going back to the rectangle K, we get :

$$|E_x^K\,(u,w)| \leqslant C\,(h(K)\,)^2 \quad |(\,(1-\sigma\,)\, \frac{\partial^2 u}{\partial y^2} - \Delta u)|_{1,K} \, |\frac{\partial w}{\partial x}|_{2,K}$$

Summing over all the elements $K \in {}_h$, we get inequality (4.15).

Applying Lemma 4.2, we easily get inequality (4.16).

We shall now consider what happens when the rectangles are equal.

We need the following notations. Let Ω be the rectangle $[0,a] \times [0,b]$. In Ω, we define the points $A_{i,j}$, $0 \leqslant i \leqslant I$, $0 \leqslant j \leqslant J$, of coordinates (x_i, y_j) where :

$$0 = x_0 < x_1 < \dots < x_I = a$$

$$0 = y_0 < y_1 < \dots < y_J = b$$

Let K = rectangle of vertices $A_{i,j}$, $A_{i+1,j}$, $A_{i,j+1}$ and $A_{i+1,j+1}$, for $0 \leqslant i \leqslant I-1$, $0 \leqslant j \leqslant J-1$ (figure 4).

For any function $w \in V_h$, we let :

$$w_{i,j} \equiv w(A_{i,j}) \quad , \text{ for } 0 \leqslant i \leqslant I, \ 0 \leqslant j \leqslant J$$

$$Dw^i_{j,j+1} = \overrightarrow{Dw} . \overrightarrow{A_{i,j} A_{i,j+1}} \ (A_{i,j}) \quad , \text{for } 0 \leqslant i \leqslant I \ , \ 0 \leqslant j \leqslant J-1 \ ,$$

$$Dw^i_{j+1,j} = \overrightarrow{Dw} . \overrightarrow{A_{i,j+1} A_{ij}} \ (A_{i,j+1}) \ , \text{for } 0 \leqslant i \leqslant I \ , \ 0 \leqslant j \leqslant J-1 \ .$$

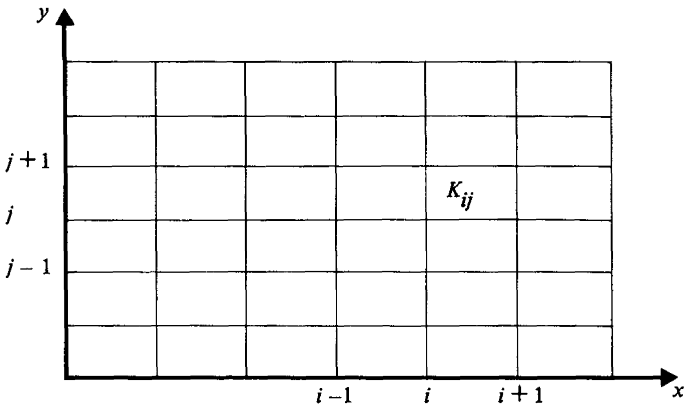


Figure 4.
A triangulation of the domain with equal rectangles.

We can write : $E_h^x(u,w) = \sum_{j=0}^{J-1} E_j(u,w)$, with

(4.17)

$$E_j(u,w) = \sum_{i=0}^{I-1} \left(\int_{\partial\text{-}K_{ij}} g_i(y) \frac{\partial w}{\partial n} \mathrm{d}y + \int_{\partial+K_{ij}} g_{i+1}(y) \frac{\partial w}{\partial n} \mathrm{d}y \right),$$

Where : $g_i(y) = \left((1-\sigma) \frac{\partial^2 u}{\partial y^2} - \Delta u \right)(x_i, y)$, $\partial - K_{ij} = A_{i,j} A_{i,j+1}$

and : $$\partial + K_{ij} = A_{i+1,j} A_{i+1,j+1} .$$

Let : $\eta = \dfrac{y - \frac{1}{2}(y_j + y_{j+1})}{y_{j+1} - y_j}$, we then have, after some calculations

Lemma 4.4 : *We have the equality*

$$E_j(u,w) = E'_j(u,w) + E''_j(u,w) \quad , \textit{ with}$$

$$E'_j(u,w) = (y_{j+1} - y_j) \sum_{i=0}^{I-1} \left(\int_{-1}^{1} \frac{\eta(1-\eta^2)}{8} \frac{\hat{g}_i(\eta) - \hat{g}_{i+1}(\eta)}{x_{i+1} - x_i} \mathrm{d}\eta \right) a_{ij}$$

$$E''_j(u,w) = (y_{j+1} - y_j) \sum_{i=0}^{I-1} \left(\int_{-1}^{1} \frac{1-\eta^2}{16} \frac{\hat{g}_i(\eta) - \hat{g}_{i+1}(\eta)}{x_{i+1} - x_i} \mathrm{d}\eta \right) b_{ij}$$

for $1 \leqslant j \leqslant J-1$, *for all* $u \in H^3(\Omega)$ *and* $w \in V_h$, *where*

$$a_{ij} = \left(w_{i,j+1} - w_{i,j} + \frac{1}{2} D w^i_{j+1,j} - \frac{1}{2} D w^i_{j,j+1}\right)$$

$$- \left(w_{i+1,j+1} - w_{i+1,j} + \frac{1}{2} D w^{i+1}_{j+1,j} - \frac{1}{2} D w^{i+1}_{j,j+1}\right) ,$$

$$b_{ij} = D w^i_{j+1,j} + D w^i_{j,j+1} - D w^{i+1}_{j+1,j} - D w^{i+1}_{j,j+1}$$

for : $0 \leqslant i,j \leqslant I-1,\ J-1$.

Lemma 4.5 : *We have the inequalities :*

(4.18) $|a_{ij}| \leqslant C \ |\hat{w}|_{3,\hat{K}_{ij}}$, *for* $0 \leqslant i,j \leqslant I\text{-}1,\ J\text{-}1$

(4.19) $|b_{ij}| \leqslant C \ |\hat{w}|_{3,\hat{K}_{ij}}$, *for* $0 \leqslant i,j \leqslant I\text{-}1,\ J\text{-}1$

(4.20) $\quad | D w^i_{j+1,j} + D w^i_{j,j+1} \quad | \leqslant C \, | \hat{w} |_{2,\hat{K}_{ij}}$, *for* $0 \leqslant i \leqslant I, 0 \leqslant j \leqslant J-1$,

for all $w \in V_h$ *where* C *is a constant independent of* h.

Proof : We can easily check that :

$$a_{ij} = \int_{\hat{K}} \eta \frac{\partial^3 \hat{w}^{i,j}}{\partial \xi \, \partial \eta^2} \, d\xi \, d\eta \quad , \quad \frac{1}{2} b_{ij} = \int_{\hat{K}} \cdot \frac{\partial^3 \hat{w}^{i,j}}{\partial \xi \, \partial \eta^2} \, d\xi \, d\eta \quad ,$$

$$D w^i_{j+1,} \quad + D w^i_{j,j+1} = -2 \int_{-1}^{1} \frac{\partial^2 w^{i,j}}{\partial \eta^2} (1, \eta) \, d\eta \quad ,$$

Where $\hat{w}^{i,j} = w_0 \, F_{K_{ij}}$.

From these equalities, we easily get inequalities (4.18) to (4.20).

Lemma 4.6 : *We have the estimates* :

(4.21)

$$| E'_j (u,w) | \leqslant C h^{k+1} \, | g |_{k,\Omega_j} \left(\sum_{K \subset \Omega_j} | w |^2_{3,K} \right)^{1/2} , \quad \textit{for } k = 1 \textit{ or } 2 ,$$

(4.22) $$| E''_j (u,w) | \leqslant C \, h^2 \quad | g |_{1,\Omega_j} \left(\sum_{K \subset \Omega_j} | w |^2_{3,K} \right)^{1/2} , \textit{ where}$$

$$\Omega_j = \cup \, (K_{ij} \; ; \; 0 \leqslant i \leqslant I-1) \quad \textit{for all } g \in H^2(\Omega) \, , \quad w \in V_h \quad .$$

If we have the equalities $x_{i+1} - x_i = \Delta x$, *for* $0 \leqslant i \leqslant I-1$, *we have* :

(4.23) $$| E''_j \; (u,w) | \leqslant C h^2 \; | g |_{2,\Omega_j} \left(\sum_{K \subset \mathscr{C} \Omega_j} | w |^2_{2,K} \right)^{1/2} ,$$

$$\textit{for all } g \in H^2(\Omega) \, , \quad w \in V_h \quad .$$

Proof : One can easily check that we have :

$$\int_{-1}^{1} \eta \, (1-\eta^2) \; (\hat{g}_i (\eta) - \hat{g}_{i+1} (\eta)) \; d\eta = 0 \quad \text{for all } \hat{g} \; (\xi, \eta) \in P \; (1) \quad ,$$

$$\int_{-1}^{1} (1 - \eta^2) \; (g_i (\eta) - \hat{g}_{i+1} \; (\eta)) \quad d\eta = 0 \quad \text{for all } \hat{g} \quad \in \quad P \; (0) \quad .$$

Using the same techniques as in [6], we get :

$$\left| \int_{-1}^{-1} \eta\,(1-\eta^2)\,(\hat{g}_i(\eta) - \hat{g}_{i+1}(\eta))\,\mathrm{d}\eta \right| \leqslant C h^{k-1}\ |g|_{k,K_{ij}} \qquad \text{for } k=1 \text{ or } 2$$

$$\left| \int_{-1}^{-1} (1-\eta^2)\,(\hat{g}\ \ (\eta) - \hat{g}\ \ _{1}(\eta))\,\mathrm{d}\eta \right| \leqslant C \qquad |g|_{1,K_{ij}} \quad ,$$

$$\text{for all } K_{ij} \subset \Omega_j \quad .$$

These two inequalities, combined with Lemma 4.5 (inequalities (4.18) and (4.19) give inequalities (4.21) and (4.22).

When $x_{i+1} - x_i = \Delta x$, $0 \leqslant i \leqslant I\text{-}1$, the expression $E''_j(u, w)$ can be written as :

$$(4.24)\quad E''_j(u,w) = \frac{y_{j+1}y_j}{\Delta x} \sum_{i=1}^{I-1} \left(\int_{-1}^{1} \frac{1-\eta}{16}\,(2\hat{g}_i(\eta) - \hat{g}_{i+1}(\eta) - \hat{g}_{i-1}(\eta))\,\mathrm{d}\eta \right).$$

$$.\ (D w^i_{j+1,j} + D w^i_{j,j+1})$$

Since we have the identity :

$$\int_{-1}^{1} (1-\eta^2)\,(2\hat{g}_i(\eta) - \hat{g}_{i+1}(\eta) - \hat{g}_{i-1}(\eta))\,\mathrm{d}\eta = 0 \quad \text{for all } \hat{g} \in P(1) \quad ,$$

we get :

$$\left| \int_{-1}^{1} (1-\eta^2)\,(2\hat{g}_i(\eta) - \hat{g}_{i+1}(\eta) - \hat{g}_{i-1}(\eta))\,\mathrm{d}\eta \right| \leqslant C h\ |g|_{2,K_{ij}}$$

This last inequality combined with inequality (4.20) yields inequality (4.23).

Remark 4.2 : Lemma 4.6 (inequality (4.21), with $k = 1$, and (4.22)) gives us a second proof for Lemma 4.3, if we replace g by its values $(1-\sigma)\dfrac{\partial^2 u}{\partial y^2} - \Delta u$ and sum over all the indices j, $0 \leqslant j \leqslant J-1$. This second proof is much more complicated, and its interest will be shown in the following result.

Lemma 4.7 : *Assume that* $x_{i+1} - x_i = \Delta x$, $0 \leqslant i \leqslant I\text{-}1$, *and* $y_{j+1} - y_j = \Delta y$, , $0 \leqslant j \leqslant J\text{-}1$, *we then have :*

(4.25) $|E_1(u,w)| \leqslant Ch^2 |u|_{4,\Omega} \|w\|_h$, *for all* $u \in H^4(\Omega)$ *and* $w \in V_h$, *where* C *is a constant* >0 *independent of* h.

Proof : We use inequalities (4.21) and (4.23) along with Lemma 4.2. Summing over the indices j , $0 \leqslant j \leqslant J-1$, we get an estimate for $E_x(u,w)$. The same work can be done for $E_y(u,w)$ and we get inequality (4.25).

Remark 4.3 : We might have expected the following estimate, instead of inequality (4.25) :

(4.26) $E_1(u,w) \leqslant Ch^3 |u|_{4,\Omega} \left(\sum_{K \in \mathcal{C}_h} |w|_{3,K}^2 \right)^{1/2}$ for all $u \in H^4(\Omega)$ and $w \in V_h$. It does not seem possible to get such an estimate, because of equality (4.24).

Remark 4.4 : Inequality (4.25) states that the patch test is passed for all $u \in P(3)$. We are now able to derive the following error bounds :

Theorem 4.1 : *Assume that* $u \in H^3(\Omega)$ *and let* $u_h \in V_h$ *be the solution of problem* (2.9). *We have :*

(4.27) $$\|u - u_h\|_h \leqslant Ch |u|_{3,\Omega} ,$$

(4.28) $$\|u - u_h\|_{1,\Omega} \leqslant Ch^2 |u|_{3,\Omega} .$$

Moreover, assume that $u \in H^4(\Omega)$ *and that* $x_{i+1} - x_i = \Delta x$, $0 \leqslant i \leqslant I-1$, $y_{j+1} - y_j = \Delta y$, $0 \leqslant j \leqslant J-1$, *then :*

(4.29) $$\|u - u_h\|_h \leqslant Ch^2 |u|_{4,\Omega} .$$

In those inequalities, C denotes a constant >0 *independent of* h.

Proof : We use Theorem 2.1, inequality (2.12). According to Lemma 4.1, we have for all $u \in H^3(\Omega)$:

$$\inf_{v \in V_h} \|u - v\|_h \leqslant \left(\sum_{K \in \mathcal{C}_h} \left| u - r_h u \right|_{2,K}^2 \right)^{1/2} \leqslant Ch |u|_{3,\Omega} .$$

According to Lemma 4.3, inequality (4.16), we have :

$$\sup_{w \in V_h} \frac{|E_h(u,w)|}{\|w\|_h} \leqslant Ch |u|_{3,\Omega} .$$

These last inequalities combined with inequality 2.12 give estimate (4.27). We shall now use Theorem 2.2, Lemma 4.3 and inequality (4.27) imply that, for all $\varphi \in H^3(\Omega)$, we have :

$$(4.30)\quad a_h(u-u_h, \varphi - r_h\varphi) \leqslant C\,\|u-u_h\|_h\,\|\varphi - r_h\varphi\|_h \leqslant Ch^2|u|_{3,\Omega}|\varphi|_{3,\Omega}.$$

We can show, using Lemmas 4.3 and 4.2, that :

$$\left(\sum_{K\in\mathcal{T}_h} |u_h|^2_{3,K}\right) \leqslant \left(\sum_{K\in\mathcal{T}_h} |u_h - r_h u|^2_{3,K}\right) + \sum_{K\in\mathcal{T}_h} |r_h u|^2_{3,K}$$

$$\leqslant Ch^{-2}\,\|u_h - r_h u\|_h^2 + C\,|u|^2_{3,\Omega}$$

$$\leqslant C\,|u|^2_{3,\Omega}\quad .$$

Using the last inequality and Lemma 4.3 (inequality 4.15), we get :

$$(4.31)\quad |E_h(\varphi, u-u_h)| = |E_h(\varphi, u_h)| \leqslant Ch^2|\varphi|_{3,\Omega}\left(\sum_{K\in\mathcal{T}_h}|u_h|^2_{3,K}\right)^{1/2}$$

$$\leqslant Ch^2|\varphi|_{3,\Omega}|u|_{3,\Omega}$$

$$(4.32)\quad |E_h(u,\varphi - r_h\varphi)| = |E_h(u, r_h\varphi)| \leqslant Ch^2|u|_{3,\Omega}|\varphi|_{3,\Omega},$$

for all $\varphi \in H^3(\Omega)$.

Inequalities (4.30), (4.31) and (4.32) give estimate (4.28).

According to Lemma 4.1, we have for all $u \in H^4(\Omega)$:

$$\inf_{v\in V_h} \|u-v\|_h \leqslant \left(\sum_{K\in\mathcal{T}_h}|u-r_h u|^2_{2,K}\right)^{1/2} \leqslant Ch^2\,|u|_{4,\Omega}$$

Lemma 4.7 implies that, if the rectangles are equal, we have :

$$\sup_{w\in V_h}\frac{|E_h(u,w)|}{\|w\|_h} \leqslant Ch^2\,|u|_{4,\Omega}\ ,\quad \text{for all } u\in H^4(\Omega)$$

These last inequalities lead to estimate (4.29).

Remark 4.5 : As we have seen in remark 4.3, it does not seem possible, because of the therms $E''_j(u,w)$, $0 \leqslant j \leqslant J-1$, to get an estimate like (4.26) for neither unequal nor equal rectangles. For this reason, we could not get a rate of convergence of order h^3 in the norm $\|\cdot\|_{0,\Omega}$.

V – CUBIC TRIANGULAR ELEMENT OF ZIENKIEWICZ

Definition 5.1 : *The cubic triangular element of Zienkiewicz* [3] *is defined as follows :*

i) The degrees of freedom are the values of the functions and of their first order derivatives at the vertices of the element (figure 5).

ii) The space P_K of the shape functions will be the space of all functions P of $P(3)$ satisfying the following relation :

$$P(G) = \frac{1}{3} \sum_{i=1}^{3} P_i + \frac{1}{18} \sum_{\substack{i \leqslant j,k \leqslant 3 \\ j \neq k}} DP_{j,k} \text{ , where}$$

$$P_i = P(A_i) \quad , \quad 1 \leqslant i \leqslant 3 \quad ,$$

$$DP_{jk} = \overrightarrow{DP} \cdot \overrightarrow{A_j \, A_k} \, (A_j) \quad , \quad 1 \leqslant j \, , k \leqslant 3 \, , j \neq k$$

We can show that the problem of finding a function of P_K assuming any given degrees of freedom has a unique solution, and we have :

$$P(x,y) = \sum_{i=1}^{3} (\lambda_i^2 (3-2\lambda_i) + 2 \lambda_1 \lambda_2 \lambda_3) P_i$$

$$+ \sum_{\substack{1 \leqslant j,k \leqslant 3 \\ j \neq k}} \frac{1}{2} \lambda_j \lambda_k (1 + \lambda_j - \lambda_k) DP_{j,k}$$

The following inclusions hold : $P(2) \subset P_K \subset P(3)$.

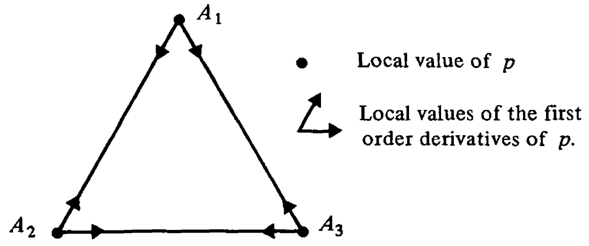


Figure 5.
ZIENKIEWICZ' element.

Now, given a function u defined and continuously differentiable on K, its *interpolate* $r_K u$ will be the function of P_K which takes on the same values as u at the degrees of freedom.

The space V_h will be the space of all functions whose restrictions to each $K \in \mathcal{T}_h$ belong to P_K, which are continuously differentiable at the vertices of the elements and equal to zero along with their first order derivatives at the vertices belonging to Γ. Given a function $u \in C^1(\bar{\Omega}) \cap H_0^2(\Omega)$, its interpolate $r_h u$ will be the function of V_h whose restriction to each $K \in \mathcal{T}_h$ is equal to $r_K u$.

The following inclusion holds : $V_h \subset C^0(\bar{\Omega})$; however the functions of V_h are not generally continuously differentiable on $\bar{\Omega}$.

Lemma 5.1 : *Problem (2.9) has a unique solution* $u_h \in V_h$, *when* V_h *is constructed with the previously defined element.*

Proof : Assume that for some $v_h \in V_h$, we have $\| v_h \|_h = 0$. Then the first derivatives of v_h are constant on each element $K \in \mathcal{T}_h$. Since they are continuous at the vertices of the elements and equal to zero at the vertices belonging to Γ, they are equal to zero. The function v_h is then constant on each element, and since it is continuous at the vertices and equal to zero at the vertices belonging to Γ, we have $v_h = 0$.

Lemma 5.2 : *Patch Test. Assume that the edges of the triangles are parallel to three given directions, then we have :*

(5.2) $$E_h(u,w) \equiv E_1(u,w) = 0 \quad \text{for all } u \in P(2) \text{ and } w \in V_h .$$

Proof : We consider figure 6 and we let $|A_2 A_3| = a$, $|A_3 A_4| = b$

$|\overrightarrow{A_1 A_3}| = c$, $\vec{s}_{ij} = \dfrac{\overrightarrow{A_i A_j}}{|\overrightarrow{A_i A_j}|}$ and A_{ij} = mid point of $A_i A_j$, for any edge $A_i A_j$,

$m_a = \overrightarrow{A_1 A_{23}} \quad \vec{s}_{23}$ and Δ = area of the triangles.

For any $u \in P(2)$, $w \in V_h$ and any edge $A_i A_j$, since the first derivatives of w belong to $P(2)$ on each element, we can write :

(5.3) $$\int_{A_i}^{A_j} \left(\Delta u - (1-\sigma)\frac{\partial^2 u}{\partial s^2}\right) \frac{\partial w}{\partial n}\, ds = A_i A_j \left(\Delta u - (1-\sigma)\frac{\partial^2 u}{\partial s^2}\right) . \left(\frac{1}{6}\frac{\partial w}{\partial n}(A_i) + \frac{2}{3}\frac{\partial w}{\partial n}(A_{ij}) + \frac{1}{6}\frac{\partial w}{\partial n}(A_j)\right)$$

where $\dfrac{\partial}{\partial n}$ denotes the first normal derivative on $A_i A_j$, along either side of $A_i A_j$.

Since the first derivatives of w are continuous at the vertices of the elements, we get :

$$(5.4)\qquad E_1(u,w) = \sum_{K\in\mathcal{T}_h} \sum_{L\subset\partial K} \frac{2}{3}\left((1-\sigma)\frac{\partial^2 u}{\partial s_L^2} - \Delta u\right)\frac{\partial w}{\partial n_L}\,\ell(L)\ ,$$

for all $u\in P(2)$ and $w\in V_h$, where s_L is a curvilinear abscissa along L, $\frac{\partial}{\partial n_L}$ is the outer normal derivative on L at the mid point of L and $\ell\,(L)$ is the length of the edge L, for all $L\subset\partial K$, $K\in\mathcal{T}_h$.

Now, consider a function $P\in P_K$, where K is the triangle $A_1\,A_2\,A_3$. Then the outer normal derivative on $A_2\,A_3$ at the point A_{23} is given by :

$$(5.5)\qquad \frac{\partial P}{\partial n} = \frac{a}{\Delta}\Big(-\frac{1}{2}P_1 + \frac{P_2+P_3}{4} - \frac{1}{8}DP_{12} - \frac{1}{8}DP_{13} - \frac{3}{8}DP_{21} - \frac{3}{8}DP_{31} + \frac{1}{4}DP_{23} + \frac{1}{4}DP_{32}\Big) + \frac{m_a}{\Delta}\Big(\frac{3}{2}(P_2 - P_3) + \frac{1}{4}(DP_{23} - DP_{32})\Big)\ .$$

In (5.4), we shall only consider the edges parallel to $A_2\,A_3$, and functions w which are equal to zero for all the degrees of freedom, except at the vertex A_1. We get :

$$(5.6)\qquad E_1^a(u,w) = \frac{2}{3}\,a\left((1-\sigma)\frac{\partial^2 u}{\partial s_a^2} - \Delta u\right)\left(\frac{\partial w^1}{\partial n}(A_{23}) + \frac{\partial w^2}{\partial n}(A_{14}) + \frac{\partial w^3}{\partial n}(A_{14}) + \frac{\partial w^4}{\partial n}(A_{56}) + \frac{\partial w^5}{\partial n}(A_{17}) + \frac{\partial w^6}{\partial n}(A_{17})\right)\ ,$$

where s_a is an abscissa in the direction of $\overrightarrow{A_2\,A_3}$, and w^i, $1\leqslant i\leqslant 6$, are the values of w in the triangles denoted by ⓘ, on figure 6. Using formula (5.5), we get :

$$\frac{\partial w^1}{\partial n}(A_{23}) = \frac{a}{\Delta}\Big(-\frac{1}{2}P_1 - \frac{1}{8}DP_{12} - \frac{1}{8}DP_{13}\Big)\ ,$$

$$\frac{\partial w^2}{\partial n}(A_{14}) = \frac{a}{\Delta}\Big(\frac{1}{4}P_1 - \frac{3}{8}DP_{13} + \frac{1}{4}DP_{14}\Big) + \frac{m_a}{\Delta}\Big(-\frac{3}{2}P_1 - \frac{1}{4}DP_{14}\Big)\ ,$$

$$\frac{\partial w^3}{\partial n}(A_{14}) = \frac{a}{\Delta}\Big(\frac{1}{4}P_1 - \frac{3}{8}DP_{15} + \frac{1}{4}DP_{14}\Big) + \frac{m_a}{\Delta}\Big(\frac{3}{2}P_1 + \frac{1}{4}DP_{14}\Big)\ ,$$

$$\frac{\partial w^4}{\partial n}(A_{56}) = \frac{a}{\Delta}\left(-\frac{1}{2}P_1 - \frac{1}{8}DP_{15} - \frac{1}{8}DP_{16}\right),$$

$$\frac{\partial w^5}{\partial n}(A_{17}) = \frac{a}{\Delta}\left(\frac{1}{4}P_1 - \frac{3}{8}DP_{16} + \frac{1}{4}DP_{17}\right) + \frac{m_a}{\Delta}\left(-\frac{3}{2}P_1 - \frac{1}{4}DP_{17}\right),$$

$$\frac{\partial w^6}{\partial n}(A_{17}) = \frac{a}{\Delta}\left(\frac{P_1}{4} - \frac{3}{8}DP_{12} + \frac{1}{4}DP_{17}\right) + \frac{m_a}{\Delta}\left(\frac{3}{2}P_1 + \frac{1}{4}DP_{17}\right).$$

Summing up all these equalities, we easily get :

$$E_1^a(u,w) = 0$$

This equality is also true for the two other directions, and for all the basis functions of the space V_h. The patch test is then passed.

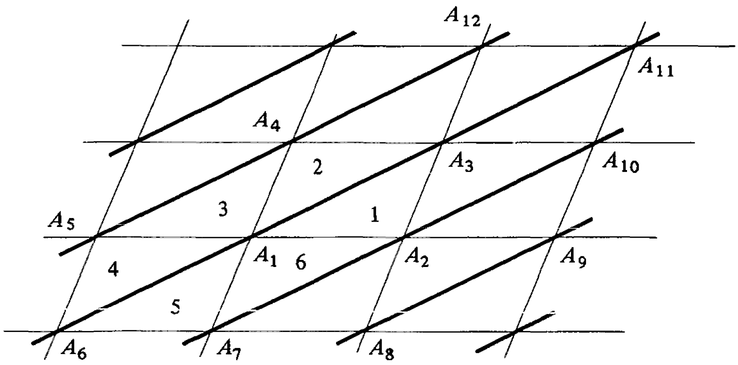


Figure 6.
An assembly of triangles with edges parallel to three given directions

Remark 5.1 : It can be seen easily that if the vertices are not parallel to three given directions, then the patch test cannot be passed and we have thus a necessary and sufficient condition to pass the patch test.

Remark 5.2 : If we add a degree of freedom at the centroid of each element, in order to get a complete cubic on each element K, then the patch test fails because the function $\lambda_1 \lambda_2 \lambda_3$ does not pass the patch test.

If we apply the results of [6] for Hermite type elements, we have :

Lemma 5.3 : *For all v defined on K and $\in H^3(K)$, for $K \in \mathcal{T}_h$, we have :*

$$(5.7) \qquad |v - r_K v|_{m,K} \leqslant C\ h\ K))^{3-m}\ |v|_{3,K}\ , \quad 0 \leqslant m \leqslant 3$$

Now write $E_1\ (u,w)$ as follows, for all $u \in H^3(\Omega)$ and $w \in V_h$:

$E_1(u,w) = E_1^a(u,w) + E_1^b(u,w) + E_1^c(u,w)$, where $E_1^a(u,w)$ is the sum of the terms relative to the normal derivatives on all the edges parallel to $A_2 A_3$, with a similar definition for E_1^b and E_1^c . We shall only consider $E_1^a(u,w)$, the two other terms being handled in the same way. We let :

$g(s) = (1-\sigma)\dfrac{\delta^2 u}{\delta s^2} - u)(s)$ where s is an abscissa in the direction parallel to $A_2 A_3$.

Let S_I and S_{II} be the sets of all edges S parallel to $A_1 A_2$ and $A_1 A_3$. We have :

Lemma 5.4 : *For all* $u \in H^3(\Omega)$ *and* $w \in V_h$, *we can write :*

$$(5.8)\qquad E_1^a(u,w) = \sum_{S \in \mathsf{S}_I} G_I(S) + \sum_{S \in \mathsf{S}_{II}} G_{II}(S)$$

where we have, for example :

$$(5.9)\quad G_I(A_1 A_2) = \frac{a^2}{\Delta}\left(\int_0^1 \alpha(1-\alpha)(g_{23}(\alpha) - g_{17}(\alpha))\,d\alpha\right) \cdot \left(w_2 - w_1 + \frac{1}{2}(Dw_{21} - Dw_{12})\right) ,$$

$$(5.10)\quad G_{II}(A_1 A_3) = \frac{a^2}{\Delta}\left(\int_0^1 \alpha(1-\alpha)(g_{23}(\alpha) - g_{14}(\alpha))\,d\alpha\right) \cdot \left(w_3 - w_1 + \frac{1}{2}(Dw_{31} - Dw_{13})\right) ,$$

where $g_{ij}(\alpha) = g(s(A_i) + a\,\alpha)$

Proof : On the edge $A_1 A_4$, we have :

$$\frac{\partial w^2}{\partial n} = (2\alpha - 1)(\alpha - 1)\frac{\partial w^2}{\partial n}(A_1) + 4\alpha(1-\alpha)\frac{\partial w^2}{\partial n}(A_{14}) + + \alpha(2\alpha - 1)\frac{\partial w^2}{\partial n}(A_4) ,$$

$$\frac{\partial w^3}{\partial n} = (2\alpha - 1)(\alpha - 1)\frac{\partial w^3}{\partial n}(A_1) + 4\alpha(1-\alpha)\frac{\partial w^3}{\partial n}(A_{14}) + + \alpha(2\alpha - 1)\frac{\partial w^3}{\partial n}(A_4) ,$$

where $\alpha = \dfrac{s - s(A_1)}{a}$. Since the first derivatives of w are continuous at the vertices, we get :

$$(5.11)\quad \int_{A_1}^{A_4} g(s)\left(\frac{\partial w^2}{\partial n} + \frac{\partial w^3}{\partial n}\right) ds = a \int_0^1 4\alpha(1-\alpha) g_{14}(\alpha)\left(\frac{\partial w^2}{\partial n}(A_{14}) + \frac{\partial w^3}{\partial n}(A_{14})\right) d\alpha \ .$$

Applying formula (5.5) , we get :

$$(5.12)\quad \frac{\partial w^2}{\partial n}(A_{14}) = \frac{a}{\Delta}\left(-\frac{1}{2} w_3 + \frac{1}{4}(w_1 + w_4) - \frac{1}{8}(Dw_{34} + Dw_{31}) - \frac{3}{8}(Dw_{43} + Dw_{13}) + \frac{1}{4}(Dw_{41} + Dw_{14})\right) + \frac{m_a}{\Delta}\left(\frac{3}{2}(w_4 - w_1) + \frac{1}{4}(Dw_{41} - Dw_{14})\right),$$

$$(5.13)\quad \frac{\partial w^3}{\partial n}(A_{14}) = \frac{a}{\Delta}\left(-\frac{1}{2} w_5 + \frac{1}{4}(w_1 + w_4) - \frac{1}{8}(Dw_{51} + Dw_{54}) - \frac{3}{8}(Dw_{15} + Dw_{45}) + \frac{1}{4}(Dw_{41} + Dw_{14})\right) + \frac{m_a}{\Delta}\left(\frac{3}{2}(w_1 - w_4) + \frac{1}{4}(Dw_{14} - Dw_{41})\right),$$

Since we have : $\overrightarrow{A_4 A_1} = \overrightarrow{A_4 A_5} + \overrightarrow{A_4 A_3}$ and $\overrightarrow{A_1 A_4} = \overrightarrow{A_1 A_5} + \overrightarrow{A_1 A_3}$, we get :

$$(5.14)\qquad Dw_{41} = Dw_{45} + Dw_{43} \text{ and } Dw_{14} = Dw_{15} + Dw_{13} \ .$$

Combining equalities (5.12) (5.13) and (5.14) , we get :

$$\frac{\partial w^2}{\partial n}(A_{14}) + \frac{\partial w^3}{\partial n}(A_{14}) = \frac{a}{\Delta}\left(\frac{1}{2}(w_1 + w_4 - w_3 - w_5) - \frac{1}{8}(Dw_{34} + Dw_{31} + Dw_{51} + Dw_{54}) + \frac{1}{8}(Dw_{43} + Dw_{13} + Dw_{15} + Dw_{45}\right),$$

which can also be written as :

$$(5.15)\quad \frac{\partial w^2}{\partial n}(A_{14}) + \frac{\partial w^3}{\partial n}(A_{14}) = \frac{a}{4\Delta}\Big[(w_1 - w_3 + \frac{1}{2}(Dw_{13} - Dw_{31})) + w_4 - w_5 + \frac{1}{2}(Dw_{45} - Dw_{54}) + w_1 - w_5 + \frac{1}{2}(Dw_{15} - Dw_{51}) + w_4 - w_3 + \frac{1}{2}(Dw_{43} - Dw_{34})\Big] .$$

Combining equalities (5.11) and (5.15) and summing over all the edges parallel to $A_2 A_3$, we get lemma 5.4.

Lemma 5.5 : *We have the following estimates :*

$$(5.16)\qquad |w_j - w_i + \frac{1}{2}(Dw_{ji} - Dw_{ij})| \leqslant C\, h\, |w|_{2,K}$$

$$(5.17)\qquad |w_j - w_i + \frac{1}{2}(Dw_{ji} - Dw_{ij})| \leqslant C\, h^2\, |w|_{3,K}$$

for all $w \in V_h$ *and for any edge* $A_i A_j$, *of any triangle* $K \in \mathcal{T}_h$.

Proof : Consider the reference triangle $\hat{K}$, with one edge $\hat{A}\hat{B}$ with length equal to 1 and the transformation F_K which maps $\hat{K}$ on K and such that $F_K(\hat{A}\hat{B}) = A_i A_j$.

Let $\hat{w} = w \circ F_K$. We have

$$|\hat{w}(\hat{B}) - \hat{w}(\hat{A}) - \frac{1}{2}(\frac{\partial \hat{w}}{\partial \xi}(\hat{B}) + \frac{\partial \hat{w}}{\partial \xi}(\hat{A}))| \leqslant \begin{cases} C\, |\hat{w}|_{2,\hat{K}} \\ C\, |\hat{w}|_{3,\hat{K}} \end{cases}$$

Going back to the element K by using the mapping F_K^{-1} , we obtain Lemma 5.5 .

Lemma 5.6 : *We have the estimates.*

$$(5.18)\qquad |E_1(u,w)| \leqslant C\, h\, |u|_{3,\Omega}\, \|w\|_h ,$$

$$(5.19)\qquad |E_1(u,w)| \leqslant C\, h^2\, |u|_{3,\Omega} \Big(\sum_{K \in \mathcal{T}_h} |w|_{3,K}^2\Big)^{1/2} ,$$ *for all*

$u \in H^3(\Omega)$ *and all* $w \in V_h$.

Proof : We shall prove estimates such as those of (5.18) and (5.19) for the expression $E_1^a\ (u,w)$, the proof being the same for E_1^b and E_1^c .

It is easy to see that :

(5.20) $$\left| \int_0^1 \alpha\,(1-\alpha)\,(g_{23}(\alpha) - g_{17}(\alpha))\, d\alpha \right| \leqslant C\, |g|_{1,K_1 \cup K_6}$$

(5.21) $$\left| \int_0^1 \alpha\,(1-\alpha)\,(g_{23}(\alpha) - g_{14}(\alpha))\, d\alpha \right| \leqslant C\, |g|_{1,K_1 \cup K_6}$$

for all $g \in H^1\ (\Omega)$, where K_i is the triangle denoted (i) on figure 6.

Combining equalities (5.8), (5.9), (5.10), inequalities (5.20), (5.21) with inequality (5.16) (resp. (5.17)), we get inequality (5.18) (resp. (5.19)).

Theorem 5.1 : *Assume that* $u \in H^3(\Omega)$ *and let* $u_h \in V_h$ *be the solution of problem* (2.9). *Assume that all the triangles* $K \in \mathcal{T}_h$ *have their edges parallel to three given directions. We have :*

$$\| u - u_h \|_h \leqslant C h\ |u|_{3,\Omega} ,$$

$$\| u - u_h \|_{1,\Omega} \leqslant C h^2\ |u|_{3,\Omega}$$

where C *is a constant* > 0 *independent of* h .

Proof : The proof is exactly the same as that of Theorem 4.1, since Lemma 4.1 with $k = 3$ and Lemma 5.3 on the one hand, and Lemma 4.3 and Lemma 5.6 on the other hand, play exactly the same role.

VI – MISCELLANEOUS REMARKS

Another type of boundary conditions :

Consider the problem of a *simply supported plate,* which can be written as follows : let $W = \left\{ v \in H^2(\Omega)\ ;\ v = 0\ on\ \Gamma \right\}$; we want to find $u \in W$ such that :

(6.1) $$a(u,v) = (f,v) \quad \text{for all } v \in W .$$

By using Green's formula, it can be shown that if the solution u of problem 6.1 is smooth enough, then u is also solution of the problem :

$$\begin{cases} \Delta^2 \; u = f \qquad \text{on} \qquad , & (6.2) \\ u = \Delta u - (1-\sigma) \dfrac{\partial^2 u}{\partial s^2} = 0 \quad \text{on } \Gamma . & (6.3) \end{cases}$$

To get an approximate solution of this problem, we may again use the method described in paragraph II, with a space W_h constructed with one of the five elements given in the preceding sections. But, in this case, the functions $v_h \in W_h$ will be equal to zero (along with their tangential derivatives, for the Ari Adini's and Zienkiewicz elements) at the vertices belonging to Γ, and no assumption is made for the normal derivatives.

As far as the existence of an approximate solution is concerned, nothing is changed, since all what we need is the fact that the tangential derivative (or its mean value along one edge) is equal to zero.

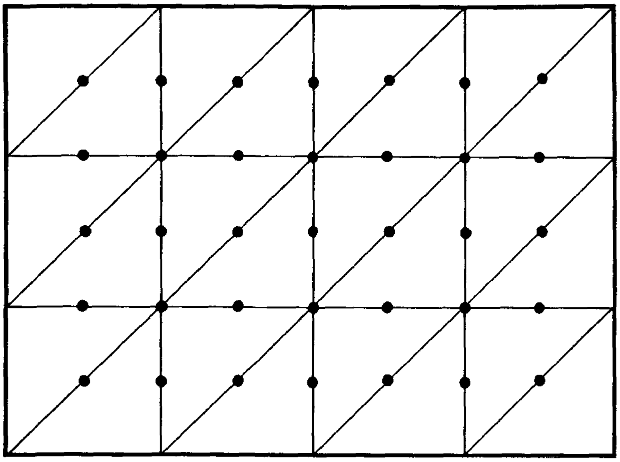
As far as the patch test is concerned, nothing is changed for the expressions $E_2\ (u,w)$ and $E_3\ (u,w)$.

But it is no longer true that $E_1\ (u,w) = 0$ for all $u \in P_2$ and $w \in W_h$. This is due to the boundary terms in E_1 which are equal to $\sum_{S \subset \Gamma} \int_S ((1-\sigma) \frac{\partial^2 u}{\partial s^2} - \Delta u) \frac{\partial w}{\partial n}\, ds$. Nevertheless, the proof of the lemmas and theorems giving error bounds are still valid since it is sufficient that the above expression vanishes for the exact solution u of (6.2), (6.3).

Existence of the approximate solution when $\sigma = 1$:

The exact solution of the clamped plate problem does not depend upon the Poisson's coefficient σ . However the approximate solution u_h does depend upon σ . The question then arises as to what should be the "best" value of the constant σ for a given h , and what happens if one sets $\sigma = 1$ to simplify the mathematical expression of $a\ (u,v)$, although physically σ has a given value less than or equal to 0.5. In the latter case, inequality (2.10) is useless and one has to reconsider the problem of the existence of an approximate solution and the validity of the error bounds.

Actually, if one sets $\sigma = 1$, one cannot use the three elements of section III, since the quadratic form $a_h\ (w,w)$ is no longer V_h elliptic. It is easy to see that setting Δw equal to zero on each triangle yields on each element one homogeneous equation in the case of element I, two equations in the case of element II and three in the case of element III ; this does not imply that $w \equiv 0$. For example, one can consider a mesh as on figure 7.



● degrees of freedom of the element TQM

$M = 4$ $N = 3$

Figure 7.
A triangulation of the domain with triangles generated by three family of parallel lines.

In table I we give the number of equations corresponding to $\Delta w = 0$ in each element and the dimension of V_h with respect to the type of the element used.

Element	Dim V_h	Number of equations
I	(2 M-1) (2 N-1)	2 MN
II	7 MN − 3 (M + N) +1	4 MN
IIII	(3 M-1) (3 N-1)	6 MN

TABLE I

If one uses the elements described in section IV and V, then writing $\Delta w \equiv 0$ on each element, for $w \in V_h$, induces that $w \equiv 0$ so that there exists a constant $C(h) > 0$ such that :

$$(6.5) \qquad \sum_{K \in \mathcal{T}_h} \int_K (\Delta w)^2 \, dx \, dy \geqslant C(h) \, \|w\|_h^2 \qquad \text{for all } w \in V_h .$$

But we are unable to decide whether this constant $C(h)$ is uniformly strictly positive with respect to h, although we know that an inequality such as that of (6.5) holds for all functions in the space $H_0^2(\Omega)$, with a constant obviously independent of h.

Case where the elements considered an section IV, are not all rectangles for a given triangulation.

This problem does not arise for the elements of section III, since no hypothesis has to be done on the shape of the elements to pass the patch test and get error bounds. For the element of section IV, consider a domain Ω as on figure 8, and consider the element $K = A_1 A_2 A_3 A_4$.

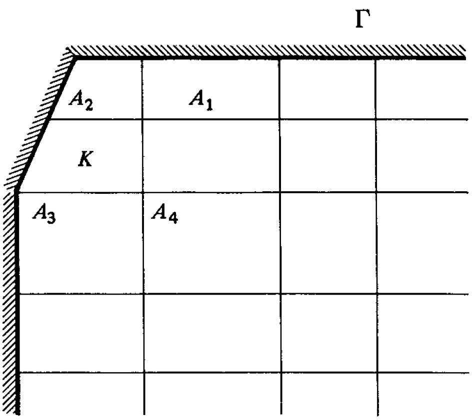


Figure 8

Particular case of boundary elements

Let $g = (1-\sigma)\dfrac{\partial^2 u}{\partial s^2} - u$ where s is a curvilinear abscisse on ∂K.

Lemma 4.3 is no longer true, because of the elements adjacent to the boundary Γ, and we have :

$$E_1(u,w) \leqslant C h^{3/2} \|u\|_{3,\Omega} \left(\sum_{K \in \mathcal{C}_h} |w|^2_{3,K}\right)^{1/2}$$

$$E_1(u,w) \leqslant C h^{1/2} \|u\|_{3,\Omega} \|w\|_h , \quad \text{for all} \quad u \in H^3(\Omega)$$

and $w \in V_h$.

It is then possible to get the following estimates :

$$\|u - u_h\|_h \leqslant C\, h^{1/2} \quad \|u\|_{3,\Omega} \quad ,$$

$$\left(\sum_{K \in \mathcal{T}_h} |u_h|^2_{3,K} \right)^{1/2} \leqslant C\, h^{-1/2} \quad \|u\|_{3,\Omega}$$

$$\|u - u_h\|_{1,\Omega} \leqslant C\, h \quad \|u\|_{3,\Omega} \quad .$$

We see that we still have convergence, but we loose upon the order of convergence, in a case where the triangulations is made up with rectangles and trapeziums for some boundary elements.

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