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FINITE ELEMENT METHODS FOR THE TRANSPORT EQUATION

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Communiqué par P.-A. RAVIART.

Abstract. — *Finite element methods for solving the two dimensional x — y transport equation are considered and some practical numerical schemes are defined. A particular emplasis is put upon the bounds for the errors due to the spatial discretization.*

I. INTRODUCTION AND POSITION OF THE PROBLEM

The neutron transport equation in plane $x — y$ geometry corresponds to the following first order problem :

$$(1.1) \qquad A\varphi \equiv \mu \frac{\partial \varphi}{\partial x} + \nu \frac{\partial \varphi}{\partial y} + \sigma\varphi = f \quad \text{for} \quad (x, y) \times (\mu, \nu) \in \Omega \times Q,$$

$$(1.2) \quad \varphi(x, y, \mu, \nu) = 0 \quad \text{for} \quad (x, y) \times (\mu, \nu) \in \Gamma \times Q \quad \text{if} \quad B = \mu n_x + \nu n_y < 0,$$

where Ω is the open square $]0, 1[\times]0, 1[$, Γ is the boundary of Ω, n_x and n_y denote the components of the outer normal on Γ, and Q is the unit disk $\mu^2 + \nu^2 \leqslant 1$. The function $\varphi(x, y, \mu, \nu)$ represents the flux of neutrons at the point (x, y) in the angular direction (μ, ν). The quantity σ denotes the cross section and f takes into account the scattering, fission and inhomogeneous sources. The boundary conditions (1.2) simply mean that the flux of neutrons entering into the system is equal to zero. Let M be defined by $M = (B^2)^{1/2}$. The boundary condition (1.2) can be written as follows

$$(1.3) \qquad (B — M)\,\varphi = 0 \qquad \text{for} \qquad (x, y) \times (\mu, \nu) \in \Gamma \times Q.$$

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Let $(\cdot, \cdot)_{L^2(\Omega \times Q)}$ denote the usual inner product in the space $L_2(\Omega \times Q)$ and let $\|\cdot\|_{L^2(\Omega \times Q)}$ be the corresponding norm. Problem (1.1), (1.3) can be considered as a symmetric positive Friedrichs' system ([5]) and one has the following result ([5], [16]) :

Theorem 1.1 : *Assume that* $f \in L^2(\Omega \times Q)$ *and that* $\sigma \in L^\infty(\Omega \times Q)$. *Then problem* (1.1), (1.3) *has a unique strong solution* φ, *in the following sense :*

There exists a sequence $\{\varphi_j\}$ *with* $\varphi_j \in H^1(\Omega \times Q)$ *and such that* φ_j *satisfies the boundary conditions* (1.3), *with the property that*

$$\lim_{j \to +\infty} \{ \|A\varphi_j - f\|_{L^2(\Omega \times Q)} + \|\varphi_j - \varphi\|_{L^2(\Omega \times Q)} \} = 0.$$

In what follows, we always assume that problem (1.1), (1.2) has a unique strong smooth solution φ (at least $\varphi \in H^1(\Omega \times Q)$) and that $f \in C^0(\bar{\Omega} \times \bar{Q})$. To solve problem (1.1), (1.3) by a Galerkin type method, we consider as in [11] the following formulation : If $\varphi \in H^1(\Omega \times Q)$, one may write

$$(A\varphi, \psi)_{L^2(\Omega \times Q)} - \left(\frac{B-M}{2}\varphi, \psi\right)_{L^2(\Gamma \times Q)} = (f, \psi)_{L^2(\Omega \times Q)},$$

for all $\psi \in H^1(\Omega \times Q)$.

As is usually done, we shall consider separately the discretizations in the angular variables (μ, ν) and in the spatial variables (x, y).

Angular discretization. Let us consider a triangulation $\mathcal{Q}$ of the domain Q in triangles T_l, $1 \leqslant l \leqslant L$, the boundary of Q being approximated by a polygonal line. Let Q_μ be the reunion of all triangles T_l, $1 \leqslant l \leqslant L$. We define the following geometrical parameter for each triangle T_l :

$h(T_l)$ = diameter of T_l,

$\rho(T_l)$ = diameter of the inscribed circle in T_l.

We assume that the triangulation $\mathcal{Q}$ is a *regular* family, i.e. there exists a constant $\alpha > 0$ independent of the triangulation such that

$$\frac{h(T_l)}{\rho(T_l)} \leqslant \alpha \quad , \quad \text{for} \quad 1 \leqslant l \leqslant L. \tag{1.5}$$

Let $\mathcal{V}_\mu$ denote the space of functions whose restriction to each triangle is a polynomial of degree $\leqslant k$ in μ and ν, the dimension N of the space $\mathcal{V}_\mu$ being then equal to $\frac{k(k+1)}{2} L$. We shall consider the following problem : we want to find $\varphi_\mu \in H_1(\Omega) \times \mathcal{V}_\mu$ which satisfies :

$$(A\varphi_\mu, \psi)_{L^2(\Omega \times Q_\mu)} - \left(\frac{B-M}{2}\varphi_\mu, \psi\right)_{L^2(\Gamma \times Q_\mu)} = (f, \psi)_{L^2(\Omega \times Q_\mu)} \tag{1.6}$$

for all $\psi \in H_1(\Omega) \times \mathcal{V}_\mu$.

Let $\{\psi_l^m(\mu, \nu)\}$, $1 \leqslant m \leqslant \frac{k(k+1)}{2}$ and $1 \leqslant l \leqslant L$, be a basis of the space $\mathcal{V}_\mu$. The functions φ_μ and ψ can be written as follows :

$$\varphi_\mu = \sum_{l,m} u_l^m(x, y)\psi_l^m(\mu, \nu) \quad , \quad \psi = \sum_{l,m} v_l^m(x, y)\psi_l^m(\mu, \nu)$$

What we need to do now is to replace ψ in expression (1.6) by all the functions $v_l^m(x, y)\psi_l^m(\mu, \nu)$, $1 \leqslant m \leqslant \frac{k(k+1)}{2}$, $1 \leqslant l \leqslant L$ and to calculate the following integrals :

$$\int_{T_l} \mu\psi_l^m\psi_l^n \, d\mu \, d\nu \quad , \quad \int_{T_l} \nu\psi_l^m\psi_l^n \, d\mu \, d\nu \quad , \quad \int_{T_l} \psi_l^m\psi_l^n \, d\mu \, d\nu \text{ and}$$

$$\int_{T_l} f\psi_l^m \, d\mu \, d\nu \quad , \quad \text{for } 1 \leqslant m, n \leqslant \frac{k(k+1)}{2} \quad , \quad 1 \leqslant l \leqslant L.$$

In what follows, we shall restrict our attention to the case where $\mathcal{V}_\mu$ is the space of functions which are constant on each triangle. Let φ_l (resp. μ_l, ν_l and f_l) denote the value of the flux φ_μ (resp. μ, ν and f) at the centroid of T_l. And let us consider the following quadrature formula :

$$\int_{T_l} g(\mu, \nu) \, d\mu \, d\nu \sim \text{area } (T_l) g_l. \tag{1.7}$$

If we use formula (1.7) to calculate the integrals arising in expression (1.6), we get the following family of problems : to find $\varphi_l \in H^1(\Omega)$ such that :

$$\int_\Omega \left(\mu_l \frac{\partial \varphi_l}{\partial x} + \nu_l \frac{\partial \varphi_l}{\partial y} + \sigma\varphi_l - f_l \right) v \, dx \, dy - \int_\Gamma \frac{B_l - M_l}{2} \varphi_l v \, d\gamma = 0, \tag{1.8}$$

for all $v \in H^1(\Omega)$, where $B_l = \mu_l n_x + \nu_l n_y$ and $M_l = (B_l^2)^{1/2}$, for $1 \leqslant l \leqslant L$.

Let $\| \cdot \|_\mu$ be the discrete norm defined by

$$\|\varphi\|_\mu^2 = \sum_{l=1}^{L} \text{area } (T_l) \int_\Omega (\varphi(x, y, \mu_l, \nu_l))^2 \, dx \, dy. \tag{1.9}$$

We have the following classical error estimate :

Theorem 1.2 : *We assume that problem* (1.1), (1.3) *has a smooth strong solution* φ. *Let* φ_μ *be the solution of problem* (1.6), *the integrals being calculated using formula* (1.7). *Then we have* :

$$\|\varphi - \varphi_\mu\|_\mu = 0(\Delta\mu^2), \tag{1.10}$$

where $\Delta\mu$ *denotes the supremum of the diameters of the triangles* T_l, $1 \leqslant l \leqslant L$.

REMARK 1.1 : The use of polynomials of degree zero leads us to a discrete ordinate method [10]. The method described above can be successfully applied when we use polynomials of higher degree on triangles [15] or on quadrilateral elements. We can then expect a better accuracy for the numerical results and we obtain some coupling between the angular directions.

REMARK 1.2 : The method described above is an example of the application of a discontinuous method ([13], [17]) to angular variables.

Let $(\cdot,\cdot)_{L^2(\Omega)}$ denote the usual inner product in $L^2(\Omega)$ and let $|\cdot|_{L^2(\Omega)}$ denote the corresponding norm (for $L^2(\Gamma)$, we shall use the same notations, with Ω replaced by Γ). We have now to consider the following problem for the spatial variables, the angular variables μ and ν being considered as parameters : *we want to find u such such that* :

$$(1.11) \qquad Au \equiv \mu \frac{\partial u}{\partial x} + \nu \frac{\partial u}{\partial y} + \sigma u = f \text{ in } \Omega,$$

$$(1.12) \qquad (B - M)u = 0 \text{ on } \Gamma.$$

When $u \in H^1(\Omega)$, this problem can be written as follows [11] :

$$(1.13) \qquad (Au, v)_{L^2(\Omega)} - \left(\frac{B-M}{2} u, v\right)_{L^2(\Gamma)} = (f, v)_{L^2(\Omega)} \text{ for all } v \in H^1(\Omega).$$

We shall use the following result :

Lemma 1.1 : *For all $v \in H^1(\Omega)$, we have* :

$$(Av, v)_{L^2(\Omega)} - \left(\frac{B-M}{2} v, v\right)_{L^2(\Gamma)} \geqslant \sigma |v|^2_{L^2(\Omega)} + \left|\left(\frac{M}{2}\right)^{1/2} v\right|^2_{L^2(\Gamma)}.$$

In what follows, we always assume that μ and ν are positive. If we want to consider the case where μ and (or) ν are negative, we just exchange x in $-x$ and (or) y in $-y$. Define Γ_i, $0 \leqslant i \leqslant 3$, by

$$\Gamma_0 = \Gamma \cap \{x = 0\}, \quad \Gamma_1 = \Gamma \cap \{y = 0\}, \quad \Gamma_2 = \Gamma \cap \{x = 1\} \quad \text{and} \quad \Gamma_3 = \Gamma \cap \{y = 1\}$$

To solve problem (1.11), (1.12), we shall use a finite element method. This approach of the problem has already been considered by several authors ([6], [14], [15], [17], ...) and gives good results [8]. To define numerical schemes, we shall use formulation (1.13) along with finite dimensional spaces of test functions constructed with four nodes isoparametric quadrilateral elements [19] whose diameter are smaller or equal to h (§ II). Existence of the approximate solutions can be shown (§ III) by using results similar to lemma 1.1. Then, generalizing results of [4], we get an error of order h^2 when the elements are equal rectangles (§ IV). When we use numerical quadrature formulas, we solve a local problem on each element, which is precisely a collacation method. Thus we get quasi-explicit numerical schemes (§ II) which are conditionally stable and accurate to the order h^2 if the quadrilaterals are not too distorted, and which are generalizations of classical schemes (D.S.N. [7]).

II. NUMERICAL SCHEMES FOR SOLVING PROBLEM (1.13)

Consider a triangulation $\mathcal{T}_h$ of Ω made up of convex quadrilateral elements Z, such that $\cup K = \bar{\Omega}$, and such that from any vertex belonging to the interior of Ω start four edges (such a triangulation may arise from the deformation of a regular grid). With each element $K \in \mathcal{T}_h$, we associate the geometrical parameters :

$h(K)$ = diameter of K,

$\rho(K)$ = sup{ diameters of the spheres contained in K }

$\theta_i(K)$ = angle of the quadrilateral K, for $1 \leqslant i \leqslant 4$.

Let h be defined by $h = \sup \{ h(K); K \in \mathcal{T}_h \}$.

We assume that the triangulation $\mathcal{T}_h$ is a *regular* family of elements [3], in the following sense : we have :

$h(K) \leqslant \beta\rho(K)$, for all $K \in \mathcal{T}_h$,

$\max \{ |\cos \theta_i(K)| \; , \; 1 \leqslant i \leqslant 4 \} \leqslant \gamma$, for all $K \in \mathcal{T}_h$

where β and γ are two constants independent of the triangulation and such that $\beta > 0$ and $0 < \gamma < 1$.

Let I (resp. J) be the number of quadrilaterals with an edge belonging to Γ_1 (resp. Γ_0). The number of quadrilaterals included in $\bar{\Omega}$ is then equal to IJ and the number of vertices in $\bar{\Omega}$ is equal to $(I+1)(J+1)$. We shall numerotate the quadrilaterals from the left to the right and from the bottom to the top so that $\bar{\Omega} = \cup K_{i,j}$, $0 \leqslant i \leqslant I-1$, $0 \leqslant j \leqslant J-1$. Consider now the quadrilateral K with vertices $A_i = (x_i, y_i)$, $1 \leqslant i \leqslant 4$ (fig. 2.1). There exists a unique invertible bilinear mapping F_K such that K is the image by F_K of the square $\hat{K} = [-1, +1] \times [-1, +1]$. This mapping is defined as follows :

(2.1)
$$x = \frac{(1+\xi)(1+\eta)}{4} x_1 + \frac{(1-\xi)(1+\eta)}{4} x_2 + \frac{(1-\xi)(1-\eta)}{4} x_3 + \frac{(1+\xi)(1-\eta)}{4} x_4 ;$$

(2.2)
$$y = \frac{(1+\xi)(1+\eta)}{4} y_i + \frac{(1-\xi)(1+\eta)}{4} y_2 + \frac{(1-\xi)(1-\eta)}{4} y_3 + \frac{(1+\xi)(1-\eta)}{4} y_4 .$$

To construct a finite dimensional space in which we shall look for an approximate solution u_h, we shall use either conforming or non-conforming elements.

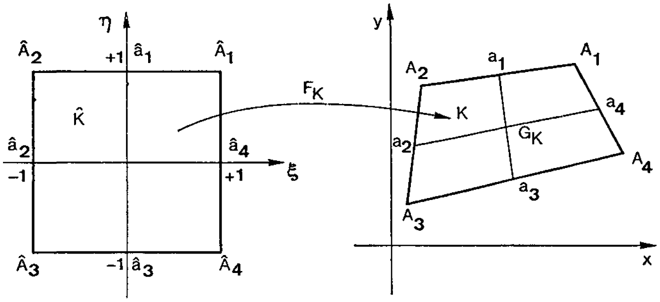

Figure 2.1

The isoparametric mapping F_K

Conforming case, definition of the space V_h : Let $\hat{Q}(1)$ be the space of polynomials defined by $\hat{Q}(1) = \{ \hat{q} : \hat{K} \rightarrow R; \hat{q} = a + b\xi + c\eta + d\xi\eta \}$. There exists a unique polynomial of $\hat{Q}(1)$ which takes given values at the vertices $\hat{A}_i$, $1 \leqslant i \leqslant 4$. The space P_K of the *shape functions* over the element K is defined by :

$$P_K = \{ p \,;\, p = \hat{p}_0 F_K^{-1} \quad , \quad \hat{p} \in \hat{Q}(1) \} \tag{2.3}$$

We shall define V_h as the space of functions defined and *continuous over* $\bar{\Omega}$ and whose restriction to each element K belongs to P_K. The dimension of V_h is equal to $(I + 1)(J + 1)$ and the degrees of freedom of V_h can be chosen as the values of the functions of V_h at the vertices belonging to $\Gamma_0 \cup \Gamma_1$ and at the centroids of the quadrilateral elements K. Let v_h be a function of V_h taking the values v_i at the vertices A_i, $1 \leqslant i \leqslant 4$ of quadrilateral K; then the restriction of v_h to the quadrilateral K can be expressed in local coordinates ξ, η as :

$$\hat{v}_h(\xi, \eta) = \frac{(1+\xi)(1+\eta)}{4} v_1 + \frac{(1-\xi)(1+\eta)}{4} v_2 + \frac{(1-\xi)(1-\eta)}{4} v_3 + \frac{(1+\xi)(1-\eta)}{4} v_4 \tag{2.4}$$

Given a function u defined and continuous over $\bar{\Omega}$, its *interpolate* $r_h u$ will be the unique function of V_h taking the same values as u at the vertices of the quadrilaterals $K \in \mathcal{C}_h$. Let V_h^0 be the subspace of V_h spanned by the functions of V_h which are equal to zero at the vertices belonging to Γ. Any function v_h of V_h can be written as :

$$v_h = v_h^0 + v_b^h \quad , \quad v_h^0 \in V_h^0 \text{ and } v_h^0 = v_h \text{ at all the vertices} \tag{2.5}$$

belonging to the interior of Ω. Such a decomposition is unique and the function v_b^h depends only on the values of v_h at the vertices belonging to Γ.

Non-conforming case, definition of the space W_h : Let $\hat{P}(1)$ be the space of polynomials defined by $\hat{P}(1) = \{ \hat{p} : \hat{K} \to R; \hat{p} = a + b\xi + c\eta \}$. There exists a unique polynomial of $\hat{P}(1)$ which takes given values $\hat{p}_i$ at the mid-points a_i, $1 \leqslant i \leqslant 4$, of the edges of $\hat{K}$ if we have :

$$\hat{p}_1 + \hat{p}_3 = \hat{p}_2 + \hat{p}_4$$

The space P_k of the shape functions over the element K is defined by :

$$P_K = \{ p ; p = \hat{p}_0 F_K^{-1} \quad , \quad \hat{p} \in \hat{P}(1) \} \tag{2.6}$$

We shall define W_h as the space of functions *continuous at the mid-points of the edges of the quadrilaterals* and whose restriction to each quadrilateral K belongs to P_K. The dimension of W_h is equal to $IJ + I + J$ and the degrees of freedom of W_h can be chosen as the values of the functions of W_h at the mid-points of the edges included in $\Gamma_0 \cup \Gamma_1$ and at the centroids of the elements $K \in \mathcal{C}_h$. The function w_h of W_h taking the values w_i at the mid-points a_i, $1 \leqslant i \leqslant 4$, of the edges of K and the value w_0 at the centroid of K, can be expressed in local coordinates ξ, η as :

$$\hat{w}_h(\xi, \eta) = w_0 + \frac{w_4 - w_2}{2}\,\xi + \frac{w_1 - w_3}{2}\,\eta, \tag{2.7}$$

$$\text{with } 2w_0 = w_1 + w_3 = w_2 + w_4$$

Given a function u defined and continuous over $\bar{\Omega}$, its *interpolate* $r_h u$ will be the unique function of W_h such that the value of $r_h u$ at the mid-point of any edge of the quadrilateral is equal to the average of the values of u at the end-points of this edge. It is still possible to write any function w_h of W_h as $w_h = w_h^0 + w_h^b$, with $w_h^0 = w_h$ at the mid-points of the edges which have no point in common with Γ and $w_h^0 = 0$ at the mid-points of the edges included in Γ. Such a decomposition is not unique and the function w_b^h does not depend only on the values of w_h on the boundary Γ.

We shall give now some numerical quadrature formulas, which will be useful to evaluate the integrals arising in the inner products. Consider the following formula on the square $\hat{K}$:

$$\int_{\hat{K}} \hat{f}(\xi, \eta)\, d\xi\, d\eta \sim 4\hat{f}(0, 0), \tag{2.8}$$

which induces on the quadrilateral K the following formula :

$$\int_K f(x, y)\, dx\, dy \sim \text{area}\,(K) \cdot f(G_K), \tag{2.9}$$

where G_K is the centroid of the quadrilateral K.

We define the two following formulas on any edge A_iA_j :

$$\int_{Ai}^{Aj} f\,\mathrm{d}t \sim \frac{A_iA_j}{2}\,(f(A_i)+f(A_j)) \tag{2.10}$$

$$\int_{Ai}^{Aj} f\,\mathrm{d}t \sim A_iA_jf(a_{ij}) \tag{2.11}$$

where t is a curvilinear abscissa along A_iA_j, and a_{ij} is the mid-point of A_iA_j. With those formulas, we can define discrete inner products between functions of V_h or W_h as follows : let v_h, $w_h \in V_h$ (or W_h), we define $(v_h, w_h)_h$ by :

$$(v_h, w_h)_h = \sum_{K\in\mathcal{T}_h} \text{area}\,(K)v_h(G_K)w_h(G_K). \tag{2.12}$$

Let $|\cdot|_h$ denote the corresponding semi norm.

Let s denote any edge belonging to Γ, for any v_h, $w_h \in V_h$, we define $\langle v_h, w_h\rangle_h$ by :

$$\langle v_h, w_h \rangle_h = \sum_{s\in\Gamma_0\cup\Gamma_1} \frac{A_sB_s}{2}\,((v_hw_h)(A_s) + (v_hw_h)(B_s)) + \sum_{s\in\Gamma_2\cup\Gamma_3} (A_sB_s)(v_hw_h)(G_s) \tag{2.13}$$

where A_s and B_s denote the end-point of s and where G_s denote the mid-point of s. The corresponding norm will be denoted by $\langle\cdot\rangle_h$. For any v_h, $w_h \in V_h$ (or W_h) we define $[.,.]_h$ by :

$$[v_h, w_h]_h = \sum_{s\in\Gamma} (A_sB_s)(v_hw_h)(G_s).$$

The corresponding semi-norm will be denoted by $[.]_h$.

Let $(.,.)_{L^2(\Omega)}$ (resp $(.,.)_{L^2(\Gamma)}$) denote the inner product in V_h or W_h induced by the inner product in $L^2(\Omega)$ (resp $L^2(\Gamma)$). In the non-conforming case, we shall use the following notation :

$$(u_h, v_h)^* = \sum_{K\in\mathcal{T}_h} (u_h, v_h)_{L^2(K)} \quad , \quad \text{for} \quad u_h, v_h \in W_h.$$

REMARK 2.1 : We have the following almost classical inequalities, where c is a constant independent of h :

$$[v_h]_h \leqslant \langle v_h \rangle_h \leqslant c\,|v_h|_{L^2(\Gamma)} \leqslant ch^{-1}\langle v_h\rangle_h \text{ for all } v_h \in V_h,$$

$$[w_h]_h \leqslant c\,|w_h|_{L^2(\Gamma)} \text{ for all } w_h \in W_h,$$

$$|v_h|_h \leqslant |v_h|_{L^2(\Omega)} \leqslant ch^{-2}\,|v_h|_h \text{ for all } v_h \in V_h^0.$$

Now we can define the following problems :

Scheme 1 : To find $u_h \in V_h$ such that :

$$(Au_h, v_h)_{L^2(\Omega)} - \left(\frac{B-M}{2} u_h, v_h\right)_{L^2(\Gamma)} = (f, v_h)_{L^2(\Omega)} \text{ for all } v_h \in V_h.$$

Scheme 2 : To find $u_h \in W_h$ such that :

$$(Au_h, w_h)^* - \left(\frac{B-M}{2} u_h, w_h\right)_{L^2(\Gamma)} = (f, w_h)_{L^2(\Omega)} \text{ for all } w_h \in W_h.$$

Scheme 3 : To find $V_h \in V_h$ such that :

$$(Au_h, v_h)_h - \langle \frac{B-M}{2} u_h, v_h \rangle_h = (f, v_h)_h \text{ for all } v_h \in V_h.$$

Scheme 4 : To find $u_h \in W_h$ such that :

$$(Au_h, w_h)_h - \left[\frac{B-M}{2} u_h, w_h\right]_h = (f, w_h)_h \text{ for all } w_h \in W_h.$$

When one wants to give a numerical solution for scheme 1, one has to invert a nine-diagonal matrix with a total bandwidth equal to $2I + 1$ (resp. $2J + 1$) if one numerotates the vertices from the left to the right and then from the bottom to the top (resp. from the bottom to the top and then from the left to the right). The situation is still more complicated for scheme 2. But we shall see that for schemes 3 and 4, one can get a quasi-explicit resolution.

Lemma 2.1 : *Scheme* 3 *can be written as follows, on each quadrilateral* K *with vertices* $A_i(x_i, y_i)$, *where* $u_i = u_h(A_i)$, $1 \leqslant i \leqslant 4$:

$$(2.14) \qquad \begin{aligned} &(u_1 - u_3)(\mu(y_2 - y_4) - \nu(x_2 - x_4)) \\ &+ (u_2 - u_4)(-\mu(y_1 - y_3) + \nu(x_1 - x_3)) + ((y_1 - y_3)(x_4 - x_2) \\ &+ (y_2 - y_4)(x_1 - x_3))\left(\sigma(G_K)\frac{u_1 + u_2 + u_3 + u_4}{4} - f(G_K)\right) = 0 \end{aligned}$$

where G_K *is the centroid of* K, $u_h = 0$ *at the vertices belonging to* $\Gamma_0 \cup \Gamma_1$.

Proof : According to the definition of scheme 3, one may write :

$$\sum_{K \in \mathcal{C}_h} \text{area}(K)(Au_h \cdot v_h)(G_K) - \sum_{s \in \Gamma_0 \cup \Gamma_1} \frac{A_s B_s}{2} \{ (B-M)u_h \cdot v_h)(A_s) + ((B-M)u_h \cdot v_h)(B_s) \} = \sum_{K \in \mathcal{C}_h} \text{area}(K)(f \cdot v_h)(G_K).$$

where A_s and B_s are the end-points of the edge s and where A_sB_s is the distance between A_s and B_s.

The values of v_h at the centroid of the elements K and at the vertices belonging to $\Gamma_0 \cup \Gamma_1$ are degrees of freedom of the space V_h. So one may write :

$$(2.15) \qquad (Au_h)(G_K) = f(G_K) \quad \text{for all} \quad K \in \mathcal{T}_h,$$

$u_h = 0$ at the vertices belonging to $\Gamma_0 \cup \Gamma_1$.

Now let J_F be the jacobian of the isoparametric transformation F_K defined by (2.1) and (2.2). One has the following classical formulas :

$$(2.16) \qquad J_F(\xi, \eta) \frac{\partial \hat{u}_h}{\partial x}(\xi, \eta) = \frac{\partial \hat{u}_h}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial \hat{u}_h}{\partial \eta} \frac{y}{\partial \xi},$$

$$(2.17) \qquad J_F(\xi, \eta) \frac{\partial \hat{u}_h}{\partial y}(\xi, \eta) = - \frac{\partial \hat{u}_h}{\partial \xi} \frac{\partial x}{\partial \eta} + \frac{\partial \hat{u}_h}{\partial \eta} \frac{\partial x}{\partial \xi}.$$

Combining equalities (2.15), (2.16) and (2.17) one easily gets equality (2.14).

In the same manner, one can prove the following result :

Lemma 2.2 : *Scheme* 4 *can be written as follows, on each quadrilateral K with vertices $A_i(x_i, y_i)$, where $u_i = u_h(a_i)$, $1 \leqslant i \leqslant 4$:*

$$\begin{aligned}(2.18) \quad & (u_4 - u_2)(\mu(y_1 + y_2 - y_3 - y_4) - \nu(x_1 + x_2 - x_3 - x_4)) + \\ & + (u_1 - u_3)(-\mu(y_1 - y_2 - y_3 + y_4) + \nu(x_1 - x_2 - x_3 + x_4)) + \\ & + ((y_1 - y_3)(x_4 - x_2) + (y_2 - y_4)(x_1 - x_3)) \\ & \qquad (\sigma(G_K)u(G_K) - f(G_K)) = 0\end{aligned}$$

$$(2.19) \qquad 2u(G_K) = u_1 + u_3 = u_2 + u_4,$$

$u_h = 0$ *at the mid-points of the edges included in* $\Gamma_0 \cup \Gamma_1$.

REMARK 2.2. : Both schemes 3 and 4 are quasi-explicit if one solves the system by starting from the elements adjacents to $\Gamma_0 \cap \Gamma_1$.

REMARK 2.3. : In the conforming case (scheme 3), if one wants to calculate the value u_1 as a function of u_2, u_3 and u_4, or in the non-conforming case (scheme 4), if one wants to calculate u_1 and u_4 as a function of u_2 and u_3, a practical necessary condition of resolution seems to be the following : $\mu(y_2 - y_4) - \nu(x_2 - x_4) > 0$. This condition means that the characteristic direction (μ, ν) makes a positive angle with the diagonal $\overrightarrow{A_4A_2}$ of the quadrilateral K. We shall see later on that this condition is not sufficient for stability.

REMARK 2.4. : Let us assume now that the domain Ω is divided into equal rectangles with edges parallel to the axes and equal respectively to $\Delta x = \frac{1}{I}$ and $\Delta y = \frac{1}{J}$. In the conforming case, we shall write $u_{i,j}$ for $u_h(x_i, y_j)$, for $0 \leqslant i \leqslant I$,

$0 \leqslant j \leqslant J$, and in both cases, we shall write $u_{i+1/2,j}$ for $u_h\left(\frac{x_i + x_{i+1}}{2} \cdot y_j\right)$, for $0 \leqslant i \leqslant I - 1$ and $0 \leqslant j \leqslant J$, and $u_{i,j+1/2}$ for $u_h\left(x_i, \frac{y_j + y_{j+1}}{2}\right)$, $0 \leqslant i \leqslant I$ and $0 \leqslant j \leqslant J - 1$. Scheme 3 can then be written as follows :

$$\mu \frac{(u_{i+1,j+1} + u_{i+1,j}) - (u_{i,j+1} + u_{i,j})}{2\Delta x} +$$

$$+ \nu \frac{(u_{i+1,j+1} + u_{i,j+1}) - (u_{i+1,j} + u_{i,j})}{2\Delta y}$$

$$+ \sigma_{i+1/2,j+1/2} \frac{u_{i+1,j+1} + u_{i+1,j} + u_{i,j+1} + u_{i,j}}{4} = f_{i+1/2,j+1/2}$$

for $0 \leqslant i \leqslant I - 1$, $0 \leqslant j \leqslant J - 1$,

$$u_{0,j} = u_{i,0} = 0 \text{ for } 0 \leqslant i \leqslant I,\ 0 \leqslant j \leqslant J.$$

Scheme 4 can be written as follows (classical D.S.N. scheme [7]) :

$$\mu \frac{u_{i+1,j+1/2} - u_{i,j+1/2}}{\Delta x} + \nu \frac{u_{i+1/2,j+1} - u_{i+1/2,j}}{\Delta y}$$

$$+ \sigma_{i+1/2,j+1/2} u_{i+1/2,j+1/2} = f_{i+1/2,j+1/2},$$

$$2u_{i+1/2,j+1/2} = u_{i+1,j+1/2} + u_{i,j+1/2} = u_{i+1/2,j+1} + u_{i+1/2,j},$$

for $0 \leqslant i \leqslant I - 1$ and $0 \leqslant j \leqslant J - 1$;

$$u_{0,j+1/2} = u_{i+1/2,0} \text{ for } 0 \leqslant j \leqslant J - 1,\ 0 \leqslant i \leqslant I - 1.$$

III. EXISTENCE AND STABILITY OF THE APPROXIMATE SOLUTIONS

We can already give the following result for scheme 1.

Lemma 3.1 : *For any $v_h \in V_h$, we have :*

(3.1) $$(Av_h, v_h)_{L^2(\Omega)} - \left(\frac{B - M}{2} v_h, v_h\right)_{L^2(\Gamma)}$$

$$\geqslant \sigma \left|v_h\right|^2_{L^2(\Omega)} + \left|\left(\frac{M}{2}\right)^{1/2} v_h\right|^2_{L^2(\Omega)}$$

Scheme 1 *has a unique solution $u_h \in V_h$ and satisfying :*

$$(3.2) \qquad |u_h|^2_{L^2(\Omega)} + \left| \left(\frac{M}{2}\right)^{1/2} u_h \right|^2_{L^2(\Gamma)} \leqslant c\,|f|_{L^2(\Omega)},$$

where c is a constant independent of h.

Proof: Since V_h is a subspace of $H^1(\Omega)$, one can apply lemma 1.1 for any $v_h \in V_h$ and we get inequality (3.1). Inequality (3.2) is a consequence of inequality (3.1).

For the other schemes, we shall need the following hypotheses :

Hypothesis 3.1 : The distance $z(K)$ between the mid-points of the diagonals of quadrilateral K satisfies the following inequality $z(K) \leqslant \lambda h(K)^2$ for all $K \in \mathcal{T}_h$, where λ is a constant independent of the triangulation.

Hypothesis 3.2 : Let $A_i(x_i, y_i)$, $1 \leqslant i \leqslant 4$, be the vertices of quadrilateral K; we have :

$$|\mu(y_1 - y_2 + y_3 - y_4) - \nu(x_1 - x_2 + x_3 - x_4)| \leqslant c_0 h(K)\,|\mu(y_1 - y_4 + y_2 - y_3) - \nu(x_1 - x_4 + x_2 - x_3)|,$$

$$|\mu(y_1 - y_2 + y_3 - y_4) - \nu(x_1 - x_2 + x_3 - x_4)| \leqslant c_0 h(K)\,|\mu(y_1 - y_2 + y_4 - y_3) - \nu(x_1 - x_2 + x_4 - x_3)|,$$

for all $v \in \mathcal{T}_h$, where c_0 is a constant independent of $\mathcal{T}_h$.

REMARK 3.1 : Let h_0 be a real positive number such that $0 < h_0 < \dfrac{1}{3c_0}\cdot$ Hypothesis 3.2 implies that, if $h \leqslant h_0$, the angles done by the characteristic direction (μ, ν) with two opposite edges $\overrightarrow{A_iA_{i+1}}$ and $\overrightarrow{A_{i+3}A_{i+2}}$, $i = 1, 2$, (with $A_5 = A_1$) have either the same sign or they are both equal to zero. When these angles are not equal to zero, any quadrilateral K has always two adjacent edges through which the flux of neutrons enters into the quadrilateral and two adjacent edges (opposite to the others) through which the flux goes outside the quadrilateral.

We have the following result for scheme 2 :

Lemma 3.2 : *We assume that hypothesis* 3.1 *is satisfied and that σ is greater than a constant depending only on μ, ν and λ (in the case where this last condition is not satisfied, one can always consider a new function v defined by $v = u \exp\left(-D\left(\frac{x}{\mu} + \frac{y}{\nu}\right)\right)$ where D is a positive constant suitably chosen; we then have :*

$$(3.3) \qquad (Av_h, v_h)^*_{L^2(\Omega)} - \left(\frac{B - M}{2} v_h, v_h\right)_{L^2(\Gamma)} \geqslant c\,|v_h|^2_{L^2(\Omega)} + |(M)^{1/2} v_h|^2_{L^2(\Gamma_0 \cup \Gamma_1)} + [M^{1/2} v_h]^2_h,$$

for all $v_h \in W_h$, where c is a constant independent of h.

Scheme 2 has a unique solution $u_h \in W_h$ and satisfying :

$$(3.4) \qquad |u_h|^2_{L^2(\Omega)} + |(M)^{1/2} u_h|^2_{L^2(\Gamma_0 \cup \Gamma_1)} + [(M)^{1/2} u_h]^2_h \leqslant c\, |f|^2_{L^2(\Omega)}.$$

Proof : Let K be the quadrilateral with vertices $A_i(x_i, y_i)$, $1 \leqslant i \leqslant 4$. We have :

$$(3.5) \qquad \int_K \left(\mu \frac{\partial v_h}{\partial x} + \nu \frac{\partial v_h}{\partial y}\right) v_h \,\mathrm{d}x\,\mathrm{d}y = \sum_{i=1}^{4} \frac{v_i^2}{2} (\mu(y_{i+1} - y_i) - \nu(x_{i+1} - x_i))$$

$$+ \frac{1}{12} (v_1^2 + v_3^2 - v_2^2 - v_4^2)(\mu(y_1 - y_2 + y_3 - y_4) - \nu(x_1 - x_2 + x_3 - x_4)),$$

with $x_5 = x_1$ and $y_5 = y_1$.

It is easy to see that :

$$(3.6) \qquad \int_K \sigma v_h^2 \,\mathrm{d}x\,\mathrm{d}y \geqslant c(h(K))^2 (v_1^2 + v_2^2 + v_3^2 + v_4^2)$$

where the constant c is independent of h.

Assume now that the quadrilateral K has an edge (for example A_3A_4) belonging to Γ_1. We have :

$$(3.7) \qquad -\int_{A_3A_4} \frac{B - M}{2} v_h^2 \,\mathrm{d}x = \nu(x_4 - x_3)\left(v_3^2 + \frac{1}{12}(v_4 - v_2)^2\right)$$

We can get the same type of equality for the edges belonging to Γ_0. Combining equalities (3.5) and (3.7), inequality (3.6) and hypothesis 3.1, we easily get inequality (3.3).

We shall now give some results for schemes 3 and 4.

Lemma 3.3 : *Let us assume that hypothesis* 3.2 *holds and that $h \leqslant h_0$. Then one can always numerotate the vertices A_i, $1 \leqslant i \leqslant 4$ of any quadrilateral $K \in \mathcal{C}_h$ in such a way that we have :*

$$\mu(y_1 - y_4) - \nu(x_1 - x_4) > 0,\ \mu(y_2 - y_1) - \nu(x_2 - x_1) > 0,$$

$$\mu(y_3 - y_2) - \nu(x_3 - x_2) < 0 \text{ and } \mu(y_4 - y_3) - \nu(x_4 - x_3) < 0.$$

Then, for scheme 3, *we have :*

(3.8)
$$\operatorname{area}(K)\left(\mu\frac{\partial v_h}{\partial x}+\nu\frac{\partial v_h}{\partial y}\right)v_h(G_K)$$
$$\geqslant \frac{1-2c_0h(K)}{1-c_0h(K)}\left(\mu\frac{y_1-y_4}{2}-\nu\frac{x_1-x_4}{2}\right)\left(\frac{v_1+v_4}{2}\right)^2$$
$$+\frac{1-2c_0h(K)}{1-c_0h(K)}\left(\mu\frac{y_2-y_1}{2}-\nu\frac{x_2-x_1}{2}\right)\left(\frac{v_1+v_2}{2}\right)^2$$
$$+\frac{1}{1-c_0h(K)}\left(\mu\frac{y_3-y_2}{2}-\nu\frac{x_3-x_2}{2}\right)\left(\frac{v_2+v_3}{2}\right)^2$$
$$+\frac{1}{1-c_0h(K)}\left(\mu\frac{y_4-y_3}{2}-\nu\frac{x_4-x_3}{2}\right)\left(\frac{v_3+v_4}{2}\right)^2$$

In the same way, for scheme 4, *we have, for all* $v_h \in W_h$ *:*

(3.9)
$$\operatorname{area}(K)\left(\mu\frac{\partial v_h}{\partial x}+\nu\frac{\partial v_h}{\partial y}\right)\cdot v_h(G_K)\geqslant$$
$$\geqslant \frac{1-3c_0h(K)}{1-c_0h(K)}\left(\mu\frac{y_1-y_4}{2}-\nu\frac{x_1-x_4}{2}\right)v_4^2$$
$$+\frac{1-3c_0h(K)}{1-c_0h(K)}\left(\mu\frac{y_2-y_1}{2}-\nu\frac{x_2-x_1}{2}\right)v_1^2$$
$$+\frac{1+c_0h(K)}{1-c_0h(K)}\left(\mu\frac{y_3-y_2}{2}-\nu\frac{x_3-x_2}{2}\right)v_2^2$$
$$+\frac{1+c_0h(K)}{1-c_0h(K)}\left(\mu\frac{y_4-y_3}{2}-\nu\frac{x_4-x_3}{2}\right)v_3^2$$

Proof : In the conforming case, we have for all $v_h \in V_h$:
$$\operatorname{area}(K)\left(\mu\frac{\partial v_h}{\partial x}+\nu\frac{\partial v_h}{\partial y}\right)v_h(G_K)=\sum_{i=1}\left(\mu\frac{y_{i+1}-y_i}{2}-\nu\frac{x_{i+1}-x_i}{2}\right)\left(\frac{v_i+v_{i+1}}{2}\right)^2$$
$$+\frac{1}{8}\left(\mu(y_1-y_2+y_3-y_4)-\nu(x_1-x_2+x_3-x_4)\right)$$
$$\left[\left(\frac{v_1+v_4}{2}-\frac{v_2+v_3}{2}\right)^2-\left(\frac{v_1+v_2}{2}-\frac{v_3+v_4}{2}\right)^2\right]$$

with $x_5 = x_1$, $y_5 = y_1$ and $v_5 = v_1$. If we combine this identity with hypothesis (3.2), we get inequality (3.8).

The proof is the same in the non-conforming case if we start from the following identity, for all $v_h \in W_h$:

$$\text{area}(K)\left(\mu \frac{\partial v_h}{\partial x} + \nu \frac{\partial v_h}{\partial y}\right) v_h(G_K) = \sum_{i=1}^{4} \left(\mu \frac{y_{i+1} - y_i}{2} - \nu \frac{x_{i+1} - x_i}{2}\right) v_i^2$$
$$+ \frac{1}{4}(\mu(y_1 - y_2 + y_3 - y_4) - \nu(x_1 - x_2 + x_3 - x_4))$$
$$((v_1 - v_3)^2 - (v_4 - v_2)^2).$$

As a consequence of Lemma 3.3, we get :

Lemma 3.4 : *We assume that hypothesis* 3.2 *holds and that* $h \leqslant h_0$, *then scheme* 3 (*resp. scheme* 4) *has a unique solution* $u_h \in V_h$ *(resp.* W_h*) and satisfying :*

(3.10) $$|u_h|_h + \langle M^{1/2} u_h \rangle_h \ (\text{resp. } |u_h|_h + [M^{1/2} u_h]_h) \leqslant c |f|_h,$$

where c is a constant independent of the triangulation.

REMARK 3.2 : If we want hypothesis (3.2) to hold for any $K \in \mathcal{T}_h$, we have to perform the calculations from the bottom to the top and from the left to

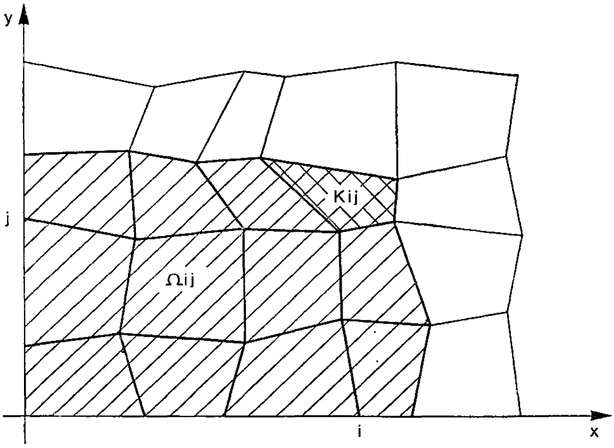

Figure 3.1
The domain Ω_{ij}

the right, starting from $\Gamma_0 \cup \Gamma_1$. The values of the flux φ in any quadrilateral K_{ij} will depend on the values of the flux in all quadrilaterals belonging to Ω_{ij}, where Ω_{jj} is defined on figure 3.1.

REMARK 3.3 : Lemma 3.4 shows that hypothesis 3.2 is a sufficient condition for stability. Numerical results show that if hypothesis 3.2 does not hold, then we do not have stability [12], and the results are meaningless.

Before we give some estimations of the error between the exact solution and the approximate solution, we shall check that the conditions of neutron conservation, as expressed in [9], are satisfied.

Neutron conservation : First of all, it is easy to see on each element K that if the solution u_h is a constant, then the approximations of the derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ cancel identically. Now we must check that spatial integration of the approximations of the derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ in equations (2.14) and (2.18) results in a balance statement involving boundary terms only. So we have to calculate the following quantities :

$$E(u_h) = \sum_{K \in \mathcal{C}_h} \int_K \left(\mu \frac{\partial u_h}{\partial x} + \nu \frac{\partial u_h}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y, \tag{3.11}$$

defined for $u_h \in V_h$ or W_h (scheme 1 or 2), and

$$E_h(u_h) = \sum_{K \in \mathcal{C}_h} \left(\mu \frac{\partial u_h}{\partial x} + \nu \frac{\partial u_h}{\partial y} \right)(G_K) \cdot \text{area } K \tag{3.12}$$

defined for $u_h \in V_h$ or W_h (scheme 3 or 4).

One may write :

$$E(u_h) = \sum_{K \in \mathcal{C}_h} \int_{\partial K} (\mu n_x^K + \nu n_y^K) u_h \, \mathrm{d}t \quad , \quad \text{and}$$

$$E_h(u_h) = \sum_{K \in \mathcal{C}_h} \int_K \left(\mu \frac{\partial u_h}{\partial x} + \nu \frac{\partial u_h}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y = \sum_{K \in \mathcal{C}_h} \int_{\partial K} (\mu n_x^K + \nu n_y^K) u_h \, \mathrm{d}t$$

where n_x^K and n_y^K are the components of the outer normal on ∂K, and where t is a curvilinear abscissa on ∂K.

Now, for schemes 1 and 3, u_h is a continuous function on $\bar{\Omega}$, so we get :

$$E(u_h) = \int_\Gamma (\mu n_x + \nu n_y) u_h \, \mathrm{d}t,$$

and

$$E_h(u_h) = \int_\Gamma (\mu n_x + \nu n_y) u_h \, \mathrm{d}t.$$

For schemes 2 and 4, we also get the same result because u_h is a polynomial of degree $\leqslant 1$ on each edge and u_h is continuous at the mid-point of the edges.

IV. ERROR BOUNDS : NOTATIONS AND FUNDAMENTAL LEMMAS

Given an integer $m \geqslant 0$ and a real number $p \geqslant 1$, we let

$$W^{m,p}(\Omega) = \{ v \, ; vL^p(\Omega), \partial^\alpha v \in L^p(\Omega), |\alpha| \leqslant m \}$$

denote the usual Sobolev space, with the following norm

$$\|v\|_{m,p,\Omega} = \left(\sum_{|\alpha| \leqslant m} \|\partial^\alpha v\|^p_{0,p,\Omega} \right)^{1/p}$$

where $\|\cdot\|_{0p,\Omega}$ represents the usual norm in $L^p(\Omega)$, and where α is a multi-index such that $\alpha = (\alpha_1, \alpha_2)$, $\alpha_i \geqslant 0$, $|\alpha| = \alpha_1 + \alpha_2$.

We shall also use the following semi-norm on $W^{m,p}(\Omega)$:

$$|v|_{m,p,\Omega} = \left(\sum_{|\alpha| = m} \|\partial^\alpha v\|^p_{0,p,\Omega} \right)^{1/p}$$

The usual modifications in the preceding definitions will be done for $p = \infty$, and we shall write $W^{m,2}(\Omega) = H^m(\Omega)$, for $m \geqslant 0$.

The same definitions will be used for Ω replaced by Γ, when $m = 0$. In what follows, c will always denote a constant independent of the triangulation $\mathcal{T}_h$. We have the following lemmas [1], [2], [18] :

Lemma 4.1 : *Let u be a function belonging to the space $W^{2,p}(\Omega)$, $p > 1$ and let $r_h u \in V_h$* (or W_h) *be its interpolate, as defined in paragraph* II. *We have, for* $0 \leqslant m \leqslant 2$, $1 < p \leqslant +\infty$ *and for all* $K \in \mathcal{T}_h$:

$$\|u - r_h u\|_{m,p,K} \leqslant ch(K)^{2-m} \, |u|_{2,p,K}.$$

Lemma 4.2 : *Let u be a function belonging to the space $H^r(\Omega)$ for $r = 2, 3$, and let $r_h u \in V_h$ be its interpolate. Then, we have :*

$$\|u - r_h u\|_{L^2(\partial K)} \leqslant c(h(K))^{r-1} \, |u|_{r-1,2,\partial K} \text{ for all } K \in \mathcal{T}_h, \text{ and}$$

$$\|u - r_h u\|_{L^2(\Gamma)} \leqslant ch^{r-1}(|u|_{r,2,\Omega} + |u|_{r-1,2,\Omega}) \text{ for } r = 2, 3$$

Lemma 4.3 : *Let u be a function belonging to the space $H^2(\Omega)$, and let $r_h u \in W_h$ be its interpolate. We have :*

$$\|u - r_h u\|_{L^2(\partial K)} \leqslant c(h(K))^{3/2} \, |u|_{2,2,K} \text{ for all } K \in \mathcal{T}_h, \text{ and}$$

$$\|u - r_h u\|_{L^2(\Gamma)} \leqslant ch^{3/2} \, |u|_{2,2,\Omega}.$$

If we assume now that $u \in W^{2,\infty}(\Omega)$, *we have* :

$$\|u - r_h u\|_{L^2(\partial K)} \leqslant c(h(K))^{5/2} \, |u|_{2,\infty,K} \text{ for all } K \in \mathcal{C}_h, \text{ and}$$

$$\|u - r_h u\|_{L^2(\Gamma)} \leqslant ch^2 \, |u|_{2,\infty,\Omega}.$$

If one uses the techniques developped in [3], one can show the following results :

Lemma 4.4 : *Let* u *be a function belonging to the space* $H^3(\Omega)$, *and let* $r_h u \in V_h$ *(or* W_h*) be its interpolate, we have* :

$$|u - r_h u|_h \leqslant ch^2 \, |u|_{2,2,\Omega}, \text{ with } r_h u \in V_h \text{ or } W_h,$$

$$[u - r_h u]_h \leqslant ch^2(|u|_{3,2,\Omega} + |u|_{2,2,\Omega}), \text{ with } r_h u \in V_h \text{ or } W_h,$$

$$\langle u - r_h u \rangle_h \leqslant ch^2(|u|_{3,2,\Omega} + |u|_{2,2,\Omega}) \text{ with } r_h u \in V_h.$$

According to lemma 4.1, we see that the L^2 norm of the first derivatives of $u - r_h u$ is of order h. We shall show that in certain circumstances, we can get an order h^2, which is a super convergence result.

Lemma 4.5 : *Let* u *be a function belonging to the space* $H^3(\Omega)$, *and let* $r_h u \in V_h$ *be its interpolate. We assume that the triangulation* $\mathcal{C}_h$ *is made up of equal rectangles whose edges are respectively equal to* Δx *and* Δy. *Let* ψ_{ij} *be the function of* V_h *equal to* 1 *at the point* $(i\Delta x, j\Delta y)$ *and equal to zero at all the other nodes, and let* Q_{ij} *be the support of* ψ_{ij}, *for* $1 \leqslant i \leqslant I - 1$, $1 \leqslant j \leqslant J - 1$. *We have* :

$$\left| \left(\frac{\partial}{\partial x}(u - r_h u), \psi_{ij} \right)_{L^2(\Omega)} \right| + \left| \left(\frac{\partial}{\partial y}(u - r_h u), \psi_{ij} \right)_{L^2(\Omega)} \right| \leqslant ch^3 |u|_{3,2,Q_{ij}}$$

for $1 \leqslant i \leqslant I - 1$, $1 \leqslant j \leqslant J - 1$.

Proof: We have $Q_{i,j} = K_{i,j} \cup K_{i-1,j} \cup K_{i,j-1} \cup K_{i-1,j-1}$. We consider the isoparametric transformation $F_{i,j}$ which maps the reference square $\hat{K}$ as defined in paragraph II on to the rectangle $Q_{i,j}$. To any function u defined on K, we let correspond a function $\hat{u}$ defined on $\hat{K}$ by $\hat{u}(\xi, \eta) = u(x, y)$ with $(x, y) = F_{i,j}(\xi, \eta)$.

We have :

$$\left(\frac{\partial}{\partial x}(u - r_h u), \psi_{ij} \right)_{L^2(Q_{ij})} = \Delta y \left(\frac{\partial}{\partial \xi}(\hat{u} - r_h \hat{u}), \hat{\psi}_{ij} \right)_{L^2(\hat{K})} \tag{4.1}$$

We can check that the application defined by $\hat{v} \to \left(\frac{\partial}{\partial \xi}(\hat{v} - r_h \hat{v}), \hat{\psi}_{ij} \right)_{L^2(\hat{K})}$ is linear and continuous from $H^3(K)$ into R and is identically equal to zero for all $\hat{v} \in \hat{P}(2)$ (the space of polynomials of degree $\leqslant 2$ in both variables ξ and η).

So we get :

$$(4.2)\qquad \left| \left(\frac{\partial}{\partial \xi} (\hat{u} - r_h \hat{u}), \hat{\psi}_{ij} \right)_{L^2(\hat{K})} \right| \leqslant c \, |\hat{u}|_{3,2,\hat{K}} \text{ for all } \hat{u} \in H^3(\hat{K}).$$

We also have :

$$(4.3)\qquad |\hat{u}|_{3,2,\hat{K}} \leqslant ch^2 \, |u|_{3,2,K} \text{ for all } u \in H^3(K).$$

Combining inequalities (4.1), (4.2) and (4.3), we get lemma 4.5.

It is easy to show the following two technical results :

Lemma 4.6 : *Let v_h^0 be any function of V_h^0. We can write v_h^0 as follows :*

$$v_h^0 = \sum_{i,j=1}^{I-1,J-1} \alpha_{ij} \psi_{ij} \quad , \quad \textit{and we have :}$$

$$|v_h^0|_{L^2(\Omega)} \geqslant ch \left(\sum_{i,j=1}^{I-1,J-1} (\alpha_{i,j})^2 \right)^{1/2} .$$

Lemma 4.7 : *Let v_h be any function of V_h, which we write, as in paragraph* II, *as follows : $v_h = v_h^0 + v_h^b$, $v_k^0 V_h^0$. We have :*

$$|v_h^0|_{L^2(\Omega)} \leqslant c \, |v_h|_{L^2(\Omega)},$$

$$|v_h^b|_{L^1(\Omega)} \leqslant ch \, |v_h|_{L^2(\Gamma)}.$$

Combining Lemmas 4.5, 4.6 and 4.7, we get :

Lemma 4.8 : *Let u be a function belonging to the space $H^3(\Omega) \cap W^{2,\infty}$ and let $r_h u \in V_h$ be its interpolate. We assume that the triangulation $\mathcal{T}_h$ is made up of equal rectangles. Then we have, for any $v_h \in V_h$:*

$$\left| \left(\frac{\partial}{\partial x} (u - r_h u), v_h \right)_{L^2(\Omega)} \right| + \left| \left(\frac{\partial}{\partial y} (u - r_h u), v_h \right)_{L^2(\Omega)} \right|$$
$$\leqslant ch^2 (|u|_{3,2,\Omega} \, |v_h|_{L^2(\Omega)} + |u|_{2,\infty,\Omega} \, |v_h|_{L^2(\Gamma)}).$$

Proof : Let v_h be any function of V_h, with $v_h = v_h^0 + v_h^b$, and

$$v_h^0 = \sum_{i,j=1}^{I-1,J-1} \alpha_{i,j} \psi_{i,j}. \text{ Let } k_{i,j} \text{ be defined by :}$$

$$k_{i,j} = \left(\frac{\partial}{\partial x} (u - r_h u), \psi_{i,j} \right)_{L^2(\Omega)} \quad , \qquad \text{for } 1 \leqslant i \leqslant I-1, \ 1 \leqslant j \leqslant J-1.$$

Combining lemmas 4.5, 4.6 and 4.7, we get :

$$(4.4)\qquad \frac{\left|\left(\dfrac{\partial}{\partial x}(u - r_h u),\, v_h^0\right)_{L^2(\Omega)}\right|}{|v_h|_{L^2(\Omega)}} \leqslant ch^{-1}\frac{\sum\limits_{i,j}\alpha_{i,j}k_{i,j}}{\left(\sum\limits_{i,j}\alpha_{i,j}^2\right)^{1/2}} \leqslant ch^2\,|u|_{3,2,\Omega}.$$

According to lemmas 4.1 and 4.7, we have :

$$(4.5)\qquad \left|\left(\frac{\partial}{\partial x}(u - r_h u),\, v_h^b\right)_{L^2(\Omega)}\right| \leqslant ch\,|u|_{2,\infty,\Omega}\,|v_h^b|_{L^1(\Omega)}$$

$$\leqslant ch^2\,|u|_{2,\infty,\Omega}\,|v_h|_{L^2(\Gamma)}.$$

Inequalities (4.4) and (4.5), along with inequalities of the same type for the term $\dfrac{\partial}{\partial y}(u - r_h u)$ give us lemma 4.8.

We shall now consider the non-conforming case. We can show the following fundamental result, using exactly the same proof as for lemma 4.5 :

Lemma 4.9 : *Let u be a function of $H^3(\Omega)$ and let $r_h u \in W_h$ be its interpolate. We assume that the triangulation $\mathcal{C}_h$ is made up of equal rectangles whose edges are respectively equal to Δx and Δy. Let $\psi_{i,j}$ be the function of W_h equal to one at the points $((i + 1/2)\Delta x, j\Delta y)$, $((i - 1/2)\Delta x, j\Delta y)$, $(i\Delta x, (j + 1/2)\Delta y)$ and $(i\Delta x, (j - 1/2)\Delta y)$ and equal to zero at all the other nodes, and let $Q_{i,j}$ be the support of $\psi_{i,j}$ for $1 \leqslant i \leqslant I - 1$, $1 \leqslant j \leqslant J - 1$. We have :*

$$\left|\left(\frac{\partial}{\partial x}(u - r_h u),\, \psi_{i,j}\right)^*_{L^2(\Omega)}\right| + \left|\left(\frac{\partial}{\partial y}(u - r_h u),\, \psi_{i,j}\right)^*_{L^2(\Omega)}\right| \leqslant ch^3\,|u|_{3,2,Q_{i,j}},$$

for $1 \leqslant i \leqslant I - 1$, $1 \leqslant j \leqslant J - 1$.

Using Lemma 4.1 with $p = + W$ and $m = 1$, it is easy to show the following result :

Lemma 4.10 : *Let u be a function of $W^{2,\infty}(\Omega)$ and let $r_h u \in W_h$ be its interpolate. We assume that the triangulation $\mathcal{C}_h$ is made up of equal rectangles whose edges are respectively equal to Δx and Δy. Let $\psi_{0,j}$ be the function of W_h equal to one at the points $(0, (j + 1/2)\Delta y)$, $(0, (j - 1/2)\Delta y)$ and $\left(\dfrac{\Delta x}{2}, j\Delta y\right)$ and equal to zero at all the other nodes, and let $Q_{0,j}$ be the support of $\psi_{0,j}$, for $1 \leqslant j \leqslant J - 1$. In the same way, we define $\psi_{I,j}$, for $1 \leqslant j \leqslant J - 1$, $\psi_{I,0}$ and $\psi_{i,J}$ for $1 \leqslant i \leqslant I - 1$. Let $\psi_{0,0}$ be the function of W_h equal to 1 at the points $\left(0, \dfrac{\Delta y}{2}\right)$ and $\left(\dfrac{\Delta x}{2}, 0\right)$ and equal to zero at all the other nodes, and let $Q_{0,0}$*

be the support of $\psi_{0,0}$. *In the same manner, we also define* $\psi_{I,0}$ *and* $\psi_{I,J}$. *We have :*

$$\left|\left(\frac{\partial}{\partial x}(u - r_h u), \psi_{i,j}\right)^*_{L^2(\Omega)}\right| + \left|\left(\frac{\partial}{\partial y}(u - r_h u), \psi_{i,j}\right)^*_{L^2(\Omega)}\right| \leqslant ch^3 \, |u|_{2,\infty,Q_{i,j}},$$

for all the indices i and j defined above.

Combining lemmas 4.9 and 4.10, we get :

Lemma 4.11 : *Let u be a function of* $H^3(\Omega) \cap W^{2,\infty}(\Omega)$ *and let* $r_h u \in W_h$ *be its interpolate. We assume that the triangulation* $\mathcal{T}_h$ *is made up of equal rectangles. Then we have, for any* $v_h \in W_h$:

$$\left|\left(\frac{\partial}{\partial x}(u - r_h u), v_h\right)^*_{L^2(\Omega)}\right| + \left|\left(\frac{\partial}{\partial y}(u - r_h u), v_h\right)^*_{L^2(\Omega)}\right|$$
$$\leqslant c(h^2 \, |u|_{3,2,\Omega} + h^{3/2} \, |u|_{2,\infty,\Omega}) \, |v_h|_{L^2(\Omega)}.$$

Proof : The set $\{ \psi_{i,j}; 0 \leqslant j \leqslant J; 0 \leqslant i \leqslant I \} - \{ \psi_{I,J} \}$ is a basis of W_h. Any function $v_h \in W_h$ can be written as $w_h = \sum_{i,j} \alpha_{i,j} w_{i,j}$, with $(i, j) \neq (I, J)$. Then we have :

$$|v_h|_{L^2(\Omega)} \geqslant ch \left(\sum_{i,j} (\alpha_{i,j})^2 \right)^{1/2}$$

Now we define $k_{i,j}$ by $k_{i,j} = \left(\frac{\partial}{\partial x}(u - r_h u), \psi_{i,j}\right)^*_{L^2(\Omega)}$, for $(i, j) \neq (I, J)$.

It is easy to show by applying lemmas 4.9 and 4.10 that :

$$\frac{\left|\left(\frac{\partial}{\partial x}(u - r_h u), v_h\right)^*_{L^2(\Omega)}\right|}{|v_h|_{L^2(\Omega)}} \leqslant ch^{-1} \left(\sum_{i,j} (k_{i,j})^2 \right)^{1/2}$$
$$\leqslant c(h^2 \, |u|_{3,2,\Omega} + h^{3/2} \, |u|_{2,\infty,\Omega}),$$

which gives us lemma 4.11.

For the sake of completness, we shall give the proof of the following results, for convex quadrilaterals [20] :

Lemma 4.12 : *Let u be a function of* $H^3(\Omega)$ *and let* $r_h u \in V_h$ *(or* W_h*) be its interpolate in the conforming (or non-conforming) case. Assume that hypothesis* 3.1 *holds. We have :*

$$\left|\frac{\partial}{\partial x}(u - r_h u)\right|_h + \left|\frac{\partial}{\partial y}(u - r_h u)\right|_h \leqslant ch^2(|u|_{3,2,\Omega} + |u|_{2,2,\Omega}).$$

Proof : We consider the conforming case. Let K be any quadrilateral of the triangulation and let F_K be the isoparametric transformation which maps the reference square $\hat{K}$ onto K. We have :

$$\text{area}\,(K)\left(\frac{\partial}{\partial x}(u - r_h u)\right)(G_K) = 4\left(\frac{\partial}{\partial \xi}(\hat{u} - r_h \hat{u})\frac{\partial y}{\partial \eta} - \frac{\partial}{\partial \eta}(\hat{u} - r_h \hat{u})\frac{\partial y}{\partial \xi}\right)(0, 0).$$

where $\hat{u}(\xi, \eta) = u(x, y)$ with $(x, y) = F_K(\xi, \eta)$.

We can easily check that the application defined by $\hat{u} \to \frac{\partial}{\partial \xi}(\hat{u} - r_h\hat{u})(0, 0)$ is linear and continuous from $H^3(\hat{K})$ into R, and is identically equal to zero for all $\hat{u} \in \hat{P}(2)$. So we get :

$$\left|\left(\frac{\partial}{\partial \xi}(\hat{u} - r_h\hat{u})\right)(0, 0)\right| \leqslant c\,|\hat{u}|_{3,2,\hat{K}}.$$

Going back to quadrilateral K, by using transformation F_K^{-1}, we get (see [3] lemma 1) :

$$(4.6) \qquad \left|\text{area}\,(K)\left(\frac{\partial}{\partial x}(u - r_h u)\right)(G_K)\right|$$
$$\leqslant c((h(K))^3\,|u|_{3,2,K} + h(K)\cdot Z(K)\,|u|_{2,2,K}).$$

for all $K \in \mathcal{T}_h$. Summing on all quadrilaterals K of $\mathcal{T}_h$, we get lemma 4.12. The proof is exactly the same in the non conforming case.

V. ERROR BOUNDS, THEOREMS

Theorem 5.1 : *Let $u_h \in V_h$ be the solution of scheme* 1. *We assume that the exact solution u belongs to $H^2(\Omega)$ and that the triangulation $\mathcal{T}_h$ is a regular family of arbitrary convex quadrilaterals. Then we have :*

$$(5.1) \qquad |u - u_h|_{L^2(\Omega)} + |(M)^{1/2}(u - u_h)|_{L^2(\Gamma)} \leqslant ch\,|u|_{2,2,\Omega}.$$

If we assume now that all the quadrilaterals are equal rectangles, and that the exact solution u belongs to $H^3(\Omega) \cap W^{2,\infty}(\Omega)$, then we have :

$$(5.2) \quad |u - u_h|_{L^2(\Omega)} + |(M)^{1/2}(u - u_h)|_{L^2(\Gamma)} \leqslant ch^2(|u|_{2,\infty,\Omega} + |u|_{3,2,\Omega}).$$

Proof : If in lemma 1.1 we replace v by $u_h - r_h u$, where $r_h u \in V_h$ is the interpolate of u, we get :

$$|u_h - r_h u|^2_{L^2(\Omega)} + |(M)^{1/2}(u_h - r_h u)|^2_{L^2(\Gamma)}$$
$$\leqslant c\left((A(u_h - r_h u), u_h - r_h u)_{L^2(\Omega)} - \left(\frac{B - M}{2}(u_h - r_h u), u_h - r_h u\right)_{L^2(\Gamma)}\right).$$

Using the definition of scheme 1, we easily get :

$$(5.3) \qquad |u_h - r_h u|^2_{L^2(\Omega)} + |(M)^{1/2}(u_h - r_h u)|^2_{L^2(\Gamma)}$$
$$\leqslant c\left(|(A(u - r_h u), u_h - r_h u)_{L^2(\Omega)}| + \left| \left(\frac{B - M}{2} (u - r_h u), u_h - r_h u \right)_{L^2(\Gamma)} \right| \right).$$

Combining inequality (5.3) with lemmas 4.1 and 4.2, we get inequality (5.1). When the quadrilaterals are equal rectangles, we use inequality (5.3) along with lemmas 4.1, 4.2 and 4.8 to get inequality (5.2).

Theorem 5.2 : *We assume that the triangulation $\mathcal{T}_h$ is a regular family of arbitrary quadrilaterals and that hypothesis* 3.1 *is satisfied. Let $u_h \in W_h$ be the solution of scheme* 2, *and let the exact solution u belong to $H^2(\Omega)$. We have :*

$$(5.4) \qquad |u - u_h|_{L^2(\Omega)} + |(M)^{1/2}(u - u_h)|_{L^2(\Gamma_0 \cup \Gamma_1)} + [(M)^{1/2}(u - u_h)]_h$$
$$\leqslant ch\, |u|_{2,2,\Omega}.$$

If we assume that all the quadrilaterals are equal rectangles and that the exact solution u belongs to $H^3(\Omega) \cap W^{2,\infty}(\Omega)$, we then have :

$$(5.5) \qquad |u - u_h|_{L^2(\Omega)} + |(M)^{1/2}(u - u_h)|^2_{L^2(\Gamma_0 \cup \Gamma_1)} + [(M)^{1/2}(u - u_h)]^2_h$$
$$\leqslant c(h^2\, |u|_{3,2,\Omega} + h^{3/2}\, |u|_{2,\infty,\Omega}).$$

Proof : Starting from lemma 3.2 and from the definition of scheme 2, we get :

$$(5.6) \qquad |u_h - r_h u|^2_{L^2(\Omega)} + |(M)^{1/2}(u_h - r_h u)^2_{L^2(\Gamma_0 \cup \Gamma_1)}| + [(M)^{1/2}(u_h - r_h u)]^2_h$$
$$\leqslant c\left(|A(u - r_h u), u_h - r_h u)_{L^2(\Omega)}| + \left| \left(\frac{B - M}{2} (u - r_h u), u_h - r_h u \right)_{L^2(\Gamma)} \right| \right).$$

where $r_h u \in W_h$ is the interpolate of u.

Combining inequality 5.6 and lemmas 4.1 and 4.3, we get inequality (5.4). When the quadrilaterals are equal rectangles, hypothesis 3.1 is automatically satisfied; we use lemmas 4.1, 4.3 and 4.11, with inequality (5.6) to get inequality (5.5).

Theorem 5.3 : *We assume that the triangulation $\mathcal{T}_h$ is a regular family of arbitrary quadrilaterals and that hypothese* 3.1 *and* 3.2 *hold. Then let $u_h \in V_h$* (*resp. W_h*) *be the solution of scheme* 3 (*resp. scheme* 4). *We assume that the exact* *solution u belongs to* $H^3(\Omega)$. *We then have :*

$$|u - u_h|_h \leqslant ch^2(|u|_{2,2,\Omega} + |u|_{3,2,\Omega}), \tag{5.7}$$

$$\max_{s \in \cup s} |(M_s)^{1/2}(u - u_h)(G_s)| \leqslant ch^{3/2}(|u|_{2,2,\Omega} + |u|_{3,2,\Omega}), \tag{5.8}$$

where $\cup s$ *denotes the set of all the edges of the quadrilaterals* K *of* $\mathcal{C}_h$, *where* G_s *is the mid-point of the edge* s *and where* $M_s = \mu n_x^s + \nu n_y^s$, n_x^s *and* n_y^s *being the components of a normal on* s.

Proof: We consider the conforming case. Inequality (3.8) of lemma 3.3 holds with v_h replaced by $u_h - r_h u$, where $r_h u$ belongs to V_h and is the interpolate of u. We consider the following expression for any K belonging to $\mathcal{C}_h$ and non adjacent to $\Gamma_0 \cup \Gamma_1$:

$$\chi_h(K) = \text{area}\,(K)\left(\mu \frac{\partial(u_h - r_h u)}{\partial x} + \nu \frac{\partial}{\partial y}(u_h - r_h u) + \sigma(u_h - r_h u)\right) \cdot (u_h - r_h u)(G_K).$$

We have, with the same notations as in lemma 3.3 :

$$\begin{aligned}\chi_h(K) \geqslant \frac{1 - 2c_0 h(K)}{1 - c_0 h(K)} \Big\{ &\left(\mu \frac{y_1 - y_4}{2} - \nu \frac{x_1 - x_4}{2}\right)((u_h - r_h u)(A_{14}))^2 \\ &+ \left(\mu \frac{y_2 - y_1}{2} - \nu \frac{x_2 - x_4}{2}\right)((u_h - r_h u)(A_{12}))^2 \Big\} \\ + \frac{1}{1 - c_0 h(K)} \Big\{ &\left(\mu \frac{y_3 - y_2}{2} - \nu \frac{x_3 - x_2}{2}\right)((u_h - r_h u)(A_{13}))^2 \\ &+ \left(\mu \frac{y_4 - y_3}{2} + \nu \frac{x_4 - x_3}{2}\right)((u_h - r_h u)(A_{34}))^2 \Big\} + \sigma((u_h - r_h u)(G_K))^2\end{aligned} \tag{5.9}$$

where A_{ij} denotes the mid-point of any edge $A_i A_j$.

According to the definition of scheme 3, we have :

$$\chi_h(K) = \text{area}\,(K)\left(\mu \frac{\partial}{\partial x}(u - r_h u) + \nu \frac{\partial}{\partial y}(u - r_h u) + \sigma(u - r_h u)\right)(u_h - r_h u)(G_K$$

When we use lemmas 4.1 and 4.8, we get :

$$\chi_h(K) \leqslant c(h(K))_2 \,(|u|_{3,2,K} + |u|_{2,2,K})\,(\text{area}\,(K) \cdot ((u_h - r_h u)(G_K))^2)^{1/2} \tag{5.10}$$

Combining inequalities (5.9) and (5.10), we get for any $K \in \mathcal{T}_h$:

$$(5.11)\quad \frac{1-2c_0(K)}{1-c_0h(K)}\left\{\left(\mu\frac{y_1-y_4}{2}-\nu\frac{x_1-x_4}{2}\right)(u_h-r_hu)^2(A_{14})\right.$$
$$\left.+\left(\mu\frac{y_2-y_1}{2}-\nu\frac{x_2-x_1}{2}\right)(u_h-r_hu)^2(A_{12})\right\}+c(u_h-r_hu)^2(G_K)$$
$$\leqslant c(h(K))^4(|u|^2_{3,2,K}+|u|^2_{2,2,K})+$$
$$+\frac{1}{1-c_0h(K)}\left\{\left(\mu\frac{y_2-y_3}{2}-\nu\frac{x_2-x_3}{2}\right)(u_h-r_hu)^2(A_{23})\right.$$
$$\left.+\left(\mu\frac{y_3-y_4}{2}-\nu\frac{x_3-x_4}{2}\right)(u_h-r_hu)^2(A_{34})\right\}.$$

If K is adjacent to $\Gamma_0 \cup \Gamma_1$, we must add up some boundary terms. If we combine inequalities (5.11) for all $K \in \mathcal{T}_h$, with appropriate weights, we get :

$$(5.12)\qquad c\,|u_h-r_hu|^2_h + ch\left(\max_{s\in\Gamma}(M_s)\right)(u_h-r_hu)^2(G_s)$$
$$\leqslant ch^4(|u|^2_{3,2,\Omega}+|u|^2_{2,2,\Omega}).$$

Inequality (5.7) follows immediately. Now we can get an inequality like (5.12) for any Ω_{ij} as defined in remark 3.2. Particularly we get :

$$(5.13)\qquad \max_{s\in\Gamma_{ij}}(M_s)(u_h-r_hu)^2(G_s)\leqslant ch^3(|u|^2_{3,2,\Omega_{ij}}+|u|^2_{2,2,\Omega_{ij}}).$$

Inequality (5.8) follows immediately from inequality (5.13) and lemma 4.4 The proof is the same in the non-conforming case.

REMARK 5.1 : We define the following discrete norm $\|\cdot\|_h$ on V_h or W_h by

$$(5.14)\qquad \|v_h\|^2_h=\sum_{K\in\mathcal{T}_h}(h(K))^2\sum_{s\in\partial K}(v_h(G_s))^2$$

where G_s is the mid-point of the edge s. We assume that hypotheses 3.1 and 3.2 hold and that we have :

$$(5.15)\qquad |M_s|\geqslant\delta>0$$

for any $s \in \cup s$, where δ is a independent of $\mathcal{T}_h$. Then it is possible to show that for scheme 3 or for scheme 4, we have :

$$(5.16)\qquad \|u-u_h\|_h\leqslant c\frac{h^2}{\sqrt{\delta}}(|u|_{3,2,\Omega}+|u|_{2,2,\Omega}).$$

Numerical results ([12]) show that we really get estimate (5.16) when hypotheses 3.1, 3.2 and 5.15 hold.

REMARK 5.2 : In the conforming case, it would seem natural to get the same estimate like (5.16) for the values at the vertices of the quadrilaterals. Numerical results give only an error of order h. In fact, one can show that :

$$|u_h - r_h u|_{L^2(\Omega)} \leqslant c h^{-1} \|u_h - r_h u\|_h$$

This last inequality combined with inequality (5.16) gives the order h for the error at the vertices of the quadrilaterals.

We shall now give an estimate for the error due to both angular and spatial discretizations. We define the following discrete norm :

$$\|\varphi\|_{h,\mu}^2 = \sum_{l=1}^{L} \text{area}\,(T_l) \sum_{K \in \mathcal{T}_h} \text{area}\,(K) \varphi^2(G_K, \mu_l, \nu_l)$$

We then have :

Theorem 5.4 : *Let* $\varphi \in H^3(\Omega \times Q)$ *be the exact solution of problem* (1.1), (1.2). *We assume that the triangulation* $\mathcal{T}_h$ *is made up of equal rectangles. Let* $\varphi_{h,\mu} \in V_h \times \mathcal{V}_\mu$ (*resp.* $W_h \times \mathcal{V}_\mu$) *be the approximate solution when we use scheme* 1 (*resp. scheme* 2). *Then we have* :

$$\|\varphi - \varphi_{h,\mu}\|_\mu = 0(h^2) + 0(\Delta\mu^2).$$

Theorem 5.5 : *Let* $\varphi \in H^3(\Omega \times Q)$ *be the exact solution of problem* (1.1), (1.2). *We assume that hypothesis* 3.2 *holds for any* (μ_l, ν_l) $1 \leqslant l \leqslant L$. *Let* $\varphi_{h,\mu} \in V_h \times \mathcal{V}_\mu$ (*resp.* $W_h \times \mathcal{V}_\mu$) *be the approximate solution when we use scheme* 3 (*resp. scheme* 4). *We then have* :

$$\|\varphi - \varphi_{h,\mu}\|_{h,\mu} = 0(h^2) + 0(\Delta\mu^2)$$

REMARK 5.2 : Hypothesis 3.2 implies that we cannot choose any value for (μ_l, ν_l). For example when the quadrilaterals K are very distorted, we cannot use a small value of $\Delta\mu$.

We shall see in a forthcoming paper [13] that this problem of stability can be handled if we use discontinuous elements in space [17] : we can get an unconditionnaly stable quasi explicit (we have to invert a sequence of 4×4 matrices when we use polynomials of degree $\leqslant 1$ in each spatial element) and rather accurate schemes.

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