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ALEXANDER ŽENÍŠEK

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A GENERAL THEOREM ON TRIANGULAR FINITE $C^{(m)}$-ELEMENTS

par Alexander ŽENÍŠEK (1)
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Summary. — *The following theorem is proved : To achieve piecewise polynomials of class C^m on an arbitrary triangulation of a polygonal domain, the nodal parameters must include all derivatives of order less than or equal to $2m$ at the vertices of the triangles.*

For the sake of brevity we shall use the expression « triangular $C^{(m)}$ element » for a polynomial on a triangle which generates piecewise polynomial and m-times continuously differentiable functions on an arbitrary triangulation. (From this point of view the Clough-Tocher element [4, p. 84] is not a triangular $C^{(1)}$-element.)

In the last few years there were constructed various types of interpolation polynomials on a triangle (see e.g., [3, 5]). All these polynomials have two following features :

1. A general triangular $C^{(m)}$-element is constructed in such a way that at the vertices of a triangle there are prescribed at least all derivatives of order less than or equal to $2m$.

2. The lowest degree of a general triangular $C^{(m)}$-element is equal to $4m + 1$.

These two features suggest the following questiones :

(i) Which derivatives should be prescribed at the vertices of a triangle to get a triangular $C^{(m)}$-element? (In other words : Is it necessary for constructing a triangular $C^{(m)}$-element to prescribe all derivatives of order less than or equal to $2m$ at the vertices of a triangle?)

(ii) What is the lowest degree of a triangular $C^{(m)}$-element?

The aim of this paper is to prove the following theorem which gives the answers to both questiones (i) and (ii).

(1) Computing Center, Technical University of Brno, Brno, Tchécoslovaquie.

Theorem 1. (i) To get a triangular $C^{(m)}$-element we must prescribe all derivatives of order less than or equal to $2m$ at the vertices of a triangle.

(ii) The lowest degree of a triangular $C^{(m)}$-element is equal to $4m + 1$.

In [4, p. 84] the first part of Theorem 1 is formulated in a little different way with reference to [6]. However, in [6] the features 1 and 2 are mentioned only.

To express ourselves in a concise form we divide the parameters uniquely determining a triangular $C^{(m)}$-element into two groups :

1. The parameters of the first kind guarantee the $C^{(m)}$-continuity of a global function on an arbitrary triangulation. These parameters are prescribed at the vertices of a triangle and at some points lying on the sides of a triangle.

In other words, the parameters of the first kind prescribed at the points of the segment P_rP_s uniquely determine the polynomials

$$(1) \qquad q_{rs,\varkappa}(\tau) = \left.\frac{\partial^\varkappa p}{\partial \nu^\varkappa}\right|_l \equiv \frac{\partial^\varkappa}{\partial \nu^\varkappa} p(x_r + (x_s - x_r)\tau, y_r + (y_s - y_r)\tau)$$

where $\varkappa = 0, \dots, m$, $P_r(x_r, y_r)$, $P_s(x_s, y_s)$ are two vertices of a triangle $\bar{T}$, l is the straight line determined by the points P_r, P_s and $p(x, y)$ is a triangular $C^{(m)}$-element on the triangle $\bar{T}$.

2. The parameters of the second kind have no influence on the smoothness of a global function; they enable together with the parameters of the first kind to determine uniquely a triangular $C^{(m)}$-element. These parameters are usually prescribed in the interior T of a triangle $\bar{T}$ but they may be prescribed also at the vertices of a triangle (see, e.g., [5, Corollary of Theorem 3]) or at some points lying on the sides of a triangle.

The basic property of the parameters of the first kind can be expressed also in the following way :

Lemma 1. Let $p(x, y)$ be a triangular $C^{(m)}$-element, P_r, P_s two vertices of the triangle $\bar{T}$ and $l(P_r, P_s)$ the straight line determined by the points P_r, P_s. If all parameters of the first kind prescribed at the points of the segment P_rP_s are equal to zero then

$$(2) \qquad D^\alpha p(P) = 0 \quad , \quad |\alpha| \leqslant m \quad , \quad \forall\, P \in l(P_r, P_s).$$

In (2) and in what follows we use the following notation for derivatives :

$$D^\alpha u = \partial^{|\alpha|} u / \partial x^{\alpha_1} \partial y^{\alpha_2} \quad , \quad \alpha = (\alpha_1, \alpha_2) \quad , \quad |\alpha| = \alpha_1 + \alpha_2 .$$

The proof of Lemma 1 is very simple : If the assumption of Lemma 1 is satisfied then

$$q_{rs,\varkappa}(\tau) \equiv 0 \qquad (\varkappa = 0, \dots, m).$$

These relations imply with respect to (1)

$$(3) \qquad {}^{\varkappa+\lambda}p(P)/\partial\nu^{\varkappa}\partial\tau^{\lambda} = 0\,;\ \varkappa, \lambda = 0, \ldots, m\,;\ \forall\ P \in l(P_r, P_s).$$

As the derivative $\partial^k p/\partial x^{k_1}\partial y^{k_2}(k = k_1 + k_2)$ can be written in the form of a linear combination of $k + 1$ derivatives

$$\partial^k p/\partial\nu^k,\ \partial^k p/\partial\nu^{k-1}\partial\tau,\ \ldots,\ \partial^k p/\partial\nu\partial\tau^{k-1},\ \partial^k p/\partial\tau^k$$

the relations (2) follow from (3).

Theorem 1 is in the case $m = 0$ trivial. In the case $m \geqslant 1$ the first part of Theorem 1 is equivalent to the assertion of Lemma 2.

Lemma 2. Let $m \geqslant 1$, $k \geqslant 1$, $l \geqslant 0$ and $\rho \geqslant 0$ be given integers. It is impossible to construct a triangular $C^{(m)}$-element the parameters of the first kind of which prescribed at the vertices P_1, P_2, P_3 of a triangle are of the form

$$(4) \qquad D^{\alpha}p(P_i) \quad , \quad \forall\ |\alpha| \in A\backslash B \qquad (i = 1, 2, 3)$$

where the sets A, B are defined by

$$(5) \qquad A = \{0, 1, \ldots, 2m + \rho\},$$

$$(6) \qquad B = \{j_1, j_2, \ldots, j_k, h_1, h_2, \ldots, h_l\}$$

and the integers from the set B satisfy the inequalities

$$(7) \qquad m < j_1 < j_2 < \ldots < j_k \leqslant 2m < h_1 < h_2 < \ldots < h_l \leqslant 2m + \rho.$$

Before proving Lemma 2 we introduce some lemmas which will be used in the proof of Lemma 2.

Lemma 3. If at every point P of the straight line $l(P_r, P_s)$ determined by the points $P_r(x_r, y_r)$, $P_s(x_s, y_s)$ the relations (2) hold then the polynomial $p(x, y)$ is divisible by the polynomial $[f_{rs}(x, y)]^{m+1}$ where

$$(8) \qquad f_{rs}(x, y) = -(y_s - y_r)(x - x_r) + (x_s - x_r)(y - y_r).$$

The proof of Lemma 3 is a modification of one device used in the proof of [2, Theorem 1].

Lemma 4. Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ be the vertices of a triangle $\bar{T}$. Let the polynomial $p(x, y)$ be of the form

$$(9) \qquad p(x, y) = g(x, y)q(x, y)$$

where

$$(10) \qquad g(x, y) = [f_{12}(x, y)f_{13}(x, y)f_{23}(x, y)]^{m+1},$$

the linear functions $f_{rs}(x, y)$ being defined by the relation (8). Then the conditions

$$D^\alpha p(P_i) = 0 \quad , \quad |\alpha| = 2m + \varkappa \quad (\varkappa \geqslant 2) \tag{11}$$

give at most $\varkappa - 1$ linearly independent homogeneous conditions for the polynomial $q(x, y)$ which are prescribed at the vertex P_i.

Proof. We prove Lemma 4 in the case $i = 3$. Let $\bar{T}_0$ be the triangle which lies in the Cartesian co-ordinate system ξ, η and has the vertices $\tilde{P}_1(0, 0)$, $\tilde{P}_2(1, 0)$, $\tilde{P}_3(0, 1)$. The transformation

$$\begin{aligned} x &= x_0(\xi, \eta) \equiv x_3 + (x_1 - x_3)\xi + (x_2 - x_3)\eta, \\ y &= y_0(\xi, \eta) \equiv y_3 + (y_1 - y_3)\xi + (y_2 - y_3)\eta \end{aligned} \tag{12}$$

maps one-to-one the triangle $\bar{T}$ on the triangle $\bar{T}_0$ and the vertex P_3 is mapped on the vertex $\tilde{P}_1$. Let us define the polynomial $\tilde{p}(\xi, \eta)$ by

$$\tilde{p}(\xi, \eta) = p(x_0(\xi, \eta), y_0(\xi, \eta)). \tag{13}$$

According to (9), (10), (12) and (13), the polynomial $\tilde{p}(\xi, \eta)$ is of the form

$$\tilde{p}(\xi, \eta) = \tilde{g}(\xi, \eta)\tilde{q}(\xi, \eta) \tag{14}$$

where

$$\tilde{g}(\xi, \eta) = J^{3m+3}\xi^{m+1}\eta^{m+1}(\xi + \eta - 1)^{m+1}, \tag{15}$$

J being the Jacobian of the transformation (12), and

$$\tilde{q}(\xi, \eta) = q(x_0(\xi, \eta), y_0(\xi, \eta)). \tag{16}$$

It follows from (15) that at the vertex $\tilde{P}_1(0,0)$ the following derivatives of the function $\tilde{g}(\xi, \eta)$ are different from zero only :

$$\frac{\partial^{2m+2+\sigma}\, \tilde{g}(\tilde{P}_1)}{\partial\xi^{m+1+\sigma-\rho}\,\partial\eta^{m+1+\rho}} \quad , \quad \rho = 0, \ldots, \sigma \quad ; \quad \sigma = 0, \ldots, m + 1.$$

This fact and the Leibnitz rule for differentiation of a product imply

$$\frac{\partial^{2m+\varkappa}\tilde{p}(\tilde{P}_1)}{\partial\xi^{\alpha_1}\partial\eta^{\alpha_2}} = 0, \ \alpha_1 + \alpha_2 = 2m + \varkappa, \ \alpha_1 \leqslant m \text{ or } \alpha_2 \leqslant m. \tag{17}$$

Let

$$\xi = \xi_0(x, y), \qquad \eta = \eta_0(x, y) \tag{18}$$

be the inverse transformation to the transformation (12). The polynomial $p(x, y)$ can be written in the form

$$p(x, y) = \tilde{p}(\xi_0(x, y), \eta_0(x, y)). \tag{19}$$

As the transformation (18) is linear we get from (19), according to the rule of differentiation of a composite function,

$$D^\alpha p(P_i) = \sum_{|\beta| = 2m+\varkappa} a_{\alpha\beta} D^\beta \tilde{p}(\tilde{P}_1) \quad , \quad |\alpha| = 2m + \varkappa \tag{20}$$

where $a_{\alpha\beta}$ are constants.

Setting (20) into (11) we get, with respect to (17), $2m + \varkappa + 1$ homogeneous linear equations for at most $\varkappa - 1$ derivatives of order $2m + \varkappa$ of the polynomial $\tilde{p}(\xi, \eta)$ at the point $\tilde{P}_1$. Omitting the linearly dependent equations we get a system of at most $\varkappa - 1$ linearly independent equations. This system is, according to (14) and (15), a system of linear equations for derivatives of the function $\tilde{q}(\xi, \eta)$ at the point $\tilde{P}_1$. Returning to the variables x, y by means of the transformation (18), we get, according to (16), a system of at most $\varkappa - 1$ linearly independent homogeneous equations for the derivatives of the polynomial $q(x, y)$ at the point P_i. Lemma 4 is proved.

Proof of Lemma 2. Lemma 2 will be proved by a contradiction. Let us suppose that the assertion of Lemma 2 is not true, i.e. that it is possible to determine uniquely a triangular $C^{(m)}$-element $p(x, y)$ the parameters of the first kind of which prescribed at the vertices of a triangle are the parameters (4) only. Let n be the degree of this triangular $C^{(m)}$-element. As the triangulation is chosen quite arbitrarily the polynomials $q_{rs,0}(\tau)$ (see (1)) are also polynomials of degree n. Thus it holds, with respect to (5) and (6),

$$n \geqslant 4m + 2\rho - 2k - 2l + 1. \tag{21}$$

Let us set

$$d = n - (4m + 2\rho - 2k - 2l + 1). \tag{22}$$

As the triangulation is quite arbitrary the polynomials $q_{rs,\varkappa}(\tau)$ are polynomials of degree $n - \varkappa$. Thus to achieve the $C^{(m)}$-continuity we must prescribe $d + \varkappa$ parameters of the first kind on each side P_rP_s for each $\varkappa$ ($\varkappa = 0, \ldots, m$). Usually these parameters are of the form

$$\partial^\varkappa p(Q_{rs}^{(\lambda, d+\varkappa)})/\partial \nu_{rs}^\varkappa \; (\lambda = 1, \ldots, d + \varkappa \,;\, \varkappa = 0, \ldots, m) \tag{23}$$

where ν_{rs} is the normal to the segment P_rP_s and $Q_{rs}^{(1,q)}, \ldots, Q_{rs}^{(q,q)}$ are the points dividing the segment P_rP_s into $q + 1$ equal parts.

Let the symbols V and S denote the numbers of the parameters of the first kind prescribed at one vertex and on one side, respectively. It follows from (4)-(7), (22) and (23) that the total number of the parameters of the first kind is given by the relation

$$3(V + S) = 3(m + 1)n - 9m(m + 1)/2 + 3\rho(\rho - 1)/2 + 6(m + 1)(k + l) - 3(k + l + j + h) \tag{24}$$

where

$$j = j_1 + j_2 + \ldots + j_k, \tag{25}$$

$$h = h_1 + h_2 + \ldots + h_l. \tag{26}$$

The polynomial $p(x, y)$ has N coefficients where

$$N = (n + 1)(n + 2)/2. \tag{27}$$

The integers N, S, V must satisfy the inequality

$$R \equiv N - 3(V + S) \geqslant 0 \tag{28}$$

which expresses the fact that the total number of the parameters of the first kind cannot be greater than N.

Let us set

$$G = 48(m + 1)(k + l) + 12\rho(\rho - 1) - 24(k + l + j + h) + 1. \tag{29}$$

If we put (24) and (27) in (28) we get a quadratic inequality in n. It follows from this inequality that

$$n \geqslant n_1 = (6m + 3 + G^{1/2})/2 \tag{30}$$

where n_1 is the first root of the quadratic polynomial in n on the left-hand side of the inequality (28). The second formal possibility $n \leqslant n_2$ does not suit because in this case, according to (22) and (33),

$$d \leqslant \max d_2 = \max n_2 - (4m + 2\rho - 2k - 2l + 1) < 0.$$

It holds, according to (7), (25) and (26),

$$\max j = 2mk - k(k - 1)/2, \tag{31}$$

$$\max h = 2ml + \rho l - l(l - 1)/2. \tag{32}$$

Thus

$$\min G = 12k(k + 1) + 12(\rho - l - 1)(\rho - l) + 1. \tag{33}$$

As $\rho \geqslant l$, $k \geqslant 1$ the relations (30) and (33) imply

$$n > 3m + 3. \tag{34}$$

The interger R defined by (28) is the number of the parameters of the second kind. Let us prescribe these parameters quite arbitrarily and set all N parameters equal to zero. Then, according to Lemmas 1 and 3, the polynomial $p(x, y)$ is of the form (9). The relations (10) and (34) imply that in this case the polynomial $q(x, y)$ is at least a polynomial of the first degree. Let the symbol M denote the total number of the coefficients of the polynomial $q(x, y)$. It is easy to find that

$$M = N - 3(m+1)n + 9m(m+1)/2. \tag{35}$$

The relations (24), (28) and (35) imply

$$M - R = 6(m+1)(k+l) - 3(k+l+j+h) + 3\rho(\rho-1)/2. \tag{36}$$

It holds with respect to (31) and (36)

$$M - R \geqslant Q \tag{37}$$

where

$$Q = 3k(k+1)/2 + 6(m+1)l - 3(l+h) + 3\rho(\rho-1)/2. \tag{38}$$

Each integer h_s can be expressed in the form

$$h_s = 2m + r_s \; (s = 1, \dots, l). \tag{39}$$

Using (26) and (39) we can write

$$h = 2ml + (r_1 + \dots + r_l). \tag{40}$$

Putting (40) in (38) we find

$$Q = 3k(k+1)/2 + H \tag{41}$$

where

$$H = 3l - 3(r_1 + \dots + r_l) + 3\rho(\rho-1)/2. \tag{42}$$

According to (5)-(7), (39) and Lemma 4, the conditions

$$D^\alpha p(P_i) = 0 \quad , \quad |\alpha| \geqslant 2m+2 \quad , \quad |\alpha| \in A \backslash B \; (i = 1, 2, 3) \tag{43}$$

give H_1 linearly independent homogeneous conditions for the polynomial $q(x, y)$ where

$$H_1 \leqslant 3\Big(1 + 2 + \dots + (r_1 - 2) + \sum_{s=1}^{l-1} [r_s + (r_s+1) + \dots + (r_{s+1} - 2)] + r_l + (r_l + 1) + \dots + \rho - 1\Big). \tag{44}$$

The right-hand side of the inequality (44) is equal to H. Thus

$$H_1 \leqslant H. \tag{45}$$

As, according to (8) and (10), the relations

$$D^\alpha g(P) = 0 \quad , \quad |\alpha| \leqslant m \quad , \quad \forall\, P \in \partial T = \bar{T} \setminus T,$$

$$D^\alpha g(P_i) = 0 \quad , \quad |\alpha| \leqslant 2m + 1 \quad (i = 1, 2, 3)$$

hold the parameters of the first kind except for the parameters (43) give no conditions for the polynomial $q(x, y)$.

The parameters of the second kind prescribed for the polynomial $p(x, y)$ give R_1 linearly independent homogeneous conditions for the polynomial $q(x, y)$ where

$$R_1 \leqslant R. \tag{46}$$

Thus we get $H_1 + R_1$ linearly independent homogeneous equations for the coefficients of the polynomial $q(x, y)$.

As it holds, according to (37), (41), (45) and (46),

$$M - R_1 - H_1 \geqslant 3k(k + 1)/2 > 0 \tag{47}$$

we can complete these $H_1 + R_1$ homogeneous equations by such $M - R_1 - H_1$ non-homogeneous equations that we get M linearly independent equations for M coefficients of a polynomial $q(x, y)$ for which it holds

$$q(x, y) \not\equiv 0. \tag{48}$$

According to (9), (10) and (48), we get a polynomial $p(x, y)$ which satisfies prescribed N homogeneous conditions and is not identically equal to zero. This is a contradiction. Lemma 2 is proved.

The proof of the second part of Theorem 1 is now now very simple : It follows from the first part of Theorem 1 that the lowest degree of a triangular $C^{(m)}$-element is greater than or equal to $4m + 1$. This fact and the result of [5] prove the second part of Theorem 1.

The assertion of the following theorem is well-known [2, 5] :

Theorem 2. A triangular $C^{(m)}$-element of degree $4m + 1$ can be uniquely determined by the parameters

$$D^\alpha p(P_i) \quad , \quad |\alpha| \leqslant 2m \quad (i = 1, 2, 3) \tag{49}$$

$$\partial^\varkappa p(Q_{rs}^{(\lambda,\varkappa)})/\partial \nu_{rs}^\varkappa \quad , \quad r = 1, 2 \,,\, s = 2, 3\ (r < s) \tag{50}$$

$$\lambda = 1, \ldots, \varkappa \,;\, \varkappa = 0, \ldots, m$$

$$D^\alpha p(P_0) \quad , \quad |\alpha| \leqslant m - 2 \tag{51}$$

where P_0 is the centre of gravity of the triangle $\bar{T}$ and the meaning of other symbols is the same as in the preceding text.

Generalizing Bell's device [1], the number of independent parameters can be reduced by imposing on $p(x, y)$ the condition that the derivatives $\partial^\varkappa p/\partial \nu^\varkappa$ be polynomials of degree $n - 2\varkappa$ along the corresponding sides of the triangle. Then the parameters (50) prescribed on the side $P_r P_s$ are linear combinations of the parameters (49) prescribed at the vertices P_r, P_s.

Setting $k = 0$ in the proof of Lemma 2 we get no contradiction. This suggests to construct triangular $C^{(m)}$-elements with $\rho > 0$ and $l > 0$. However, these polynomials are not useful for applications because their degrees are too high. Only one exception can be mentioned : A triangular $C^{(0)}$-element of the fourth degree can be determined by the parameters

$$(52) \qquad D^\alpha p(P_i) \quad , \quad |\alpha| = 0,2 \quad ; \quad p(Q_i) \quad (i = 1, 2, 3)$$

where Q_1, Q_2, Q_3 are the mid-points of the sides of a triangle. This element can be used when we do not need the first derivatives and want to get from some reasons continuous second derivatives at the nodal points of a triangulation.

REMARK. A family of triangular $C^{(m)}$-elements with arbitrary $\rho > 0$ and $l = 0$ is studied in [3].

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