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# CONFORMING AND NONCONFORMING FINITE ELEMENT METHODS FOR SOLVING THE STATIONARY STOKES EQUATIONS I

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Communiqué par P.-A. RAVIART

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*Abstract. — The paper is devoted to a general finite element approximation of the solution of the Stokes equations for an incompressible viscous fluid. Both conforming and nonconforming finite element methods are studied and various examples of simplicial elements well suited for the numerical treatment of the incompressibility condition are given. Optimal error estimates are derived in the energy norm and in the  $L^2$ -norm.*

## 1. INTRODUCTION

Let  $\Omega$  be a bounded domain of  $R^N$  ( $N = 2$  or  $3$ ) with boundary  $\Gamma$ . We consider *the stationary Stokes problem* for an incompressible viscous fluid confined in  $\Omega$  : Find functions  $\vec{u} = (u_1, \dots, u_N)$  and  $p$  defined over  $\Omega$  such that

$$(1.1) \quad \begin{aligned} -\nu \Delta \vec{u} + \overrightarrow{\text{grad}} p &= \vec{f} \text{ in } \Omega, \\ \text{div } \vec{u} &= 0 \text{ in } \Omega, \\ \vec{u} &= \vec{0} \text{ on } \Gamma, \end{aligned}$$

where  $\vec{u}$  is the fluid velocity,  $p$  is the pressure,  $\vec{f}$  are the body forces per unit mass and  $\nu > 0$  is the dynamic viscosity.

This paper is devoted to the numerical approximation of problem (1.1) by finite element methods using triangular elements ( $N = 2$ ) or tetrahedral

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elements ( $N = 3$ ). Clearly, the main difficulty stems from the numerical treatment of the incompressibility condition  $\operatorname{div} \vec{u} = 0$ . Because of this additional constraint, and except in some special cases, standard finite elements as those described in Zienkiewicz [16, Chapter 7] appear to be rather unsuitable. Thus, it has been found worthwhile to generate special finite elements which are well adapted to the numerical treatment of the divergence condition.

Indeed, one can construct finite element methods where the incompressibility condition is exactly satisfied (cf. Fortin [8], [9]) but this leads to the use of complex elements of limited applicability. Thus, in this paper, we shall construct and study finite element methods using simpler elements where the incompressibility condition is only approximatively satisfied.

On the other hand, we have found it very convenient to use nonconforming finite elements which violate the interelement continuity condition of the velocities. Thus, we shall develop in this paper both conforming and nonconforming finite element methods for solving the Stokes problem (1.1).

An outline of the paper is as follows. In § 2, we shall recall some standard results on the continuous problem and we shall give a general formulation of the finite element approximation. Section 3 will be devoted to the derivation of general error bounds for the velocity both in the energy norm and in the  $L^2$ -norm. In §§ 4 and 5, we shall give examples of conforming and nonconforming elements, respectively. In § 6, we shall derive general error bounds for the pressure in the  $L^2$ -norm. Finally, we shall consider in § 7 the approximation of the Stokes problem with *inhomogeneous* boundary conditions

$$(1.2) \quad \vec{u} = \vec{g} \text{ on } \Gamma.$$

For the sake of simplicity, we have confined ourselves to *polyhedral* domains  $\Omega$  but it is very likely that our results can be extended to the case of general curved domains by using isoparametric finite elements, as analyzed in Ciarlet and Raviart [6], [7]. Similarly, we have not considered the effect of numerical integration since this effect has been already studied : see Ciarlet and Raviart [7], Strang and Fix [15].

In a subsequent paper, we shall describe and study both direct and iterative matrix methods for numerically finding the finite element approximation of the Stokes problem. Finally, let us mention that all the methods and results of this paper can be extended to some nonlinear problems. In this respect, we refer to a forthcoming paper of Jamet and Raviart [11] where the stationary Navier-Stokes equations are considered.

## 2. NOTATIONS AND PRELIMINARIES

We shall consider real-valued functions defined on  $\Omega$ . Let us denote by

$$(2.1) \quad (u, v) = \int_{\Omega} u(x)v(x) \, dx$$

the scalar product in  $L^2(\Omega)$  and by

$$(2.2) \quad \|v\|_{0,\Omega} = (v,v)^{1/2}$$

the corresponding norm. Consider also the quotient space  $L^2(\Omega)/R$  provided with the quotient norm

$$(2.3) \quad \|v\|_{L^2(\Omega)/R} = \inf_{c \in R} \|v + c\|_{0,\Omega}.$$

For simplicity, we shall denote also by  $v$  any function in the class  $v \in L^2(\Omega)/R$ .

Given any integer  $m \geq 0$ , let

$$(2.4) \quad H^m(\Omega) = \{v \mid v \in L^2(\Omega), \partial^\alpha v \in L^2(\Omega), |\alpha| \leq m\}$$

be the usual Sobolev space provided with the norm

$$(2.5) \quad \|v\|_{m,\Omega} = \left( \sum_{|\alpha| \leq m} \|\partial^\alpha v\|_{0,\Omega}^2 \right)^{1/2}.$$

We shall need the following seminorm

$$(2.6) \quad |v|_{m,\Omega} = \left( \sum_{|\alpha|=m} \|\partial^\alpha v\|_{0,\Omega}^2 \right)^{1/2}.$$

In (2.4), ..., (2.6),  $\alpha$  is a multiindex :  $\alpha = (\alpha_1, \dots, \alpha_N)$ ,  $\alpha_i \geq 0$ ,

$$|\alpha| = \alpha_1 + \dots + \alpha_N \text{ and } \partial^\alpha = \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \dots \left( \frac{\partial}{\partial x_n} \right)^{\alpha_N}.$$

Let

$$(2.7) \quad H_0^1(\Omega) = \{v \mid v \in H^1(\Omega), v|_{\Gamma} = 0\}.$$

Note that  $|v|_{1,\Omega}$  is a norm over  $H_0^1(\Omega)$  which is equivalent to the  $H^1(\Omega)$ -norm.

Let  $(L^2(\Omega))^N$  (resp.  $(H^m(\Omega))^N$ ) be the space of vector functions  $\vec{v} = (v_1, \dots, v_N)$

with components  $v_i$  in  $L^2(\Omega)$  (resp. in  $H^m(\Omega)$ ). The scalar product in  $(L^2(\Omega))^N$  is given by

$$(2.8) \quad (\vec{u}, \vec{v}) = \int_{\Omega} \vec{u}(x) \cdot \vec{v}(x) \, dx = \sum_{i=1}^N \int_{\Omega} u_i(x) v_i(x) \, dx.$$

We consider the following norm and seminorm on the space  $(H^m(\Omega))^N$ :

$$(2.9) \quad \|\vec{v}\|_{m,\Omega} = \left( \sum_{i=1}^N \|v_i\|_{m,\Omega}^2 \right)^{1/2},$$

$$(2.10) \quad |v|_{m,\Omega} = \left( \sum_{i=1}^N |v_i|_{m,\Omega}^2 \right)^{1/2}$$

Introduce now the space

$$(2.11) \quad V = \{ \vec{v} \mid \vec{v} \in (H_0^1(\Omega))^N, \operatorname{div} \vec{v} = 0 \}.$$

We extend the scalar product in  $(L^2(\Omega))^N$  to represent the duality between  $V$  and its dual space  $V'$ .

Let

$$(2.12) \quad a(\vec{u}, \vec{v}) = \sum_{i,j=1}^N \int_{\Omega} \frac{\partial u_i}{\partial x_j}(x) \frac{\partial v_i}{\partial x_j}(x) \, dx, \quad \vec{u}, \vec{v} \in (H^1(\Omega))^N,$$

be the bilinear form associated with the operator  $-\Delta$ . A weak form of problem (1.1) is as follows: *Given a function  $\vec{f} \in V'$ , find functions  $\vec{u} \in V$  and  $p \in L^2(\Omega)/R$  such that*

$$(2.13) \quad va(\vec{u}, \vec{v}) + (\overrightarrow{\operatorname{grad}} p, \vec{v}) = (\vec{f}, \vec{v}) \text{ for all } \vec{v} \in (H_0^1(\Omega))^N$$

or equivalently

$$(2.14) \quad va(\vec{u}, \vec{v}) - (p, \operatorname{div} \vec{v}) = (\vec{f}, \vec{v}) \text{ for all } \vec{v} \in (H_0^1(\Omega))^N.$$

Clearly, if  $(\vec{u}, p) \in V \times L^2(\Omega)/R$  is a solution of equation (2.13) (or 2.14), then  $\vec{u} \in V$  is a solution of

$$(2.15) \quad va(\vec{u}, \vec{v}) = (\vec{f}, \vec{v}) \text{ for all } \vec{v} \in V.$$

In fact, one can prove the following result (cf. Ladyzhenskaya [12], Lions [13]).

**Theorem 1.** *There exists a unique pair of functions  $(\vec{u}, p) \in V \times L^2(\Omega)/R$  solution of equation (2.13). Moreover, the function  $\vec{u} \in V$  can be characterized as the unique solution of equation (2.15).*

For the sake of simplicity, we shall always assume in the sequel that  $\Omega$  is a polyhedral domain of  $R^N$  and that  $\vec{f}$  belongs to the space  $(L^2(\Omega))^N$ .

In order to approximate problems (2.13) or (2.15), we first construct a triangulation  $\mathcal{T}_h$  of the set  $\bar{\Omega}$  with nondegenerate  $N$ -simplices  $K$  (i.e. triangles if  $N = 2$  or tetrahedrons if  $N = 3$ ) with diameters  $\leq h$ . For any  $K \in \mathcal{T}_h$ , we let :

$$(2.16) \quad \begin{aligned} h(K) &= \text{diameter of } K, \\ \rho(K) &= \text{diameter of the inscribed sphere of } K, \\ \sigma(K) &= \frac{h(K)}{\rho(K)}, \quad \sigma = \sup_{K \in \mathcal{T}_h} \sigma(K), \end{aligned}$$

Note that, in the case  $N = 2$ , we have the estimate

$$\sigma(K) \leq \frac{2}{\sin \theta(K)}, \quad \sigma \leq \frac{2}{\sin \theta},$$

where  $\theta(K)$  is the smallest angle of the triangle  $K$  and  $\theta$  is the smallest angle of the triangulation  $\mathcal{T}_h$ . In the following, we shall refer to  $h$  and  $\sigma$  as parameters associated with the triangulation  $\mathcal{T}_h$ .

Let  $k \geq 1$  be a *fixed* integer. With any  $N$ -simplex  $K \in \mathcal{T}_h$ , we associate a finite-dimensional space  $P_K$  of functions defined on  $K$  and satisfying the inclusions

$$(2.17) \quad P_k \subset P_K \subset C^1(K),$$

where  $P_k$  is the space of all polynomials of degree  $k$  in the  $N$  variables  $x_1, \dots, x_N$ . Next, we are given two finite-dimensional spaces  $W_h$  and  $W_{0,h} \subset W_h$  of functions  $v_h$  defined on  $\Omega$  and such that  $v_h|_K \in P_K$  for all  $K \in \mathcal{T}_h$ . We provide the space  $W_h$  with the following seminorm

$$(2.18) \quad \|v_h\|_h = \left( \sum_{K \in \mathcal{T}_h} |v_h|_{1,K}^2 \right)^{1/2}.$$

REMARK 1. The spaces  $W_h$  and  $W_{0,h}$  will appear in the sequel as finite-dimensional approximations of the spaces  $H^1(\Omega)$  and  $H_0^1(\Omega)$  respectively. The inclusions  $W_h \subset H^1(\Omega)$ ,  $W_{0,h} \subset H_0^1(\Omega)$  occur when *conforming* finite elements are used and we get  $\|v_h\|_h = |v_h|_{1,\Omega}$  for all  $v_h \in W_h$ . But, in the general case of *nonconforming* finite elements, these inclusions are no longer true and we shall need some appropriate compatibility conditions : see Hypothesis H.2 below.

Let  $(W_h)^N$  (resp.  $(W_{0,h})^N$ ) be the space of vector functions  $\vec{v}_h = (v_{1,h}, \dots, v_{N,h})$  with components  $v_{i,h}$  in  $W_h$  (resp. in  $W_{0,h}$ ). We provide  $(W_h)^N$  with the seminorm

$$(2.19) \quad \|\vec{v}_h\|_h = \left( \sum_{i=1}^N \|v_{i,h}\|_h^2 \right)^{1/2}.$$

Consider now the space  $\Phi_h$  of functions  $\varphi_h$  defined on  $\Omega$  and such that  $\varphi_h|_K \in P_{k-1}$  for all  $K \in \mathcal{T}_h$ . Let us introduce the operator

$$\operatorname{div}_h \in \mathcal{L}((W_h)^N; \Phi_h) \cap \mathcal{L}((H^1(\Omega))^N; \Phi_h)$$

by

$$(2.20) \quad (\operatorname{div}_h \vec{v}, \varphi_h) = \sum_{K \in \mathcal{T}_h} \int_K \operatorname{div} \vec{v} \cdot \varphi_h \, dx \text{ for all } \varphi_h \in \Phi_h.$$

Then, define the space

$$(2.21) \quad V_h = \{ \vec{v}_h \mid \vec{v}_h \in (W_{0,h})^N, \operatorname{div}_h \vec{v}_h = 0 \}.$$

With the bilinear form  $a(\vec{u}, \vec{v})$ , we associate

$$(2.22) \quad a_h(\vec{u}, \vec{v}) = \sum_{K \in \mathcal{T}_h} \sum_{i,j=1}^N \int_K \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, dx, \quad \vec{u}, \vec{v} \in (H^1(\Omega))^N \cup (W_h)^N$$

Notice that  $a_h(\vec{u}_h, \vec{v}_h) = a(\vec{u}_h, \vec{v}_h)$ ,  $\vec{u}_h, \vec{v}_h \in (W_h)^N$ , when  $W_h \subset H^1(\Omega)$ . Then the approximate problem is the following : *Find a function  $\vec{u}_h \in V_h$  such that*

$$(2.23) \quad \forall a_h(\vec{u}_h, \vec{v}_h) = (\vec{f}, \vec{v}_h) \text{ for all } \vec{v}_h \in V_h.$$

**Theorem 2.** *Assume that  $\|\vec{v}_h\|_h$  is a norm over  $W_{0,h}$ . Then, problem (2.23) has a unique solution  $\vec{u}_h \in V_h$ .*

*Proof.* Since  $\|\vec{v}_h\|_h$  is a norm over  $(W_{0,h})^N$ , this result is an easy consequence of the Lax-Milgram Theorem.

### 3. GENERAL ERROR ESTIMATES FOR THE VELOCITY

Now, we want to derive bounds for the error  $\vec{u}_h - \vec{u}$  when the solution  $\vec{u} \in V$  of (2.15) is smooth enough (For regularity properties of the solution  $\vec{u}$ , we refer to [12]). We begin with an estimate for  $\|\vec{u}_h - \vec{u}\|_h$ . We may write for all  $\vec{v}_h \in V_h$

$$a_h(\vec{u}_h - \vec{v}_h, \vec{u}_h - \vec{v}_h) = a_h(\vec{u}_h - \vec{u}, \vec{u}_h - \vec{v}_h) + a_h(\vec{u} - \vec{v}_h, \vec{u}_h - \vec{v}_h)$$

and

$$\|\vec{u}_h - \vec{v}_h\|_h \leq \|\vec{u} - \vec{v}_h\|_h + \sup_{\vec{w}_h \in V_h} \frac{|a_h(\vec{u}_h - \vec{u}, \vec{w}_h)|}{\|\vec{w}_h\|_h}.$$

Thus, we get

$$(3.1) \quad \|\vec{u}_h - \vec{u}\|_h \leq 2 \inf_{\vec{v}_h \in V_h} \|\vec{u} - \vec{v}_h\|_h + \sup_{\vec{w}_h \in V_h} \frac{|a_h(\vec{u}_h - \vec{u}, \vec{w}_h)|}{\|\vec{w}_h\|_h}.$$

In order to evaluate the term  $\inf_{\vec{v}_h \in V_h} \|\vec{u} - \vec{v}_h\|_h$  appearing in (3.1), we need some approximability assumption :

**Hypothesis H.1.** *There exists an operator*

$$r_h \in \mathcal{L}((H^2(\Omega))^N; (W_h)^N) \cap \mathcal{L}((H^2(\Omega) \cap H_0^1(\Omega))^N; (W_{0,h})^N)$$

uch that

$$(i) \quad (3.2) \quad \operatorname{div}_h r_h \vec{v} = \operatorname{div}_h \vec{v} \text{ for all } \vec{v} \in (H^2(\Omega))^N;$$

(ii) for some integer  $l \geq 1$

$$(3.3) \quad \|r_h \vec{v} - \vec{v}\|_h \leq C \sigma^l h^m |\vec{v}|_{m+1, \Omega} \text{ for all } \vec{v} \in (H^{m+1}(\Omega))^N, 1 \leq m \leq k,$$

where the constant  $C$  is independent of  $h$  and  $\sigma$ .

By (2.20), condition (3.2) is equivalent to the following property :

$$(3.4) \quad \int_K q \operatorname{div} r_h \vec{v} \, dx = \int_K q \operatorname{div} \vec{v} \, dx \text{ for all } q \in P_{k-1} \text{ and all } K \in \mathcal{T}_h.$$

**Lemma 1.** *Assume that Hypothesis H.1 holds. Then  $r_h \in \mathcal{L}(V \cap (H^2(\Omega))^N; V_h)$  and we have the estimate*

$$(3.5) \quad \inf_{\vec{v}_h \in V_h} \|\vec{v}_h - \vec{v}\|_h \leq C \sigma^l h^m |\vec{v}|_{m+1, \Omega} \text{ for all } \vec{v} \in V \cap (H^{m+1}(\Omega))^N, 1 \leq m \leq k,$$

where the constant  $C$  is independent of  $h$  and  $\sigma$ .

Now, for estimating the term  $a_h(\vec{u}_h - \vec{u}, \vec{w}_h)$ ,  $\vec{w}_h \in V_h$ , we assume that the solution  $(\vec{u}, p)$  of (2.13) satisfies the smoothness assumptions :

$$\vec{u} \in V \cap (H^2(\Omega))^N, p \in H^1(\Omega).$$

From (2.23), we obtain

$$a_h(\vec{u}_h - \vec{u}, \vec{w}_h) = \frac{1}{\nu} \int_{\Omega} \vec{f} \cdot \vec{w}_h \, dx - a_h(\vec{u}, \vec{w}_h).$$

Clearly

$$\int_{\Omega} \vec{f} \cdot \vec{w}_h \, dx = -\nu \int_{\Omega} \Delta \vec{u} \cdot \vec{w}_h \, dx + \int_{\Omega} \overline{\operatorname{grad} p} \cdot \vec{w}_h \, dx$$

and, by using Green's formula on each  $K \in \mathcal{T}_h$ , we get

$$\begin{aligned} \int_{\Omega} \vec{f} \cdot \vec{w}_h \, dx &= \nu a_h(\vec{u}, \vec{w}_h) - \sum_{K \in \mathcal{T}_h} \int_K p \operatorname{div} \vec{w}_h \, dx \\ &\quad - \nu \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial \vec{u}}{\partial n} \cdot \vec{w}_h \, d\sigma + \sum_{K \in \mathcal{T}_h} \int_{\partial K} p \vec{w}_h \cdot \vec{n} \, d\sigma \end{aligned}$$



where  $\vec{n}$  denotes the exterior (with respect to  $K$ ) unit vector normal to the boundary  $\partial K$  of  $K$ . Thus, we have

$$(3.6) \quad \begin{aligned} a_h(\vec{u}_h - \vec{u}, \vec{w}_h) = & -\frac{1}{v} \sum_{K \in \mathcal{T}_h} \int_K p \operatorname{div} \vec{w}_h \, dx - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial \vec{u}}{\partial n} \cdot \vec{w}_h \, d\sigma \\ & + \frac{1}{v} \sum_{K \in \mathcal{T}_h} \int_{\partial K} p \vec{w}_h \cdot \vec{n} \, d\sigma \end{aligned}$$

In order to evaluate the surface integrals which appear in (3.6) (and which are identically zero when  $W_{0,h} \subset H_0^1(\Omega)$ , i.e. for conforming finite element methods), we need first some compatibility assumption.

**Hypothesis H.2.** *We assume the following compatibility conditions :*

(i) *For any  $(N-1)$ -dimensional face  $K'$  which separates two  $N$ -simplices  $K_1, K_2 \in \mathcal{T}_h$ , we have*

$$(3.7) \quad \int_{K'} q(v_{h,1} - v_{h,2}) \, d\sigma = 0 \text{ for all } q \in P_{k-1} \text{ and all } v_h \in W_h,$$

where  $v_{h,i}$  is the restriction of  $v_h$  to  $K_i$ ,  $i = 1, 2$ ;

(ii) *For any  $(N-1)$ -dimensional face  $K'$  of a  $N$ -simplex  $K \in \mathcal{T}_h$  such that  $K'$  is a portion of the boundary  $\Gamma$ , we have*

$$(3.8) \quad \int_{K'} qv_h \, d\sigma = 0 \text{ for all } q \in P_{k-1} \text{ and all } v_h \in W_{0,h}.$$

**REMARK 2.** Clearly, Hypothesis H.2 is satisfied when  $W_h \subset H^1(\Omega)$  and  $W_{0,h} \subset H_0^1(\Omega)$ . When  $W_h \not\subset H^1(\Omega)$  and  $W_{0,h} \not\subset H^1(\Omega)$ , i.e. for nonconforming finite element methods, Hypothesis H.2 implies that, for second order elliptic problems, all polynomials of degree  $k$  pass the « patch test » of Irons (cf. [1], [10] and [15] for a more mathematical point of view) so that the right order of convergence can be reasonably expected.

As a consequence of Hypothesis H.2, we can prove :

**Lemma 2.** *Assume that Hypothesis H.2 holds. Then  $\|v_h\|_h$  is a norm over the space  $W_{0,h}$ .*

*Proof.* Let  $v_h$  be a function of  $W_{0,h}$  such that  $\|v_h\|_h = 0$ . From (2.18), we get  $\frac{\partial v_h}{\partial x_i} = 0$ ,  $1 \leq i \leq N$ , in each  $K \in \mathcal{T}_h$ . Thus,  $v_h$  is constant in each  $N$ -simplex  $K \in \mathcal{T}_h$ . Using Hypothesis H.2 (i) with  $q \equiv 1$ , we find that  $v_h$  is constant over  $\Omega$ . Finally, by using Hypothesis H.2 (ii), we get  $v_h = 0$ .

Besides Hypothesis H.2, we need an essential technical result. Let  $K$  be a nondegenerate  $N$ -simplex of  $R^N$  and let  $K'$  be a  $(N-1)$ -dimensional face

of  $K$ . Let us denote by  $P'_\mu$  the space of the restrictions to  $K'$  of all polynomials of degree  $\mu$  and  $\mathcal{M}^\mu_{K'}$  the projection operator from  $L^2(K')$  onto  $P'_\mu$  :

$$(3.9) \quad \int_{K'} q \cdot \mathcal{M}^\mu_{K'} v d\sigma = \int_{K'} q v d\sigma \text{ for all } q \in P'_\mu.$$

**Lemma 3.** *For any integer  $m$  with  $0 \leq m \leq \mu$ , there exists a constant  $C > 0$  independent of  $K$  such that*

$$(3.10) \quad \left| \int_{K'} \varphi(v - \mathcal{M}^\mu_{K'} v) d\sigma \right| \leq C\sigma(K)(h(K))^{m+1} |\varphi|_{1,K} |v|_{m+1,K}$$

for all  $\varphi \in H^1(K)$  and all  $v \in H^{m+1}(K)$ .

*Proof.* Let  $\hat{K}$  be a nondegenerate  $N$ -simplex of  $R^N$  and let  $\hat{K}'$  be a  $(N - 1)$ -dimensional face of  $\hat{K}$ . Just for convenience, we shall assume that  $K'$  and  $\hat{K}'$  have the same supporting hyperplane  $x_N = 0$ . Let us denote by

$$F : x \rightarrow F(x) = Bx + b, \quad B \in \mathcal{L}(R^N), \quad b \in R^N,$$

an affine invertible mapping such that  $K = F(\hat{K})$ ,  $K' = F(\hat{K}')$ , and by  $B'$  the  $(N - 1) \times (N - 1)$  matrix obtained by crossing out the  $N^{\text{th}}$  row and the  $N^{\text{th}}$  column of the  $N \times N$  matrix  $B$ .

For any function  $f$  defined on  $K$  (or on  $K'$ ), we let :  $\hat{f} = f \circ F$ . Then, we have

$$\widehat{\mathcal{M}^\mu_{K'} v} = \mathcal{M}^\mu_{\hat{K}'} \hat{v}.$$

We may write for all  $\varphi \in H^1(K)$  and all  $v \in H^{m+1}(K)$ .

$$(3.11) \quad \int_{K'} \varphi(v - \mathcal{M}^\mu_{K'} v) d\sigma = |\det(B')| \int_{\hat{K}'} \hat{\varphi}(\hat{v} - \mathcal{M}^\mu_{\hat{K}'} \hat{v}) d\sigma$$

Consider, for fixed  $\hat{v} \in H^{m+1}(\hat{K})$ ,  $0 \leq m \leq \mu$ , the linear functional

$$\hat{\varphi} \rightarrow \int_{\hat{K}'} \hat{\varphi}(\hat{v} - \mathcal{M}^\mu_{\hat{K}'} \hat{v}) d\sigma$$

which is continuous over  $H^1(\hat{K})$  with norm  $\leq \|\hat{v} - \mathcal{M}^\mu_{\hat{K}'} \hat{v}\|_{0,\hat{K}'}$  and which vanishes over  $P_0$  by (3.9). By the Bramble-Hilbert lemma [3] in the form given in [5, Lemma 6], we get

$$(3.12) \quad \left| \int_{\hat{K}'} \hat{\varphi}(\hat{v} - \mathcal{M}^\mu_{\hat{K}'} \hat{v}) d\sigma \right| \leq c_1 |\hat{\varphi}|_{1,\hat{K}} \|\hat{v} - \mathcal{M}^\mu_{\hat{K}'} \hat{v}\|_{0,\hat{K}'}$$

for some constant  $c_1 = c_1(\hat{K})$ . Since  $\mathcal{M}_{\hat{K}}^{\mu, \hat{v}} = \hat{v}$  for all  $\hat{v} \in P_m$ , we get as an easy consequence of the Bramble-Hilbert lemma (see also [5, Lemma 7]).

$$(3.13) \quad \|\hat{v} - \mathcal{M}_{\hat{K}}^{\mu, \hat{v}}\|_{0, \hat{K}'} \leq c_2 |\hat{v}|_{m+1, \hat{K}}$$

for some constant  $c_2 = c_2(\hat{K})$ . Combining (3.11), ..., (3.13), we obtain

$$(3.14) \quad \left| \int_{K'} \varphi(v - \mathcal{M}_{\hat{K}}^{\mu, v}) d\sigma \right| \leq c_1 c_2 |\det(B')| |\hat{\varphi}|_{1, \hat{K}} |\hat{v}|_{m+1, \hat{K}}.$$

Since (cf. [5, formula (4.15)])

$$(3.15) \quad |\hat{v}|_{1, \hat{K}} \leq |\det(B)|^{-1/2} \|B\|^t |v|_{1, K} \quad \text{for all } v \in H^1(K),$$

we get

$$(3.16) \quad \left| \int_{K'} \varphi(v - \mathcal{M}_{\hat{K}}^{\mu, v}) d\sigma \right| \leq c_1 c_2 |\det(B')| |\det(B)|^{-1} \|B\|^{m+2} |\varphi|_{1, K} |v|_{m+1, K}$$

where  $\|B\|$  is the norm of the matrix  $B$  subordinate to the Euclidean vector norm.

Denote by  $e_N$  the  $N^{\text{th}}$  vector of the canonical basis of  $R^N$ . Then, the  $N^{\text{th}}$  component of the vector  $B^{-1}e_N$  is given by

$$(B^{-1}e_N)_N = \det(B') (\det(B))^{-1}$$

so that

$$(3.17) \quad |\det(B')| \leq |\det(B)| \|B^{-1}\|.$$

By (3.16) and (3.17), we may write

$$(3.18) \quad \left| \int_{K'} \varphi(v - \mathcal{M}_{\hat{K}}^{\mu, v}) d\sigma \right| \leq c_1 c_2 \|B\|^{m+2} \|B^{-1}\| |\varphi|_{1, K} |v|_{m+1, K}.$$

Since

$$(3.19) \quad \|B\| \leq \frac{h(K)}{\rho(\hat{K})}, \quad \|B^{-1}\| \leq \frac{h(\hat{K})}{\rho(K)}$$

(cf. [5, Lemma 2]), the desired inequality follows with  $C = c_1 c_2 \frac{h(\hat{K})}{(\rho(\hat{K}))^{m+2}}$  which depends only on  $\hat{K}$ .

In the sequel, we shall denote by  $C$  or  $C_i$  various constants independent of  $h$  and  $\sigma$ . We are now able to derive a bound for the error  $\|\vec{u}_h - \vec{u}\|_h$ .

**Theorem 3.** Assume that Hypotheses H.1 and H.2 hold. Assume, in addition, that the solution  $(\vec{u}, p)$  of problem (2.13) satisfies the smoothness properties

$$(3.20) \quad \vec{u} \in V \cap (H^{k+1}(\Omega))^N, p \in H^k(\Omega).$$

Then, problem (2.23) has a unique solution  $\vec{u}_h \in V_h$  and we have the estimate

$$(3.21) \quad \|\vec{u}_h - \vec{u}\|_h \leq C\sigma^l h^k (|\vec{u}|_{k+1, \Omega} + |p|_{k, \Omega})$$

*Proof.* Existence and uniqueness of the solution  $\vec{u}_h \in V_h$  follow from Hypothesis H.2, Lemma 2 and Theorem 2. Consider now equation (3.6) : we begin with an estimate for the term

$$(3.22) \quad \sum_{K \in \mathcal{T}_h} \int_K \frac{\partial \vec{u}}{\partial n} \cdot \vec{w}_h \, d\sigma, \vec{w}_h \in (W_{0,h})^N.$$

Let  $K'$  be a  $(N-1)$ -dimensional face which separates two  $N$ -simplices  $K_1, K_2 \in \mathcal{T}_h$ . For  $i = 1, 2$ , let us denote by  $\vec{w}_{h,i}$  the restriction  $\vec{w}_h$  to  $K_i$  and by  $\vec{n}_i$  the unit vector normal to  $K'$  and pointing out of  $K_i$ . The contribution of  $K'$  in the expression (3.22) is given by

$$\int_{K'} \left( \frac{\partial \vec{u}}{\partial n_1} \cdot \vec{w}_{h,1} + \frac{\partial \vec{u}}{\partial n_2} \cdot \vec{w}_{h,2} \right) d\sigma = \int_{K'} \frac{\partial \vec{u}}{\partial n_1} \cdot (\vec{w}_{h,1} - \vec{w}_{h,2}) \, d\sigma.$$

According to Hypothesis H.2 (i), we have

$$\int_{K'} \left( \mathcal{M}_{K'}^{k-1} \frac{\partial \vec{u}}{\partial n_1} \right) \cdot (\vec{w}_{h,1} - \vec{w}_{h,2}) \, d\sigma = 0, \vec{w}_h \in (W_{0,h})^N,$$

and we may write

$$\begin{aligned} \int_{K'} \left( \frac{\partial \vec{u}}{\partial n_1} \cdot \vec{w}_{h,1} + \frac{\partial \vec{u}}{\partial n_2} \cdot \vec{w}_{h,2} \right) d\sigma &= \int_{K'} \left( \frac{\partial \vec{u}}{\partial n_1} - \mathcal{M}_{K'}^{k-1} \frac{\partial \vec{u}}{\partial n_1} \right) \cdot (\vec{w}_{h,1} - \vec{w}_{h,2}) \, d\sigma \\ &= \int_{K'} \left\{ \left( \frac{\partial \vec{u}}{\partial n_1} - \mathcal{M}_{K'}^{k-1} \frac{\partial \vec{u}}{\partial n_1} \right) \cdot \vec{w}_{h,1} - \left( \frac{\partial \vec{u}}{\partial n_2} - \mathcal{M}_{K'}^{k-1} \frac{\partial \vec{u}}{\partial n_2} \right) \cdot \vec{w}_{h,2} \right\} d\sigma. \end{aligned}$$

Now let  $K'$  be a  $(N-1)$ -dimensional face of a  $N$ -simplex  $K \in \mathcal{T}_h$  such that  $K'$  is a portion of the boundary  $\Gamma$ . According to Hypothesis H.2 (ii), we have

$$\int_{K'} \left( \mathcal{M}_{K'}^{k-1} \frac{\partial \vec{u}}{\partial n} \right) \cdot \vec{w}_h \, d\sigma = 0, \vec{w}_h \in (W_{0,h})^N.$$

Thus, we may write

$$\int_{K'} \frac{\partial \vec{u}}{\partial n} \cdot \vec{w}_h \, d\sigma = \int_{K'} \left( \frac{\partial \vec{u}}{\partial n} - \mathcal{M}_{K'}^{k-1} \frac{\partial \vec{u}}{\partial n} \right) \cdot \vec{w}_h \, d\sigma.$$

In conclusion, we get as a consequence of Hypothesis H.2

(3.23)

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial \vec{u}}{\partial n} \cdot \vec{w}_h \, d\sigma = \sum_{K \in \mathcal{T}_h} \sum_{K' \in \partial K} \int_{K'} \left( \frac{\partial \vec{u}}{\partial n} - \mathcal{M}_{K'}^{k-1} \frac{\partial \vec{u}}{\partial n} \right) \cdot \vec{w}_h \, d\sigma, \quad \vec{w}_h \in (W_{0,h})^N.$$

By using Lemma 3 with  $m = k - 1$ , we get the estimate

$$(3.24) \quad \left| \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial \vec{u}}{\partial n} \cdot \vec{w}_h \, d\sigma \right| \leq c_1 \sigma h^k |\vec{u}|_{k+1, \Omega} \|\vec{w}_h\|_h \text{ for all } \vec{w}_h \in (W_{0,h})^N.$$

Similarly, we get

(3.25)

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} p \vec{w}_h \cdot \vec{n} \, d\sigma = \sum_{K \in \mathcal{T}_h} \sum_{K' \in \partial K} \int_{K'} (p - \mathcal{M}_{K'}^{k-1} p) \vec{w}_h \cdot \vec{n} \, d\sigma, \quad \vec{w}_h \in (W_{0,h})^N,$$

and by Lemma 3

$$(3.26) \quad \left| \sum_{K \in \mathcal{T}_h} \int_{\partial K} p \vec{w}_h \cdot \vec{n} \, d\sigma \right| \leq c_2 \sigma h^k |p|_{k, \Omega} \|\vec{w}_h\|_h \text{ for all } \vec{w}_h \in (W_{0,h})^N.$$

It remains to estimate the term

$$(3.27) \quad \sum_{K \in \mathcal{T}_h} \int_K p \operatorname{div} \vec{w}_h \, dx, \quad \vec{w}_h \in V_h.$$

By definition of the space  $V_h$ , we have :

$$(3.28) \quad \vec{w}_h \in V_h \Leftrightarrow \int_K q \operatorname{div} \vec{w}_h \, dx = 0 \text{ for all } q \in P_{k-1} \text{ and all } K \in \mathcal{T}_h.$$

Thus, we may write

$$\int_K p \operatorname{div} \vec{w}_h \, dx = \int_K (p - q) \operatorname{div} \vec{w}_h \, dx \text{ for all } q \in P_{k-1} \text{ and all } K \in \mathcal{T}_h.$$

By applying [5, Theorem 5], we get the estimate

$$\min_{q \in P_{k-1}} \|p - q\|_{0,K} \leq c_3 (h(K))^k |p|_{k,K}$$

and therefore

$$(3.29) \quad \left| \sum_{K \in \mathcal{T}_h} \int_K p \operatorname{div} \vec{w}_h \, dx \right| \leq c_4 h^k |p|_{k, \Omega} \|\vec{w}_h\|_h \text{ for all } \vec{w}_h \in V_h$$

Combining (3.6), (3.24), (3.26), (3.29), we obtain for all  $\vec{w}_h \in V_h$

$$(3.30) \quad |a_h(\vec{u}_h - \vec{u}, \vec{w}_h)| \leq c_5 \sigma h^k (|\vec{u}|_{k+1, \Omega} + |p|_{k, \Omega}) \|\vec{w}_h\|_h.$$

Then, the desired inequality (3.21) follows from (3.1), (3.30) and lemma 1.

**REMARK 3.** In the case of conforming finite element methods, the proof of Theorem 3 reduces to the proof of inequality (3.29).

**REMARK 4.** When the solution  $(\vec{u}, p)$  verifies only

$$(3.31) \quad \vec{u} \in V \cap (H^{m+1}(\Omega))^N, p \in H^m(\Omega),$$

for some integer  $m$  with  $1 \leq m \leq k$ , we similarly get the estimate

$$(3.32) \quad \|\vec{u}_h - \vec{u}\|_h \leq C \sigma^l h^m (|\vec{u}|_{m+1, \Omega} + |p|_{m, \Omega}).$$

Assume now that (3.31) holds with  $m = 0$ , i.e.  $(\vec{u}, p)$  does not satisfy any smoothness property. Then, by using the density of  $V \cap (\mathcal{D}(\Omega))^N$  in  $V$ , one can easily show that for bounded  $\sigma$

$$(3.33) \quad \lim_{h \rightarrow 0} \|\vec{u}_h - \vec{u}\|_h = 0.$$

We now come to an  $L^2$ -estimate for the error  $\vec{u}_h - \vec{u}$ . To do this, we need the following regularity property for the Stokes problem :

$$(3.34) \quad \begin{aligned} &\text{The mapping } (\vec{\varphi}, \chi) \rightarrow -\nu \Delta \vec{\varphi} + \overrightarrow{\text{grad}} \chi \text{ is an} \\ &\text{isomorphism from } [V \cap (H^2(\Omega))^N] \times [H^1(\Omega)/R] \text{ onto } (L^2(\Omega))^N. \end{aligned}$$

Since  $\Omega$  is a polyhedral domain, this property holds for example when  $\Omega$  is convex.

**Theorem 4.** Assume that Hypotheses H.1, H.2, (3.20) and (3.34) hold. Then we have the estimate

$$(3.35) \quad \|\vec{u}_h - \vec{u}\|_{0, \Omega} \leq C \sigma^{2l} h^{k+1} (|\vec{u}|_{k+1, \Omega} + |p|_{k, \Omega}).$$

*Proof.* We use and generalize to the nonconforming case the now classical Aubin-Nitsche's duality argument. We may write

$$(3.36) \quad \|\vec{u}_h - \vec{u}\|_{0, \Omega} = \sup_{\vec{g} \in (L^2(\Omega))^N} \frac{|(\vec{u}_h - \vec{u}, \vec{g})|}{\|\vec{g}\|_{0, \Omega}}.$$

Given  $\vec{g} \in (L^2(\Omega))^N$ , we let  $(\vec{\varphi}, \chi)$  be the solution of the Stokes problem

$$(3.37) \quad \begin{aligned} &-\nu \Delta \vec{\varphi} + \overrightarrow{\text{grad}} \chi = \vec{g} \text{ in } \bar{\Omega}, \\ &\text{div } \vec{\varphi} = 0 \text{ in } \bar{\Omega}, \\ &\vec{\varphi} = 0 \text{ on } \Gamma. \end{aligned}$$

According to (3.34), we have  $\vec{\varphi} \in V \cap (H^2(\Omega))^N$ ,  $\chi \in H^1(\Omega)$  and

$$(3.38) \quad \|\vec{\varphi}\|_{2,\Omega} + |\chi|_{1,\Omega} \leq C \|\vec{g}\|_{0,\Omega}.$$

By using Green's formula over each  $K \in \mathcal{T}_h$ , we get

$$(3.39) \quad \begin{aligned} (\vec{u}_h - \vec{u}, \vec{g}) &= -\nu \sum_{K \in \mathcal{T}_h} \int_K (\vec{u}_h - \vec{u}) \cdot \Delta \vec{\varphi} \, dx \\ &\quad + \sum_{K \in \mathcal{T}_h} \int_K (\vec{u}_h - \vec{u}) \cdot \overrightarrow{\text{grad}} \chi \, dx \\ &= \nu a_h(\vec{u}_h - \vec{u}, \vec{\varphi}) - \sum_{K \in \mathcal{T}_h} \left\{ \int_K \chi \operatorname{div} (\vec{u}_h - \vec{u}) \, dx \right. \\ &\quad \left. + \nu \int_{\partial K} (\vec{u}_h - \vec{u}) \cdot \frac{\partial \vec{\varphi}}{\partial n} \, d\sigma - \int_{\partial K} (\vec{u}_h - \vec{u}) \cdot n \chi \, d\sigma \right\}. \end{aligned}$$

Combining (3.6) and (3.37), we obtain for all  $\vec{\varphi}_h \in (W_{0,h})^N$

$$(3.40) \quad \begin{aligned} (\vec{u}_h - \vec{u}, \vec{g}) &= \nu a_h(\vec{u}_h - \vec{u}, \vec{\varphi} - \vec{\varphi}_h) \\ &\quad - \sum_{K \in \mathcal{T}_h} \left\{ \int_K p \operatorname{div} \vec{\varphi}_h \, dx + \int_K \chi \operatorname{div} (\vec{u}_h - \vec{u}) \, dx \right. \\ &\quad \left. + \nu \int_{\partial K} \frac{\partial \vec{u}}{\partial n} \cdot \vec{\varphi}_h \, d\sigma - \int_{\partial K} p \vec{\varphi}_h \cdot \vec{n} \, d\sigma \right. \\ &\quad \left. + \nu \int_{\partial K} (\vec{u}_h - \vec{u}) \cdot \frac{\partial \vec{\varphi}}{\partial n} \, d\sigma - \int_{\partial K} (\vec{u}_h - \vec{u}) \cdot \vec{n} \chi \, d\sigma \right\}. \end{aligned}$$

Let us consider first the expression

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial \vec{u}}{\partial n} \cdot \vec{\varphi}_h \, d\sigma = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial \vec{u}}{\partial n} \cdot (\vec{\varphi}_h - \vec{\varphi}) \, d\sigma.$$

Using Hypothesis H.2, we may write as in the proof of theorem 3

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial \vec{u}}{\partial n} \cdot \vec{\varphi}_h \, d\sigma = \sum_{K \in \mathcal{T}_h} \sum_{K' \subset \partial K} \int_{K'} \left( \frac{\partial \vec{u}}{\partial n} - \mathcal{M}_K^{k-1} \frac{\partial \vec{u}}{\partial n} \right) \cdot (\vec{\varphi}_h - \vec{\varphi}) \, d\sigma.$$

By using Lemma 3 with  $m = k - 1$ , we get for all  $\vec{\varphi}_h \in (W_{0,h})^N$

$$(3.41) \quad \left| \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial \vec{u}}{\partial n} \cdot \vec{\varphi}_h \, d\sigma \right| \leq c_1 \sigma h^k |\vec{u}|_{k+1,\Omega} \|\vec{\varphi}_h - \vec{\varphi}\|_h.$$

Consider now

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} (\vec{u}_h - \vec{u}) \cdot \frac{\partial \vec{\varphi}}{\partial n} \, d\sigma.$$

Using again Hypothesis H.2, we may write

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} (\vec{u}_h - \vec{u}) \cdot \frac{\partial \vec{\varphi}}{\partial n} d\sigma = \sum_{K \in \mathcal{T}_h} \sum_{K' \subset \partial K} \int_{K'} (\vec{u}_h - \vec{u}) \cdot \left( \frac{\partial \vec{\varphi}}{\partial n} - \mathcal{M}_{K'}^{k-1} \frac{\partial \vec{\varphi}}{\partial n} \right) d\sigma$$

and therefore by using Lemma 3 with  $m = 0$

$$(3.42) \quad \left| \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\vec{u}_h - \vec{u}) \cdot \frac{\partial \vec{\varphi}}{\partial n} d\sigma \right| \leq c_2 \sigma h \|\vec{u}_h - \vec{u}\|_h |\vec{\varphi}|_{2,\Omega}.$$

Similarly, we can prove

(3.43)

$$\left| \sum_{K \in \mathcal{T}_h} \int_{\partial K} p \vec{\varphi}_h \cdot \vec{n} d\sigma \right| \leq c_3 \sigma h^k |p|_{k,\Omega} \|\vec{\varphi}_h - \vec{\varphi}\|_h \text{ for all } \vec{\varphi}_h \in (W_{0,h})^N,$$

$$(3.44) \quad \left| \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\vec{u}_h - \vec{u}) \cdot \vec{n} \chi d\sigma \right| \leq c_4 \sigma h \|\vec{u}_h - \vec{u}\|_h |\chi|_{1,\Omega}.$$

Finally, we want to estimate

$$\sum_{K \in \mathcal{T}_h} \int_K p \operatorname{div} \vec{\varphi}_h dx \text{ and } \sum_{K \in \mathcal{T}_h} \int_K \chi \operatorname{div} (\vec{u}_h - \vec{u}) dx.$$

Since  $\vec{\varphi} \in V$ , we get for all  $\vec{\varphi}_h \in V_h$  (cf. (3.28))

$$\int_K p \operatorname{div} \vec{\varphi}_h dx = \int_K p \operatorname{div} (\vec{\varphi}_h - \vec{\varphi}) dx = \int_K (p - q) \operatorname{div} (\vec{\varphi}_h - \vec{\varphi}) dx$$

for all  $q \in P_{k-1}$  and all  $K \in \mathcal{T}_h$ , and therefore

$$\left| \int_K p \operatorname{div} \vec{\varphi}_h dx \right| \leq c_5 \inf_{q \in P_{k-1}} \|p - q\|_{0,K} |\vec{\varphi}_h - \vec{\varphi}|_{1,K} \leq c_6 h^k |p|_{k,K} |\vec{\varphi}_h - \vec{\varphi}|_{1,K}.$$

Thus, we obtain

$$(3.45) \quad \left| \sum_{K \in \mathcal{T}_h} \int_K p \operatorname{div} \vec{\varphi}_h dx \right| \leq c_6 h^k |p|_{k,\Omega} \|\vec{\varphi}_h - \vec{\varphi}\|_h \text{ for all } \vec{\varphi}_h \in V_h.$$

On the other hand,

$$\begin{aligned} \int_K \chi \operatorname{div} (\vec{u}_h - \vec{u}) dx \\ = \int_K (\chi - q) \operatorname{div} (\vec{u}_h - \vec{u}) dx \text{ for all } q \in P_{k-1} \text{ and all } K \in \mathcal{T}_h \end{aligned}$$



so that

$$\begin{aligned} \left| \int_K \chi \operatorname{div} (\vec{u}_h - \vec{u}) \, dx \right| &\leq c_7 \inf_{q \in P_0} \|\chi - q\|_{0,K} |\vec{u}_h - \vec{u}|_{1,K} \\ &\leq c_8 h |\chi|_{1,K} |\vec{u}_h - \vec{u}|_{1,K} \end{aligned}$$

and therefore

$$(3.46) \quad \left| \sum_{K \in \mathcal{T}_h} \int_K \chi \operatorname{div} (\vec{u}_h - \vec{u}) \, dx \right| \leq c_8 h \|\vec{u}_h - \vec{u}\|_h |\chi|_{1,\Omega}.$$

Now, combining (3.40), ..., (3.46), we obtain

$$(3.47) \quad \begin{aligned} |(\vec{u}_h - \vec{u}, \vec{g})| &\leq c_9 \left\{ \|\vec{u}_h - \vec{u}\|_h \inf_{\vec{\varphi}_h \in \mathcal{V}_h} \|\vec{\varphi} - \vec{\varphi}_h\|_h \right. \\ &\quad + \sigma h \|\vec{u}_h - \vec{u}\|_h (|\vec{\varphi}|_{2,\Omega} + |\chi|_{1,\Omega}) \\ &\quad \left. + \sigma h^k (|\vec{u}|_{k+1,\Omega} + |p|_{k,\Omega}) \inf_{\vec{\varphi}_h \in \mathcal{V}_h} \|\vec{\varphi} - \vec{\varphi}_h\|_h \right\}. \end{aligned}$$

Then, applying Lemma 1 and Theorem 3 gives

$$(3.48) \quad |(\vec{u}_h - \vec{u}, \vec{g})| \leq c_{10} \sigma^{2l} h^{k+1} (|\vec{u}|_{k+1,\Omega} + |p|_{k,\Omega}) (|\vec{\varphi}|_{2,\Omega} + |\chi|_{1,\Omega}).$$

The conclusion follows from (3.36), (3.38) and (3.48).

#### 4. APPLICATIONS I : CONFORMING FINITE ELEMENTS

Let us recall some general definitions [5]. Let  $K$  be a  $N$ -simplex belonging to  $\mathcal{T}_h$  with vertices  $a_{i,K}$ ,  $1 \leq i \leq N + 1$ ; we denote by  $\lambda_i(x) = \lambda_{i,K}(x)$ ,  $1 \leq i \leq N + 1$ , the *barycentric coordinates* of a point  $x \in R^N$  with respect to the vertices of  $K$ . Let  $\Sigma_K = \{b_{i,K}\}_{i=1}^M$  be a set of  $M$  distinct points of  $K$ . We shall say that the set  $\Sigma_K$  is  $P_K$ -*unisolvent* if the Lagrange interpolation problem : « Find  $p \in P_K$  such that  $p(b_{i,K}) = \alpha_i$ ,  $1 \leq i \leq M$  » has a unique solution for any given set  $\{\alpha_i\}_{i=1}^M$  of real numbers. If  $\Sigma_K$  is  $P_K$ -unisolvent, we denote by  $p_{i,K}$ ,  $1 \leq i \leq M$ , the *basis functions* over the set  $K$  (i.e.  $p_{i,k} \in P_K$  and  $p_{i,K} = \delta_{ij}$ ,  $1 \leq j \leq M$ ).

We shall consider now examples of conforming finite element methods corresponding to the cases  $k = 1, 2, 3$ .

**EXAMPLE 1.** Just for simplicity, we shall restrict ourselves to the case  $N = 2$ . Denote by  $a_{ij,K}$  the midpoint of the side  $[a_{i,K}, a_{j,K}]$ ,  $1 \leq i < j \leq 3$ . Then, as is well known  $\Sigma_K = \{a_{i,K}\}_{1 \leq i \leq 3} \cup \{a_{ij,K}\}_{1 \leq i < j \leq 3}$  is a  $P_2$ -unisolvent set.

Moreover, the basis functions are given by

$$(4.1) \quad \begin{aligned} p_{i,K} &= \lambda_i(2\lambda_i - 1), \quad 1 \leq i \leq 3, \\ p_{ij,K} &= 4\lambda_i\lambda_j, \quad 1 \leq i < j \leq 3. \end{aligned}$$

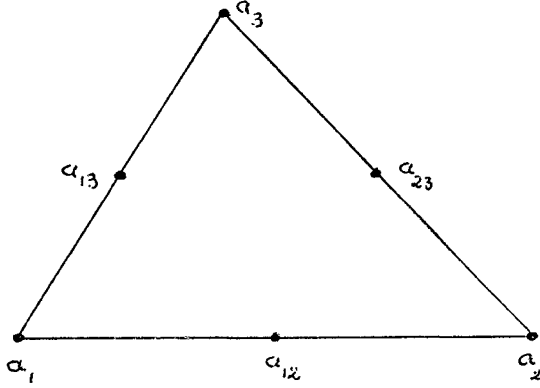


Figure 1

Then, define the spaces :

$$(4.2) \quad P_K = P_2$$

$$(4.3) \quad W_h = \{ v_h \mid v_h \in C^0(\bar{\Omega}), v_h|_K \in P_2 \text{ for all } K \in \mathcal{T}_h \} \subset H^1(\Omega).$$

Clearly, a function  $v_h \in W_h$  is uniquely determined by its values  $v_h(a_{i,K})$ ,  $1 \leq i \leq 3$ , and  $v_h(a_{ij,K})$ ,  $1 \leq i < j \leq 3$ ,  $K \in \mathcal{T}_h$ . We let

$$(4.4) \quad W_{0,h} = \{ v_h \mid v_h \in W_h, v_h|_{\Gamma} = 0 \} = W_h \cap H_0^1(\Omega).$$

Let us prove now that *Hypothesis H.1 holds with  $k = l = 1$* . First, for any  $K \in \mathcal{T}_h$ , we define the operator  $\Pi_K \in \mathcal{L}(H^2(K); P_2)$  by

$$(4.5) \quad \begin{aligned} \Pi_K v(a_{i,K}) &= v(a_{i,K}), \quad 1 \leq i \leq 3, \\ \int_{[a_{i,K}, a_{j,K}]} \Pi_K v \, d\sigma &= \int_{[a_{i,K}, a_{j,K}]} v \, d\sigma, \quad 1 \leq i < j \leq 3. \end{aligned}$$

By the Sobolev's imbedding theorem, we have  $H^2(K) \subset C^0(K)$  so that the first condition (4.5) makes sense. On the other hand, the restriction  $p|_{[a_{i,K}, a_{j,K}]}$  to the side  $[a_{i,K}, a_{j,K}]$  of any polynomial  $p \in P_2$  depends only on  $p(a_{i,K})$ ,  $p(a_j, K)$ ,  $p(a_{ij, K})$ . Thus, since  $\int_{[a_{i,K}, a_{j,K}]} p_{ij,K} \, d\sigma$  is  $> 0$ , the last condition (4.5)

determines  $\Pi_K v(a_{ij, K})$ . Clearly,

$$(4.6) \quad \Pi_K v \Big|_{[a_i, K, a_j, K]} \text{ depends only on } v \Big|_{[a_i, K, a_j, K]}, \quad 1 \leq i < j \leq 3, K \in \mathcal{T}_h.$$

Since  $\Pi_K v = v$  for all  $v \in P_2$ , we get by applying [5, theorem 5]

$$(4.7) \quad \left| \Pi_K v - v \right|_{1, K} \leq C\sigma(K)(h(K))^m |v|_{m+1, K} \text{ for all } v \in H^{m+1}(K), m = 1, 2,$$

for some constant  $C > 0$  which is independent of  $K$ .

Now, for any  $\vec{v} \in (H^2(\Omega))^2$ , we let  $r_h \vec{v}$  be the function in  $(W_h)^2$  such that  $(r_h \vec{v})_i \Big|_K = \Pi_K v_i$  for all  $K \in \mathcal{T}_h, i = 1, 2$ . This definition makes sense because of property (4.6). Obviously, we have

$$r_h \in \mathcal{L}((H^2(\Omega))^2; (W_h)^2) \cap \mathcal{L}((H^2(\Omega) \cap H_0^1(\Omega))^2; (W_{0,h})^2).$$

Moreover, using (4.5) and Green's formula, we obtain

$$(4.8) \quad \int_K \operatorname{div} r_h \vec{v} \, dx = \int_{\partial K} r_h \vec{v} \cdot \vec{n} \, d\sigma = \int_{\partial K} \vec{v} \cdot \vec{n} \, d\sigma = \int_K \operatorname{div} \vec{v} \, dx$$

so that (3.4) (and (3.2)) holds with  $k = 1$ . On the other hand, by (4.7), we have

$$(4.9) \quad \left\| r_h \vec{v} - \vec{v} \right\|_h = \left| r_h \vec{v} - \vec{v} \right|_{1, \Omega} \leq C\sigma h^m |\vec{v}|_{m+1, \Omega} \text{ for all } \vec{v} \in (H^{m+1}(\Omega))^2, m = 1, 2,$$

so that (3.3) holds with  $k = 1$  (and also with  $k = 2$ ),  $l = 1$ .

Since we are using conforming finite elements, Hypothesis H.2 is trivially satisfied. Thus, defining

$$(4.10) \quad V_h = \{ \vec{v}_h \mid \vec{v}_h \in (W_{0,h})^2, \int_K \operatorname{div} \vec{v}_h \, dx = 0 \text{ for all } K \in \mathcal{T}_h \}$$

and applying Theorems 3 and 4, we obtain when  $\vec{u} \in \mathcal{V} \cap (H^2(\Omega))^2, p \in H^1(\Omega)$

$$(4.11) \quad \left| \vec{u}_h - \vec{u} \right|_{1, \Omega} \leq C\sigma h (|\vec{u}|_{2, \Omega} + |p|_{1, \Omega})$$

and

$$(4.12) \quad \left\| \vec{u}_h - \vec{u} \right\|_{0, \Omega} \leq C\sigma^2 h^2 (|\vec{u}|_{2, \Omega} + |p|_{1, \Omega})$$

provided (3.34) holds. The bound (4.11) is a slight improvement of a result of Fortin [8] who first discussed this type of approximation. Notice however that these error estimates are quite disappointing since we use polynomials of degree 2 in each triangle  $K \in \mathcal{T}_h$ . This comes from the low order of accuracy in approximating the divergence condition. In fact, it is impossible to construct

an operator  $r_h \in \mathcal{L}((H^2(\Omega))^2, (W_h)^2)$  such that (3.4) holds with  $k = 2$ . We shall see in § 5 how the use of nonconforming finite elements enables us to obtain the same asymptotic error estimates by using polynomials of degree 1 only in each triangle  $K \in \mathcal{T}_h$ .

EXAMPLE 2. Now, we show that we can raise by one the asymptotic order of convergence of the previous method by slightly increasing the corresponding number of degrees of freedom. Assume for the moment that  $N = 2$ . We introduce the centroid  $a_{123,K}$  of the triangle  $K$  with vertices  $a_{i,K}$ ,  $1 \leq i \leq 3$ . Let us denote by  $P_K$  the space of polynomials spanned by

$$\lambda_1^2, \lambda_2^2, \lambda_3^2, \lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_3\lambda_1, \lambda_1\lambda_2\lambda_3.$$

Then,  $P_2 \subset P_K$  and  $\Sigma_K = \{a_{i,K}\}_{1 \leq i \leq 3} \cup \{a_{ij,K}\}_{1 \leq i < j \leq 3} \cup \{a_{123,K}\}$  is a  $P_K$ -unisolvent set. Moreover, we can easily compute the basis functions :

$$(4.13) \quad \begin{aligned} p_{i,K} &= \lambda_i(2\lambda_i - 1) - 3\lambda_1\lambda_2\lambda_3, \quad 1 \leq i \leq 3, \\ p_{ij,K} &= 4\lambda_i\lambda_j - 12\lambda_1\lambda_2\lambda_3, \quad 1 \leq i < j \leq 3, \\ p_{123,K} &= 27\lambda_1\lambda_2\lambda_3. \end{aligned}$$

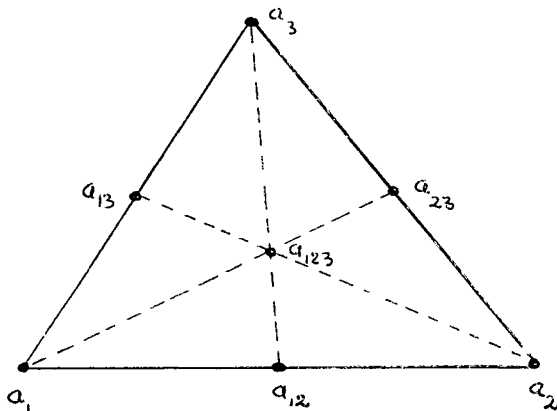


Figure 2

Let us define the space

$$(4.14) \quad W_h = \{v_h \mid v_h \in C^0(\bar{\Omega}), v_h|_K \in P_K \text{ for all } K \in \mathcal{T}_h\} \subset H^1(\Omega).$$

Here again, a function  $v_h \in W_h$  is uniquely determined by its values  $v_h(a_{i,K})$ ,  $1 \leq i \leq 3$ ,  $v_h(a_{ij,K})$ ,  $1 \leq i < j \leq 3$ , and  $v_h(a_{123,K})$ ,  $K \in \mathcal{T}_h$ . We let

$$(4.15) \quad W_{0,h} = \{v_h \mid v_h \in W_h, v_h|_{\Gamma} = 0\} = W_h \cap H_0^1(\Omega).$$

Let us show that *Hypothesis H.1 holds with  $k = l = 2$* . We begin with a preliminary result.

**Lemma 4.** *For any  $K \in \mathcal{C}_h$ , the equations*

$$(4.16) \quad \begin{aligned} & \text{(i)} \quad \Pi_K \vec{v}(a_{i,K}) = \vec{v}(a_{i,K}), \quad 1 \leq i \leq 3, \\ & \text{(ii)} \quad \int_{[a_{i,K}, a_{j,K}]} \Pi_K \vec{v} \, d\sigma = \int_{[a_{i,K}, a_{j,K}]} \vec{v} \, d\sigma, \quad 1 \leq i < j \leq 3, \\ & \text{(iii)} \quad \int_K x_i \operatorname{div} \Pi_K \vec{v} \, dx = \int_K x_i \operatorname{div} \vec{v} \, dx, \quad i = 1, 2, \end{aligned}$$

define an operator  $\Pi_K \in \mathcal{L}((H^2(K))^2; (P_K)^2)$ . Moreover,  $\Pi_K \vec{v} \Big|_{[a_{i,K}, a_{j,K}]}$  depends only on  $\vec{v} \Big|_{[a_{i,K}, a_{j,K}]}$ ,  $1 \leq i < j < 3$ , and we have the estimate :

$$(4.17) \quad \left| \Pi_K \vec{v} - \vec{v} \right|_{1,K} \leq C(\sigma(K))^2 (h(K))^m \left| \vec{v} \right|_{m+1,K} \quad \text{for all } \vec{v} \in (H^{m+1}(K))^2, \quad m = 1, 2$$

*Proof.* Let  $\vec{v}$  be in  $(H^2(K))^2$ . Since  $\lambda_1 \lambda_2 \lambda_3$  vanishes on  $\partial K$ , the restriction to any side of  $K$  of a polynomial  $p \in P_K$  coincides with a polynomial of degree  $\leq 2$ . Thus, as in example 1, the first two equations (4.16) determine

$$\Pi_K \vec{v}(a_{i,K}), \quad 1 \leq i \leq 3, \quad \Pi_K \vec{v}(a_{ij,K}), \quad 1 \leq i < j \leq 3,$$

and consequently the restriction  $\Pi_K \vec{v} \Big|_{\partial K}$  of  $\Pi_K \vec{v}$  to  $\partial K$ . Moreover,  $\Pi_K \vec{v} \Big|_{[a_{i,K}, a_{j,K}]}$  depends only on  $\vec{v} \Big|_{[a_{i,K}, a_{j,K}]}$ . By using Green's formula, equation (4.16) (iii) becomes

$$(4.18) \quad - \int_K (\Pi_K \vec{v})_i \, dx + \int_{\partial K} x_i \Pi_K \vec{v} \cdot \vec{n} \, d\sigma = - \int_K v_i \, dx + \int_{\partial K} x_i \vec{v} \cdot \vec{n} \, d\sigma.$$

Since  $\int_K p_{123,K} \, dx$  is  $> 0$ , this equation (4.18) determines  $(\Pi_K \vec{v})_i(a_{123,K})$ . Thus, equations (4.16) uniquely define a function  $\Pi_K \vec{v} \in (P_K)^2$ . Clearly, the operator  $\Pi_K$  belongs to  $\mathcal{L}((H^2(K))^2; (P_K)^2)$ .

Now, notice that conditions (4.16) (ii) imply

$$\int_K \operatorname{div} \Pi_K \vec{v} \, dx = \int_K \operatorname{div} \vec{v} \, dx.$$

Thus, we get as a consequence of equations (4.16) (ii), (iii)

$$(4.19) \quad \int_K q \operatorname{div} \Pi_K \vec{v} \, dx = \int_K q \operatorname{div} \vec{v} \, dx \quad \text{for all } q \in P_1.$$

Finally, let us derive the estimate (4.17). We shall denote by  $\hat{K}$  a nondegenerate reference triangle of  $R^2$  and we shall use the notations given in the proof of Lemma 3. Since  $\widehat{\Pi_K \vec{v}} \neq \Pi_{\hat{K}} \vec{v}$ , inequality (4.17) is not a direct consequence of [5, Theorem 5]. Thus, let us introduce the operator  $\tilde{\Pi}_K \in \mathcal{L}((H^2(K))^2, (P_K)^2)$  defined by

$$(4.20) \quad \begin{aligned} \tilde{\Pi}_K \vec{v}(a_{i,K}) &= \vec{v}(a_{i,K}), \quad 1 \leq i \leq 3, \\ \int_{[a_{i,K}, a_j, K]} \tilde{\Pi}_K \vec{v} \, d\sigma &= \int_{[a_{i,K}, a_j, K]} \vec{v} \, d\sigma, \quad 1 \leq i < j \leq 3, \\ \int_K \tilde{\Pi}_K \vec{v} \, dx &= \int_K \vec{v} \, dx. \end{aligned}$$

Observe now that

$$\tilde{\Pi}_K \vec{v} = \tilde{\Pi}_K \hat{\vec{v}} \text{ for all } \vec{v} \in (H^2(K))^2.$$

Thus, by using [5, Theorem 5], we get

$$(4.21) \quad |\tilde{\Pi}_K \vec{v} - \vec{v}|_{1,K} \leq c_1 \sigma(K) (h(K))^m |\vec{v}|_{m+1,K} \text{ for all } \vec{v} \in (H^{m+1}(K))^2, \quad m = 1, 2.$$

It remains to estimate

$$|\Pi_K \vec{v} - \tilde{\Pi}_K \vec{v}|_{1,K}, \quad \vec{v} \in (H^{m+1}(K))^2 \quad m = 1, 2.$$

Clearly, we have

$$(4.22) \quad \tilde{\Pi}_K \vec{v} \Big|_{\partial K} = \Pi_K \vec{v} \Big|_{\partial K} \text{ for all } v \in (H^2(K))^2.$$

Thus, we obtain

$$(4.23) \quad \Pi_K \vec{v} - \tilde{\Pi}_K \vec{v} = p_{123,K} (\Pi_K \vec{v} - \tilde{\Pi}_K \vec{v})(a_{123,K})$$

and therefore

$$(4.24) \quad |\Pi_K \vec{v} - \tilde{\Pi}_K \vec{v}|_{1,K} = |p_{123,K}|_{1,K} \|(\Pi_K \vec{v} - \tilde{\Pi}_K \vec{v})(a_{123,K})\|$$

where  $\|\cdot\|$  is the Euclidean vector norm in  $R^2$ . By a change of variable, we get

$$(4.25) \quad |p_{123,K}|_{1,K} \leq |\det(B)|^{1/2} \|B^{-1}\| |\hat{p}_{123,K}|_{1,\hat{K}} \leq c_2 |\det(B)|^{1/2} \|B^{-1}\|.$$

On the other hand, using (4.18), (4.20) and (4.22), we obtain

$$\int_K (\Pi_K \vec{v} - \tilde{\Pi}_K \vec{v})_i \, dx = \int_{\partial K} x_i (\tilde{\Pi}_K \vec{v} - \vec{v}) \cdot \vec{n} \, d\sigma, \quad i = 1, 2,$$

and by (4.23)

$$\left( \int_K p_{123,K} dx \right) (\Pi_K \vec{v} - \tilde{\Pi}_K \vec{v})_i(a_{123,K}) = \int_{\partial K} x_i (\tilde{\Pi}_K \vec{v} - \vec{v}) \cdot \vec{n} d\sigma.$$

Since

$$\left| \int_K p_{123,K} dx \right| = |\det(B)| \left| \int_{\hat{K}} \hat{p}_{123,K} dx \right| \geq c_3 |\det(B)|,$$

we have

$$(4.26) \quad |(\Pi_K \vec{v} - \tilde{\Pi}_K \vec{v})_i(a_{123,K})| \leq c_4 |\det(B)|^{-1} \left| \int_{\partial K} x_i (\tilde{\Pi}_K \vec{v} - \vec{v}) \cdot \vec{n} d\sigma \right|, i = 1, 2$$

Let  $K'$  be a side of the triangle  $K$ . We may write

$$\begin{aligned} \int_{K'} x_i (\tilde{\Pi}_K \vec{v} - \vec{v}) \cdot \vec{n} d\sigma &= |\det(B')| \int_{\hat{K}'} (Bx + b)_i (\tilde{\Pi}_{\hat{K}'} \vec{v} - \vec{v}) \cdot \vec{n} d\sigma \\ &= |\det(B')| \int_{\hat{K}'} (Bx)_i (\tilde{\Pi}_{\hat{K}'} \vec{v} - \vec{v}) \cdot \vec{n} d\sigma \end{aligned}$$

and therefore

$$(4.27) \quad \left| \int_{K'} x_i (\tilde{\Pi}_K \vec{v} - \vec{v}) \cdot \vec{n} d\sigma \right| \leq c_5 |\det(B')| \|B\| \|\tilde{\Pi}_{\hat{K}'} \vec{v} - \vec{v}\|_{0,\hat{K}'}$$

Since  $\tilde{\Pi}_{\hat{K}'} \vec{v} = \vec{v}$  for all  $\vec{v} \in (P_2)^2$ , we get as a consequence of the Bramble-Hilbert Lemma

$$(4.28) \quad \|\tilde{\Pi}_{\hat{K}'} \vec{v} - \vec{v}\|_{0,\hat{K}'} \leq c_6 |\vec{v}|_{m+1,\hat{K}'} \leq c_6 |\det(B)|^{-1/2} \|B\|^{m+1} |\vec{v}|_{m+1,K}.$$

Thus, using (3.17), (4.27) and (4.28), we obtain

$$\left| \int_{K'} x_i (\tilde{\Pi}_K \vec{v} - \vec{v}) \cdot \vec{n} d\sigma \right| \leq c_7 |\det(B)|^{1/2} \|B\|^{m+2} \|B^{-1}\| |\vec{v}|_{m+1,K}$$

and by (4.26)

$$(4.29) \quad \|(\Pi_K \vec{v} - \tilde{\Pi}_K \vec{v})_i(a_{123,K})\| \leq c_8 |\det(B)|^{-1/2} \|B\|^{m+2} \|B^{-1}\| |\vec{v}|_{m+1,K}.$$

From (4.24), (4.25) and (4.29), we get

$$\|\Pi_K \vec{v} - \tilde{\Pi}_K \vec{v}\|_{1,K} \leq c_9 \|B\|^{m+2} \|B^{-1}\|^2 |\vec{v}|_{m+1,K}$$

and by (3.19)

$$(4.30) \quad \left| \Pi_K \vec{v} - \tilde{\Pi}_K \vec{v} \right|_{1,K} \leq c_{10} (\sigma(K))^2 (h(K))^m \left| \vec{v} \right|_{m+1,K} \quad \text{for all } \vec{v} \in (H^{m+1}(K))^2,$$

$m = 1, 2$ . The desired estimate (4.17) follows from (4.21) and (4.30).

Now, for any  $\vec{v} \in (H^2(\Omega))^2$ , we let  $r_h \vec{v}$  be the function in  $(W_h)^2$  such that  $r_h \vec{v}|_K = \Pi_K \vec{v}$  for all  $K \in \mathcal{T}_h$ . Again, we have

$r_h \in \mathcal{L}((H^2(\Omega))^2, (W_h)^2) \cap \mathcal{L}((H^2(\Omega) \cap H_0^1(\Omega))^2; (W_{0,h})^2)$ . By (4.19), we have

$$(4.31) \quad \int_K q \operatorname{div} r_h \vec{v} \, dx = \int_K q \operatorname{div} \vec{v} \, dx \quad \text{for all } q \in P_1 \text{ and all } K \in \mathcal{T}_h,$$

so that (3.2) holds with  $k = 2$ . On the other hand, we get from (4.17)

$$(4.32)$$

$$\|r_h \vec{v} - \vec{v}\|_h = |r_h \vec{v} - \vec{v}|_{1,\Omega} \leq c \sigma^2 h^m \left| \vec{v} \right|_{m+1,\Omega} \quad \text{for all } \vec{v} \in (H^{m+1}(\Omega))^2, m = 1, 2$$

so that (3.3) holds with  $k = l = 2$ .

Defining

$$(4.33) \quad V_h = \left\{ \vec{v}_h \mid \vec{v}_h \in (W_{0,h})^2, \int_K q \operatorname{div} \vec{v}_h \, dx = 0 \text{ for all } q \in P_1 \right. \\ \left. \text{and all } K \in \mathcal{T}_h \right\}.$$

and applying Theorems 3 and 4, we obtain when  $\vec{u} \in V \cap (H^3(\Omega))^2, p \in H^2(\Omega)$

$$(4.34) \quad \left| \vec{u}_h - \vec{u} \right|_{1,\Omega} \leq c \sigma^2 h^2 \left( \left| \vec{u} \right|_{3,\Omega} + |p|_{2,\Omega} \right)$$

and

$$(4.35) \quad \left\| \vec{u}_h - \vec{u} \right\|_{0,\Omega} \leq c \sigma^4 h^3 \left( \left| \vec{u} \right|_{3,\Omega} + |p|_{2,\Omega} \right)$$

provided (3.34) holds.

The previous analysis can be now extended to *3-dimensional problems*. Here  $K$  is a tetrahedron with vertices  $a_{i,K}, 1 \leq i \leq 4$ . Let us denote by  $a_{ij,K}$  the midpoint of the edge  $[a_{i,K}, a_{j,K}], 1 \leq i < j \leq 4$ , by  $a_{ijm,K}$  the centroid of the triangle with vertices  $a_{i,K}, a_{j,K}, a_{m,K}$ , and  $a_{1234,K}$  the centroid of the tetrahedron  $K$ . Let  $P_K$  be the space of polynomials spanned by

$$\lambda_1^2, \lambda_2^2, \lambda_3^2, \lambda_4^2, \quad \lambda_1 \lambda_2, \lambda_2 \lambda_3, \lambda_3 \lambda_4, \lambda_4 \lambda_1, \lambda_1 \lambda_3, \lambda_2 \lambda_4, \\ \lambda_1 \lambda_2 \lambda_3, \lambda_2 \lambda_3 \lambda_4, \lambda_3 \lambda_4 \lambda_1, \lambda_4 \lambda_1 \lambda_2, \quad \lambda_1 \lambda_2 \lambda_3 \lambda_4.$$

Then  $P_2 \subset P_K$  and

$$\Sigma_K = \{ a_{i,K} \}_{1 \leq i \leq 4} \cup \{ a_{ij,K} \}_{1 \leq i < j \leq 4} \cup \{ a_{ijm,K} \}_{1 \leq i < j < m \leq 4} \cup \{ a_{1234,K} \}$$



is a  $P_K$ -unisolvant set of  $R^3$ . Moreover, the basis functions are :

$$\begin{aligned}
 p_{1,K} &= \lambda_1(2\lambda_1 - 1) - 3\lambda_1(\lambda_2\lambda_3 + \lambda_3\lambda_4 + \lambda_4\lambda_2) + 68\lambda_1\lambda_2\lambda_3\lambda_4, \dots \\
 (4.36) \quad p_{12,K} &= 4\lambda_1\lambda_2 - 12\lambda_1\lambda_2(\lambda_3 + \lambda_4) + 128\lambda_1\lambda_2\lambda_3\lambda_4, \dots \\
 p_{123,K} &= 27\lambda_1\lambda_2\lambda_3 - 108\lambda_1\lambda_2\lambda_3\lambda_4, \dots \\
 p_{1234,K} &= 256\lambda_1\lambda_2\lambda_3\lambda_4.
 \end{aligned}$$

Let us define the spaces  $W_h$  and  $W_{0,h}$  by (4.14) and (4.15) respectively. Then, we can similarly show that Hypothesis H.1 holds with  $k = l = 2$ .

This is easily done by introducing for any  $K \in \mathfrak{C}_h$  the operator

$$\Pi_K \in \mathcal{L}((H^2(K))^3; (P_K)^3)$$

such that

$$\begin{aligned}
 (4.37) \quad \Pi_K \vec{v}(a_{i,K}) &= \vec{v}(a_{i,K}), \quad 1 \leq i \leq 4, \\
 \Pi_K \vec{v}(a_{ij,K}) &= \vec{v}(a_{ij,K}), \quad 1 \leq i < j \leq 4, \\
 \int_{K'} \Pi_K \vec{v} \, d\sigma &= \int_{K'} \vec{v} \, d\sigma \text{ for each 2-dimensional face } K' \text{ of } K, \\
 \int_K x_i \operatorname{div} \Pi_K \vec{v} \, dx &= \int_K x_i \operatorname{div} \vec{v} \, dx, \quad i = 1, 2, 3.
 \end{aligned}$$

Thus, defining again  $V_h$  by (4.33), the estimates (4.34) and (4.35) still remain valid when  $\vec{u} \in V \cap (H^3(\Omega))^3$ ,  $p \in H^2(\Omega)$ .

**EXAMPLE 3.** We can easily extend the ideas given in Example 2 in order to construct conforming finite element methods of higher-order of accuracy. We shall consider only a 2-dimensional example corresponding to  $k = 3$ . With any triangle  $K \in \mathfrak{C}_h$ , we associate the points  $a_{i,K}$ ,  $1 \leq i \leq 12$ , whose barycentric coordinates are :

$$\begin{aligned}
 a_{1,K} &= (1, 0, 0), & a_{2,K} &= (0, 1, 0), & a_{3,K} &= (0, 0, 1), \\
 a_{4,K} &= \left(\frac{2}{3}, \frac{1}{3}, 0\right), & a_{5,K} &= \left(\frac{1}{3}, \frac{2}{3}, 0\right), & a_{6,K} &= \left(0, \frac{2}{3}, \frac{1}{3}\right), \\
 a_{7,K} &= \left(0, \frac{1}{3}, \frac{2}{3}\right), & a_{8,K} &= \left(\frac{1}{3}, 0, \frac{2}{3}\right), & a_{9,K} &= \left(\frac{2}{3}, 0, \frac{1}{3}\right), \\
 a_{10,K} &= \left(\alpha, \frac{1-\alpha}{2}, \frac{1-\alpha}{2}\right), & a_{11,K} &= \left(\frac{1-\alpha}{2}, \alpha, \frac{1-\alpha}{2}\right), \\
 & & a_{12,K} &= \left(\frac{1-\alpha}{2}, \frac{1-\alpha}{2}, \alpha\right),
 \end{aligned}$$

where  $0 < \alpha < 1, \alpha \neq \frac{1}{3}$ . Let  $P_K$  be the space of polynomials spanned by

$$\lambda_1^3, \lambda_2^3, \lambda_3^3, \lambda_1^2\lambda_2, \lambda_2^2\lambda_3, \lambda_3^2\lambda_1, \lambda_1\lambda_2^2, \lambda_2\lambda_3^2, \lambda_3\lambda_1^2, \lambda_1^2\lambda_2\lambda_3, \lambda_2^2\lambda_3\lambda_1, \lambda_3^2\lambda_1\lambda_2.$$

Then  $P_3 \subset P_K$  and  $\Sigma_K = \{ a_{i,K} \}_{1 \leq i \leq 12}$  is a  $P_K$ -unisolvent set of  $R^2$ .

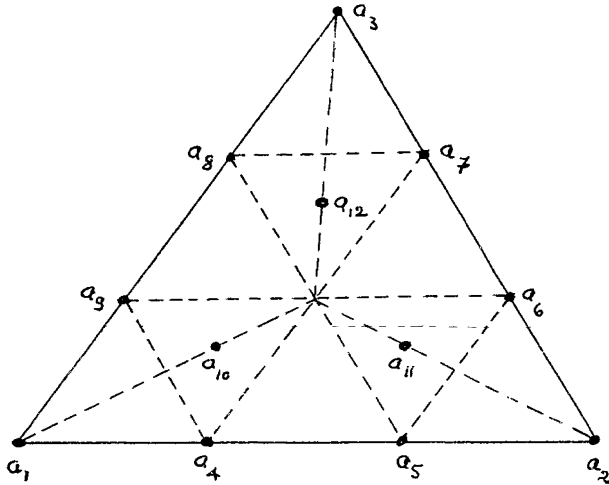


Figure 3

By an elementary calculation, one can verify that the basis functions are

$$\begin{aligned}
 p_{1,K} &= \frac{1}{2} \lambda_1(3\lambda_1 - 1)(3\lambda_1 - 2) - \frac{9}{2} \lambda_1\lambda_2\lambda_3 \\
 &\quad + \frac{-27\alpha^2 + 18\alpha + 1}{2\alpha(1 - \alpha)^2} \lambda_1^2\lambda_2\lambda_3, \dots \\
 (4.38) \quad p_{4,K} &= \frac{9}{2} \lambda_1\lambda_2(3\lambda_1 - 1) - \frac{9(3\alpha + 1)(\alpha + 1)}{4\alpha(1 - \alpha)} \lambda_1^2\lambda_2\lambda_3 \\
 &\quad - \frac{9(3\alpha^2 - 8\alpha + 1)}{4\alpha(1 - \alpha)} \lambda_2^2\lambda_3\lambda_1 \\
 &\quad + \frac{9(3\alpha + 1)}{4\alpha} \lambda_3^2\lambda_1\lambda_2, \dots \\
 p_{10,K} &= \frac{8}{\alpha(1 - \alpha)^2(3\alpha - 1)} \lambda_1\lambda_2\lambda_3 \left( \lambda_1 - \frac{1 - \alpha}{2} \right), \dots
 \end{aligned}$$

Let us define the spaces  $W_h$  and  $W_{0,h}$  by (4.14) and (4.15) respectively. Then, a function  $v_h \in W_h$  is uniquely determined by its values  $v_h(a_{i,K})$ ,  $1 \leq i \leq 12$ ,  $K \in \mathcal{T}_h$ . Moreover, one can easily show that Hypothesis H.1 holds with  $k = 3$ ,  $l = 2$ . The proof is similar to that given in Example 2 : for each  $K \in \mathcal{T}_h$ , we consider the operator  $\Pi_K \in \mathcal{L}((H^2(K))^2; (P_K)^2)$  defined by

$$\begin{aligned}
 \Pi_K \vec{v}(a_{i,K}) &= \vec{v}(a_{i,K}), \quad 1 \leq i \leq 3, \\
 \int_{[a_{i,K}, a_{j,K}]} q \Pi_K \vec{v} \, d\sigma &= \int_{[a_{i,K}, a_{j,K}]} q \vec{v} \, d\sigma \text{ for all } q \in P_1, \quad 1 \leq i < j \leq 3, \\
 \int_K q \operatorname{div} \Pi_K \vec{v} \, dx &= \int_K q \operatorname{div} \vec{v} \, dx \text{ for all } q \in P_2, \\
 \int_K [x_1(\Pi_K \vec{v})_2 - x_2(\Pi_K \vec{v})_1] \, dx &= \int_K (x_1 v_2 - x_2 v_1) \, dx.
 \end{aligned}
 \tag{4.39}$$

Thus, putting

$$\begin{aligned}
 V_h = \left\{ \vec{v}_h \mid \vec{v}_h \in (W_{0,h})^2, \int_K q \operatorname{div} \vec{v}_h \, dx = 0 \text{ for all } q \in P_2 \right. \\
 \left. \text{and all } K \in \mathcal{T}_h \right\}
 \end{aligned}
 \tag{4.40}$$

we obtain when  $\vec{u} \in V \cap (H^4(\Omega))^2$ ,  $p \in H^3(\Omega)$

$$\|\vec{u}_h - \vec{u}\|_{1,\Omega} \leq c\sigma^2 h^3 (\|\vec{u}\|_{4,\Omega} + |p|_{3,\Omega})
 \tag{4.41}$$

and

$$\|\vec{u}_h - \vec{u}\|_{0,\Omega} \leq c\sigma^4 h^4 (\|\vec{u}\|_{4,\Omega} + |p|_{3,\Omega})
 \tag{4.42}$$

if property (3.4) holds.

### 5. APPLICATIONS II : NONCONFORMING FINITE ELEMENTS

We now come to nonconforming finite element methods for solving the stationary Stokes equations. We shall give two examples which correspond to the cases  $k = 1$  and  $k = 3$ .

**EXAMPLE 4.** For  $N = 2, 3$ , let  $K \in \mathcal{T}_h$  be a  $N$ -simplex with vertices  $a_{i,K}$ ,  $1 \leq i \leq N + 1$ . Denote by  $K'_i$  the  $(N - 1)$ -dimensional face of  $K$  which is opposite to  $a_{i,K}$  and by  $b_{i,K}$  the centroid of  $K'_i$ ,  $1 \leq i \leq N + 1$ . Then,  $\Sigma_K = \{ b_{i,K} \}_{1 \leq i \leq N+1}$  is a  $P_1$ -unisolvent set and the basis function  $p_{i,K}$  asso-

ciated with the point  $b_{i,K}$  is given by

$$(5.1) \quad p_{i,K} = 1 - N\lambda_i, \quad 1 \leq i \leq N + 1.$$

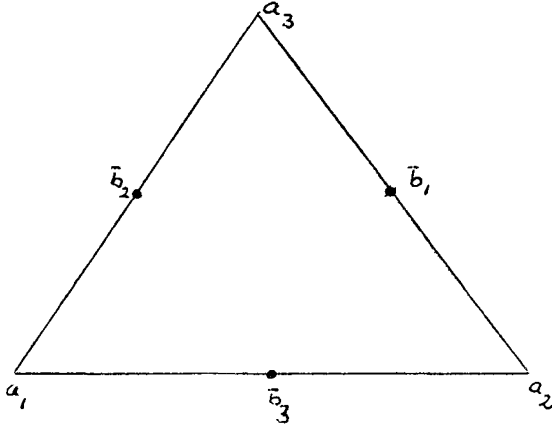


Figure 4

Let us define the space  $W_h$  as follows : a function  $v_h$  defined on  $\Omega$  belongs to  $W_h$  if and only if

$$(5.2) \quad v_h|_K \in P_1 \text{ for any } K \in \mathcal{T}_h,$$

$$(5.3) \quad v_h \text{ is continuous at the points } b_{i,K} \in \Omega, \quad 1 \leq i \leq N + 1, \quad K \in \mathcal{T}_h.$$

Thus, a function  $v_h \in W_h$  is uniquely determined by its values  $v_h(b_{i,K})$ ,  $1 \leq i \leq N + 1$ ,  $K \in \mathcal{T}_h$ . We let

$$(5.4) \quad W_{0,h} = \{ v_h | v_h \in W_h, v_h(b_{i,K}) = 0 \text{ for all } b_{i,K} \in \Gamma \}.$$

Observe that  $W_h \not\subset H^1(\Omega)$  and  $W_{0,h} \not\subset H_0^1(\Omega)$ .

Let us show that *Hypothesis H.1 holds with  $k = l = 1$* . For any  $K \in \mathcal{T}_h$ , we define the operator  $\Pi_K \in \mathcal{L}(H^1(K); P_1)$  by

$$(5.5) \quad \int_{K'_i} \Pi_K v \, d\sigma = \int_{K'_i} v \, d\sigma, \quad 1 \leq i \leq N + 1.$$

Since

$$\int_{K'_i} p_{j,K} \, d\sigma = \left( \int_{K'_i} d\sigma \right) \delta_{ij},$$

the equation (5.5) determines  $\Pi_K v(b_{i,K})$  :

$$(5.6) \quad \Pi_K v(b_{i,K}) = \frac{\int_{K'_i} v \, d\sigma}{\int_{K'_i} d\sigma}, \quad 1 \leq i \leq N+1.$$

Since  $\Pi_K v = v$  for all  $v \in P_1$ , we get by applying [5, Theorem 5]

$$(5.7) \quad |\Pi_K v - v|_{1,K} \leq c\sigma(K)(h(K))^m |v|_{m+1,K} \quad \text{for all } v \in H^{m+1}(K), m = 0, 1,$$

for some constant  $c > 0$  which is independent of  $K$ .

Now, for any  $\vec{v} \in (H^1(\Omega))^N$ , we let  $r_h \vec{v}$  be the function in  $(W_h)^N$  such that  $(r_h v)_i|_K = \Pi_K v_i$  for all  $K \in \mathfrak{T}_h$ ,  $1 \leq i \leq N$ . Then

$$r_h \in \mathfrak{L}((H^1(\Omega))^N; (W_h)^N) \cap \mathfrak{L}((H_0^1(\Omega))^N; (W_{0,h})^N)$$

and, by (5.5), we have

$$(5.8) \quad \begin{aligned} \int_K \operatorname{div} r_h \vec{v} \, dx &= \int_{\partial K} r_h \vec{v} \cdot \vec{n} \, d\sigma = \sum_{i=1}^{N+1} \int_{K'_i} r_h \vec{v} \cdot \vec{n} \, d\sigma \\ &= \sum_{i=1}^{N+1} \int_{K'_i} \vec{v} \cdot \vec{n} \, d\sigma = \int_K \operatorname{div} \vec{v} \, dx \end{aligned}$$

so that (3.4) holds with  $k = 1$ . Using (5.7), we obtain

$$(5.9) \quad \|r_h v - v\|_h \leq c\sigma h^m |\vec{v}|_{m+1,\Omega} \quad \text{for all } \vec{v} \in (H^{m+1}(\Omega))^N, m = 0, 1,$$

so that (3.3) holds with  $k = l = 1$ .

On the other hand, let us show that *Hypothesis H.2 holds with  $k = 1$* . For any  $(N-1)$ -dimensional face  $K'_i$  of a  $N$ -simplex  $K \in \mathfrak{T}_h$  and any function  $v_h \in W_h$ , we have

$$v_h(b_{i,K}) = 0 \Leftrightarrow \int_{K'_i} v_h \, d\sigma = 0$$

so that (3.7) and (3.8) hold with  $k = 1$ .

Thus, defining

$$(5.10) \quad W_h = \left\{ \vec{v}_h \mid \vec{v}_h \in (W_{0,h})^N, \int_K \operatorname{div} \vec{v}_h \, dx = 0 \text{ for all } K \in \mathfrak{T}_h \right\}$$

as in Example 1 and applying Theorems 3 and 4, we obtain when

$$(5.11) \quad \begin{aligned} \vec{u} &\in V \cap (H^2(\Omega))^N, p \in H^1(\Omega) \\ \|\vec{u}_h - \vec{u}\|_h &\leq C\sigma h \left( |\vec{u}|_{2,\Omega} + |p|_{1,\Omega} \right) \end{aligned}$$

and

$$(5.12) \quad \|\vec{u}_h - \vec{u}\|_{0,\Omega} \leq C\sigma^2 h^2 (|\vec{u}|_{2,\Omega} + |p|_{1,\Omega})$$

provided (3.14) holds.

In conclusion, the nonconforming finite element method discussed here appears to be more attractive than the conforming method of Example 1 which involves more degrees of freedom without improving the asymptotic order of accuracy.

EXAMPLE 5. Here again, it is possible to construct nonconforming finite element methods of higher-order of accuracy. For the sake of simplicity, we shall confine ourselves to a specific 2-dimensional example corresponding to  $k = 3$ . With any triangle  $K \in \mathcal{T}_h$ , we associate the points  $b_{i,K}$ ,  $1 \leq i \leq 12$ , whose barycentric coordinates with respect to the vertices  $a_{i,K}$ ,  $1 \leq i \leq 3$ , of  $K$  are given by

$$\begin{aligned} b_{1,K} &= (g_1, g_2, 0), & b_{2,K} &= \left(\frac{1}{2}, \frac{1}{2}, 0\right), & b_{3,K} &= (g_2, g_1, 0), \\ b_{4,K} &= (0, g_1, g_2), & b_{5,K} &= \left(0, \frac{1}{2}, \frac{1}{2}\right), & b_{6,K} &= (0, g_2, g_1), \\ b_{7,K} &= (g_2, 0, g_1), & b_{8,K} &= \left(\frac{1}{2}, 0, \frac{1}{2}\right), & b_{9,K} &= (g_1, 0, g_2), \\ b_{10,K} &= \left(\alpha, \frac{1-\alpha}{2}, \frac{1-\alpha}{2}\right), & b_{11,K} &= \left(\frac{1-\alpha}{2}, \alpha, \frac{1-\alpha}{2}\right), \\ & & b_{12,K} &= \left(\frac{1-\alpha}{2}, \frac{1-\alpha}{2}, \alpha\right) \end{aligned}$$

where  $0 < \alpha < 1$ ,  $\alpha \neq \frac{1}{3}$  and

$$g_1 = \frac{1}{2} \left(1 + \sqrt{\frac{3}{5}}\right), \quad g_2 = \frac{1}{2} \left(1 - \sqrt{\frac{3}{5}}\right).$$

Notice that  $g_2, \frac{1}{2}, g_1$  are Gaussian quadrature points on  $(0,1)$ .

As in Example 3, let us denote by  $P_K$  the space of polynomials spanned by

$$\lambda_1^3, \lambda_2^3, \lambda_3^3, \lambda_1^2\lambda_2, \lambda_2^2\lambda_3, \lambda_3^2\lambda_1, \lambda_1\lambda_2^2, \lambda_2\lambda_3^2, \lambda_3\lambda_1^2, \lambda_1^2\lambda_2\lambda_3, \lambda_2^2\lambda_3\lambda_1, \lambda_3^2\lambda_1\lambda_2.$$

By an easy but tedious calculation, one can verify that  $\Sigma_K = \{b_{i,K}\}_{1 \leq i \leq 12}$  is a  $P_K$ -unisolvent set of  $R^2$ . Moreover, one can compute explicitly the corresponding basis functions.

Let  $W_h$  be the space of functions  $v_h$  defined on  $\Omega$  and such that

$$(5.13) \quad v_h|_K \in P_K \text{ for any } K \in \mathcal{T}_h,$$

$$(5.14) \quad v_h \text{ is continuous at the points } b_{i,K} \in \Omega, 1 \leq i \leq 9, K \in \mathcal{T}_h.$$

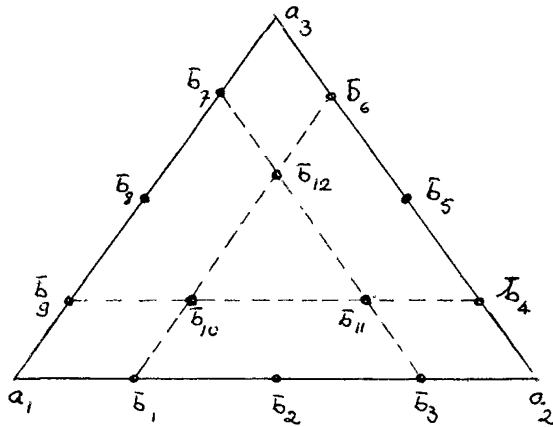


Figure 5 ( $\alpha = 1 - 2g_2 = \frac{1}{3}$ )

Then clearly, a function  $v_h \in W_h$  is uniquely determined by its values  $v_h(b_{i,K}), 1 \leq i \leq 12, K \in \mathcal{T}_h$ . We let

$$(5.15) \quad W_{0,h} = \{ v_h | v_h \in W_h, v_h(b_{i,K}) = 0 \text{ for all } b_{i,K} \in \Gamma \}$$

Let us prove first that *Hypothesis H.1 holds with  $k = 3, l = 1$ .*

**Lemma 5.** *For any  $K \in \mathcal{T}_h$ , the equations*

$$(5.16) \quad \begin{aligned} (i) \quad & \int_{[a_i, K, a_j, K]} q \Pi_K v \, d\sigma = \int_{[a_i, K, a_j, K]} q v \, d\sigma \text{ for all } q \in P_2, 1 \leq i < j \leq 3, \\ (ii) \quad & \int_K q \Pi_K v \, d\sigma = \int_K q v \, d\sigma \text{ for all } q \in P_1, \end{aligned}$$

define an operator  $\Pi_K \in \mathcal{L}(H^1(K), P_K)$ . Moreover  $\Pi_K v|_{[a_i, K, a_j, K]}$  depends only on  $v|_{[a_i, K, a_j, K]}$ ,  $1 \leq i < j \leq 3$ , and we have the estimate

$$(5.17) \quad |\Pi_K v - v|_{1,K} \leq C\sigma(K)(h(K))^m |v|_{m+1,K} \text{ for all } v \in H^{m+1}(K), 0 \leq m \leq 3.$$

*Proof.* We only sketch the proof. First, we remark that the restriction to each side of  $K$  of any polynomial of  $P_K$  is a polynomial of degree  $\leq 3$ . Thus, if  $p \in P_K$  vanishes at points  $b_{i,K}$ ,  $1 \leq i \leq 3$  for instance, we get

$$\int_{[a_{1,K}, a_{2,K}]} qp \, d\sigma = 0 \text{ for all } q \in P_2$$

since the points  $b_{i,K}$ ,  $1 \leq i \leq 3$ , are Gaussian quadrature points. Then, for fixed  $q \in P_2$ , the integral  $\int_{[a_{1,K}, a_{2,K}]} qp \, d\sigma$  depends on  $p(b_{i,K})$ ,  $1 \leq i \leq 3$ ,  $p \in P_K$ . Now, one can easily verify that

$$\left. \begin{array}{l} p \in P_K \\ \int_{[a_{1,K}, a_{2,K}]} qp \, d\sigma = 0 \text{ for all } q \in P_2 \end{array} \right\} \Rightarrow p(b_{i,K}) = 0, 1 \leq i \leq 3.$$

Thus, equations (5.16) (i) uniquely determine  $\Pi_{K^0} v(b_{i,K})$ ,  $1 \leq i \leq 9$ . Finally, equations (5.16) (ii) determine  $\Pi_{K^0} v(b_{i,K})$ ,  $10 \leq i \leq 12$ .

Since  $\Pi_{K^0} v = v$  for any  $p \in P_3$ , the estimate (5.17) is obtained by applying again [5, Theorem 5].

For any  $\vec{v} \in (H^1(\Omega))^2$ , we let  $r_h \vec{v}$  be the function in  $(W_h)^2$  such that  $(r_h \vec{v})_i|_K = \Pi_K v_i$  for all  $K \in \mathcal{T}_h$ ,  $i = 1, 2$ . Then

$$r_h \in \mathcal{L}((H^1(\Omega))^2; (W_h)^2) \cap \mathcal{L}((H_0^1(\Omega))^2; (W_{0,h})^2)$$

and, by (5.16), we get for all  $q \in P_2$  and all  $K \in \mathcal{T}_h$

$$\begin{aligned} (5.18) \quad \int_K q \operatorname{div} r_h \vec{v} \, dx &= - \int_K \overrightarrow{\operatorname{grad}} q \cdot r_h \vec{v} \, dx + \int_K q r_h \vec{v} \cdot \vec{n} \, d\sigma \\ &= - \int_K \overrightarrow{\operatorname{grad}} q \cdot \vec{v} \, dx + \int_{\partial K} q \vec{v} \cdot \vec{n} \, d\sigma = \int_K q \operatorname{div} \vec{v} \, dx. \end{aligned}$$

Using (5.17), we obtain

$$(5.19) \quad \|r_h \vec{v} - \vec{v}\|_h \leq C\sigma h^m |\vec{v}|_{m+1, \Omega} \text{ for all } \vec{v} \in (H^{m+1}(\Omega))^2, 0 \leq m \leq 3.$$

Now, *Hypothesis H.2 holds with  $k = 3$*  since for any  $K \in \mathcal{T}_h$  and any function  $v_h \in W_h$  we have :

$$v_h(b_{i,K}) = 0, 1 \leq i \leq 3 \Leftrightarrow \int_{[a_{1,K}, a_{2,K}]} qv_h \, d\sigma = 0 \text{ for all } q \in P_2.$$

Thus, defining

$$(5.20) \quad V_h = \left\{ \vec{v}_h \mid \vec{v}_h \in (W_{0,h})^2, \int_K q \operatorname{div} \vec{v}_h \, dx = 0 \text{ for all } q \in P_2 \text{ and all } K \in \mathcal{T}_h \right\}$$



(as in Example 3) and applying Theorems 3 and 4, we obtain when

$$(5.21) \quad \begin{aligned} & \vec{u} \in V \cap (H^4(\Omega))^2, p \in H^3(\Omega) \\ & \|\vec{u}_h - \vec{u}\|_h \leq C\sigma h^3 (|\vec{u}|_{4,\Omega} + |p|_{3,\Omega}) \end{aligned}$$

and

$$(5.22) \quad \|\vec{u}_h - \vec{u}\|_{0,\Omega} \leq C\sigma^2 h^4 (|\vec{u}|_{4,\Omega} + |p|_{3,\Omega})$$

provided (3.14) holds.

## 6. ERROR ESTIMATES FOR THE PRESSURE

Let us consider again the general finite element approximation of the Stokes equations as it has been described and studied in §§ 2 and 3. A discrete analogue of problem (2.13) is the following : *Find functions  $\vec{u}_h \in V_h$  and  $p_h \in \Phi_h/R$  such that*

$$(6.1) \quad \forall_h(\vec{u}_h, \vec{v}_h) - (p_h, \operatorname{div}_h \vec{v}_h) = (f, \vec{v}_h) \text{ for all } \vec{v}_h \in (W_{0,h})^N.$$

In order to prove that problems (2.23) and (6.1) are equivalent and to estimate the error  $p_h - p$ , we need the following assumption.

**Hypothesis H.3.** *With any function  $\varphi_h \in \Phi_h$  such that  $\int_{\Omega} \varphi_h \, dx = 0$ , we can associate a function  $\vec{v}_h \in (W_{0,h})^N$  with the following properties :*

$$(i) \quad (6.2) \quad \operatorname{div}_h \vec{v}_h = \varphi_h;$$

(ii) *for some integer  $\lambda \geq 0$*

$$(6.3) \quad \|\vec{v}_h\|_h \leq C\sigma^\lambda \|\varphi_h\|_{0,\Omega}$$

where the constant  $C$  is independent of  $h$  and  $\sigma$ .

**REMARK 5.** Denote by  $V^\perp$  the orthogonal complement of  $V$  in  $(H_0^1(\Omega))^N$ , i.e. the space of vector functions  $\vec{v} \in (H^1(\Omega))^N$  such that

$$a(\vec{v}, \vec{w}) = 0 \text{ for all } \vec{w} \in V.$$

Then, Hypothesis H.3. appears to be a discrete analogue of

**Lemma 6.** *Given any function  $\varphi \in L^2(\Omega)$  such that  $\int_{\Omega} \varphi \, dx = 0$ , there exists a unique function  $\vec{v} \in V^{\perp}$  such that*

$$(6.4) \quad \operatorname{div} \vec{v} = \varphi \text{ in } \Omega.$$

*Moreover, there exists a constant  $C > 0$  such that*

$$(6.5) \quad \|\vec{v}\|_{1,\Omega} \leq C \|\varphi\|_{0,\Omega}.$$

*Proof.* We sketch the proof. First, we show that a function  $\vec{v} \in (H_0^1(\Omega))^N$  is in  $V^{\perp}$  if and only if there exists a function  $\varphi \in L^2(\Omega)$  such that

$$(6.6) \quad \Delta \vec{v} = \overrightarrow{\operatorname{grad}} \varphi \text{ in } \Omega.$$

Since  $a(\vec{v}, \vec{w}) = -(\Delta \vec{v}, \vec{w})$  for all  $\vec{w} \in (H_0^1(\Omega))^N$ , a function  $\vec{v} \in (H_0^1(\Omega))^N$  is in  $V^{\perp}$  if and only if  $(\Delta \vec{v}, \vec{w}) = 0$  for all  $\vec{w} \in V$ . By de Rham's duality theorem (cf. [14]), this exactly means that there exists a distribution  $\varphi$  in  $\Omega$  ( $\varphi \in \mathcal{D}'(\Omega)$ ) such that (6.6) holds. Moreover, one can prove (cf. [2]) that

$$\left. \begin{array}{l} \varphi \in \mathcal{D}'(\Omega) \\ \overrightarrow{\operatorname{grad}} \varphi \in (H^{-1}(\Omega))^N \end{array} \right\} \Rightarrow \varphi \in L^2(\Omega).$$

Thus,  $\overrightarrow{\operatorname{grad}}$  is a one to one operator on  $L^2(\Omega)/R$  onto  $\Delta(V^{\perp})$ . By Banach's theorem,  $\operatorname{grad}$  is an isomorphism on  $L^2(\Omega)/R$  onto  $\Delta(V^{\perp})$ . But it is an easy matter to verify that the dual space  $\Delta(V^{\perp})'$  of  $\Delta(V^{\perp})$  can be identified with  $V^{\perp}$ . So, the adjoint operator-div of the operator  $\operatorname{grad}$  is an isomorphism on  $V^{\perp}$  onto  $(L^2(\Omega)/R)'$  and the lemma is proved.

Denote by  $V_h^{\perp}$  the space of vector functions  $\vec{v}_h \in (W_{0,h})^N$  such that

$$a_h(\vec{v}_h, \vec{w}_h) = 0 \text{ for all } \vec{w}_h \in V_h.$$

Clearly, we might assume that in (6.2) and (6.3)  $\vec{v}_h \in V_h^{\perp}$  but this is unnecessary.

**Lemma 7.** *Assume that Hypothesis H.2 and H.3 (i) hold. Then, a linear functional  $\vec{v}_h \rightarrow L(\vec{v}_h)$  defined on  $(W_{0,h})^N$  vanishes on  $V_h$  if and only if there exists a unique function  $\varphi_h \in \Phi_h/R$  such that*

$$(6.7) \quad L(\vec{v}_h) = (\varphi_h, \operatorname{div}_h \vec{v}_h) \text{ for all } \vec{v}_h \in (W_{0,h})^N.$$

*Proof.* The «if» part is obvious. Let us prove the «only if» part. By definition of  $V_h$ , the space of linear functionals defined on  $(W_{0,h})^N$  and vanishing on  $V_h$  is spanned by the linear functionals

$$\vec{v}_h \rightarrow \int_K q \operatorname{div} \vec{v}_h \, dx, \quad q \in P_{k-1}, \quad K \in \mathcal{T}_h.$$

Thus, given a linear functional  $\vec{v}_h \rightarrow L(\vec{v}_h)$  on  $(W_{0,h})^N$  which vanishes on  $V_h$ , there exists a function  $\varphi_h \in \Phi_h$  such that (6.7) holds. Let us show that this function  $\varphi_h$  is uniquely determined up to an additive constant. On the one hand, using Hypothesis H.2, we obtain for all  $\vec{v}_h \in (W_{0,h})^N$

$$(1, \operatorname{div}_h \vec{v}_h) = \sum_{K \in \mathcal{T}_h} \int_K \operatorname{div} \vec{v}_h \, dx = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \vec{v}_h \cdot \vec{n} \, d\sigma = 0.$$

On the other hand, let  $\varphi_h \in \Phi_h$  be such that

$$\int_{\Omega} \varphi_h \, dx = 0,$$

$$(\varphi_h, \operatorname{div}_h \vec{v}_h) = 0 \text{ for all } \vec{v}_h \in (W_{0,h})^N.$$

By Hypothesis H.3 (i), we may choose  $\vec{v}_h \in (W_{0,h})^N$  such that

$$\operatorname{div}_h \vec{v}_h = \varphi_h.$$

Then, we obtain  $\varphi_h = 0$ .

**Theorem 5.** *There exists a unique pair of functions  $(\vec{u}_h, p_h) \in V_h \times \Phi_h/R$  solution of problem (6.1). Moreover, the function  $\vec{u}_h \in V_h$  can be characterized as the unique solution of problem (2.23).*

*Proof.* Let  $(\vec{u}_h, p_h) \in V_h \times \Phi_h/R$  be a solution of problem (6.1). Then, clearly  $\vec{u}_h$  is the solution of problem (2.23). Conversely let  $\vec{u}_h \in V_h$  be the solution of (2.23). Then, the linear functional defined on  $(W_{0,h})^N$

$$\vec{v}_h \rightarrow \nu a_h(\vec{v}_h, \vec{u}_h) - (f, \vec{v}_h)$$

vanishes on  $V_h$ . By the previous lemma, there exists a unique function  $p_h \in \Phi_h/R$  such that

$$\nu a_h(\vec{u}_h, \vec{v}_h) - (f, \vec{v}_h) = (p_h, \operatorname{div}_h \vec{v}_h) \text{ for all } \vec{v}_h \in (W_{0,h})^N.$$

Therefore  $(\vec{u}_h, p_h)$  is the solution of (6.1).

We now estimate the error  $\|p_h - p\|_{L^2(\Omega)/R}$

**Theorem 6.** *Assume that Hypotheses H.1, H.2 and H.3 hold. Assume, in addition, that the solution  $(\vec{u}, p)$  of problem (2.13) satisfies the smoothness properties (3.20). Then, we get the estimate*

$$(6.8) \quad \|p_h - p\|_{L^2(\Omega)/R} \leq C \sigma^{l+\lambda} h^k (|\vec{u}|_{k+1, \Omega} + |p|_{k, \Omega}).$$

*Proof.* We may assume that

$$\int_{\Omega} p_h \, dx = \int_{\Omega} p \, dx = 0.$$

Let  $\rho_h p$  be the orthogonal projection in  $L^2(\Omega)$  of  $p$  upon  $\Phi_h$ . Then we have

$$\int_{\Omega} \rho_h p \, dx = \int_{\Omega} p \, dx = 0$$

and

$$\|p - \rho_h p\|_{0,K} = \min_{q \in P_{k-1}} \|p - q\|_{0,K} \leq c_1 h^k |p|_{k,K} \quad \text{for all } K \in \mathcal{T}_h$$

so that

$$(6.9) \quad \|p - \rho_h p\|_{0,\Omega} \leq c_1 h^k |p|_{k,\Omega}.$$

Let  $\vec{v}_h$  be in  $(W_{0,h})^N$ . Applying (6.1) and Green's formula, we obtain

$$\begin{aligned} (p_h, \operatorname{div}_h \vec{v}_h) &= \nu a_h(\vec{u}_h, \vec{v}_h) - (f, \vec{v}_h) \\ &= (p, \operatorname{div}_h \vec{v}_h) + \nu a_h(\vec{u}_h - \vec{u}, \vec{v}_h) \\ &\quad + \nu \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial \vec{u}}{\partial n} \cdot \vec{v}_h \, d\sigma - \sum_{K \in \mathcal{T}_h} \int_{\partial K} p \vec{v}_h \cdot \vec{n} \, d\sigma. \end{aligned}$$

Thus, we may write for all  $\vec{v}_h \in (W_{0,h})^N$

$$\begin{aligned} (p_h - \rho_h p, \operatorname{div}_h \vec{v}_h) &= (p - \rho_h p, \operatorname{div}_h \vec{v}_h) + \nu a_h(\vec{u}_h - \vec{u}, \vec{v}_h) \\ &\quad + \nu \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial \vec{u}}{\partial n} \cdot \vec{v}_h \, d\sigma - \sum_{K \in \mathcal{T}_h} \int_{\partial K} p \vec{v}_h \cdot \vec{n} \, d\sigma. \end{aligned}$$

Now  $\int_{\Omega} (p_h - \rho_h p) \, dx = 0$ . Hence, by Hypothesis H.3, we may choose  $\vec{v}_h \in (W_{0,h})^N$  such that

$$\begin{aligned} \operatorname{div}_h \vec{v}_h &= p_h - \rho_h p, \\ \|\vec{v}_h\|_h &\leq c_2 \sigma^\lambda \|p_h - \rho_h p\|_{0,\Omega}. \end{aligned}$$

Then, using Theorem 3 and the estimates (3.21), (3.24) and (3.26), we obtain

$$(6.10) \quad \|p_h - \rho_h p\|_{0,\Omega} \leq \|p - \rho_h p\|_{0,\Omega} + c_3 \sigma^{l+\lambda} h^k (|\vec{u}|_{k+1,\Omega} + |p|_{k,\Omega}).$$

The desired inequality follows from (6.9) and (6.10).

REMARK 6. More generally, we can easily prove the following result. Assume that  $(\vec{u}, p)$  satisfies the smoothness properties (2.31) for some integer  $m$  with  $1 \leq m \leq k$ . Then, we get the estimate

$$(6.11) \quad \|p_h - p\|_{L^2(\Omega)/R} \leq C \sigma^{l+\lambda} h^m (|\vec{u}|_{m+1,\Omega} + |p|_{m,\Omega}).$$

In order to apply the previous theorem to the examples considered in §§ 4 and 5, it remains to verify that Hypothesis H.3 holds. This is easily done for the nonconforming examples of § 5. On the other hand, for the conforming examples of § 4, the derivation of the inequality (6.3) appears to be rather technical. Thus, we begin with the

(i) *Nonconforming Case.* In each Example 4 or 5, we have built an operator  $r_h \in \mathcal{L}((H_0^1(\Omega))^N; (W_{0,h})^N)$  which satisfies for all  $\vec{v} \in (H_0^1(\Omega))^N$

$$\begin{aligned} \operatorname{div}_h r_h \vec{v} &= \operatorname{div}_h \vec{v}, \\ \|r_h \vec{v} - \vec{v}\|_h &\leq c_1 \sigma |\vec{v}|_{1,\Omega}. \end{aligned}$$

Then, given a function  $\varphi_h \in \Phi_h$  with  $\int_{\Omega} \varphi_h \, dx = 0$ , there exists by Lemma 6 a unique function  $\vec{v} \in \mathcal{V}^\perp$  such that

$$\begin{aligned} \operatorname{div}_h \vec{v} &= \operatorname{div} \vec{v} = \varphi_h, \\ |\vec{v}|_{1,\Omega} &\leq c_2 \|\varphi_h\|_{0,\Omega}. \end{aligned}$$

Therefore, the function  $\vec{v}_h = r_h \vec{v} \in (W_{0,h})^N$  satisfies

$$\begin{aligned} \operatorname{div}_h \vec{v}_h &= \varphi_h, \\ \|\vec{v}_h\|_h &\leq c_3 \sigma \|\varphi_h\|_{0,\Omega} \end{aligned}$$

so that Hypothesis H.3 holds with  $\lambda = 1$ . We may now conclude that the following estimates hold :

$$(6.12) \quad \|p_h - p\|_{L^2(\Omega)/R} \leq C \sigma^2 h (|\vec{u}|_{2,\Omega} + |p|_{1,\Omega}) \text{ in Example 4}$$

and

$$(6.13) \quad \|p_h - p\|_{L^2(\Omega)/R} \leq C \sigma^2 h^3 (|\vec{u}|_{4,\Omega} + |p|_{3,\Omega}) \text{ in Example 5.}$$

We next consider the

(ii) *Conforming case.* The previous analysis cannot be used here since in each conforming example of § 4 the operator  $r_h$  is defined on  $(H^2(\Omega) \cap H_0^1(\Omega))^N$  only. For the sake of brevity, we shall confine ourselves to a specific example, namely Example 2 with  $N = 2$ , but the corresponding analysis can be easily extended to the other conforming examples. Let  $\varphi_h \in \Phi_h$  satisfy  $\int_{\Omega} \varphi_h \, dx = 0$ .

We construct a function  $\vec{v}_h \in (W_{0,h})^2$  in the following way. By Lemma 6, there exists a unique function  $\vec{v} \in V^1$  such that

$$(6.14) \quad \operatorname{div}_h \vec{v} = \varphi_h,$$

$$(6.15) \quad |\vec{v}|_{1,\Omega} \leq c_1 \|\varphi_h\|_{0,\Omega}.$$

Let  $\vec{w}_h$  be the orthogonal projection in  $(H_0^1(\Omega))^2$  of  $\vec{v}$  upon  $(W_{0,h})^2$ , i.e.

$$(6.16) \quad a(\vec{w}_h - \vec{v}, \vec{z}_h) = 0 \text{ for all } \vec{z}_h \in (W_{0,h})^2.$$

Then, we define the function  $\vec{v}_h$  by

$$(6.17) \quad \begin{aligned} \text{(i)} \quad & \vec{v}_h(a_{i,K}) = \vec{w}_h(a_{i,K}), \quad 1 \leq i \leq 3, \\ \text{(ii)} \quad & \int_{[a_i,K,a_j,x]} \vec{v}_h \, d\sigma = \int_{[a_i,K,a_j,x]} \vec{v} \, d\sigma, \quad 1 \leq i < j \leq 3, \\ \text{(iii)} \quad & \int_K x_i \operatorname{div} \vec{v}_h \, dx = \int_K x_i \operatorname{div} \vec{v} \, dx, \quad i = 1, 2, \end{aligned}$$

for any triangle  $K \in \mathfrak{T}_h$ .

Clearly, using (6.17) (ii) and (iii), we obtain

$$\int_K q \operatorname{div} (\vec{v}_h - \vec{v}) \, dx = 0 \text{ for all } q \in P_1 \text{ and all } K \in \mathfrak{T}_h,$$

i.e.

$$\operatorname{div}_h \vec{v}_h = \operatorname{div}_h \vec{v}.$$

Thus, by (6.14) we get

$$(6.18) \quad \operatorname{div}_h \vec{v}_h = \varphi_h.$$

The next step consists in proving the

**Lemma 8.** *Assume that the following hypotheses hold :*

$$(6.19) \quad h \leq Ch(K) \text{ for all } K \in \mathfrak{T}_h;$$

$$(6.20) \quad -\Delta_{\underline{j}} \text{ is an isomorphism from } H^2(\Omega) \cap H_0^1(\Omega) \text{ onto } L^2(\Omega)$$

*Then, we have the estimate*

$$(6.21) \quad |\vec{v}_h|_{1,\Omega} \leq C\sigma^2 \|\varphi_h\|_{0,\Omega}.$$

*Proof.* Letting

$$(6.22) \quad \vec{z} = \vec{v} - \vec{w}_h, \quad \vec{z}_h = \vec{v}_h - \vec{w}_h,$$

we get from (6.17)

$$\begin{aligned}
 (i) \quad & \vec{z}_h(a_{i,K}) = 0, \quad 1 \leq i \leq 3, \\
 (6.23) \quad (ii) \quad & \int_{[a_i, K, a_j, K]} \vec{z}_h \, d\sigma = \int_{[a_i, K, a_j, K]} \vec{z} \, d\sigma, \quad 1 \leq i < j \leq 3, \\
 (iii) \quad & \int_K x_i \operatorname{div} \vec{z}_h \, dx = \int_K x_i \operatorname{div} \vec{z} \, dx, \quad i = 1, 2,
 \end{aligned}$$

for all  $K \in \mathcal{T}_h$ . In order to estimate  $|\vec{z}_h|_{1,\Omega}$ , we write

$$(6.24) \quad \vec{z}_h = \sum_{K \in \mathcal{T}_h} \left\{ \sum_{1 \leq i < j \leq 3} p_{ij,K} \vec{z}_h(a_{ij,K}) + p_{123,K} \vec{z}_h(a_{123,K}) \right\}.$$

Let  $K$  be a triangle of  $\mathcal{T}_h$ . We have to estimate

$$\|\vec{z}_h(a_{ij,K})\|, \quad 1 \leq i < j \leq 3, \quad \|\vec{z}_h(a_{123,K})\|.$$

Denote by  $\hat{K}$  a nondegenerate reference triangle of  $R^2$  with vertices  $\hat{a}_i, 1 \leq i \leq 3$ , and use the notations given in the proof of Lemma 4. According to (6.23) (i) and (ii), we may write

$$\begin{aligned}
 \int_{[a_i, K, a_j, K]} \vec{z}_h \, d\sigma &= \left( \int_{[a_i, K, a_j, K]} p_{ij,K} \, d\sigma \right) \vec{z}_h(a_{ij,K}) \\
 &= \int_{[a_i, K, a_j, K]} \vec{z} \, d\sigma.
 \end{aligned}$$

Then, by a change of variable, we get

$$\vec{z}_h(a_{ij,K}) = \left( \int_{[\hat{a}_i, \hat{a}_j]} \hat{p}_{ij,K} \, d\sigma \right)^{-1} \int_{[\hat{a}_i, \hat{a}_j]} \vec{z} \, d\sigma$$

so that

$$(6.25) \quad \|\vec{z}_h(a_{ij,K})\| \leq c_2 \|\vec{z}\|_{1,\hat{K}} = c_2 (|\hat{z}|_{1,\hat{K}}^2 + \|\vec{z}\|_{0,\hat{K}}^2)^{1/2}.$$

Using (3.15), we obtain

$$(6.26) \quad \|\vec{z}_h(a_{ij,K})\| \leq c_2 |\det(B)|^{-1/2} (\|B\|^2 |\vec{z}|_{1,K}^2 + \|\vec{z}\|_{0,K}^2)^{1/2}, \quad 1 \leq i < j \leq 3.$$

Let us estimate now  $\|\vec{z}_h(a_{123,K})\|$ . According to (6.23) (ii) and (iii), we get by using Green's formula

$$\int_K \vec{z}_h \, dx = \int_K \vec{z} \, dx$$

and

$$\left( \int_K p_{123,K} dx \right) \vec{z}_h(a_{123,K}) = \int_K \vec{z} dx - \sum_{1 \leq i < j \leq 3} \left( \int_K p_{ij,K} dx \right) \vec{z}_h(a_{ij,K}).$$

By a change of variable, we get

$$\vec{z}_h(a_{123,K}) = \left( \int_{\hat{K}} \hat{p}_{123,K} dx \right)^{-1} \left\{ \int_{\hat{K}} \vec{z} dx - \sum_{1 \leq i < j \leq 3} \left( \int_{\hat{K}} \hat{p}_{ij,K} dx \right) \vec{z}_h(a_{ij,K}) \right\}$$

so that

$$\|\vec{z}_h(a_{123,K})\| \leq c_3 \left( \|\vec{z}\|_{1,\hat{K}} + \sum_{1 \leq i < j \leq 3} \|\vec{z}_h(a_{ij,K})\| \right)$$

and by (6.25)

$$\|\vec{z}_h(a_{123,K})\| \leq c_4 \left( |\vec{z}|_{1,\hat{K}}^2 + \|\vec{z}\|_{0,\hat{K}}^2 \right)^{1/2}.$$

Thus, we have

$$(6.27) \quad \|\vec{z}_h(a_{123,K})\| \leq c_4 |\det(B)|^{-1/2} \left( \|B\|^2 |\vec{z}|_K + \|\vec{z}\|_{0,K}^2 \right)^{1/2}.$$

On the other hand, we may write

$$(6.28) \quad |p_{ij,K}|_{1,K} \leq |\det(B)|^{-1/2} \|B^{-1}\| |\hat{p}_{ij,K}|_{1,\hat{K}} \leq c_5 |\det(B)|^{1/2} \|B^{-1}\|$$

and similarly

$$(6.29) \quad |p_{123,K}|_{1,K} \leq c_6 |\det(B)|^{1/2} \|B^{-1}\|.$$

Combining (6.24), (6.26), ..., (6.29), we obtain

$$|\vec{z}_h|_{1,K} \leq c_7 \|B^{-1}\| \left( \|B\|^{-2} |\vec{z}|_{1,K}^2 + \|\vec{z}\|_{0,K}^2 \right)^{1/2}$$

and by (3.19)

$$(6.30) \quad |\vec{z}_h|_{1,K} \leq c_8 \sigma(K) \left( |\vec{z}|_{1,K}^2 + (h(K))^{-2} \|\vec{z}\|_{0,K}^2 \right)^{1/2} \text{ for all } K \in \mathcal{T}_h.$$

Thus, using the hypothesis (6.19), we obtain

$$(6.31) \quad |\vec{z}_h|_{1,\Omega} \leq c_9 \sigma \left( |\vec{z}|_{1,\Omega}^2 + h^{-2} \|\vec{z}\|_{0,\Omega}^2 \right)^{1/2}.$$

Finally, by using (6.16), the hypothesis (6.20) and the classical Aubin-Nitsche's duality argument, we easily get

$$(6.32) \quad \|\vec{z}\|_{0,\Omega} \leq c_{10} \sigma h |\vec{z}|_{1,\Omega}.$$



Then, applying (6.16), (6.31) and (6.32) gives

(6.33)

$$|\vec{v}_h|_{1,\Omega} \leq |\vec{z}_h|_{1,\Omega} + |\vec{w}_h|_{1,\Omega} \leq c_{11}\sigma^2 |\vec{w}_h - \vec{v}|_{1,\Omega} + |\vec{w}_h|_{1,\Omega} \leq c_{12}\sigma^2 |\vec{v}|_{1,\Omega}.$$

Thus, the desired inequality follows from (6.15) and (6.33).

*In conclusion*, assume that the triangulation  $\mathcal{T}_h$  verifies the uniformity condition (6.19) and that  $\Omega$  is convex for instance (so that (6.20) holds). Then, in Example 2 with  $N = 2$ , Hypothesis H.3 holds with  $\lambda = 2$  and we get the estimate

$$(6.34) \quad \|p_h - p\|_{L^2(\Omega)_R} \leq c\sigma^4 h^2 (|\vec{u}|_{3,\Omega} + |p|_{2,\Omega}).$$

## 7. THE CASE OF INHOMOGENEOUS BOUNDARY DATA

Consider the stationary Stokes problem with inhomogeneous boundary data

$$(7.1) \quad \begin{aligned} -\nu\Delta\vec{u} + \overrightarrow{\text{grad}} p &= \vec{f} \text{ in } \Omega, \\ \text{div } \vec{u} &= 0 \text{ in } \Omega, \\ \vec{u} &= \vec{g} \text{ on } \Gamma. \end{aligned}$$

Assume that the vector-valued function  $\vec{g}$  can be extended inside  $\Omega$  as a function  $\vec{u}_0 \in (H^1(\Omega))^N$  such that  $\text{div } \vec{u}_0 = 0$ . In other words, the function  $\vec{g}$  satisfies the two conditions :

$$(7.2) \quad \vec{g} \in (H^{1/2}(\Gamma))^N,$$

$$(7.3) \quad \int_{\Gamma} \vec{g} \cdot \vec{n} \, d\sigma = 0$$

(see. Cattabriga [4]). Then, a weak form of problem (7.1) is as follows : *Given a function  $\vec{f} \in (L^2(\Omega))^N$  (or  $\vec{f} \in V'$ ), find functions  $\vec{u} \in (H^1(\Omega))^N$  and  $p \in L^2(\Omega)/R$  such that*

$$(7.4) \quad \begin{aligned} \vec{u} - \vec{u}_0 &\in V, \\ \nu a(\vec{u}, \vec{v}) + (\overrightarrow{\text{grad}} p, \vec{v}) &= (\vec{f}, \vec{v}) \text{ for all } \vec{v} \in (H_0^1(\Omega))^N. \end{aligned}$$

As a corollary of Theorem 1, we get the following result. *If  $\vec{g}$  satisfies the conditions (7.2), (7.3), there exists a unique pair of functions*

$$(\vec{u}, p) \in (H^1(\Omega))^N \times L^2(\Omega)/R$$

*solution of problem (7.4).*

Let us introduce the space  $\gamma_0 W_h$  of the restrictions over  $\Gamma$  of the functions of  $W_h$  and the space  $G_h$  of the restrictions over  $\Gamma$  of the functions  $\vec{v}_h \in (W_h)^N$  such that  $\operatorname{div}_h \vec{v}_h = 0$ . Then, we have the following characterization of the space  $G_h$ .

**Lemma 9.** *Assume that Hypotheses H.2 (i) and H.3 (i) hold. Then*

$$(7.5) \quad G_h = \left\{ \vec{g}_h \mid \vec{g}_h \in (\gamma_0 W_h)^N, \int_{\Gamma} \vec{g}_h \cdot \vec{n} \, d\sigma = 0 \right\}.$$

*Proof.* Let  $\vec{v}_h \in (W_h)^N$ . Using Green's formula, we may write

$$\int_{\Omega} \operatorname{div}_h \vec{v}_h \, dx = \sum_{K \in \mathcal{G}_h} \int_K \operatorname{div} \vec{v}_h \, dx = - \sum_{K \in \mathcal{G}_h} \int_{\partial K} \vec{v}_h \cdot \vec{n} \, d\sigma$$

and by Hypothesis H. 2 (i)

$$(7.6) \quad \int_{\Omega} \operatorname{div}_h v_h \, dx = - \int_{\Gamma} \vec{v}_h \cdot \vec{n} \, d\sigma.$$

Thus, any function  $\vec{g}_h \in G_h$  satisfies

$$(7.7) \quad \int_{\Gamma} \vec{g}_h \cdot \vec{n} \, d\sigma = 0.$$

Conversely, let  $\vec{g}_h \in (\gamma_0 W_h)^N$  verify condition (7.7). Let  $\vec{w}_h$  be a function in  $(W_h)^N$  such that  $\vec{w}_h|_{\Gamma} = \vec{g}_h$ . Using (7.6), we obtain

$$\int_{\Gamma} \operatorname{div}_h \vec{w}_h \, dx = 0.$$

Then, by Hypothesis H.3 (i), there exists a function  $\vec{z}_h \in (W_{0,h})^N$  such that

$$\operatorname{div}_h \vec{z}_h = - \operatorname{div}_h \vec{w}_h.$$

Now, the function  $\vec{v}_h = \vec{w}_h + \vec{z}_h$  satisfies

$$\operatorname{div}_h \vec{v}_h = 0, \vec{v}_h|_{\Gamma} = \vec{g}_h.$$

A discrete analogue of problem (7.4) is the following : *Given a function  $\vec{g}_h \in G_h$ , find functions  $\vec{u}_h \in (W_h)^N$  and  $p_h \in \Phi_h/R$  such that*

$$(7.8) \quad \begin{aligned} & \nu a_h(\vec{u}_h, \vec{v}_h) - (p_h, \operatorname{div}_h \vec{v}_h) = (f, \vec{v}_h) \text{ for all } \vec{v}_h \in (W_{0,h})^N, \\ & \operatorname{div}_h \vec{u}_h = 0, \\ & \vec{u}_h|_{\Gamma} = \vec{g}_h. \end{aligned}$$

Let us introduce a function  $\vec{u}_{0,h} \in (W_h)^N$  such that

$$(7.9) \quad \operatorname{div}_h \vec{u}_{0,h} = 0, \quad \vec{u}_{0,h} \Big|_{\Gamma} = \vec{g}_h.$$

Then, problem (7.8) can be equivalently stated as follows : *Find functions  $\vec{u}_h \in (W_h)^N$  and  $p_h \in \Phi_h/R$  such that*

$$(7.10) \quad \begin{aligned} &\vec{u}_h - \vec{u}_{0,h} \in V_h, \\ &\forall a_h(\vec{u}_h, \vec{v}_h) - (p_h, \operatorname{div}_h \vec{v}_h) = (\vec{f}, \vec{v}_h) \text{ for all } \vec{v}_h \in (W_{0,h})^N. \end{aligned}$$

We rewrite the 2nd equation (7.10) in the form

$$(7.11) \quad \forall a_h(\vec{u}_h - \vec{u}_{0,h}, \vec{v}_h) - (p_h, \operatorname{div}_h \vec{v}_h) = (\vec{f}, \vec{v}_h) - \forall a_h(\vec{u}_{0,h}, \vec{v}_h).$$

Since  $\vec{v}_h \rightarrow (\vec{f}, \vec{v}_h) - \forall a_h(\vec{u}_{0,h}, \vec{v}_h)$  is a linear functional on  $(W_{0,h})^N$ , we get as a corollary of Theorem 5 : *Given a function  $\vec{g}_h \in G_h$ , there exists a unique pair of functions  $(\vec{u}_h, p_h) \in (W_h)^N \times \Phi_h/R$  solution of problems (7.8).*

It remains to choose the function  $\vec{g}_h \in G_h$ . To this purpose, we assume that  $\vec{u}_0 \in (H^2(\Omega))^N$ . Then, by Hypothesis H.1, the function  $r_h \vec{u}_0$  satisfies  $\operatorname{div}_h r_h \vec{u}_0 = \operatorname{div}_h \vec{u}_0 = 0$ . So, we may choose

$$(7.12) \quad \vec{g}_h = r_h \vec{u}_0 \Big|_{\Gamma}.$$

Note that, in each example considered in §§ 4 and 5,  $r_h \vec{u}_0 \Big|_{\Gamma}$  depends only upon  $\vec{g}$ . Thus, the choice (7.12) appears to be practically relevant. Moreover, we obtain if  $\vec{u} \in (H^2(\Omega))^N$

$$\vec{g}_h = r_h \vec{u} \Big|_{\Gamma}.$$

Now, using equation (7.11) with  $\vec{u}_{0,h} = r_h \vec{u}$ , it is an easy matter to prove that *Theorems 3, 4 and 5 hold without any modification.*

Since, in all the examples of §§ 4 and 5, the determination of  $\vec{g}_h = r_h \vec{u}_0 \Big|_{\Gamma}$  involves the exact computation of surface integrals, the choice (7.12) can be inconvenient in some cases. An alternative procedure consists in defining first an approximation  $\tilde{\vec{g}}_h$  of  $\vec{g}$  in  $(\gamma_0 W_h)^N$  (for example,  $\tilde{\vec{g}}_h$  can be a suitable interpolate of  $\vec{g}$ ). Then, we let  $\vec{g}_h$  to be the orthogonal projection in  $(L^2(\Gamma))^N$  of  $\tilde{\vec{g}}_h$  upon  $G_h$ . We shall not give here the corresponding error analysis since it involves further technical results.

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