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GENERALIZED FUNCTIONAL EQUATION OF TWO VARIABLES IN INFORMATION THEORY *

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Abstract. — *Darôczy* [1] *generalized the functional equation given by Kendall* [4] *as*

$$f(x) + (1-x)^{\beta} f\left(\frac{y}{1-x}\right) = f(y) + (1-y)^{\beta} f\left(\frac{x}{1-y}\right), \ \beta > 0, \ \beta \neq 1 \qquad (0.1)$$

and introduced the concept of information functions of type-β *which are the solutions of the functional equation* (0.1) *under suitable boundary conditions. Authors* [9] *have formed the functional equation of two variables*

$$f(x_1; y_1) + (1-x_1) f\left(\frac{x_2}{1-x_1}; \frac{y_2}{1-y_1}\right) = f(x_2; y_2) + (1-x_2) f\left(\frac{x_1}{1-x_2}; \frac{y_1}{1-y_2}\right) \qquad (0.2)$$

This is a generalization of Kendall's functional equation in two variables. The solutions of (0.2) *are inaccuracy functions under the conditions*

$$f(0; 0) = f(1; 1); f(1/2; 1/2) = 1. \qquad (0.3)$$

In this paper we have generalized the functional equation (0.2) *by introducing two parameters* β *and* γ *such that* $\beta, \gamma > 0$ *and* $\beta \neq 1$ *when* $\gamma = 1$ *as*

$$f(x_1; y_1) + (1-x_1)^{\gamma}(1-y_1)^{\beta-\gamma} f\left(\frac{x_2}{1-x_1}; \frac{y_2}{1-y_1}\right) = f(x_2; y_2) + (1-x_2)^{\gamma}(1-y_2)^{\beta-\gamma} f\left(\frac{x_1}{1-x_2}; \frac{y_1}{1-y_2}\right). \qquad (0.4)$$

This equation is a generalization of Darôczy equation (0.1) *in more than one way because it is a functional equation in two variables and involves two parameters* β *and* γ. *The solutions of this functional equation under boundary conditions* (0.3) *are defined as the inaccuracy functions of type* (β, γ). *These inaccuracy functions of type* (β, γ) *give rise to new measure of inaccuracy* $H_n^{(\beta,\gamma)}(P; Q)$ *of type* (β, γ). *Also when* $\gamma = 1$ *and* $\beta \to 1$ *then* (0.4) *is same as* (0.2). *Some properties of* $H_n^{(\beta,\gamma)}(P; Q)$ *are studied.*

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I. INTRODUCTION

Authors [9] have earlier formed a functional equation of two variables given by

$$f(x_1\,;y_1) + (1 - x_1)f\left(\frac{x_2}{1 - x_1}\,;\frac{y_2}{1 - y_1}\right)$$
$$= f(x_2\,;y_2) + (1 - x_2)f\left(\frac{x_1}{1 - x_2}\,;\frac{y_1}{1 - y_2}\right) \quad (1.1)$$

where $x_1,\ y_1,\ x_2,\ y_2,\ \in [0,1),\ x_1 + x_2 \leqslant 1$ and $y_1 + y_2 \leqslant 1$.

Solutions of this functional equation under the boundary conditions

$$f(0\,;0) = f(1\,;1) \quad , \quad f\left(\frac{1}{2}\,;\frac{1}{2}\right) = 1 \quad (1.2)$$

are defined as inaccuracy functions.

The functional equation (1.1) generalizes the functional equation

$$f(x) + (1 - x)f\left(\frac{y}{1 - x}\right) = f(y) + (1 - y)f\left(\frac{x}{1 - y}\right) \quad (1.3)$$

for all $(x,\ y)$ such that $x,\ y \in [0,1)$ and $x + y \leqslant 1$, given by Kendall [4] and we note that $f(x;\ x) = f(x)$ is the Kendall's information function.

Darôczy [1] generalized the functional equation (1.3) as

$$f(x) + (1 - x)^{\beta}f\left(\frac{y}{1 - x}\right) = f(x) + (1 - y)^{\beta}f\left(\frac{x}{1 - y}\right) \quad , \quad \beta > 0 \quad (1.4)$$

and obtained that under the boundary conditions

$$f(1) = f(0)\,; f\left(\frac{1}{2}\right) = 1 \quad (1.5)$$

$$f(x) = [x^{\beta} + (1 - x)^{\beta} - 1](2^{1-\beta} - 1)^{-1} \quad , \quad \beta \neq 1 \quad (1.6)$$

which according to Darôczy is information function of type-β.

When $\beta \to 1$, under the regularity condition

$$f(x) \to - x \log_2 x - (1 - x) \log_2(1 - x) \quad (1.7)$$

which is Shannon's [7] information function [4].

For a probability distribution $P = (p_1, \ldots, p_n)$, $\sum_{i=1}^{n} p_i = 1$, the entropy of type-β is defined as

$$H_n^\beta(p_1, \ldots, p_n) = \sum_{i=2}^{n} s_i^\beta f\left(\frac{p_i}{s_i}\right) \tag{1.8}$$

where $s_i = p_1 + p_2 + \ldots + p_i$, $i = 2, 3, \ldots, n$ and f is an information function (1.6).

A generalization of (1.4) in two variables studied by Sharma [8] is

$$f(x_1 ; y_1) + (1 - x_1)(1 - y_1)^{\beta-1} f\left(\frac{x_2}{1 - x_1} ; \frac{y_2}{1 - y_1}\right)$$
$$= f(x_2 ; y_2) + (1 - x_2)(1 - y_2)^{\beta-1} f\left(\frac{x_1}{1 - x_2} ; \frac{y_1}{1 - y_2}\right) \tag{1.9}$$

where $0 \leqslant x_1, y_1, x_2, y_2 < 1$, $x_1 + x_2 \leqslant 1$ and $y_1 + y_2 \leqslant 1$.

This contains only one parameter $\beta(> 0)$. In the present paper we introduce a generalization including two parameters $\beta > 0$ and $\gamma > 0$. This generalization contains all earlier ones as particular cases.

In section 2 of this paper we shall give generalized functional equation in two variables and define both inaccuracy function and measure of inaccuracy of type (β, γ).

We shall study some properties of new measure of inaccuracy of type (β, γ) in section 3. In section 4, we shall define the joint and marginal measures of inaccuracy of type (β, γ) and discuss their properties.

2. GENERALIZED FUNCTIONAL EQUATION IN TWO VARIABLES

In what follows, a generalized functional equation would mean,

$$f(x_1 ; y_1) + (1 - x_1)^\gamma(1 - y_1)^{\beta-\gamma} f\left(\frac{x_2}{1 - x_1} ; \frac{y_2}{1 - y_1}\right)$$
$$= f(x_2 ; y_2) + (1 - x_2)^\gamma(1 - y_2)^{\beta-\gamma} f\left(\frac{x_1}{1 - x_2} ; \frac{y_1}{1 - y_2}\right) \tag{2.1}$$

where $\beta, \gamma > 0$, $x_1, y_1, x_2, y_2 \in [0,1)$, $x_1 + x_2 \leqslant 1$ and $y_1 + y_2 \leqslant 1$. Here β and γ are two parameters. It would be noted that if $x_1 = y_1$, $x_2 = y_2$ and $f(x; x) = f(x)$ then (2.1) gives rise to (1.4) for all values of γ.

Definitions

I. An Inaccuracy function $f(x; y)$ of two real variables x and y where $x, y \in (0,1)$ of type (β, γ) is defined to be a function satisfying the functional equation (2.1) and the boundary conditions (1.2).

II. Measure of Inaccuracy of type (β, γ) of the discrete probability distribution $Q = (q_1, ..., q_n)$ with respect to probability distribution $P = (p_1, ..., p_n)$ where $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$, obtained from the inaccuracy function f of type (β, γ) is defined by the quantity

$$H_n^{(\beta,\gamma)}(P ; Q) = \sum_{i=2}^{n} s_i^{\gamma} t_i^{\beta-\gamma} f\left(\frac{p_i}{s_i} ; \frac{q_i}{t_i}\right) \tag{2.2}$$

where $s_i = p_1 + p_2 + ... + p_i$, $t_i = q_1 + q_2 + ... + q_i$ and $i = 2, 3, ..., n$.

Elsewhere we have given two characterizations of inaccuracy function of type (β, γ) [10] under the specified conditions.

The forms of the functions obtained are

$$f(x ; y) = K^{(\beta,\gamma)}(x ; y) = [x^{\gamma} y^{\beta-\gamma} + (1-x)^{\gamma}(1-y)^{\beta-\gamma} - 1](2^{1-\beta} - 1)^{-1} \tag{2.3}$$

for all $x, y \in [0,1]$, $\beta > 0$, $\gamma > 0$ and $\beta \neq 1$ when $\gamma = 1$.

Also when $\beta = \gamma = 1$. we have

$$f(x ; y) = K(x ; y) = \lim_{\beta \to 1} K^{(\beta, 1)}(x ; y) = -x \log_2 y - (1-x) \log_2 (1-y).$$

This is Kerridge's [5] Inaccuracy function.

3. NEW MEASURES OF INACCURACY

Let X be a random variate assuming the values $x_1, ..., x_n$. If an experimenter asserts that the probability of the ith event is q_i while p_i is its true probability, then by definition II, the Measure of Inaccuracy of type (β, γ) is obtained by the quantity

$$H_n^{(\beta,\gamma)}(P ; Q) = (2^{1-\beta} - 1)^{-1}\left(\sum_{i=1}^{n} p_i^{\gamma} q_i^{\beta-\gamma} - 1\right) \tag{3.1}$$

where $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$, $\beta, \gamma > 0$ and $\beta \neq 1$.

If $\gamma \neq 1$, $\gamma > 0$ and $\beta \to 1$, then $H_n^{(\beta,\gamma)}(P ; Q)$ is in general an infinite quantity.

Further if $\gamma = 1$ and $\beta \to 1$ then

$$H(P\,;Q) = \lim_{\beta \to 1} H_n^{(\beta,1)}(P\,;Q) = -\sum_{i=1}^{n} p_i \log_2 q_i \tag{3.2}$$

which is Measure of Inaccuracy defined by Kerridge.

If $p_i = q_i$ for each i, then we have

$$H_n^{\beta}(P) = (2^{1-\beta} - 1)^{-1}\left(\sum_{i=1}^{n} p_i^{\beta} - 1\right) \tag{3.3}$$

which is entropy of type-β defined by Darôczy [1] and Havrada and Charuat [3]. This, in the limiting case when $\beta \to 1$, gives Shannon's entropy for a complete probability distribution $P = (p_1, \ldots, p_n)$.

It would be observed that

$$H_n^{(\beta,\gamma)}(P\,;Q) = 0 \tag{3.4}$$

if $p_i = q_i = 1$ for one value of $i = k$ and consequently $p_i = q_i = 0$ for $i \neq k$ provided $\beta > 0$ and $\beta \neq 1$, which implies that in the absence of inaccuracy there is a correct statement made with complete certainty.

In the following theorem we prove some properties of $H_n^{(\beta,\gamma)}(P\,;Q)$.

Theorem 1. The measure of inaccuracy $H_n^{(\beta,\gamma)}(P\,;Q)$ has the following properties :

(i) *Symmetric:* $H_n^{(\beta,\gamma)}(P\,;Q)$ is a symmetric function of its arguments provided the same probabilities p_i and q_i correspond, that is

$$H_n^{(\beta,\gamma)}(p_1, \ldots, p_{n-1}, p_n\,; q_1, \ldots, q_{n-1}, q_n) = H_n^{(\beta,\gamma)}(p_n, p_1, \ldots, p_{n-1}\,; q_n, q_1, \ldots, q_{n-1}).$$

(ii) *Normalised:* $H_2^{(\beta,\gamma)}\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}\right) = 1.$

(iii) *Expansible:*

$$H_{n+1}^{(\beta,\gamma)}(p_1, \ldots, p_n, 0\,; q_1, \ldots, q_n, 0) = H_n^{(\beta,\gamma)}(p_1, \ldots, p_n\,; q_1, \ldots, q_n).$$

(iv) *Recursive type* (β, γ):

$$H_n^{(\beta,\gamma)}(p_1, \ldots, p_n\,; q_1, \ldots, q_n) - H_{n-1}^{(\beta,\gamma)}(p_1 + p_2, p_3, \ldots, p_n\,; q_1 + q_2, q_3, \ldots, q_n)$$
$$= (p_1 + p_2)^{\gamma}(q_1 + q_2)^{\beta-\gamma} H_2^{(\beta,\gamma)}\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}; \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2}\right) \tag{3.5}$$

(v) *Strongly additive type* (β, γ):

$$H_{mn}^{(\beta,\gamma)}(p_1p_{11}, \ldots, p_1p_{1n}, \ldots, p_mp_{m1}, \ldots, p_mp_{mn}; q_1q_{11}, \ldots, q_1q_{1n}, \ldots, q_mq_{m1}, \ldots, q_mq_{mn})$$
$$= H_m^{(\beta,\gamma)}(p_1, \ldots, p_m; q_1, \ldots, q_m) + \sum_{j=1}^{m} p_j^{\gamma} q_j^{\beta-\gamma} H_n^{(\beta,\gamma)}(p_{j1}, \ldots, p_{jn}; q_{j1}, \ldots, q_{jn}) \quad (3.6)$$

for all $(p_1, \ldots, p_m)$ and $(q_1, \ldots, q_m) \in \Delta_m$; $(p_{j1}, \ldots, p_{jn})$ and $(q_{j1}, \ldots, q_{jn}) \in \Delta_n$ where $\Delta_m = \left\{ (p_1, \ldots, p_m) : p_j \geqslant 0, \sum_{j=1}^{m} p_j = 1 \right\}$.

Proof. The properties (i) to (iii) are obvious and can be verified easily. We, however prove (iv) and (v) by direct computation.

(iv)
$$H_n^{(\beta,\gamma)}(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) - H_{n-1}^{(\beta,\gamma)}(p_1 + p_2, p_3, \ldots, p_n; q_1 + q_2, q_3, \ldots, q_n)$$
$$= (2^{1-\beta} - 1)^{-1}\left[\left(\sum_{i=1}^{n} p_i^{\gamma} q_i^{\beta-\gamma} - 1\right) - (p_1 + p_2)^{\gamma}(q_1 + q_2)^{\beta-\gamma} - \sum_{i=3}^{n} p_i^{\gamma} q_i^{\beta-\gamma} + 1\right]$$
$$= (2^{1-\beta} - 1)^{-1}(p_1 + p_2)^{\gamma}(q_1 + q_2)^{\beta-\gamma} \left[\left(\frac{p_1}{p_1 + p_2}\right)^{\gamma}\left(\frac{q_1}{q_1 + q_2}\right)^{\beta-\gamma} + \left(\frac{p_2}{p_1 + p_2}\right)^{\gamma}\left(\frac{q_2}{q_1 + q_2}\right)^{\beta-\gamma} - 1\right]$$
$$= (p_1 + p_2)^{\gamma}(q_1 + q_2)^{\beta-\gamma} H_2^{(\beta,\gamma)} \left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}; \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2}\right).$$

(v) We have

$$H_m^{(\beta,\gamma)}(p_1, \ldots, p_m; q_1, \ldots, q_m) + \sum_{j=1}^{m} p_j^{\gamma} q_j^{\beta-\gamma} H_n^{(\beta,\gamma)}(p_{j1}, \ldots, p_{jn}; q_{j1}, \ldots, q_{jn})$$
$$= (2^{1-\beta} - 1)^{-1}\left[\left(\sum_{i=1}^{m} p_i^{\gamma} q_i^{\beta-\gamma} - 1\right) + \sum_{j=1}^{m} p_j^{\gamma} q_j^{\beta-\gamma}\left(\sum_{i=1}^{n} p_{ji}^{\gamma} q_{ji}^{\beta-\gamma} - 1\right)\right]$$
$$= (2^{1-\beta} - 1)^{-1}\left[\sum_{j=1}^{m} \sum_{i=1}^{n} (p_j p_{ji})^{\gamma}(q_j q_{ji})^{\beta-\gamma} - 1\right]$$
$$= H_{mn}^{(\beta,\gamma)}(p_1p_{11}, \ldots, p_1p_{1n}, \ldots, p_mp_{m1}, \ldots, p_mp_{mn}; q_1q_{11}, \ldots, q_1q_{1n}, \ldots, q_mq_{m1}, \ldots, q_mq_{mn}).$$

Theorem 2. Let $P_1 = (p_{11}, p_{12}, \ldots, p_{1n}) \in \Delta_n$ and $P_2 = (p_{21}, p_{22}, \ldots, p_{2m}) \in \Delta_m$ and similar notations for Q_1 and Q_2. Define

$$P_1 * P_2 = (p_{11}p_{21}, \ldots, p_{11}p_{2m}; \ldots; p_{1n}p_{21}, \ldots, p_{1n}p_{2m}) \tag{3.7}$$

then for $\beta > 0$, $\gamma > 0$ and $\beta \neq 1$, we have

$$H_{mn}^{(\beta,\gamma)}(P_1 * P_2 ; Q_1 * Q_2) = H_m^{(\beta,\gamma)}(P_1 ; Q_1) + H_m^{(\beta,\gamma)}(P_2 ; Q_2) + (2^{1-\beta} - 1)H_n^{(\beta,\gamma)}(P_1 ; Q_1)H_m^{(\beta,\gamma)}(P_2 ; Q_2) \tag{3.8}$$

and $$H_{mn}^{(\beta,\gamma)}(P_1 * P_2 ; Q_1 * Q_2) = H_n^{(\beta,\gamma)}(P_1 ; Q_1) + H_m^{(\beta,\gamma)}(P_2 ; Q_2) \tag{3.9}$$

if $\beta \to 1$ while $\gamma > 0$.

Proof.

$$H_{mn}^{(\beta,\gamma)}(P_1 * P_2 ; Q_1 * Q_2) = \left[\sum_{j=1}^{m}\sum_{i=1}^{n}(p_{1i}p_{2j})^{\gamma}(q_{1i}q_{2j})^{\beta-\gamma} - 1\right](2^{1-\beta} - 1)^{-1}$$

$$= \left[\left(\sum_{i=1}^{n} p_{1i}^{\gamma} q_{1i}^{\beta-\gamma}\right)\left(\sum_{j=1}^{m} p_{2j}^{\gamma} q_{2j}^{(\beta-\gamma)}\right) - 1\right] (2^{1-\beta} - 1)^{-1}$$

$$= [\{(2^{1-\beta} - 1)H_n^{(\beta,\gamma)}(P_1 ; Q_1) + 1\} \{(2^{1-\beta} - 1)H_m^{(\beta,\gamma)}(P_2 ; Q_2) + 1\}] \times (2^{1-\beta} - 1)^{-1}$$

$$= H_n^{(\beta,\gamma)}(P_1 ; Q_1) + H_m^{(\beta,\gamma)}(P_2 ; Q_2) + (2^{1-\beta} - 1)H_n^{(\beta,\gamma)}(P_1 ; Q_1)H_m^{(\beta,\gamma)}(P_2 ; Q_2).$$

When $\beta \to 1$, then last term vanishes and we have

$$H_{mn}^{(\beta,\gamma)}(P_1 * P_2 ; Q_1 * Q_2) = H_n^{(\beta,\gamma)}(P_1 ; Q_1) + H_m^{(\beta,\gamma)}(P_2 ; Q_2)$$

and when $\beta \neq 1$, then

$$H_{mn}^{(\beta,\gamma)}(P_1 * P_2 \cdot Q_1 * Q_2) \gtrless H_n^{(\beta,\gamma)}(P_1 ; Q_1) + H_m^{(\beta,\gamma)}(P_2 ; Q_2)$$

according as

$$(2^{1-\beta} - 1)H_n^{(\beta,\gamma)}(P_1 ; Q_1)H_m^{(\beta,\gamma)}(P_2 ; Q_2) \gtrless 0. \tag{3.10}$$

Theorem 3. For $(p_1, ..., p_n) \in \Delta_n$, $(q_{1i}, ..., q_{mi})$ and $(q_1, ..., q_m) \in \Delta_m$, $i = 1, 2, ..., n$, we have

$$H_m^{(\beta,\gamma)}\left(\sum_{i=1}^n p_i q_{1i}, ..., \sum_{i=1}^n p_i q_{mi} ; q_1, ..., q_m\right) \geqslant \sum_{i=1}^n p_i H_m^{(\beta,\gamma)}(q_{1i}, ..., q_{mi} ; q_1, ..., q_m) \quad (3.11)$$

where β and γ are positive numbers such that $\beta \neq 1$ and $\gamma < 1$.

The inequality is reversed if $\gamma > 1$.

Proof. We have

$$H_m^{(\beta,\gamma)}\left(\sum_{i=1}^n p_i q_{1i}, ..., \sum_{i=1}^n p_i q_{mi} ; q_1, ..., q_m\right)$$

$$= (2^{1-\beta} - 1)^{-1}\left[\sum_{k=1}^m \left(\sum_{i=1}^n p_i q_{ki}\right)^\gamma q_k^{\beta-\gamma} - 1\right]$$

$$\geqslant (2^{1-\beta} - 1)^{-1}\left[\sum_{k=1}^m \sum_{i=1}^n p_i q_{ki}^\gamma q_k^{\beta-\gamma} - 1\right] \text{ if } \gamma < 1 \text{ [2].}$$

$$= (2^{1-\beta} - 1)^{-1}\left(\sum_{i=1}^n p_i\right)\left[\sum_{k=1}^m q_{ki}^\gamma q_k^{(\beta-\gamma)} - 1\right]$$

$$= \sum_{i=1}^n p_i H_m^{(\beta,\gamma)}(q_{1i}, ..., q_{mi} ; q_1, ..., q_m).$$

Since $\left(\sum_{i=1}^n p_i q_{ki}\right)^\gamma \leqslant \sum_{i=1}^n p_i q_{ki}^\gamma$ if $\gamma > 1$ [2], therefore, the inequality in (3.11) is reversed if $\gamma > 1$.

Theorem 4. Let β and γ be the positive numbers such that $\beta \neq 1$. For $n \geqslant N + 1$ where $N \geqslant 2$, $(p_1, ..., p_n)$ and $(q_1, ..., q_n) \in \Delta_n$, we have

$$H_n^{(\beta,\gamma)}(p_1, ..., p_n ; q_1, ..., q_n) - H_{n-N+1}^{(\beta,\gamma)}\left(\sum_{i=1}^N p_i, p_{N+1}, ..., p_n ; \sum_{i=1}^N q_i, q_{N+1}, ..., q_n\right)$$

$$= \left(\sum_{i=1}^N p_i\right)^\gamma \left(\sum_{i=1}^N q_i\right)^{\beta-\gamma} H_N^{(\beta,\gamma)}\left(\frac{p_1}{\sum_{i=1}^N p_i}, ..., \frac{p_N}{\sum_{i=1}^N p_i} ; \frac{q_1}{\sum_{i=1}^N q_i}, ..., \frac{q_N}{\sum_{i=1}^N q_i}\right) \quad (3.12)$$

$$\gtrless \left(\sum_{i=1}^{N} p_i^{\gamma}\right)\left(\sum_{i=1}^{N} q_i^{\beta-\gamma}\right) H_N^{(\beta,\gamma)}\left(\frac{p_1}{\sum_{i=1}^{N} p_i}, \ldots, \frac{p_N}{\sum_{i=1}^{N} p_i}; \frac{q_1}{\sum_{i=1}^{N} q_i}, \ldots, \frac{q_N}{\sum_{i=1}^{N} q_i}\right) \tag{3.13}$$

according as

$$\gamma > 1,\ \beta-\gamma > 1 \text{ or } \gamma < 1,\ \beta-\gamma < 1. \tag{3.14}$$

Proof. The theorem is a generalization of property (iv) of the theorem I. We have

$$H_n^{(\beta,\gamma)}(p_1, \ldots, p_n; q_1, \ldots, q_n) - H_{n-N+1}^{(\beta,\gamma)}\left(\sum_{i=1}^{N} p_i; p_{N+1}, \ldots, p_n; \sum_{i=1}^{N} q_i, q_{N+1}, \ldots, q_n\right).$$

$$= (2^{1-\beta}-1)^{-1}\left[\left(\sum_{i=1}^{n} p_i^{\gamma} q_i^{\beta-\gamma} - 1\right) - \left\{\left(\sum_{i=1}^{N} p_i\right)^{\gamma}\left(\sum_{i=1}^{N} q_i\right)^{\beta-\gamma} + \sum_{i=N+1}^{n} p_i^{\gamma} q_i^{\beta-\gamma} - 1\right\}\right]$$

$$= (2^{1-\beta}-1)^{-1}\left[\sum_{i=1}^{N} p_i^{\gamma} q_i^{\beta-\gamma} - \left(\sum_{i=1}^{N} p_i\right)^{\gamma}\left(\sum_{i=1}^{N} q_i\right)^{\beta-\gamma}\right]$$

$$= (2^{1-\beta}-1)^{-1}\left(\sum_{i=1}^{N} p_i\right)^{\gamma}\left(\sum_{i=1}^{N} q_i\right)^{\beta-\gamma}\left[\sum_{i=1}^{N}\left(\frac{p_i}{\sum_{i=1}^{N} p_i}\right)^{\gamma}\left(\frac{q_i}{\sum_{i=1}^{N} q_i}\right)^{\beta-\gamma} - 1\right]$$

$$= \left(\sum_{i=1}^{N} p_i\right)^{\gamma}\left(\sum_{i=1}^{N} q_i\right)^{\beta-\gamma} H_N^{(\beta,\gamma)}\left(\frac{p_1}{\sum_{i=1}^{N} p_i}, \ldots, \frac{p_N}{\sum_{i=1}^{N} p_i}; \frac{q_1}{\sum_{i=1}^{N} q_i}, \ldots, \frac{q_N}{\sum_{i=1}^{N} q_i}\right)$$

Now to prove the inequalities we recall [2] that

$$\left(\sum_{i=1}^{N} a_i\right)^{\gamma} \gtrless \left(\sum_{i=1}^{N} a_i^{\gamma}\right) \text{ according to as } \gamma \gtrless 1.$$

From this inequality, we have

$$\left(\sum_{i=1}^{N} p_i\right)^{\gamma} \gtrless \left(\sum_{i=1}^{N} p_i^{\gamma}\right) \text{ according to as } \gamma \gtrless 1 \tag{3.15}$$

and

$$\left(\sum_{i=1}^{N} q_i\right)^{\beta-\gamma} \gtrless \left(\sum_{i=1}^{N} q_i^{\beta-\gamma}\right) \text{ according to as } \beta-\gamma \gtrless 1. \tag{3.16}$$

Combining (3.15) and (3.16) with (3.12) we get the inequalities (3.13). It may be seen that inequalities can also be obtained by combining either (3.15) or (3.16) with (3.12).

Corollaries :

(i) $$H_n^{(\beta,\gamma)}(p_1, \ldots, p_n ; q_1, \ldots, q_n) - H_2^{(\beta,\gamma)}\left(\sum_{i=1}^{n-1} p_i, p_n ; \sum_{i=1}^{n-1} q_i, q_n\right)$$

$$= \left(\sum_{i=1}^{n-1} p_i\right)^{\gamma}\left(\sum_{i=1}^{n-1} q_i\right)^{\beta-\gamma} H_{n-1}^{(\beta,\gamma)}\left(\frac{p_1}{\sum_{i=1}^{n-1} p_i}, \ldots, \frac{p_{n-1}}{\sum_{i=1}^{n-1} p_i} ; \frac{q_1}{\sum_{i=1}^{n-1} q_i}, \ldots, \frac{q_{n-1}}{\sum_{i=1}^{n-1} q_i}\right) \quad (3.17)$$

$$\gtrless \left(\sum_{i=1}^{n-1} p_i^{\gamma}\right)\left(\sum_{i=1}^{n-1} q_i^{\beta-\gamma}\right) H_{n-1}^{(\beta,\gamma)}\left(\frac{p_1}{\sum_{i=1}^{n-1} p_i}, \ldots, \frac{p_{n-1}}{\sum_{i=1}^{n-1} p_i} ; \frac{q_1}{\sum_{i=1}^{n-1} q_i}, \ldots, \frac{q_{n-1}}{\sum_{i=1}^{n-1} q_i}\right) \quad (3.18)$$

according as (3.14).

(ii) $$H_n^{(\beta,\gamma)}\left(\frac{p_{11}}{p_{21}}, \ldots, \frac{p_{1n}}{p_{2n}} ; p_{11}, \ldots, p_{1n}\right) - H_2^{(\beta,\gamma)}\left(\sum_{i=1}^{n-1} \frac{p_{1i}}{p_{2i}}, \frac{p_{1n}}{p_{2n}} ; \sum_{i=1}^{n-1} p_{1i}, p_{1n}\right)$$

$$= \left(\sum_{i=1}^{n-1} \frac{p_{1i}}{p_{2i}}\right)^{\gamma}\left(\sum_{i=1}^{n-1} p_{1i}\right)^{\beta-\gamma} H_{n-1}^{(\beta,\gamma)}\left(\frac{p_{11}}{p_{21}}\Bigg|\sum_{i=1}^{n-1} \frac{p_{1i}}{p_{2i}}, \ldots, \frac{p_{1,n-1}}{p_{2,n-1}}\Bigg|\sum_{i=1}^{n-1} \frac{p_{1i}}{p_{2i}} ; \frac{p_{11}}{\sum_{i=1}^{n-1} p_{1i}}, \ldots, \frac{p_{1,n-1}}{\sum_{i=1}^{n-1} p_{1i}}\right) \quad (3.19)$$

$$\gtrless \left[\sum_{i=1}^{n-1} \left(\frac{p_{1i}}{p_{2i}}\right)^{\gamma}\right]\left[\sum_{i=1}^{n-1} p_{1i}^{\beta-\gamma}\right] H_{n-1}^{(\beta,\gamma)}\left(\frac{p_{11}}{p_{21}}\Bigg|\sum_{i=1}^{n-1} \frac{p_{1i}}{p_{2i}}, \ldots, \frac{p_{1,n-1}}{p_{2,n-1}}\Bigg|\sum_{i=1}^{n-1} \frac{p_{1i}}{p_{2i}} ; \frac{p_{11}}{\sum_{i=1}^{n-1} p_{1i}}, \ldots, \frac{p_{1,n-1}}{\sum_{i=1}^{n-1} p_{1i}}\right) \quad (3.20)$$

according as (3.14).

4. MEASURES OF INACCURACY OF TYPE (β, γ) FOR BIVARIATE DISTRIBUTIONS

Let X and Y be two discrete finite random variables assuming the values $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_m)$ respectively. We define the measure of Inaccuracy of type (β, γ) of the probability distribution $Q(X)$ with respect to probability distribution $P(X)$ by

$$H_n^{(\beta,\gamma)}[P(X) ; Q(X)] = H_n^{(\beta,\gamma)}[(p_1, ..., p_n ; q_1, ..., q_n)] \tag{4.1}$$

where $p_i = P(X = x_1)$, $q_i = Q(X = x_i)$, $i = 1, 2, ..., n$ and

$$\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1.$$

Similarly for Y, we define

$$H_m^{(\beta,\gamma)}[P(Y) ; Q(Y)] = H_m^{(\beta,\gamma)}(p'_1, ..., p'_m ; q'_1, ..., q'_m)] \tag{4.2}$$

where

$$p'_j = P(Y = y_j) \quad , \quad q'_j = Q(Y = y_j) \quad , \quad j = 1, 2, ..., m,$$

and $\sum_{j=1}^{m} p'_j = \sum_{j=1}^{m} q'_j = 1.$

Next if $p(x_i, y_j)$ and $q(x_i, y_j)$ are the joint probability of (x_i, y_j) for the distribution P and Q of (X, Y) respectively, then the joint Inaccuracy of type (β, γ) of $Q(X, Y)$ with respect to $P(X, Y)$ is defined as

$$H_{mn}^{(\beta,\gamma)}\begin{bmatrix} P(X, Y) ; \\ Q(X, Y) \end{bmatrix} = H_{mn}^{(\beta,\gamma)}[p(x_1, y_1), ..., p(x_1, y_m), ..., p(x_n, y_1), ..., p(x_n, y_m) ;$$

$$q(x_1, y_1), ..., q(x_1, y_m), ..., q(x_n, y_1), ..., q(x_n, x_m)] \tag{4.3}$$

such that $\sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} q(x_i, y_j) = 1.$

Further since $P(Y/x_i)$ and $Q(Y/x_i)$ are complete probability distributions of Y, given x_i, we may define

$$H_m^{(\beta,\gamma)}[P(Y/x_i) ; Q(Y/x_i)] = (2^{1-\beta} - 1)^{-1} \left[\sum_{j=1}^{m} p^{\gamma}(y_j/x_i) q^{\beta-\gamma}(y_j/x_i) - 1 \right] \tag{4.4}$$

And now the marginal inaccuracy of Y given X can be defined as

$$H_m^{(\beta,\gamma)}[P(Y/X) ; Q(Y/X)] = \sum_{i=1}^{n} p^{\gamma}(x_i) q^{\beta-\gamma}(x_i) H_m^{\beta-\gamma}[P(Y/x_i) ; Q(Y/x_i)] \tag{4.5}$$

Similarly the marginal inaccuracy of X given Y is defined by

$$H_n^{(\beta,\gamma)}[P(X/Y) ; Q(X/Y)] = \sum_{j=1}^{m} p^{\gamma}(y_j)q^{\beta-\gamma}(y_j)H_n^{(\beta,\gamma)}[P(X/y_j) ; Q(X/y_j)] \quad (4.6)$$

It may be noted that if $p_i = q_i$ for each i, then we get the corresponding entropies of type β [1] and in the limiting case $\beta \to 1$, these give corresponding Shannon's entropies for the bivariate case.

Theorem 5. Let β and γ be positive numbers with $\beta \neq 1$, then

$$H_{mn}^{(\beta,\gamma)}[P(X, Y) ; Q(X, Y)] = H_n^{(\beta,\gamma)}[P(X) ; Q(X)] + H_m^{(\beta,\gamma)}[P(Y/X) ; Q(Y/X)] \quad (4.7)$$

$$= H_m^{(\beta,\gamma)}[P(Y) ; Q(Y)] + H_n^{(\beta,\gamma)}[P(X/Y) ; Q(X/Y)] \quad (4.8)$$

Proof. By the definitions given above, we have

$$H_n^{(\beta,\gamma)}[P(X) ; Q(X)] + H_m^{(\beta,\gamma)}[P(Y/X) ; Q(Y/X)]$$

$$= (2^{1-\beta} - 1)^{-1}\left[\left\{ \sum_{i=1}^{n} p^{\gamma}(x_i)q^{\beta-\gamma}(x_i) - 1 \right\} + \sum_{i=1}^{n} p^{\gamma}(x_i)q^{\beta-\gamma}(x_i) \left\{ \sum_{j=1}^{m} p^{\gamma}(y_j / x_i)q^{\beta-\gamma}(y_j / x_i) - 1 \right\} \right]$$

$$= (2^{1-\beta} - 1)^{-1}\left[\sum_{i=1}^{n} \sum_{j=1}^{m} p^{\gamma}(x_i, y_j)q^{\beta-\gamma}(x_i, y_j) - 1 \right]$$

as $p(x_i, y_j) = p(x_i)p(y_j/x_i)$ and $q(x_i, y_j) = q(x_i)q(y_j/x_i)$

$$= H_{mn}^{(\beta,\gamma)}[P(X, Y) ; Q(X, Y)].$$

The other result can similarly be proved.

Corollary 1. When X and Y are statistically independent discrete random variables, then

$$H_{mn}^{(\beta,\gamma)}[P(X, Y) ; Q(X, Y)] = H_n^{(\beta,\gamma)}[P(X) ; Q(X)] + H_m^{(\beta,\gamma)}[P(Y) ; Q(Y)] + (2^{1-\beta} - 1)H_n^{(\beta,\gamma)}[P(X) ; Q(X)]H_m^{(\beta,\gamma)}[P(Y) ; Q(Y)] \quad (4.9)$$

Proof. From above theorem,

$$H_{mn}^{(\beta,\gamma)}[P(X, Y) ; Q(X, Y)] = H_n^{(\beta,\gamma)}[P(X) ; Q(X)] + H_m^{(\beta,\gamma)}[P(Y/X) ; Q(Y/X)]$$

$$= H_n^{(\beta,\gamma)}[P(X) ; Q(X)] + \sum_{i=1}^{n} p^{\gamma}(x_i)q^{\beta-\gamma}(x_i) \left[\sum_{j=1}^{m} p^{\gamma}(y_j)q^{\beta-\gamma}(y_j) - 1) \right] (2^{1-\beta} - 1)^{-1}$$

as X and Y are statistically independent.

$$= H_n^{(\beta,\gamma)}[P(X)\,;\,Q(X)] + H_m^{(\beta,\gamma)}[P(Y)\,;\,Q(Y)]\left[\sum_{i=1}^{n} p^\gamma(x_i)q^{\beta-\gamma}(x_i)\right]$$

$$= H_n^{(\beta,\gamma)}[P(X)\,;\,Q(X)] + H_m^{(\beta,\gamma)}[P(Y)\,;\,Q(Y)] + (2^{1-\beta}-1)H_n^{(\beta,\gamma)}[P(X)\,;\,Q(X)]H_m^{(\beta,\gamma)}[P(Y)\,;\,Q(Y)].$$

When $\beta \to 1$, the last term in (4.9) vanishes and we have

$$H_{mn}^{(\beta,\gamma)}[P(X,Y)\,;\,Q(X,Y)] = H_n^{(\beta,\gamma)}[P(X)\,;\,Q(X)] + H_m^{(\beta,\gamma)}[P(Y)\,;\,Q(Y)]. \quad (4.10)$$

In case $\beta \neq 1$, we have the inequalities,

$$H_{mn}^{(\beta,\gamma)}[P(X,Y)\,;\,Q(X,Y)] \gtrless H_n^{(\beta,\gamma)}[P(X)\,;\,Q(X)] + H_m^{(\beta,\gamma)}[P(Y)\,;\,Q(Y)] \quad (4.11)$$

according as

$$(2^{1-\beta}-1)H_n^{(\beta,\gamma)}[P(X)\,;\,Q(X)]H_m^{(\beta,\gamma)}[P(Y)\,;\,Q(Y)] \gtrless 0 \quad (4.12)$$

REMARKS : These results correspond to Shannon's result $H(X;\,Y) \leqslant H(X) + H(Y)$, etc.

Corollary 2. We have

$$H_m^{(\beta,\gamma)}[P(Y/X)\,;\,Q(Y/X)] = H_m^{(\beta,\gamma)}[P(Y)\,;\,Q(Y)] + (2^{1-\beta}-1)H_n^{(\beta,\gamma)}[P(X)\,;\,Q(X)]H_m^{(\beta,\gamma)}[P(Y)\,;\,Q(Y)] \quad (4.13)$$

and $$H_n^{(\beta,\gamma)}[P(X/Y)\,;\,Q(X/Y)] = H_n^{(\beta,\gamma)}[P(X)\,;\,P(X)] + (2^{1-\beta}-1)H_n^{(\beta,\gamma)}[P(X)\,;\,Q(X)]H_m^{(\beta,\gamma)}[P(Y)\,;\,Q(Y)] \quad (4.14)$$

if X and Y are satistically independent.

Proof. By definitions we have

$$H_m^{(\beta,\gamma)}[P(Y/X)\,;\,Q(Y/X)]$$

$$= \sum_{i=1}^{n} p^\gamma(x_i)q^{\beta-\gamma}(x_i)\left[\sum_{j=1}^{m} p^\gamma(y_j/x_i)q^{\beta-\gamma}(y_j/x_i) - 1\right](2^{1-\beta}-1)^{-1}$$

$$= \sum_{i=1}^{n} p^\gamma(x_i)q^{\beta-\gamma}(x_i)\left[\sum_{j=1}^{m} p^\gamma(y_j)q^{\beta-\gamma}(y_j) - 1\right](2^{1-\beta}-1)^{-1}$$

as X and Y are stastistically independent.

$$= H_m^{(\beta,\gamma)}[P(Y)\,;\,Q(Y)]\{(2^{1-\beta}-1)H_n^{(\beta,\gamma)}[P(X)\,;\,Q(X)]+1\}$$

$$= H_m^{(\beta,\gamma)}[P(Y)\,;\,Q(Y)]+(2^{1-\beta}-1)H_n^{(\beta,\gamma)}[P(X)\,;\,Q(X)]H_m^{(\beta,\gamma)}[P(Y)\,;\,Q(Y)]$$

and similarly we can prove (4.14).

When $\beta \to 1$, the last terms in (4.13) and (4.14) vanish and we have

$$H_m^{(\beta,\gamma)}[P(Y/X)\,;\,Q(Y/X)] = H_m^{(\beta,\gamma)}[P(Y)\,;\,Q(Y)] \tag{4.15}$$

and

$$H_n^{(\beta,\gamma)}[P(X/Y)\,;\,Q(X/Y)] = H_n^{(\beta,\gamma)}[P(X)\,;\,Q(X)] \tag{4.16}$$

Furthermore if $\beta \neq 1$, then

$$H_m^{(\beta,\gamma)}[P(Y/X)\,;\,Q(Y/X)] \gtrless H_m^{(\beta,\gamma)}[P(Y)\,;\,Q(Y)] \tag{4.17}$$

and

$$H_n^{(\beta,\gamma)}[P(X/Y)\,;\,Q(X/Y)] \gtrless H_n^{(\beta,\gamma)}[P(X)\,;\,Q(X)] \tag{4.18}$$

according as $(2^{1-\beta}-1)H_n^{(\beta,\gamma)}[P(X)\,;\,Q(X)]H_m^{(\beta,\gamma)}[P(Y)\,;\,Q(Y)] \gtrless 0.$ (4.19)

REMARKS. These results correspond to Shannon's results

$$H(X) \geqslant H(X/Y) \text{ and } H(Y) \geqslant H(Y/X).$$

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