REVUE FRANÇAISE D'INFORMATIQUE ET DE RECHERCHE OPÉRATIONNELLE

T. O. M. KRONSJÖ

Centralization and decentralization of decision making. The decomposition of any linear programme in primal and dual directions - to obtain a primal and a dual master solved in parallel and one or more common subproblems

Revue française d'informatique et de recherche opérationnelle, tome 2, nº R2 (1968), p. 73-113

<http://www.numdam.org/item?id=M2AN_1968__2_2_73_0>

R.I.R.O.
(2e année, N° 10, 1968, p. 73-114)

CENTRALIZATION AND DECENTRALIZATION OF DECISION MAKING

THE DECOMPOSITION OF ANY LINEAR PROGRAMME IN PRIMAL AND DUAL DIRECTIONS — TO OBTAIN A PRIMAL AND A DUAL MASTER SOLVED IN PARALLEL AND ONE OR MORE COMMON SUBPROBLEMS —

by T.O.M. KRONSJÖ *
University of Birmingham, Great Britain

Résumé. — *La méthode de la double décomposition d'un programme linéaire ayant une solution optimale finie de M. D. Pigot est généralisée au traitement de* n'importe quel *programme. On démontre que la méthode peut être employée en faisant des itérations* simultanées *ou séquentielles dans les directions primale et duale et que, sous réserve de certaines conditions, la méthode* converge *en un nombre fini d'itérations.*

On peut signaler que l'application à la résolution de problèmes non linéaires convexes séparables a été entreprise par l'auteur dans une autre recherche.

On peut attendre des résultats intéressants de l'application de la méthode de décomposition proposée à la résolution de gros problemes de programmation convexes ayant une structure triangulaire ou quelqu'autre structure à l'aide d'un réseau de calculateurs interconnectés.

In this investigation the first proof is given that the double decomposition method proposed by D. Pigot (1) may be generalized to deal with *any* type of linear programming problem by considering two or three related linear programmes each of which may be decomposed into a primal and a dual master and one or more common subproblems, the solution of which may be undertaken by simultaneous or sequential iterations in primal and dual directions, and that, if certain conditions are observed, the method will converge in a finite number of iterations to the optimal solution.

The generalization is based upon the following concepts of the author :

i) That any linear programme (no matter whether it has some type of solution or not) may be evaluated by the solution of two or three related programmes with *finite* optimal primal and dual solutions;

---

* This research was supported by the Swedish Council for Social Science Research, Stockholm, Sweden.

(1) D. PIGOT, Double décomposition d'un programme linéaire, *in Actes de la 3e Conférence Internationale de Recherche Opérationnelle* (Proceedings of the 3rd International Conference on Operational Research held in Oslo, 1-5, July 1963), Dunod, Paris, 1964, pp. 72-78.

ii) that any linear programme with finite optimal primal and dual solutions, may be extended into an equivalent extended problem to which *initial* feasible primal and dual solutions may easily be found;

iii) that the extended problem may be constructed so as *to prevent* the possibility of *infinite solutions of some subproblem* obtained by disregarding certain constraints;

iv) that *the primal and dual master problems may contain activities of their own*, in which case the simplex multipliers must form a dual feasible solution to the corresponding dual constraints, before information may be transferred from the masters to the common subproblem(s);

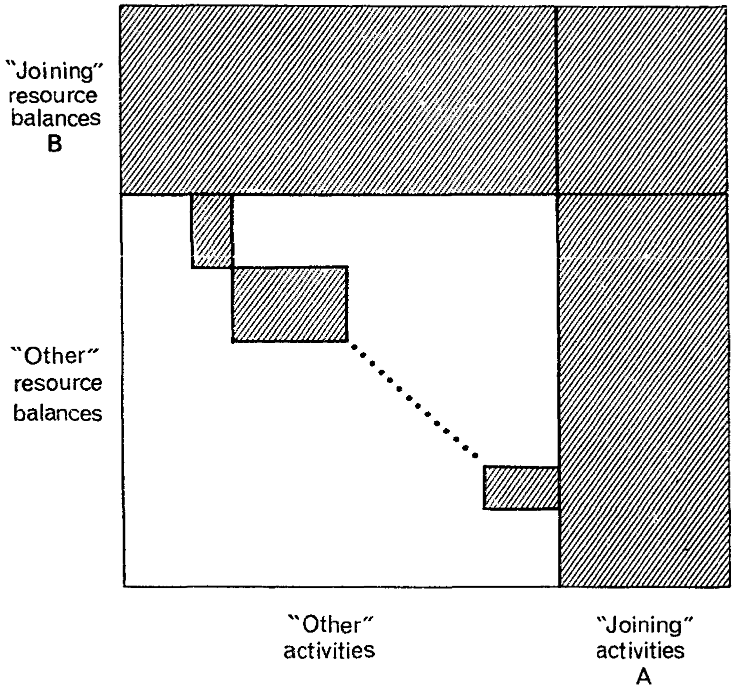

Figure 1

This linear economic problem involves problems of centralization and decentralization of decision making, a convergent scheme for which is the aim of the proposed method.

v) that obtaining *bounds upon the further improvement of the primal and dual feasible solutions of the common subproblem(s)* may be useful in deciding whether an improved primal or dual feasible solution, or both, should be sought to this (these) problem(s);

vi) that *formal* consideration of several independent common subproblems in certain cases may be useful;

vii) that *not only sequential but also simultaneous iterations* in primal and dual directions may be undertaken and that the process *converges* to the optimal solution in a *finite* number of iterations.

As for certain purposes an economic system may be approximated in the form of a linear programming problem of large dimensions, this decomposition procedure is of profound theoretical and practical importance in indicating a possible system for optimal planning based upon a combination of central price parameters [cp. the Dantzig-Wolfe primal decomposition method (1)] *and* central quantity parameters [cp. the Bender's dual

(1) G. B. Dantzig, *Linear Programming and Extensions*, Princeton University Press, Princeton, New Jersey, 1963, Chapter 23.

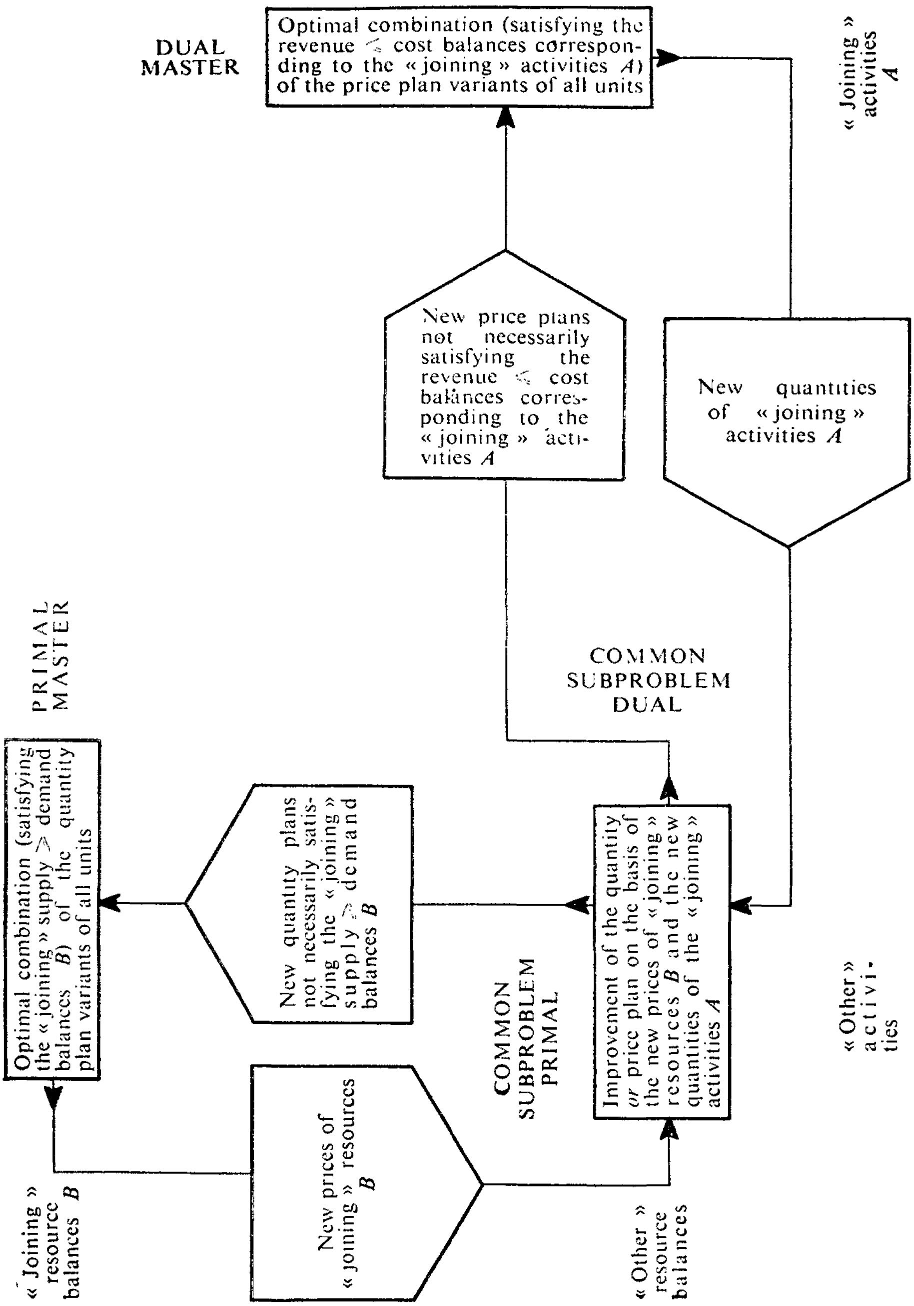

Figure 2

Economic interpretation of the decomposition in primal and dual directions of the proceding problem resulting in a primal and dual master and a common subproblem.

decomposition method [1]]. An economic interpretation of the process is given in Fig. 1 and 2.

It is of interest to observe that another decompositional scheme in primal and dual directions may be devised which leads to a common master and a primal and dual subproblem and that both of these schemes may be generalized to nonlinear convex separable programmes [2].

0. THE PROBLEM

It is desired to establish the existence or non-existence of a finite or infinite optimal primal and dual solution of any linear programming problem by decomposing some related problems with finite primal and dual solutions into a primal master problem and one dual master problem with one common subproblem. Any linear programming problem may be considered to be a special case of the primal problem [3].

$$(1) \qquad \min_{x,y,z} \{ Ax + By + Cz \;|\; \begin{aligned} Hx + Ky + Lz &\geqslant P \\ Dy + Mz &\geqslant Q \\ Nz &\geqslant R \\ x \geqslant 0\, y \geqslant 0\, z &\geqslant 0 \end{aligned} \}$$

with the corresponding dual problem

$$(2) \qquad \max_{s,u,v} \{ sP + uQ + vR \;|\; \begin{aligned} sH &\leqslant A \\ sK + uD &\leqslant B \\ sL + uM + vN &\leqslant C \\ s \geqslant 0\, u \geqslant 0\, v &\geqslant 0 \end{aligned} \}$$

It is not known whether some primal or dual solution exists at all to both problems.

For the solution of the above problem, the author establishes the first generalization and proof of convergence of the decomposition method proposed and practically tested by the French operational analyst Daniel Pigot [4].

The generalization and proof of convergence are based upon the following concepts of the author.

1. Any linear programme (no matter whether it has some type of solution or not) may be evaluated by the successive solution of at most three related

---

(1) Cf. M. L. Balinski, Integer Programming : Methods, Uses, Computation, in *Management Science*, Vol. 12, n° 3, Nov. 1965, pp. 271-274.

(2) Cf. Forthcoming papers by the author in University of Birmingham, CREES, Discussion Papers, Series RC/A, mimeographed, Birmingham, Great Britain, 1968.

(3) The most important special case is when the matrices H and N with corresponding vectors x, v, A and R, do not exist.

(4) D. Pigot, *op. cit.* in the introduction.

linear programmes with *finite* optimal primal and dual solutions (cf. section 1).

2. Any linear programme with finite optimal primal and dual solutions, may be extended into an equivalent extended problem ([1]) to which *initial* feasible primal and dual solutions may easily be found. The consequences of this theorem are that the same solution method may be used from the initiation of computations to the obtention of the near-optimal or optimal solution and that the proof of convergence probably is facilitated ([2]) (cf. section 2).

3. The extended problem may be constructed so as to *prevent the possibility of infinite solutions of some subproblem* obtained by disregarding certain constraints (cf. sections 2 and 3 together with the note of section 8).

4. The decomposition of the linear programme may lead to *a primal and a dual master which contain activities of their own,* in which case the appropriate simplex multipliers must form a dual feasible solution to the corresponding dual constraints, before information may be transferred from one master to the common subproblem(s) and to the other master (cf. section 3).

5. The possible usefulness of obtaining bounds upon the further improvement of the primal and dual feasible solutions of a common subproblem in deciding upon whether an improved primal or dual feasible solution or both should be sought (cf. section 4).

6. *Formal* consideration of several independent common subproblems (cf. section 5).

7. *Not only sequential but also simultaneous iterations* may be undertaken in primal and dual directions and the method still be proven to *converge* to the optimal solution in a *finite* number of iterations (cf. section 3).

1. ESTABLISHING THE TYPE AND SOLUTIONS OF ANY LINEAR PROGRAMME BY THE SOLUTION OF TWO OR THREE DERIVED LINEAR PROGRAMMES WITH FINITE OPTIMAL SOLUTIONS

Theorem. The optimal primal and dual solution, if any, of *any* linear programme with primal

$$(1) \qquad \min_x \left\{ cx \;\middle|\; \begin{array}{l} Ax \geqslant b \\ x \geqslant 0 \end{array} \right\}$$

([1]) When feasible primal and dual solutions have been found without using variables or constraints of the extended problem, the extended formulation may be dropped, if so desired, with the consequence that the various types of subproblem may have infinite solutions, though the complete linear programme may not have any.

([2]) Note that the primal and dual objective functions of a subproblem then always assume the same optimal value.

and dual

(2)
$$\operatorname*{Max}_{u} \{ ub \mid uA \leqslant c,\ u \geqslant 0 \}$$

may be established by the sequential solution of three related primal (dual) linear programming problems with finite optimal solutions, viz. :

i) the primal phase I or dual bounded homogeneous problem to establish the existence of a feasible primal or an infinite dual homogeneous solution;

ii) the primal bounded homogeneous or dual phase I problem to establish the existence of an infinite primal homogeneous solution or a feasible dual solution;

iii) if both a feasible primal and a feasible dual solution exists then continue by solving the primal and dual phase II problem.

The proof will be outlined in the continuation.

i. **The primal phase I or the dual bounded homogeneous problem** involves minimizing the sum of artificial variables y to establish whether the primal problem has a feasible solution or not. The sum of artificial variables $\sum_{i=1}^{m} y_i$ may be denoted by the inner product $1y$ of the row vector $1 = (1, 1, ..., 1)$ and the column vector $y = (y_1, y_2, ..., y_m)$.

(3)
$$\operatorname*{Min}_{x,y} \{ 0x + 1y \mid Ax + y \geqslant b,\ x \geqslant 0\ y \geqslant 0 \} =$$

(4)
$$= \operatorname*{Max}_{u} \{ ub \mid uA \leqslant 0,\ u \leqslant 1,\ u \geqslant 0 \} =$$
$$= w$$

Definition of max (a, b) : If a, b, c are vectors with elements a_i, b_i, c_i $(i = 1, ..., m)$ we define $c = \max(a, b)$ to mean that $c_i = \max(a_i, b_i)$.

As the primal problem (3) has a feasible primal solution

$$x = 0$$
$$y = \max(b, 0)$$

and as the primal objective function may never become negative, there exists a finite optimal primal solution [1]. The dual (4) must therefore have the same finite optimal dual solution [2].

The dual problem (4) may be termed the *bounded homogeneous problem* as it is related to the dual homogeneous problem

$$(5) \qquad \operatorname*{Max}_{u} \{ ub \mid uA \leqslant 0,\ u \geqslant 0 \}$$

An illustration of the homogeneous and the bounded homogeneous problem is given in fig. 3.

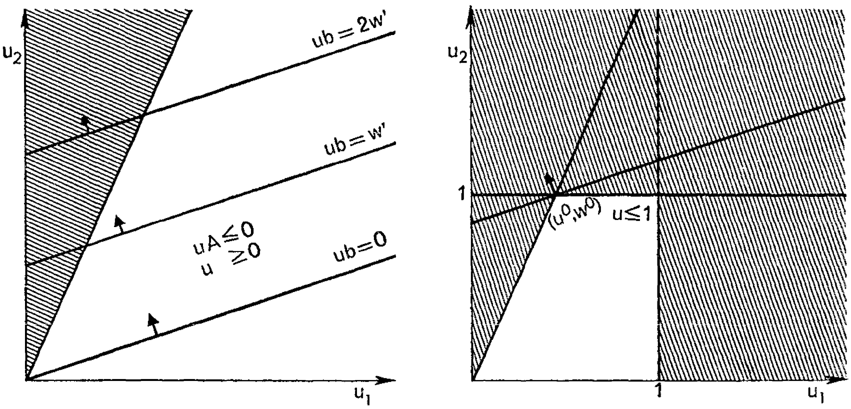

Figure 3.
A homogeneous problem and the corresponding bounded homogeneous problem.

RELATIONSHIPS BETWEEN THE BOUNDED HOMOGENEOUS PROBLEM AND THE HOMOGENEOUS PROBLEM

If no solution exists to the homogeneous problem for which the objective form is greater than zero then no such solution exists to the bounded homogeneous problem, because its solution space is part of that of the homogeneous problem.

If one or more solutions exist to the homogeneous problem for which the objective form is greater than zero, then one such solution must exist to the bounded homogeneous problem, because on the basis of any particular solution $u = u^*$ of the homogeneous problem, a solution may be constructed to the bounded homogeneous problem, if $u^* \leqslant 1$ by using the homogeneous

(1) Cf. G. B. Dantzig, opus cit., section 6-4, Theorem 2.
(2) Opus cit., section 6-4, Theorem 3 and section 6-3, Theorem 1.

solution else by dividing the u^* vector by its largest element $\|u^*\|$; the objective form of the bounded homogeneous problem must in either case similarly be greater than zero.

Conversely, if no solution exists to the bounded homogeneous problem with the objective function greater than zero, the same must be true of the homogeneous problem because of the following reasons.

The homogeneous problem consists of points $u \in U$ which either belong to the bounded homogeneous problem, i.e. the points $u = u^+ \in U^+$ or do not belong to the bounded homogeneous problem, i.e. the points $u = u^- \in U^-$. No solution with the objective function greater than zero exists for points belonging to the bounded homogeneous problem. Any solution point u^- of the homogeneous problem which does not belong to the bounded homogeneous problem may be obtained by multiplying any one of the points $u^+ = u^+(u^-)$ of the bounded homogeneous problem which lie upon the ray joining u^- with the origin by a positive factor $k = u^-/u^+$. If the objective function corresponding to any u^+ point is nonpositive the same must then be true about any u^- point, as the value of the objective function of the point u^- is that of any corresponding u^+ point times the positive factor k, i.e.

$$u^-b = (u^-/u^+)u^+b = ku^+b$$

Finally, if a solution u^0 exists to the bounded homogeneous problem for which the objective function is greater than zero, $0 \leqslant u^0 \leqslant 1$, $u^0A \leqslant 0$, $u^0b = w^0 > 0$, then the homogeneous problem has a solution $u = ku^0$ where $k \to \infty$ for which the objective function becomes infinitely large, because

$$\begin{aligned}\operatorname*{Max}_u \{ ub = ku^0b = kw^0 \mid uA = ku^0A \leqslant 0, u = ku^0 \geqslant 0 \} = \\ = \lim_{k\to\infty} \{ kw^0 \mid w^0 > 0 \} = \infty\end{aligned}$$

Conclusions from solving the primal phase I or dual bounded homogeneous problem

Case i — a

The optimal objective function $w > 0$ implies no primal feasible solution, and a dual infinite homogeneous solution.

Case i — b

The optimal objective function $w = 0$ implies a primal feasible solution, and no dual infinite homogeneous solution.

ii. The primal bounded homogeneous or dual phase I problem

Independently of the outcome of problem *i*, we may proceed to solve a problem ii, which is usually part of the primal phase II problem, but here considered independently as a bounded problem, with finite optimal primal solution to assure that a finite optimal dual solution will exist.

$$(6) \qquad \underset{x}{\text{Min}} \left\{ \begin{array}{rcl} cx & & | \\ Ax & \geqslant & 0 \\ -x & \geqslant & -1 \\ x & \geqslant & 0 \end{array} \right\} =$$

$$(7) \qquad = \underset{u,v}{\text{Max}} \left\{ \begin{array}{l} u0 - v1 \quad | \\ uA - v \leqslant c \\ u \geqslant 0 \; v \geqslant 0 \end{array} \right\} =$$

$$= e$$

The above primal bounded homogeneous problem (6) is similarly related o the primal h omogeneous problem

$$(8) \qquad \underset{x}{\text{Min}} \left\{ \begin{array}{rl} cx & | \\ Ax & \geqslant 0 \\ x & \geqslant 0 \end{array} \right\}$$

and the conclusions above concerning the relations between the bounded homogeneous problem and the homogeneous problem apply with appropriate changes.

The dual (7) of the primal bounded homogeneous problem (6) is identical with the dual phase I problem.

CONCLUSIONS FROM SOLVING THE PRIMAL BOUNDED HOMOGENEOUS OR DUAL PHASE I PROBLEM

Case ii-a

The optimal objective function $e < 0$ implies a primal infinite homogeneous solution, and no dual feasible solution.

Case ii-b

The optimal objective function $e = 0$ implies no primal infinite homogeneous solution, and a dual feasible solution.

CONCLUSIONS FROM THE SOLUTIONS OF BOTH PROBLEMS

Case i-a and ii-a

Neither a primal nor a dual feasible solution.

Case i-a and ii-b

No primal but an infinite dual solution.

Case i-b and ii-a

An infinite primal solution and no dual solution.

If any of these cases apply the solution of the linear programming problem is concluded. Finally :

Case i-b and ii-b

Both a primal and a dual feasible solution
implies that there exists a finite optimal solution which may be obtained by solving the following problem iii.

iii. The primal or dual phase II problem

This problem is identical with problems (1) and (2), and will give the finite optimal primal and dual solutions.

Thereby the proof of the theorem is completed.

2. INITIAL PRIMAL AND DUAL SOLUTIONS OF A LINEAR PROGRAMME

Theorem. A linear programme with finite optimal primal and dual solutions, x^0, u^0

$$(1) \qquad \underset{x}{\text{Min}} \left\{ cx \;\middle|\; \begin{array}{l} Ax \geqslant b \\ x \geqslant 0 \end{array} \right\} = cx^0 =$$

$$(2) \qquad = \underset{u}{\text{Max}} \left\{ ub \;\middle|\; \begin{array}{l} uA \leqslant c \\ u \geqslant 0 \end{array} \right\} = u^0 b$$

to which we find difficulties in immediately constructing feasible primal and dual solutions may be extended into the linear programme

$$(3) \qquad \underset{x,y}{\text{Min}} \left\{ cx + \bar{u}y \;\middle|\; \begin{array}{ll} Ax + y \geqslant b & \\ -x \geqslant -\bar{x} & \\ x \geqslant 0 \quad y \geqslant 0 & \end{array} \right\} =$$

$$(4) \qquad = \underset{u,v}{\text{Max}} \left\{ \begin{array}{llll} ub & -v\bar{x} & & \Big| \\ uA & -v & \leqslant & c \\ u & & \leqslant & \bar{u} \\ u \geqslant 0 & v \geqslant & & 0 \end{array} \right\}$$

which has the same optimal x^0, u^0 solution if the vector of constants $\bar{x}$ is greater than the x^0 vector

$$(5) \qquad \bar{x} > x^0 \geqslant 0$$

and the vector $\bar{u}$ greater than the u^0 vector

$$(6) \qquad \bar{u} > u^0 \geqslant 0$$

To the extended problem a feasible primal solution may be found by putting

$$(7) \qquad \begin{array}{l} x = x^* \quad \text{where} \quad 0 \leqslant x^* \leqslant \bar{x} \\ y = y^* = \max(b - Ax^*, 0) \end{array}$$

and a feasible dual solution by putting

$$(8) \qquad \begin{array}{l} u = u^* \quad \text{where} \quad 0 \leqslant u^* \leqslant \bar{u} \\ v = v^* = \max(-c + u^*A, 0) \end{array}$$

Proof. The primal optimal solution of (1) $x = x^0$ together with $y = 0$ is because of (1) and (5) a primal feasible solution of (3) with the primal objective function equal to that of (1). The dual optimal solution of (2) $u = u^0$ together with $v = 0$ is because of (2) and (6) a dual feasible solution of (4) with the dual objective function equal to that of (2). As the optimal primal objective function (1) equals the optimal dual objective function (2) it follows that (3) equals (4) for the feasible primal and dual solutions used, which therefore ([1]) must be optimal primal and dual solutions. Consequently the extended problem has the same optimal x and u solution(s) as the original problem.

Feasible primal and dual solutions to the original problem

If, as will be the case in the following, some solution method is used which alternately solves the primal (3) and the dual problem (4), then feasible primal and dual solutions to the original primal (1) and dual problem (2) will be available when all $y = 0$ and all $v = 0$.

([1]) Consequence of section 6-3, theorem 1, in G. B. Dantzig, *op. cit.*

3. DECOMPOSING THE PROBLEM INTO A PRIMAL AND A DUAL MASTER WITH A COMMON SUBPROBLEM

The results of the preceding sections 1 and 2 may be used to *transform the problem of section* 0 *into some related linear programmes viz. the extended Primal Phase I, the Dual Phase I, the Primal or Dual Phase II problems with feasible primal and dual solutions.* These problems are of the type

(1)

$$
\underset{a,b,c,x,y,z}{\text{Min}} \left\{ \begin{array}{cccccccc}
\bar{s}a & +\bar{u}b & +\bar{v}c & +Ax & +By & +Cz & & \Big| \\
a & & & +Hx & +Ky & +Lz & \geqslant & P \\
 & b & & & +Dy & +Mz & \geqslant & Q \\
 & & c & & & +Nz & \geqslant & R \\
 & & & -x & & & \geqslant & -\bar{x} \\
 & & & & -y & & \geqslant & -\bar{y} \\
 & & & & & -z & \geqslant & -\bar{z} \\
a \geqslant 0 & b \geqslant 0 & c \geqslant 0 & x \geqslant 0 & y \geqslant 0 & z & \geqslant & 0
\end{array} \right\} =
$$

(2)

$$
= \underset{s,u,v,p,q,r}{\text{Max}} \left\{ \begin{array}{cccccccc}
sP & +uQ & +vR & -p\bar{x} & -q\bar{y} & -r\bar{z} & & \Big| \\
s & & & & & & \leqslant & \bar{s} \\
 & u & & & & & \leqslant & \bar{u} \\
 & & v & & & & \leqslant & \bar{v} \\
sH & & & -p & & & \leqslant & A \\
sK & +uD & & & -q & & \leqslant & B \\
sL & +uM & +vN & & & -r & \leqslant & C \\
s \geqslant 0 & u \geqslant 0 & v \geqslant 0 & p \geqslant 0 & q \geqslant 0 & r \geqslant 0 & &
\end{array} \right\}
$$

where $\bar{s}$, $\bar{u}$, $\bar{v}$, $\bar{x}$, $\bar{y}$, $\bar{z}$ may be considered to be nonnegative vectors of upper bounds upon the corresponding vectors of variables.

A problem of the above type may be decomposed into *the primal master problem*

(3)
$$\operatorname*{Min}_{a,x,t_i} \left\{ \bar{s}a + Ax + \sum_i (\bar{u}b^i + \bar{v}c^i + By^i + Cz^i)t_i \;\middle|\; \begin{array}{l} a + Hx + \sum_i (Ky^i + Lz^i)t_i \geqslant P \\ -x \geqslant -\bar{x} \\ \sum_i t_i = 1 \\ a \geqslant 0 \quad x \geqslant 0 \quad t_i \geqslant 0 \end{array} \right\} =$$

(4)
$$= \operatorname*{Max}_{s,p,w} \left\{ sP - p\bar{x} + w \;\middle|\; \begin{array}{l} s \leqslant \bar{s} \\ sH - p \leqslant A \\ s(Ky^i + Lz^i) + w \leqslant \bar{u}b^i + \bar{v}c^i + By^i + Cz \\ s \geqslant 0 \quad p \geqslant 0 \quad w \text{ unrestricted} \end{array} \right\}$$

and *the primal subproblem*

(5)
$$\operatorname*{Min}_{b,c,y,z} \left\{ \bar{u}b + \bar{v}c + (B - s^kK)y + (C - s^kL)z - w^k \;\middle|\; \begin{array}{l} b + Dy + Mz \geqslant Q \\ c + Nz \geqslant R \\ -y \geqslant -\bar{y} \\ -z \geqslant -\bar{z} \\ b \geqslant 0 \quad c \geqslant 0 \quad y \geqslant 0 \quad z \geqslant 0 \end{array} \right\} =$$

$$= -\mathrm{d}f \leqslant 0$$

No infinite solutions may exist to this subproblem (the coefficients of b and c in the objective function being nonnegative, and all other variables being bounded).

A feasible or optimal primal solution of the primal master is assumed to be known.

$$a^k, x^k, t_i^k$$

with corresponding simplex multipliers, representing an infeasible or optimal dual solution

$$s^k, p^k, w^k$$

The objective function f of the primal master equals the corresponding simplex multipliers times the right hand constants ([1])

$$f = \bar{s}a^k + Ax^k + \sum_i (\bar{u}b^i + \bar{v}c^i + By^i + Cz^i)t_i^k = s^kP - p^k\bar{x} + w^k \tag{6}$$

The problem may also be decomposed into *the dual master problem*

$$\max_{t_j, v, r} \left\{ \begin{array}{llll} \sum_j (s^jP + u^jQ - p^j\bar{x} - q^j\bar{y})t_j & + \; vR & - \; r\bar{z} & \Big| \\ & \quad v & & \leqslant \bar{v} \\ \sum_j (s^jL + u^jM)t_j & + \; vN & - \; r & \leqslant C \\ \sum_j t_j & & & = 1 \\ t_j \geqslant 0 & \quad v \geqslant 0 & \quad r \geqslant 0 & \end{array} \right\} = \tag{7}$$

$$= \min_{c, z, m} \left\{ \begin{array}{llll} \bar{v}c & + \; Cz & + \; m & \Big| \\ & (s^jL + u^jM)z & + \; m & \geqslant s^jP + u^jQ - p^j\bar{x} - q^j\bar{y} \\ c & + \; Nz & & \geqslant R \\ & - \; z & & \geqslant -\bar{z} \\ c \geqslant 0 & \quad z \geqslant 0 & m \text{ unrestricted} & \end{array} \right\} \tag{8}$$

---

[1] This may be demonstrated on the basis of the linear programme

$$\min_{x', x''} \{ c'x' + c''x'' \mid A'x' + A''x'' = b, x' > 0, x'' = 0 \},$$

where x' and x'' denote respectively basic and non-basic variables in the current iteration with prime and biss indicating the corresponding parts of the vector c and the matrix A.

The values of the basic variables are $x' = A'^{-1}b$ and of the simplex multipliers $u = c'A'^{-1}$. The primal objective form is $c'x' = c'A'^{-1}b = ub$, which is the simplex multipliers times the right hand constants.

Alternatively, this may be demonstrated by considering the revised simplex method, in which the value of the basic variable corresponding to the objective function is obtained by summing the product of each simplex multiplier times the corresponding original right hand side constant. As the simplex multiplier of the equation corresponding to the objective function is 1 and the corresponding original right hand constant is 0, the above equality follows.

and *the dual subproblem*

$$(9)\quad \max_{s,u,p,q}\left\{ \begin{array}{llllll} s(P - Lz^l) & + u(Q - Mz^l) & - p\bar{x} & - q\bar{y} & - m^l & \Bigg| \\ s & & & & \leqslant \bar{s} & \\ & u & & & \leqslant \bar{u} & \\ sH & & -p & & \leqslant A & \\ sK & + uD & & -q & \leqslant B & \\ s \geqslant 0 & u \geqslant 0 & p \geqslant 0 & q \geqslant 0 & & \end{array} \right\} =$$

$$= \mathrm{d}h \geqslant 0$$

No infinite solutions may exist to this subproblem (the coefficients of p and q in the objective form being non-positive, and all other variables being bounded).

A feasible or optimal solution of the dual master is assumed to be known

$$t_j^l,\ v^l,\ r^l$$

with corresponding simplex multipliers representing an infeasible or optimal feasible dual solution

$$c^l,\ z^l,\ m^l$$

The objective function h of the dual master equals the simplex multipliers times the right hand constants

$$(10)\quad h = \sum_j (s^jP + u^jQ - p^j\bar{x} - q^j\bar{y})t_j^l + v^lR - r^l\bar{z} = \bar{v}c^l + Cz^l + m^l$$

It is advantageous to note that every inequality of the primal or dual master may be formulated as an equation by subtracting or adding a nonnegative slack variable.

The dual constraints corresponding to the v, r and slack variables of the dual master are the relations $c + Nz \geqslant R$, $-z \geqslant -\bar{z}$, $c \geqslant 0$, $z \geqslant 0$, which are part of the constraints of the primal subproblem. Therefore the simplex multipliers c^1 and z^1 of the dual master may be used to construct an improved solution to the primal subproblem, *provided that none of the v, r and slack variables is a candidate for introduction into the basis of the dual master.*

The dual constraints corresponding to the a, x and slack variables of the primal master are the relations $s \leqslant \bar{s}$, $sH - p \leqslant A$, $s \geqslant 0$, $p \geqslant 0$, which are part of the constraints of the dual subproblem.

The simplex multipliers satisfy the dual constraints, except for those dual constraints which correspond to variables which are candidates for introduction into the basis [1].

Therefore, the simplex multipliers s^k and p^k of the primal master may be used to construct an improved solution to the dual subproblem *provided that none of the a, x and slack variables is a candidate for introduction into the basis of the primal master.*

In the case that only improved feasible solutions of the primal and dual masters are sought, the above conditions may be satisfied by slight modifications of the linear programming algorithm used.

For an outline of the algorithm in greater detail the following definitions are required.

The primal common subproblem is defined as the problem obtained when $z = z^l$ and $c = c^l$ have been inserted into the primal subproblem (5) and all constant terms in the objective function dropped, i.e.

$$(11) \qquad \operatorname*{Min}_{b,y} \left\{ \bar{u}b + (B - s^k K)y \;\middle|\; \begin{array}{ll} b + Dy \geqslant Q - Mz^l \\ \quad -y \geqslant -\bar{y} \\ b \geqslant 0 \quad y \geqslant 0 \end{array} \right\} = -\mathrm{d}f^0$$

which equals in value that of *the dual common subproblem* similarly defined as

---

[1] Cf. G. B. DANTZIG, *op. cit.*, section 8-5.

The dual constraints corresponding to the linear programme given there may be formulated as

$$uP_j \leqslant c_j \qquad (j = 1, \ldots, n)$$

where u is a row vector of unrestricted dual variables. The reduced cost coefficient of a variable is according to (16)

$$\bar{c}_j = c_j - \pi P_j$$

A variable which is not a candidate for introduction has a nonnegative reduced cost coefficient, hence

$$c_j - \pi P_j \geqslant 0$$

or

$$\pi P_j \leqslant c_j \qquad (j = \text{non candidates for introduction})$$

A variable which is a candidate for introduction will have a negative reduced cost coefficient, hence

$$c_j - \pi P_j < 0$$

or

$$\pi P_j > c_j \qquad (j = \text{candidates for introduction})$$

It follows, that the simplex multipliers will only satisfy those dual constraints for which the corresponding primal variable is not a candidate for introduction into the basis.

the problem obtained when $s = s^k$ and $p = p^k$ have been inserted into the dual subproblem (9) and all constant terms in the objective function dropped, i.e.

$$(12)\qquad \underset{u,q}{\text{Max}} \left\{ u(Q - Mz^l) - q\bar{y} \;\middle|\; \begin{array}{lll} u & & \leqslant \bar{u} \\ uD & -q & \leqslant B - s^k K \\ u \geqslant 0 & q & \geqslant 0 \end{array} \right\} = \mathrm{d}h^0$$

where

$$(13)\qquad -\mathrm{d}f^0 = \mathrm{d}h^0$$

Further definitions of importance for following the remaining of this section are

$-\mathrm{d}f$ optimal value of the primal subproblem

$-\mathrm{d}f^0$ optimal value of the primal common subproblem

$-\mathrm{d}f'$ value of the primal subproblem with $c = c^l$, $z = z^l$ for an achieved feasible solution

$-\mathrm{d}f''$ possible improvement of the value of the primal subproblem with $c = c^l$, $z = z^l$ by achieving the optimal instead of the current feasible solution

$-\mathrm{d}f'''$ optimal value of the primal subproblem with $c = c^l$, $z = z^l$, thus

$$(14)\qquad -\mathrm{d}f' - \mathrm{d}f'' = -\mathrm{d}f'''$$

or from (5) and (11)

$$(15)\qquad -\mathrm{d}f''' = -\mathrm{d}f^0 + \bar{v}c^l + (C - s^k L)z^l - w^k$$

$\mathrm{d}h$ optimal value of the dual subproblem

$\mathrm{d}h^0$ optimal value of the dual common subproblem

$\mathrm{d}h'$ value of the dual subproblem with $s = s^k$, $p = p^k$ for an achieved feasible solution

$\mathrm{d}h''$ possible improvement of the value of the dual subproblem with $s = s^k$, $p = p^k$ by achieving the optimal instead of the current feasible solution

$\mathrm{d}h'''$ optimal value of the dual subproblem with $s = s^k$, $p = p^k$, thus

$$(16)\qquad \mathrm{d}h' + \mathrm{d}h'' = \mathrm{d}h'''$$

or from (9) and (12)

$$(17)\qquad \mathrm{d}h''' = \mathrm{d}h^0 + s^k(P - Lz^l) - p^k\bar{x} - m^l$$

Asteriks may replace apostrophes to indicate an estimate absolutely greater than or equal to the value concerned. This estimate may, whenever necessary, be successively improved until it equals the value being estimated.

The algorithm may then be described as follows.

O. Initiation

The solution process is initiated by reading the data including $\varepsilon \geqslant 0$, formulating the initial parts of the common subproblem, the primal and the dual masters, setting $f = w = \infty$, $h = m = -\infty$, assigning some suitable (arbitrary) values to $s = s^k$ $(0 \leqslant s^k \leqslant \bar{s})$, $z = z^l$ $(0 \leqslant z^l \leqslant \bar{z})$, and $p = p^k = \max(-A + s^k H, 0)$, $c = c^l = \max(R - Nz^l, 0)$.

GC. General control of the process

If $f - h > \varepsilon$ then step 1, 2, 3 are solved *in parallel or in some sequence* else step 4. To ensure convergence the results of the following theorem should be taken into account.

1. Common subproblem and information transfer decision

On the basis of the latest received information concerning s^k, p^k, c^l, z^l the modified objective function and the modified constants of the common subproblem (11) are obtained.

The common subproblem is then solved for i) a primal, or ii) a dual, or iii) a primal and a dual feasible solution. The decision as to which type of solution (i, ii, iii) that is desired may be made in the course of the solution process in order to minimize the computational work necessary to produce a primal or a dual solution or both which may improve the primal or the dual master or both. The optimization of the primal or the dual or both solution(s) of the common subproblem must at least be continued until i) $-\mathrm{d}f' < 0$, or ii) $\mathrm{d}h' \geqslant 0$, or iii) $-\mathrm{d}f' \leqslant 0$ and $\mathrm{d}h' \geqslant 0$.

During the solution process the estimates $-\mathrm{d}f^{**}$ and $\mathrm{d}h^{**}$ may be obtained and used together with $-\mathrm{d}f'$, $\mathrm{d}h'$ for determining which type of solution should be aimed at in solving the common subproblem (cf. section 4).

If $-\mathrm{d}f' < 0$ information concerning the vector of t_i coefficients corresponding to the achieved feasible solution b^i, y^i of the primal common subproblem is sent to the primal master (3).

If $\mathrm{d}h' > 0$ information concerning the vector of t_j coefficients corresponding to the achieved feasible solution u^j, q^j of the dual common subproblem is sent to the dual master (7).

2. Primal master problem

An improved primal feasible solution is obtained to the primal master (3), *such that none of the a, x and the slack variables is a possible candidate for introduction into the basis.* The value of the objective function f^k provides an *upper bound* upon the optimal solution. The information $s = s^k, p = p^k$, $w = w^k$, $f = f^k$ is sent to the dual master and to the common subproblem.

3. Dual master problem

An improved primal feasible solution is obtained to the dual master (7), *such that none of the v, r and slack variables is a possible candidate for introduction in the basis.* The value of the objective function h^l provides a *lower bound* upon the optimal solution. The information $c = c^l$, $z = z^l$, $m = m^l$, $h = h^l$ is sent to the primal master and to the common subproblem.

4. Final solution

Establishing as may be required

i) the achieved primal feasible (ε-optimal) solution

$$a = a^k, \qquad b = \sum_i b^i t_i^k, \qquad c = \sum_i c^i t_i^k,$$

$$x = x^k, \qquad y = \sum_i y^i t_i^k, \qquad z = \sum_i z^i t_i^k$$

and/or

ii) the achieved dual feasible (ε-optimal) solution

$$s = \sum_j s^j t_j^l, \qquad u = \sum_j u^j t_j^l, \qquad v = v^l,$$

$$p = \sum_j p^j t_j^l, \qquad q = \sum_j q^j t_j^l, \qquad r = r^l$$

Theorem. *The process above will converge to an optimal primal and dual solution provided that*

either

i) *a primally feasible solution to the common subproblem is only obtained when*

$$-(f-h) + \mathrm{d}h' + \mathrm{d}h'' + \mathrm{d}f'' < 0$$

and this solution is improved until at least

$$-\mathrm{d}f' < 0 \qquad \text{i.e.} \qquad -\mathrm{d}f^0 < -\bar{v}c^l - (C - s^k L)z^l + w^k$$

before the primal master is entered;

ii) *a dually feasible solution to the common subproblem is only obtained when*

$$f - h - \mathrm{d}f' - \mathrm{d}f'' - \mathrm{d}h'' > 0$$

and this solution is improved until at least

$$\mathrm{d}h' > 0 \qquad \text{i.e.} \qquad \mathrm{d}h^0 > -s^k(P - Lz^l) + p^k\bar{x} + m^l$$

before the dual master is entered;

iii) *transfer is compulsory*

a) *if* $f - h > 0$ *and* $\mathrm{d}h''' = 0$ *then go to step* 1 (i) *and then to* 2,

b) *if* $f - h > 0$ *and* $-\mathrm{d}f''' = 0$ *then go to step* 1 (ii) *and then to* 3,

c) *if* $f - h = 0$ *then go to step* 4;

iv) *the terms* $\mathrm{d}f''$ *and* $\mathrm{d}h''$ *above represent an envisaged inoptimality of the corresponding subproblem solution which successively must be decreased to zero;*

or

a scheme of iterations is used which will cause transfer from primal to dual iterations if $-\mathrm{d}f''' \geqslant 0$, *and from dual to primal iterations if* $\mathrm{d}h''' \leqslant 0$, *and finish if* $f = h$.

Proof

THE INITIATION OF ITERATIONS.

The initial primal subproblem solution must always enter the primal master to fulfil the constraint $\sum_{i=1} t_i = 1$; similarly, the initial dual subproblem solution must always enter the dual master to fulfil the constraint $\sum_{j=1} t_j = 1$, hence the process may always be started.

The optimal solution of the primal and dual common subproblem must lead to an improvement of either the primal or the dual master or both.

The value of the primal subproblem is $-\mathrm{d}f'''$ and that of the dual subproblem $\mathrm{d}h'''$. If $-\mathrm{d}f''' < 0$ (or $\mathrm{d}f''' > 0$) then the primal subproblem solution must improve the primal master (degeneracy being handled by the lexicographic method). Similarly, if $\mathrm{d}h''' > 0$ then the dual subproblem solution must improve the dual master (degeneracy being handled by the lexicographic method). A positive value of $\mathrm{d}f''' + \mathrm{d}h'''$ means that either $\mathrm{d}f'''$ or $\mathrm{d}h'''$ or both are positive and hence that the corresponding primal and dual subproblem solutions may improve either the primal or the dual or both masters.

Thus it is of importance to consider the value of

(18)

$$\mathrm{d}f''' + \mathrm{d}h''' =$$

$$\begin{array}{ll|lr}
= -(-\mathrm{d}f^0 + \bar{v}c^l + (C - s^kL)z^l - w^k) + & & = \mathrm{d}f''' & \text{from (15)} \\
\quad + (\mathrm{d}h^0 + s^k(P - Lz^l) - p^k\bar{x} - m^l & = & = \mathrm{d}h''' & \text{from (17)} \\
= \quad s^kP - p^k\bar{x} + w^k - & & = f & \text{from (6)} \\
\quad -(\bar{v}c^l + Cz^l + m^l) - & & = -h & \text{from (10)} \\
\quad -(-\mathrm{d}f^0) + \mathrm{d}h^0 = & & = 0 & \text{from (13)} \\
= \quad f - h & & &
\end{array}$$

According to the Duality Theorem [1] the primal objective function will always be greater than or equal to the dual objective function, hence $f - h \geqslant 0$. Unless an optimal primal and an optimal dual solution have been found the strict inequality will apply, thus

$$\text{(19)} \qquad \mathrm{d}f''' + \mathrm{d}h''' = f - h > 0$$

Therefore, the optimal primal and dual solution of the common subproblem may be used to improve either the primal master or the dual master or both until an optimal primal and an optimal dual solution have been obtained to the entire problem, in which case $f - h = 0$.

The cases in which the optimal primal and the optimal dual common subproblem solution may not improve one of the masters.

It follows from (19) that

$$\text{(20)} \qquad \text{if} \quad \mathrm{d}f''' \geqslant f - h > 0 \quad \text{then} \quad \mathrm{d}h''' \leqslant 0$$

in which case it would be of no avail to obtain an improved feasible or optimal dual common subproblem solution as neither would be able to improve the dual master.

Similarly, it follows from (19) that

$$\text{(21)} \qquad \text{if} \quad \mathrm{d}h''' \geqslant f - h > 0 \quad \text{then} \quad \mathrm{d}f''' \leqslant 0$$

in which case it would be of no avail to obtain an improved feasible or optimal primal common subproblem solution as neither would be able to improve the primal master.

The corresponding conditions for the case of a certain inoptimality of the primal and dual common subproblem solutions.

If instead of considering a completely optimal common subproblem solution, a certain amount of inoptimality is envisaged formula (19) may be expressed using (14) and (16) as

$$\text{(22)} \qquad \mathrm{d}f' + \mathrm{d}f'' + \mathrm{d}h' + \mathrm{d}h'' = f - h$$

The primal common subproblem solution may only improve the primal master *if*

$$\text{(23)} \qquad -\mathrm{d}f' < 0$$

which condition using (22) may be expressed as

$$\text{(24)} \qquad -\mathrm{d}f' = -(f - h) + \mathrm{d}h' + \mathrm{d}h'' + \mathrm{d}f'' < 0$$

from which condition i) of the theorem is obtained.

---

(1) Cf. G. B. Dantzig, *op. cit.*, section 6-3-(6).

Similarly, the dual common subproblem solution may only improve the dual master *if*

(25) $$\mathrm{d}h' > 0$$

which condition using (22) may be expressed as

(26) $$\mathrm{d}h' = f - h - \mathrm{d}f' - \mathrm{d}f'' - \mathrm{d}h'' > 0$$

from which condition ii) of the theorem is obtained.

The possibility of reducing $\mathrm{d}h'$, $\mathrm{d}h''$, $\mathrm{d}f''$ *to zero in a finite number of iterations will always enable further improvement of the primal master unless an optimal solution has already been attained.*

By iterating between the dual master and dual subproblem (s, p unchanged) we may reduce in a finite number of iterations

$$\mathrm{d}h' + \mathrm{d}h'' = \mathrm{d}h''' = 0$$

Similarly, by finding a completely optimal primal common subproblem solution we may reduce in a finite number of pivots

$$\mathrm{d}f'' = 0$$

Therefore we may always make the expression (24) less than zero and thus enable an improvement of the primal master unless

$$f - h = 0$$

in which case both an optimal primal and an optimal dual solution have been found.

Similarly, the possibility of reducing $\mathrm{d}f'$, $\mathrm{d}f''$, $\mathrm{d}h''$ *to zero in a finite number of iterations will always enable further improvement of the dual master unless an optimal dual solution has already been attained.*

As $\mathrm{d}f'$, $\mathrm{d}f''$, $\mathrm{d}h''$ may always be reduced to zero in a finite number of operations, the expression (26) may always be fulfilled, and hence an improved dual solution always be possible unless both an optimal primal and an optimal dual solution have been reached.

THE CONDITIONS FOR TRANSFER NOT FULFILLED AT EVERY STAGE.

That the conditions (24) and (26) are not redundant may be seen by considering the case of the primal master having reached the optimal primal solution and simplex multipliers, though the dual master has one or more subproblem solutions to complete before the optimal dual solution may be reached. As then $f = f^0$, $h + \mathrm{d}h''' \geqslant h^0 = f^0$ it follows that $-\mathrm{d}f''' \geqslant 0$ and expression (24) will therefore correctly prevent transfer to the primal master. Concurrently expression (26) would permit transfer from the primal master as $\mathrm{d}f''' = 0$.

The conditions (24) and (26) explain the interesting experience of D. Pigot [1]

[1] Cp. D. PIGOT, *op. cit.*, end of section 2.3.

of the impossibility of affecting an improvement of one of the master problems (the primal) after only one iteration had been made between the other master (the dual) and its subproblem, and that on the average three (dual) iterations proved necessary before a successful transfer to the other (primal) iteration direction was possible.

CONVERGING TOWARDS THE OPTIMUM.

If alternating transfers are made from primal to dual iterations and from dual to primal iterations fulfilling conditions i-iv, this will lead to strictly monotonically improved primal and dual master solutions with their final value equal to the optimal solution of the problem.

As no non-improving subproblem solution may ever enter a master, any alternating transfer between the various problems making some iterations at each stage, will lead to monotonically improved primal and dual masters solutions with their final value equal to the optimal solution of the problem.

4. AN UPPER BOUND $-\mathrm{d}f^{**}$ OR $\mathrm{d}h^{**}$ UPON THE POSSIBLE FURTHER IMPROVEMENT $-\mathrm{d}f''$ OR $\mathrm{d}h''$

A feasible solution is assumed to be available to the primal subproblem with inserted c^l, z^l solution, thus to the problem 3-(11) such that no slack variable is a candidate for introduction into the basis.

As longs as the $\bar{u}$ coefficients have been chosen large enough, no b variable will ever become a candidate for introduction. Therefore, only y variables with negative reduced cost coefficients

$$\bar{\bar{B}} = (B - s^k K) - u^i D - q^i(-I) \tag{1}$$

where u^i, q^i represent the simplex multipliers corresponding to the b^i, y^i solution, may be candidates for introduction into the basis. The y variables are, however, subject to the upper bound $\bar{y}$.

A lower bound upon the possible improvement of the primal subproblem with unchanged c, z, is therefore given by

$$\begin{aligned} -\mathrm{d}f^{**} &= \sum_j (\mathit{if}\ \bar{\bar{B}}_j < 0\ \mathit{then}\ \bar{\bar{B}}_j \bar{y}_j\ \mathit{else}\ 0) \\ &\leqslant -\mathrm{d}f'' \end{aligned} \tag{2}$$

assuming that no b and no slack variable is a candidate for introduction into the basis

An upper bound upon the possible improvement of the dual subproblem with unchanged s, p, is similarly derived by defining the reduced revenue coefficients

$$\bar{\bar{Q}} = (Q - Mz^l) - Ib^j - Dy^j \tag{3}$$

where b^j, y^j represent the simplex multipliers corresponding to the u^j, q^j solution, and obtaining

$$\text{(4)} \qquad \mathrm{d}h^{**} = \sum_j (\textit{if } \bar{\bar{Q}}_j > 0 \textit{ then } \bar{u}_j\bar{\bar{Q}}_j \textit{ else } 0) \geqslant \mathrm{d}h''$$

assuming that no q and no slack variable is a candidate for introduction into the basis.

If an optimal solution is obtained to the corresponding subproblem, the estimates become

$$-\mathrm{d}f^{**} = -\mathrm{d}f'' = 0$$

and

$$\mathrm{d}h^{**} = \mathrm{d}h'' = 0$$

as no variable is a candidate for introduction.

The above lower and upper bounds may be used to avoid unnecessary improvement of a primal or dual feasible solution of the common subproblem if $-\mathrm{d}f' - \mathrm{d}f^{**} > 0$ or $\mathrm{d}h' + \mathrm{d}h^{**} < 0$, respectively.

If $-\mathrm{d}f' - \mathrm{d}f^{**} > 0$ then it follows from (2) and 3-(14) that even an optimal primal common subproblem solution will not be able to improve the primal master as

$$-\mathrm{d}f''' = -\mathrm{d}f' - \mathrm{d}f'' \geqslant -\mathrm{d}f' - \mathrm{d}f^{**} > 0$$

in which case further computations aiming at improving the primal common subproblem are of no use.

Similarly, if $\mathrm{d}h' + \mathrm{d}h^{**} < 0$ then it follows from (4) and 3-(16) that even an optimal dual common subproblem solution will not be able to improve the dual master as

$$\mathrm{d}h''' = \mathrm{d}h' + \mathrm{d}h'' \leqslant \mathrm{d}h' + \mathrm{d}h^{**} < 0$$

in which case further computations aiming at improving the dual common subproblem solution are of no use.

5. SUITABLE MODIFICATION IN THE CASE OF BLOCK-DIAGONAL STRUCTURE OF THE COMMON SUBPROBLEM

If the common subproblem D in section 3-(11)-(12) has a block-diagonal structure (as in the introductory illustration), then it may be considered to consist of as many independent part problems as there are blocks and the solution of each of these may take place in parallel.

In the primal phase, different solutions $n \in \mathcal{N}(g, l)$ of the part problems $g \in \mathcal{G}$

corresponding to the same quantity solution $l \in \mathfrak{L}$ of the joining activities may be combined to form a solution of the entire primal subproblem or some fraction thereof.

It may therefore be useful to reformulate the constraints of the primal master

$$\text{(1)} \qquad \sum_i t_i = 1$$
$$t_i \geqslant 0 \qquad (i \in \mathfrak{J})$$

by defining nonnegative fractions t_l denoting by which amount the quantity solutions $l \in \mathfrak{L}$ of the joining activities are combined together with the quantity solutions i of the complete primal subproblem

$$\text{(2)} \qquad \sum_i t_i + \sum_l t_l = 1$$
$$t_i \geqslant 0 \qquad (i \in \mathfrak{J})$$
$$t_l \geqslant 0 \qquad (l \in \mathfrak{L})$$

and other nonnegative fractions t_{lgn} denoting by which amount the n^{th} solution of the g^{th} part problem corresponding to the l^{th} quantity solution of the joining activities are taken.

The combinations of fractions of part problem solutions corresponding to a particular fraction of a solution of joining activities must fulfil the following conditions, if the overall combination should give a feasible solution of the whole primal subproblem.

$$\text{(3)} \qquad \sum_{n \in \mathcal{N}(l,g)} t_{lgn} = t_l \qquad (l \in \mathfrak{L}, g \in \mathfrak{G})$$
$$t_{lgn} \geqslant 0 \qquad (l \in \mathfrak{L}, g \in \mathfrak{G}, n \in \mathcal{N}(l, g))$$

The number of equations in (3) would be equal to the number of solutions for the joining activities times the number of part problems. As the number of solutions $\mathfrak{L}$ for the joining activities would increase with every dual iteration, the number of such solutions separately considered by using the constraints (3) may have to be kept limited.

This may be achieved by exchanging some apparently less important subproblem solution based upon $t_{l'}$ and corresponding $t_{l'gn}$ variables into a complete subproblem solution i' based upon a fixed combination of the different solutions $n \in \mathcal{N}(l', g)$ of the part problems $g \in \mathfrak{G}$. The corresponding fixed solution of each part problem g could then be selected on the basis of the current values of the fractions

$$\text{(4)} \qquad \frac{t_{l'gn}}{t_{l'}} \qquad (g \in \mathfrak{G}, n \in \mathcal{N}(l', g))$$

where $l'gn$ denotes the n^{th} solution of the g^{th} part problem corresponding to the l'^{th} quantity solution of the joining activities.

In exchanging a $t_{l'}$ variable for an additional $t_{i'}$ variable, we shall always abolish $|\mathcal{G}|$ (1) of the constraints (3) and $|\mathcal{G}|$ or more variables $t_{l'gn}$, and still obtain the same solution to the master problem with the previous values of other variables and $t_{i'} = t_{l'}$. In the case that $t_{l'}$ is basic and more than $|\mathcal{G}|$ basic variables $t_{l'gn}$ are abolished the corresponding solution of the master problem will become degenerate.

In the dual phase, different solutions $m \in \mathcal{M}(k, g)$ of the part problems $g \in \mathcal{G}$ corresponding to the same price solution $k \in \mathcal{K}$ for the joining resource balances may be combined in the same way to form a solution of the entire dual sub-problem.

If the above approach of combining different solutions of part problems is adopted the linear programme procedure used must be able to deal with degenerate solutions arising from

i) the replacement of several variables t_{lgn}, t_l (t_{kgm}, t_k) by one variable t_i (t_j) ;

ii) all the constraints (3) have constant terms equal to zero (2).

6. REQUIREMENTS ON THE LP ALGORITHMS FOR THE SOLUTION OF THE MASTER PROBLEMS AND THE COMMON SUBPROBLEM

The linear programme procedure(s) used to solve the master problems and the subproblem should deal in a special way with the added variables and the upper bounds, and before information is transferred from one master problem to the other and to the common subproblem(s) ensure that no variable from certain groups of variables may be a candidate for introduction into the basis.

The method used for solving the common subproblem should give a feasible primal and/or a feasible dual solution. This may be obtained by using

1) any linear programming procedure to determine both an improved primal feasible and an improved dual feasible solution, or to determine an optimal solution, thereby obtaining both an optimal primal feasible and an optimal dual feasible solution; or

---

(1) The number of elements of the set $\mathcal{G}$, i.e. the number of part problems is here denoted by $|\mathcal{G}|$.

(2) It seems likely that a useful way of handling the latter degeneracy is : to calculate a joint reduced cost coefficient based upon any particular quantity solution for the joining activities and the most favourable of the corresponding quantity solutions of each of the part problems; after having found the most favourable of these joint reduced cost coefficients to introduce the corresponding group of variables in sequence, the improvement in the objective function only occurring when the last of these variables has entered the basis.

2) some linear programming procedure which provides at each iteration both a feasible primal and a dual feasible solution ([1]), like the logarithmic potential method by R. Frisch ([2]) possibly modified as proposed by G. R. Parisot ([3]).

The method used should probably make use of the previous common sub-problem solution in obtaining a new solution to this problem after the objective function and constants have been modified. Possibly this may best be achieved by using a parametric programming approach ([4]) in conjunction with one of the above mentioned methods.

If as a rule only improved feasible primal or dual solutions of the common subproblem are sought in the iteration process, then it may be required that the algorithm used should give a bound upon the possible improvement of the corresponding primal or dual objective function.

7. INFINITE SOLUTIONS OF THE PRIMAL AND DUAL SUBPROBLEMS

Even though the overall problem may be assumed to have a finite optimal solution, the consequences of permitting the primal and dual subproblems to have infinite solutions (dropping the extended formulation of section 2) may be investigated with respect to the three possibilities that only $y(u)$, both y and $z(u$ and $s)$, only $z(s)$ variables may give rise to infinite solutions of the primal (dual) subproblem.

The general consequences would be the following.

i) The primal and dual masters would have to have columns corresponding to the infinite directions, which would not be constrained by the requirements

$$\sum_i t_i = 1 \quad \text{and} \quad \sum_j t_j = 1.$$

The above constraints would then have to be formulated as

$$\sum_i \delta_i t_i = 1 \quad \text{and} \quad \sum_j \delta_j t_j = 1$$

where $\delta_i = 0$ or 1 and $\delta_j = 0$ or 1 depending upon whether the ith and the jth column correspond to a finite solution or an infinite direction, respectively.

([1]) Cf. P. Wolfe, *Methods of Nonlinear Programming*, in J. Abadie (Editor), Nonlinear Programming, North-Holland Publishing Company, Amsterdam, 1967, pp. 117-119.

([2]) R. Frisch, *The logarithmic potential method for solving linear programming problems*, Memorandum from the University Institute of Economics, Oslo, 1955.

([3]) G. R. Parisot, *Résolution numérique approchée du problème de programmation linéaire par application de la programmation logarithmique*, Thesis, Université de Lille, 1961.

([4]) Cf. G. B. Dantzig, *Linear Programming and Extensions, op. cit.*, section 11-3; and Sverre Spurkland, *The Parametric Descent Method of Linear Programming*, Norwegian Computing Center, Oslo.

ii) The primal and dual subproblems would have to be solved for these infinite directions.

iii) As long as there remains an infinite direction to the primal (dual) of a subproblem there exists no dual (primal) solution to it. Therefore iterations must be continued between the primal (dual) master and the primal (dual) subproblems until no further infinite direction may improve the primal (dual) master.

Only infinite $y(u)$ solutions possible

The finding of infinite primal (dual) subproblem solutions may in this case easily be achieved in the course of solving the primal phase II problem. If the subproblem is not optimally solved before a switch is made to dual iterations, some yet undiscovered infinite directions may make a feasible dual solution impossible.

Both infinite y and z (u and s) solutions possible

The finding of infinite primal (dual) subproblem solutions becomes then fairly complicated because of difficulty of finding improved $z(s)$ solutions due to the size and structure of the problem. In the case of the primal subproblem the corresponding bounded homogeneous primal subproblem [1] would be

$$(1) \quad \min_{y,z} \left\{ (B - s^k K)y + (C - s^k L)z \;\middle|\; \begin{array}{rl} Dy + Mz & \geqslant 0 \\ Nz & \geqslant 0 \\ -y & \geqslant -1 \\ -z & \geqslant -1 \\ y \geqslant 0 \quad z & \geqslant 0 \end{array} \right\} =$$

$$(2) \quad = \max_{u,v,q,r} \left\{ u0 + v0 - q1 - r1 \;\middle|\; \begin{array}{rl} uM + vN - r & \leqslant C - s^k L \\ uD - q & \leqslant B - s^k K \\ u \geqslant 0 \; v \geqslant 0 \; q \geqslant 0 \; r & \geqslant 0 \end{array} \right\} \leqslant 0$$

---

[1] The structure of this problem seemingly suggests solution by dual decomposition. As the coefficients $B - s^k K$, $C - s^k L$ of the modified objective function have changed every time when a new solution y, z is required, the previous dual solutions u, q of the corresponding subproblem(s) are no longer true solutions of them, and therefore it is impermissible to use the previous vectors of the dual master problem. The previous dual subproblem solutions may possibly be brought up to date by only making a change in the q vector, which will only affect the $q1$ terms of the dual master problem. The vectors of the previous master problem may then so modified be used in finding an improved solution to the present master problem.

Only infinite $z(s)$ solutions possible

As it is the finding of infinite $z(s)$ solutions, which seems to lead to increased computational difficulties, no greater relief occurs apparently from restricting the possibility of infinite solutions only to the $z(s)$ variables.

8. SUMMARY OF THE GENERALIZED METHOD

Fig. 4, 5 and 6 summarize the main features of the generalized method.

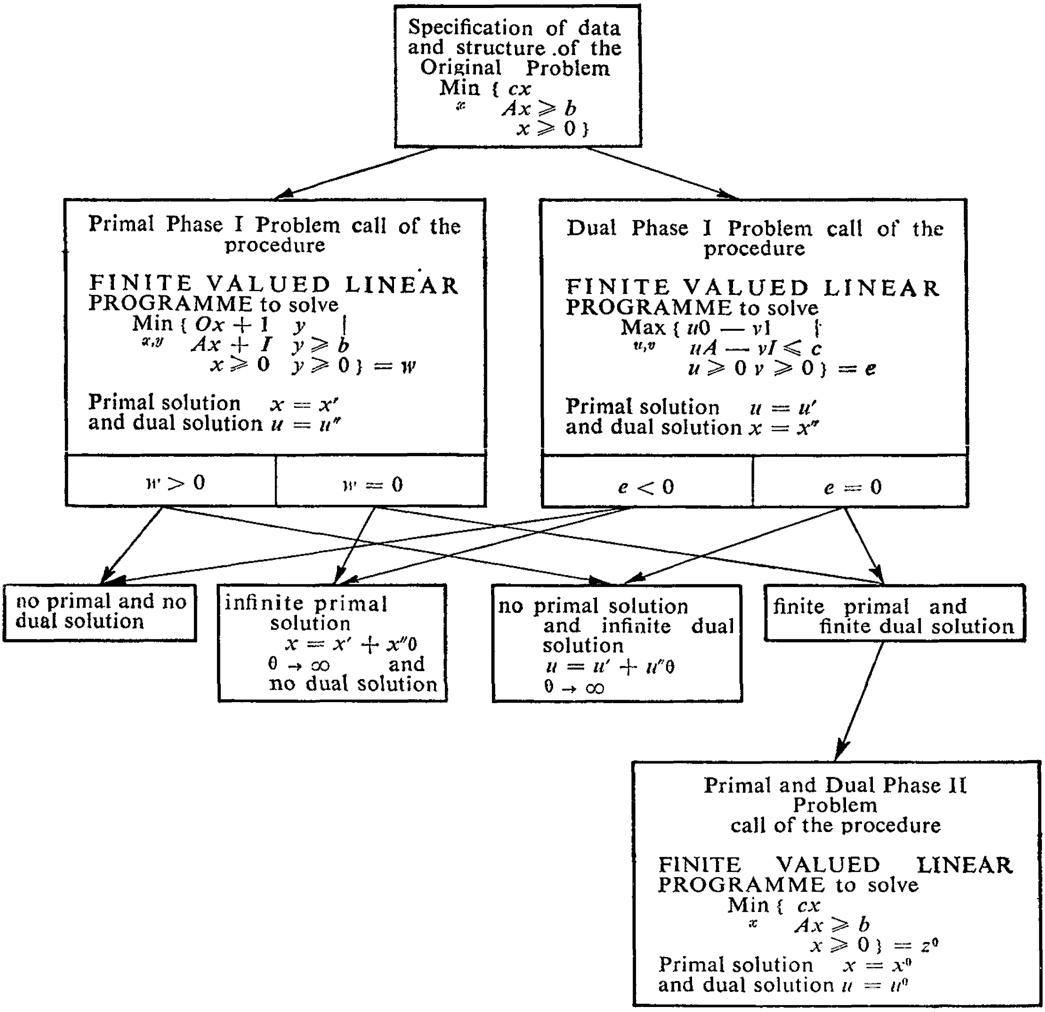

Figure 4

The method of establishing the solution of any linear programme by considering three related finite valued linear programmes.

Note : An advantage from a computational point of view may be to replace the upper bounds upon the original primal (dual) variables by a few constraints.

A formally superflous identity matrix I has therefore been inserted into all the tables of the following summary. By changing the definition of the *vectors* y, $\mathbf{v}$, 1 of figure 4 and a, b, c, $\bar{s}$, $\bar{u}$, $\bar{v}$, p, q, r, $-\bar{x}$, $-\bar{y}$, $-\bar{z}$ of figures 5 and 6 to

	$\bar{s}$	$\bar{u}$	$\bar{v}$	A	B	C	≥ Min	
s	I			H	K	L	≥	P
u		I			D	M	≥	Q
v			I			N	≥	R
p				−I			≥	$-\bar{x}$
q					−I		≥	$-\bar{y}$
r						−I	≥	$-\bar{z}$
	a	b	c	x	y	z		

Figure 5

Extension of the finite valued primal phase I or dual phase I or primal-or-dual phase II linear programme (in bold frames) with upper bounds upon the original primal and dual variables.

become *scalars* — I to mean a row vector (— 1, — 1, ..., — 1) and I the corresponding column vector (1, 1, ..., 1) then all the artificial variables $y(\mathbf{v})$ will be replaced by one artificial variable, and the upper bounds upon the original primal (dual) variables $x(s)$ by a single constraint of the type

$$-\sum_j x_j \geqslant -\bar{x} \quad \Big(\sum_j s_j \leqslant \bar{s}\Big), \text{ etc.}$$

As a necessary preparation in developing a computer programme based upon the above decomposition mathod, A. C. McKay of the Faculty of Commerce and Social Science, University of Birmingham, has constructed

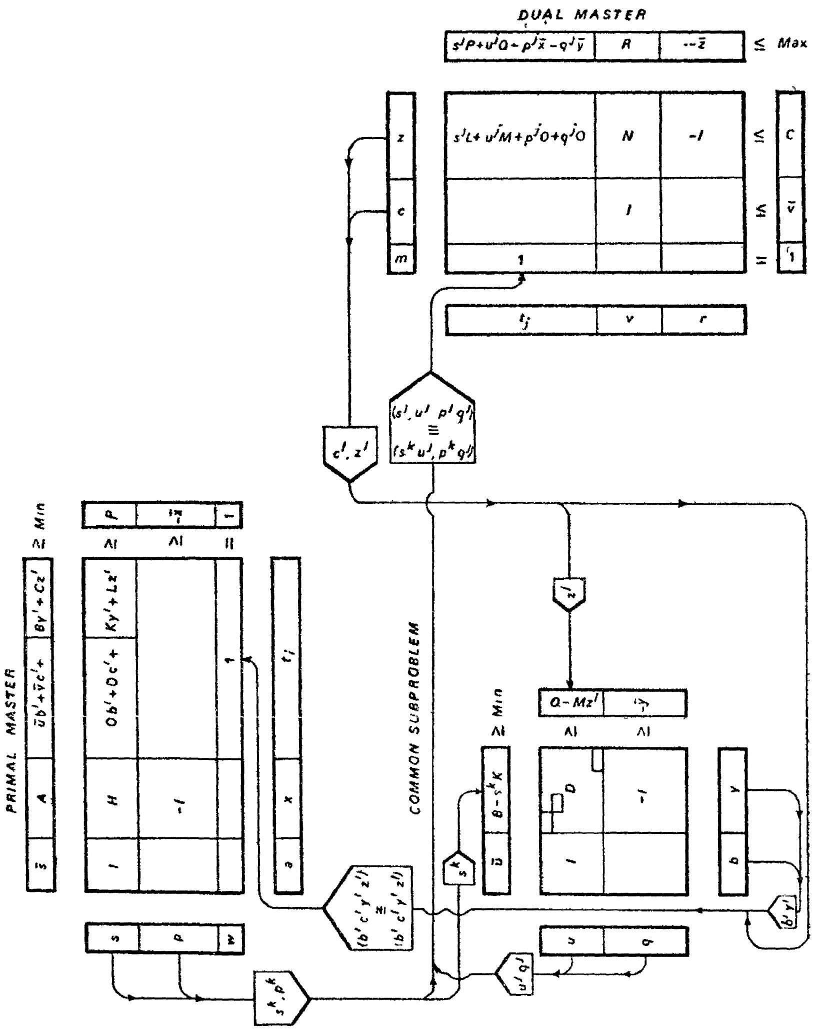

Figure 6

Decomposition of a finite valued linear programme in primal and dual directions to obtain a primal and a dual master solved in parallel and one or more common subproblems (information flow).

It is of importance to observe that neither master problems nor subproblems make detailed use of the transmitted values variables but only use them in formulating new columns or rows of net effects. Therefore the information flow may be reduced by transmitting corresponding net effects instead of values of variables wherever this would lead to a lesser amount of information.

numerical test examples [1], one of which completes this paper by illustrating in full detail the functioning of the method. This numerical study was undertaken in order to provide a test example for a computer programme.

An extremely simplified computer programme to illustrate the main computational aspects of the method has been elaborated [2]. An advanced computer programme by A. C. McKay has recently been successfully tested by him upon a numerical problem involving formal consideration of two common subproblems [1]. A systematic study of factors influencing the practical speed of convergence of the method is currently being undertaken.

9. CONCLUSIONS

The favourable computational experiences of Beale, Small and Hughes [3] with their large primal decomposition programme may probably be taken as an indication that the above generalization of the double decomposition method by D. Pigot may offer not only an important theoretical but also a forceful practical tool for achieving a near optimal solution of very large economic planning systems, especially for optimal international/interregional interunit economic planning [4].

A great advantage of the method is that the solution of a common subproblem of block diagonal structure may take place in parallel, each block independently of the other. A very large economic planning problem containing reasonable number of joining resource balances and joining activities and a great number N of diagonal blocks each of reasonable size, may therefore, in principle, be solved on N computers each solving one block of the common subproblem plus 2 computers solving the primal and dual master problems.

This should make the practical formulation and solution of very large economic planning systems possible by only relying upon available computing resources from high speed computers to pencil and paper.

---

(1) Cf. A. C. McKay, Centralization and Decentralization of Decision Making, The Double Decomposition Method, A Numerical Example with One Common Subproblem and Finite Optimal Solution ; A Numerical Example with Two Common Subproblems and Finite Optimal Solution ; University of Birmingham, CREES, Discussion Papers, Series RC/A, Nos. 14 and 17, Birmingham, Great Britain, 10th August, 1967 and Ist April, 1968.

(2) T. O. M. Kronsjö, *Computer Problems in Mathematical Programming (1-9)* University of Birmingham, CREES, Discussion Paper, Series RC/A No. 21, Birmingham, Great Britain, forthcoming.

(3) E. M. L. Beale, P. A. B. Hughes, and R. E. Small, Experiences in using a Decomposition Program, *The Computer Journal*, Vol. 8, No. 1, April, 1965, pp. 13-18. Recently this report has been followed by an investigation by P. Broise, P. Huard and J. Sentenac, *Décomposition des programmes mathématiques*, Dunod, Paris, 1968.

(4) Cf. T. O. M. Kronsjö, *Internationnal/Interregional/Interunit Economic Co-operation by Linked Computers*, University of Birminghan, CREES, Discussion Papers, Series RC/A, No. 20, Birmingham, Great Britain, forthcoming.

APPENDIX

A NUMERICAL EXAMPLE WITH ONE COMMON SUBPROBLEM AND FINITE OPTIMAL SOLUTION

by A C McKay*

1 General Remarks Concerning the Numerical Calculations

To achieve greater clarity of exposition all the simplex tableaux are given in the standard canonical form[1] with empty spaces for those elements which the Simplex Method Using Multipliers[2] makes it unnecessary to calculate As the initial solution at one stage or another is both primally and dually infeasible the Self-Dual Parametric Simplex Algorithm[3] is implicitly used The Upper Bounding Technique[4] is used to deal with upper bounds a bar denoting a non-basic variable at its upper bound

It will often be necessary to solve a common subproblem for which one or both of the objective function and the right hand side have changed but for which the inverse of a previous basis is known Suppose the previous problem was

$$x_o = \min \{ cx + dy \mid Ax + By \geq b,\ x \geq 0\ \ y \geq 0 \}$$

and that x is the vector of variables that were in the basis The problem would be set out as

basic variable	x	y	e	x_o	constant
x_o	$-c$	$-d$	0	1	0
e	$-A$	$-B$	1	0	$-b$
x_o	0	$-d+cA^{-1}B$	$-cA^{-1}$	1	$cA^{-1}b$
x	1	$A^{-1}B$	$-A^{-1}$	0	$A^{-1}b$

If the problem is changed to

$$x_o = \min \{ c x + d y \mid Ax + By \geq b,\ x \geq 0\ \ y \geq 0 \}$$

it is clear that the tableau with the vector x basic will be

basic variable	x	y	e	x_o	constant
x_o	0	$-d + c A^{-1}B$	$-c A^{-1}$	1	$c A^{-1}b'$
x	1	$A^{-1}B$	$-A^{-1}$	0	$A^{-1}b'$

Thus the tableau can be constructed from a knowledge of A^{-1} The result will be extensively used to avoid unnecessary operations

The possibility arises that in using an earlier inverse with a new objective function or right hand side, an upper bound may be exceeded As a first stage of the simplex method it would then be necessary to remove this variable from the basis and include it as a non-basic variable at its upper bound

The dual of the common subproblem is not solved because the solution is available in the tableau of the primal problem The simplex multipliers give the solution for the dual variables corresponding to the constraints of the original problem The values of the dual variables corresponding to upper bounds on any primal variable are given by the relative cost factor of a variable which is at its upper bound otherwise they are zero

In a problem of the size used here an optimal solution to the common subproblem is easily found In larger problems it would be necessary only to find a primally feasible solution and a dually feasible solution to the common subproblem It is then possible to calculate upper bounds upon the possible further improvement of the subproblem solutions these upper bounds being used in the decision of which master problem to enter

---

* The Swedish Council for Social Research has sponsored this research by a grant awarded to T O M Kronsjö University of Birmingham under whose supervision it has been undertaken

1) Cf G B Dantzig Linear Programming and Extensions Princeton University Press Princeton New Jersey 1963 section 5-1

2) Cf opus cit section 9-1

3) Cf opus cit section 11 3

4) Cf opus cit section 18-1

2. The Problem Constructed for Test Calculations

To investigate the behaviour of the generalization of the double decomposition method elaborated by T.O.M. Kronsjö in the preceding paper the following numerical example was constructed.

$$\min_{x,y,z} \{ x_1 + x_2 + y_1 + y_2 - y_3 + z_1 - z_2 \mid$$
$$x_1 + x_2 + y_1 + y_2 + y_3 + z_1 + z_2 \geq 3$$
$$2x_1 - x_2 + 2y_1 - y_2 + y_3 - 2z_1 - z_2 \geq 4$$
$$- y_1 - y_2 + z_1 - 2z_2 \geq 2$$
$$2y_1 + y_2 - y_3 - z_1 - z_2 \geq 1$$
$$z_1 - 2z_2 \geq 0$$
$$x_1 \geq 0, x_2 \geq 0,\ y_1 \geq 0, y_2 \geq 0, y_3 \geq 0,\ z_1 \geq 0,\ z_2 \geq 0 \}$$

which has the optimal solution: $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$

The Extended Formulation of the Problem

An extended formulation is used, where bars denote upper bounds upon variables. The notation follows that of T.O.M. Kronsjö as given in the preceding paper 3-(1), and thus

$$\bar{s} = (10\ 10)\quad \bar{u} = (10\ 10)\quad \bar{v} = (10)\quad A = (1\ \ 1)\quad B = (1\ \ 1\ -1)\quad C = (1\ -1)$$

$$H = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\quad K = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix}\quad L = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}\quad P = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$D = \begin{pmatrix} -1 & -1 & 0 \\ 2 & 1 & -1 \end{pmatrix}\quad M = \begin{pmatrix} 1 & -2 \\ -1 & -1 \end{pmatrix}\quad Q = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$N = (1\ -2)\quad R = (0)$$

$$\bar{x} = \begin{pmatrix} 20 \\ 20 \end{pmatrix}$$

$$\bar{y} = \begin{pmatrix} 20 \\ 20 \\ 20 \end{pmatrix}$$

$$\bar{z} = \begin{pmatrix} 20 \\ 20 \end{pmatrix}$$

Figure 1: The solution process (see facing page)

3. The Solution Process

Although the calculations are here set out in sequence, they can actually be carried out in parallel as shown in Figure.1. The transfer decision in Step 1 is made on the basis of primally and dually feasible solutions to the Common Subproblem.

Step 0 Initial Values

$f = w = 100\quad s^k = (1\ 1)\quad p^k = \max(-A + s^k K, 0) = (2\ 0)\quad h = m = -100\quad z^1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}\quad c^1 = \max(R - Nz^1, 0) = (2)$

Step 1.0 Transfer Decicion

(a) Common Subproblem as in 3-(11)-(12), thus

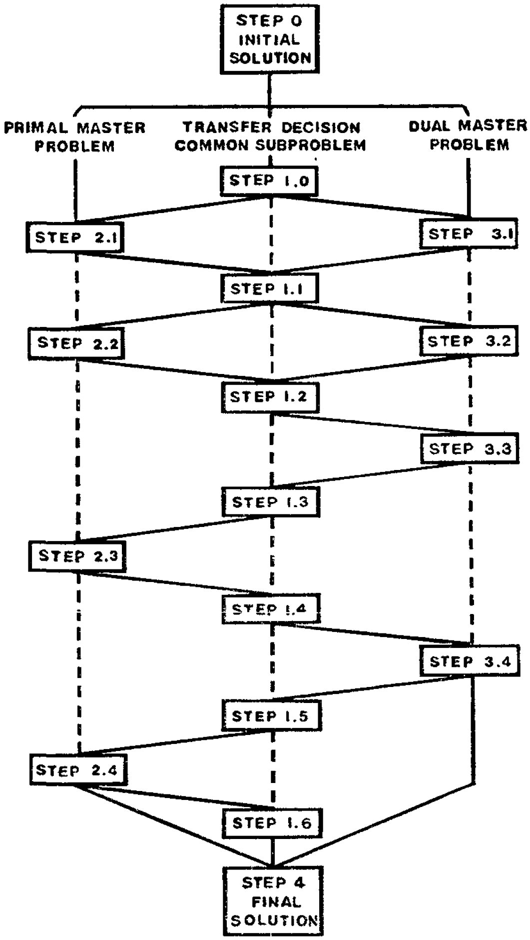
STEP 0
INITIAL
SOLUTION
PRIMAL MASTER PROBLEM
TRANSFER DECISION COMMON SUBPROBLEM
DUAL MASTER PROBLEM
STEP 1.0
STEP 2.1
STEP 3.1
STEP 1.1
STEP 2.2
STEP 3.2
STEP 1.2
STEP 3.3
STEP 1.3
STEP 2.3
STEP 1.4
STEP 3.4
STEP 1.5
STEP 2.4
STEP 1.6
STEP 4
FINAL
SOLUTION

Figure 1

$$
\begin{aligned}
-df^{o} = \min_{b,y} \{ & 10b_1 + 10b_2 + (1-3)y_1 + (1-0)y_2 + (-1-2)y_3 \mid \\
& b_1 \quad - y_1 - y_2 \geq 2 + 2 \\
& b_2 \quad + 2y_1 + y_2 - y_3 \geq 1 + 4 \\
& - y_1 \geq -20 \\
& - y_2 \geq -20 \\
& - y_3 \geq -20 \\
& b_1 \geq 0 \;\; b_2 \geq 0 \quad y_1 \geq 0 \quad y_2 \geq 0 \quad y_3 \geq 0 \}
\end{aligned}
$$

basic variable	b_1	b_2	y_1	y_2	y_3	e_1	e_2	constant
upper bounds			20	20	20			
$-df^{o}$	-10	-10	2	-1	3	0	0	0
e_1	-1	0	1	1	0	1	0	-4
e_2	0	-1	-2*	-1	1	0	1	-5
$-df^{o}$	-10	-11	0	-2	4	0	1	-5
e_1	-1*	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{13}{2}$
y_1	0					0	$-\frac{1}{2}$	$\frac{5}{2}$
$-df^{o}(=dh^{o})$	0	-6	0	-7	-1	-10	-4	60
b_1						-1	$-\frac{1}{2}$	$\frac{13}{2}$
y_1						0	$-\frac{1}{2}$	$\frac{5}{2}$

(b) Primal Subproblem

The objective function is given by 3-(15), thus

$-df^{o\prime} = 60 + 20 + 2 - 100 = -18 < 0$ hence the decision is taken to

TRANSFER INFORMATION TO THE PRIMAL MASTER

consisting of $b^1 = \begin{pmatrix} b_1^1 \\ b_2^1 \end{pmatrix} = \begin{pmatrix} \frac{13}{2} \\ 0 \end{pmatrix}$ $y^1 = \begin{pmatrix} y_1^1 \\ y_2^1 \\ y_3^1 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ 0 \\ 0 \end{pmatrix}$ in the form of the more compact information $\begin{pmatrix} \underline{ub^1 + By^1} \\ Ky^1 \end{pmatrix} = \begin{pmatrix} \underline{65 + \frac{5}{2}} \\ \frac{5}{2} \\ 5 \end{pmatrix} = \begin{pmatrix} 67\frac{1}{2} \\ \frac{5}{2} \\ 5 \end{pmatrix}$

(c) Dual Subproblem

The objective function is given by 3-(17), thus

$-dh^{\prime *} = 60 + 9 - 40 + 100 = 129 > 0$, hence the decision is taken to

TRANSFER INFORMATION TO THE DUAL MASTER

consisting of $u^1 = (u_1^1 \; u_2^1) = (10 \; 4)$ $q^1 = (q_1^1 \; q_2^1 \; q_3^1) = (0 \; 0 \; 0)$

in the form of the more compact information

$(u \; K \quad u^1Q - q^1\bar{y}) = (6 \; -24 \quad 24 - 0) = (6 \; -24 \; 24)$

Step 2.1 Primal Master Problem as in 3-(3), thus

basic variable	a_1	a_2	x_1	x_2	t_1	e_1	e_2	e_3	constant
upper bounds			20	20					
f	-10	-10	-1	-1	$-87\frac{1}{2}$	0	0	0	0
e_1	-1	0	-1	-1	$-\frac{13}{2}$	1	0	0	-3
e_2	0	-1	-2	1	1	0	1	0	-4
e_3	0	0	0	0	-1*	0	0	1	-1
f	-10	-10	-1	-1	0	0	0	$-87\frac{1}{2}$	$87\frac{1}{2}$
e_1			-1			1	0	$-\frac{13}{2}$	$\frac{7}{2}$
e_2	0	-1	-2*	1	0	0	1	1	-5
t_1			0			0	0	-1	1
f	-10	$-\frac{19}{2}$	0	$-\frac{3}{2}$	0	0	$-\frac{1}{2}$	-88	90
e_1						1	$-\frac{1}{2}$	-7	6
x_1						0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{5}{2}$
t_1						0	0	-1	1

$s = (0\ \frac{1}{2})$
$p = (0\ 0)$
$v = (88)$
$f = 90$

Step 3.1 Dual Master Problem as in 3-(7), thus

$$h = \max_{t_j, v, r} \left\{ \begin{array}{lr} (7 - 40 + 24)t \qquad - 20r_1 - 20r_2 & \\ v & \leq 10 \\ (-1 + 6)t_1 + v - r_1 & \leq 1 \\ (0 - 24)t_1 - 2v \qquad - r_2 & \leq -1 \\ t_1 & = 1 \\ t_1 \geq 0 \quad v \geq 0 \quad r_1 \geq 0 \quad r_2 \geq 0 & \end{array} \right\}$$

basic variable	t_1	v	r_1	r_2	e_1	e_2	e_3	constant
upper bounds		10						
h	9	0	20	20	0	0	0	0
e_1	5	1	-1	0	1	0	0	1
e_2	-24	-2	0	-1	0	1	0	-1
e_3	1*	0	0	0	0	0	1	1
h	0	0	20	20	0	0	-9	-9
e_1	0	1	-1*	0	1	0	-5	-4
e_2			0		0	1	24	23
t_1			0		0	0	1	1
h	0	20	0	20	20	0	-109	-89
r_1					-1	0	5	4
e_2					0	1	24	23
t_1					0	0	1	1

$c = (0)$
$z = (20\ 0)$
$m = (109)$
$h = -89$

Step 1.1 Transfer Decision

(a) Common Subproblem (The inverses of previous problems are used to update the modified objective function and constants in common subproblems, and the entering column in master problems)

Modified objective function $10b_1 + 10b_2 + (1 - 1)y_1 + (1 + \frac{1}{2})y_2 + (-1 - \frac{1}{2})y_3$

Basic variable	b_1	b_2	y_1	y_2	y_3	e_1	e_2	constant
upper bounds			20	20	20			
$-df^o$	0	-5	0	$-\frac{13}{2}$	$-\frac{7}{2}$	-10	-5	-75
b_1	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$*	-1	$-\frac{1}{2}$	$-\frac{15}{2}$
y_1					$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{21}{2}$
$-df^o(=dh^o)$	-7	$-\frac{17}{2}$	0	-3	0	-3	$-\frac{3}{2}$	$-\frac{45}{2}$
y_3						2	1	15
y_1						1	0	18

Modified constants
(2 - 20)
(1 + 20)

(b) Primal Subproblem

$$-df'' = -\frac{45}{2} + 0 + 40 - 88 = -70\tfrac{1}{2} < 0 \quad \text{hence}$$

TRANSFER INFORMATION TO THE PRIMAL MASTER

consisting of $\begin{pmatrix} \underline{ub^i + By^i} \\ Ky^i \end{pmatrix} = \begin{pmatrix} \underline{0 + 3} \\ 33 \\ 51 \end{pmatrix} = \begin{pmatrix} 3 \\ 33 \\ 51 \end{pmatrix}$

(c) Dual Subproblem

$$dh'' = -\frac{45}{2} + 22 - 0 + 109 = 108\tfrac{1}{2} > 0 \quad \text{hence}$$

TRANSFER INFORMATION TO THE DUAL MASTER

consisting of $(u^jM \,\vdots\, u^jQ - q^j\bar{y}) = (\tfrac{3}{2}\ -\tfrac{15}{2} \,\vdots\, \tfrac{15}{2} - 0) = (\tfrac{3}{2}\ -\tfrac{15}{2}\ \tfrac{15}{2})$

Step 2.2 Primal Master Problem

basic variable	a_1	a_2	x_1	x_2	t_1	t_2	e_1	e_2	e_3	constant	Additional column (original form)
upper bounds			20	20							
f	-10	$-\frac{19}{2}$	0	$-\frac{3}{2}$	0	$70\frac{1}{2}$	0	$-\frac{1}{2}$	-88	90	$(3 + 0 + 20)t_2$
e_1						$-40\frac{1}{2}$	1	$-\frac{1}{2}$	-7	6	$(33 \quad + 20)t_2$
x_1						6*	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{5}{2}$	$(51 \quad - 40)t_2$
t_1						1	0	0	-1	1	$1\ t_2$
f	-10	$-15\frac{3}{8}$	$-11\frac{3}{4}$	$\frac{19}{8}$	0	0	0	$\frac{43}{8}$	$-82\frac{1}{8}$	$60\frac{5}{8}$	
e_1							1	$-\frac{31}{8}$	$-\frac{83}{8}$	$22\frac{7}{8}$	
t_2							0	$-\frac{1}{12}$	$-\frac{1}{12}$	$\frac{5}{12}$	
t_1							0	$\frac{1}{12}$*	$-\frac{11}{12}$	$\frac{7}{12}$	
f	-10	-10	-1	-1	$-64\frac{1}{2}$	0	0	0	-23	23	s = (0 0)
e_1							1	0	-53	50	p = (0 0)
t_2							0	0	-1	1	w = (23)
e_2							0	1	-11	7	f = 23

Step 3.2 Dual Master Problem

basic variable	t_1	t_2	v	r_1	r_2	e_1	e_2	e_3	constant	Additional column (original form)
upper bounds			10							
h	0	$-108\frac{1}{2}$	20	0	20	20	0	-109	-89	$(2 - 0 + \frac{15}{2})t_2$
r_1		$\frac{3}{2}$*				-1	0	5	4	$(-1 \quad + \frac{3}{2})t_2$
e_2		16				0	1	24	23	$(-\frac{1}{2} \quad - \frac{15}{2})t_2$
t_1		1				0	0	1	1	$1\ t_2$
h	0	0	$-\frac{37}{9}$	$24\frac{1}{9}$	20	$-\frac{37}{9}$	0	$11\frac{5}{9}$	$\frac{67}{9}$	
t_2						$-\frac{2}{9}$	0	$\frac{19}{9}$	$\frac{8}{9}$	
e_2						$\frac{32}{9}$	1	$\frac{56}{9}$	$8\frac{7}{9}$	
t_1						$\frac{2}{9}$*	0	$-\frac{1}{9}$	$\frac{1}{9}$	
h	$18\frac{1}{2}$	0	0	20	20	0	0	$\frac{19}{2}$	$\frac{19}{2}$	c = (0)
t_2						0	0	1	1	z = (0 0)
e_2						0	1	8	7	m = $(\frac{19}{2})$
e_1						1	0	$-\frac{1}{2}$	$\frac{1}{2}$	h = $\frac{19}{2}$

Step 1.2 Transfer Decision

(a) Common Subproblem

Modified objective function $10b_1 + 10b_2 + (1 - 0)y_1 + (1 + 0)y_2 + (-1 - 0)y_3$

NEOCSFOS-6-R1

basic variable	b_1	b_2	y_1	y_2	y_3	e_1	e_2	constant	Modified constants
upper bounds			2						
$-df^o$	-9	-9	0	-1	0	-1	-1	-3	
y_3	-2*	-1	0	1	1	2	1	-5	2 - 0
y_1	-1					1	0	-2	1 - 0
$-df^o(=dh^o)$	0	$-\frac{9}{2}$	0	$-\frac{11}{2}$	$-\frac{9}{2}$	-10	$-\frac{11}{2}$	$25\frac{1}{2}$	
b_1						-1	$-\frac{1}{2}$	$\frac{5}{2}$	
y_1						0	$-\frac{1}{2}$	$\frac{1}{2}$	

(b) Primal Subproblem

$-df''' = 25\frac{1}{2} + 0 + 0 - 23 = \frac{5}{2} > 0$ hence

TRANSFER NO INFORMATION TO THE PRIMAL MASTER

(c) Dual Subproblem

$dh''' = 25\frac{1}{2} + 0 - 0 - \frac{19}{2} = 16 > 0$ hence

TRANSFER INFORMATION TO THE DUAL MASTER

consisting of $(u^jM \vdots u^jQ - g^j\bar{y}) = (\frac{9}{2} \; -25\frac{1}{2} \vdots 25\frac{1}{2} - 0) = (\frac{9}{2} \; -25\frac{1}{2} \; 25\frac{1}{2})$

Step 3.3 Dual Master Problem

basic variable	t_1	t_2	t_3	v	r_1	r_2	e_1	e_2	e_3	constant
upper bounds				10						
h	$18\frac{1}{2}$	0	-16	0	20	20	0	0	$\frac{19}{2}$	$\frac{19}{2}$
t_2			1				0	0	1	1
e_2			$-17\frac{1}{2}$				0	1	8	7
e_1			4*				1	0	$-\frac{1}{2}$	$\frac{1}{2}$
h	$36\frac{1}{2}$	0	0	4	16	20	4	0	$\frac{15}{2}$	$\frac{23}{2}$
t_2							$-\frac{1}{4}$	0	$\frac{9}{8}$	$\frac{7}{8}$
e_2							$\frac{35}{8}$	1	$5\frac{13}{16}$	$9\frac{3}{16}$
r_3							$\frac{1}{4}$	0	$-\frac{1}{8}$	$\frac{1}{8}$

Additional column (original form)

$(0 - 0 + 25\frac{1}{2})t_3$
$(0 \quad + \frac{9}{2})t_3$
$(0 \quad - 25\frac{1}{2})t_3$
$1 \; t_3$

$c = (0)$
$z = (4 \; 0)$
$w = (\frac{15}{2})$
$h = \frac{23}{2}$

Step 1.3 Transfer Decision

(a) Common Subproblem

Modified objective function $10b_1 + 10b_2 + (1 - 0)y_1 + (1 + 0)y_2 + (-1 - 0)y_3$

basic variable	b_1	b_2	y_1	y_2	y_3	e_1	e_2	constant	Modified constants
upper bounds			20	20	20				
$-df^o(=dh^o)$	0	$-\frac{9}{2}$	0	$-\frac{11}{2}$	$-\frac{9}{2}$	-10	$-\frac{11}{2}$	$\frac{15}{2}$	
b_1						-1	$-\frac{1}{2}$	$\frac{1}{2}$	2 - 4
y_1						0	$-\frac{1}{2}$	$\frac{5}{2}$	1 + 4

(b) Primal Subproblem

$-df''' = \frac{15}{2} + 0 + 4 - 23 = -\frac{23}{2} < 0$ hence

TRANSFER INFORMATION TO THE PRIMAL MASTER

consisting of $\begin{pmatrix} \bar{u}b^i + By^i \\ Ky^i \end{pmatrix} = \begin{pmatrix} 5 + \frac{5}{2} \\ \frac{5}{2} \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{15}{2} \\ \frac{5}{2} \\ 5 \end{pmatrix}$

(c) Dual Subproblem

$-dh''' = \frac{15}{2} + 0 - 0 - \frac{15}{2} = 0$ hence

TRANSFER NO INFORMATION TO THE DUAL MASTER

Step 2.3 Primal Master Problem

basic variable	a_1	a_2	x_1	x_2	t_1	t_2	t_3	e_1	e_2	e_3	constant	Additional Column (original form)
upper bounds			20	20								
f	-10	-10	-1	-1	$-64\frac{1}{2}$	0	$11\frac{1}{2}$	0	0	-23	23	$(\frac{15}{2} + 0 + 4)t_3$
e_1							$46\frac{1}{2}$	1	0	-53	50	$(\frac{5}{2} + 4)t_3$
t_2							1	0	0	-1	1	$(5 - 8)t_3$
e_2							14*	0	1	-11	7	$1\ t_3$
f	-10	$-9\frac{5}{28}$	$-\frac{9}{14}$	$-\frac{51}{28}$	$-74\frac{5}{14}$	0	0	0	$-\frac{23}{28}$	$-13\frac{27}{28}$	$17\frac{1}{4}$	
e_1			$5\frac{9}{14}$					1	$-\frac{93}{28}$	$-16\frac{13}{28}$	$26\frac{3}{4}$	
t_2			$-\frac{1}{7}$*					0	$-\frac{1}{14}$	$-\frac{3}{14}$	$\frac{1}{2}$	
t_3			$-\frac{1}{7}$					0	$\frac{1}{14}$	$-\frac{11}{14}$	$\frac{1}{2}$	
f	-10	$-\frac{19}{2}$	0	$-\frac{1}{2}$	-75	$-\frac{9}{2}$	0	0	$-\frac{1}{2}$	-13	15	$s = (0\ \frac{1}{2})$
e_1								1	$-\frac{1}{2}$	-8	7	$p = (0\ 0)$
x_1								0	$-\frac{1}{2}$	$-\frac{3}{2}$	$\frac{7}{2}$	$w = (13)$
t_3								0	0	-1	1	$f = 15$

Step 1.4 Transfer Decision

(a) Common Subproblem

Modified objective function $10b_1 + 10b_2 + (1 - 1)y_1 + (1 + \frac{1}{2})y_2 + (-1 - \frac{1}{2})y_3$

basic variable	b_1	b_2	y_1	y_2	y_3	e_1	e_2	constant
upper bounds			20	20	20			
$-df^o(=dh^o)$	0	-5	0	$-\frac{11}{2}$	$-\frac{7}{2}$	-10	-5	5
b_1						-1	$-\frac{1}{2}$	$\frac{1}{2}$
y_1						0	$-\frac{1}{2}$	$\frac{5}{2}$

Modified constants unchanged

(b) Primal Subproblem

$-df'^n = 5 + 0 + 8 - 13 = 0$ hence

TRANSFER NO INFORMATION TO THE PRIMAL MASTER

(c) Dual Subproblem

$dh'^n = 5 + 6 - 0 - \frac{15}{2} = \frac{7}{2} > 0$ hence

TRANSFER INFORMATION TO THE DUAL MASTER

consisting of $(u^J M \quad u^J Q - q^J \bar{y}) = (5\ -25,\ 25 - 0) = (5\ -25\ 25)$

Step 3.4 Dual Master Problem

basic variable	t_1	t_2	t_3	t_4	v	r_1	r_2	e_1	e_2	e_3	constant	Additional Column (original form)
upper bounds					10							
h	$36\frac{1}{2}$	0	0	$-\frac{7}{2}$	4	16	20	4	0	$\frac{15}{2}$	$\frac{23}{2}$	$(2 - 0 + 25)t_4$
t_2				$\frac{1}{8}$				$-\frac{1}{4}$	0	$\frac{9}{8}$	$\frac{7}{8}$	$(-1 + 5)t_4$
e_2				$-\frac{35}{16}$				$\frac{35}{8}$	1	$5\frac{13}{16}$	$9\frac{3}{16}$	$(-\frac{1}{2} - 25)t_4$
t_3				$\frac{7}{8}$*				$\frac{1}{4}$	0	$-\frac{1}{8}$	$\frac{1}{8}$	$1\ t_4$
h	41	0	4	0	5	20	13	5	0	7	12	$c = (0)$
t_2								$-\frac{2}{7}$	0	$\frac{8}{7}$	$\frac{6}{7}$	$z = (5\ 0)$
e_2								5	1	$\frac{11}{2}$	$\frac{19}{2}$	$m = (7)$
t_4								$\frac{2}{7}$	0	$-\frac{1}{7}$	$\frac{1}{7}$	$h = 12$

Step 1.5 Transfer Decision

(a) Common Subproblem

Modified objective function unchanged

basic variable	b_1	b_2	y_1	y_2	y_3	e_1	e_2	constant	modified constants
upper bounds			20	20	20				
$-df^0(=dh^0)$	0	-5	0	$-\frac{13}{2}$	$-\frac{7}{2}$	-10	-5	0	
b_1						-1	$-\frac{1}{2}$	0	2 - 5
y_1						0	$-\frac{1}{2}$	3	1 + 5

(b) Primal Subproblem

$-df'' = 0 + 0 + 10 - 13 = -3 < 0$ hence

TRANSFER INFORMATION TO THE PRIMAL MASTER

consisting of $\begin{pmatrix} \underline{ab^i + By^i} \\ Ky^i \end{pmatrix} = \begin{pmatrix} \underline{0 + 3} \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix}$

(c) Dual Subproblem

$dh''' = 0 + 7 + 0 - 7 = 0$ hence

TRANSFER NO INFORMATION TO THE DUAL MASTER

Step 2.4 Primal Master Problem

basic variable	a_1	a_2	x_1	x_2	t_1	t_2	t_3	t_4	e_1	e_2	e_3	constant	Additional column (original form)
upper bounds			20	20									
f	-10	$-\frac{19}{2}$	0	$-\frac{1}{2}$	-75	$-\frac{9}{2}$	0	3	0	$-\frac{1}{2}$	-13	15	$(3 + 0 + 5)t_4$
e_1								-2	1	$-\frac{1}{2}$	-8	7	$(3 + 5)t_4$
x_1								$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{3}{2}$	$\frac{7}{2}$	$(6 -10)t_4$
t_3								1*	0	0	-1	1	$1\ t_4$
f	-10	$-\frac{19}{2}$	0	$-\frac{3}{2}$	-78	$-\frac{15}{2}$	-3	0	0	$-\frac{1}{2}$	10	12	
e_1									1	$-\frac{1}{2}$	-10	9	$s = (0\ \frac{1}{2})$
x_1									0	$-\frac{1}{2}$	-2	4	$p = (0\ 0)$
t_4									0	0	-1	1	$v = (10)$
													$f = 12$

Step 1.6 Transfer Decision

$f = h = 12 - 12 = 0$ hence go to Step 4

Step 4 Final Solution

Intermediate solutions corresponding to each t_i are given by:

for primal direction

	b_1	b_2	c	y_1	y_2	y_3	z_1	z_2
t_1	$\frac{13}{2}$	0	2	$\frac{5}{2}$	0	0	2	2
t_2	0	0	0	18	0	15	20	0
t_3	$\frac{1}{2}$	0	0	$\frac{5}{2}$	0	0	4	0
t_4	0	0	0	3	0	0	5	0

(i) Achieved primal feasible solution

$a = a^k = (0\ 0)$

$b = \sum_i b^i t_i^k = (0\ 0)$

$c = \sum_i c^i t_i^k = (0)$

$x = x^k = (4\ 0)$

$y = \sum_i y^i t_i^k = (3\ 0\ 0)$

$z = \sum_i z^i t_i^k = (5\ 0)$

for dual direction

	s_1	s_2	u_1	u_2	p_1	p_2	q_1	q_2	q_3
t_1	1	1	10	4	2	0	0	0	0
t_2	0	$\frac{1}{2}$	3	$\frac{3}{2}$	0	0	0	0	0
t_3	0	0	10	$\frac{11}{2}$	0	0	0	0	0
t_4	0	$\frac{1}{2}$	10	5	0	0	0	0	0

(ii) Achieved dual feasible solution

$s = \sum_j s^j t_j^l = \frac{6}{7}(0\ \frac{1}{2}) + \frac{1}{7}(0\ \frac{1}{2}) = (0\ \frac{1}{2})$

$u = \sum_j u^j t_j^l = \frac{6}{7}(3\ \frac{3}{2}) + \frac{1}{7}(10\ 5) = (4\ 2)$

$u = v^l = (0)$

$p = \sum_j p^j t_j^l = \frac{6}{7}(0\ 0) + \frac{1}{7}(0\ 0) = (0\ 0)$

$q = \sum_j q^j t_j^l = \frac{6}{7}(0\ 0\ 0) + \frac{1}{7}(0\ 0\ 0) = (0\ 0\ 0)$

$r = r^l = (0\ 0)$