Journal de Théorie des Nombres de Bordeaux

Jean-Marc Deshouillers
François Hennecart
Bernard Landreau

# Waring's problem for sixteen biquadrates. Numerical results

*Journal de Théorie des Nombres de Bordeaux*, tome 12, n$^o$ 2 (2000), p. 411-422

<http://www.numdam.org/item?id=JTNB_2000__12_2_411_0>

© Université Bordeaux 1, 2000, tous droits réservés.

L'accès aux archives de la revue « Journal de Théorie des Nombres de Bordeaux » (http://jtnb.cedram.org/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

Numdam

*Article numérisé dans le cadre du programme*
*Numérisation de documents anciens mathématiques*
http://www.numdam.org/

*Journal de Théorie des Nombres*
*de Bordeaux* **12** (2000), 411–422

# Waring's problem for sixteen biquadrates - Numerical results

par Jean-Marc DESHOUILLERS, François HENNECART et Bernard LANDREAU

*À Jacques Martinet, pour ses soixante ans*

Résumé. Nous expliquons les algorithmes qui nous ont permis de vérifier que tout entier congru à 4 modulo 80 dans l'intervalle $[6\times 10^{12}\ ;\ 2.17\times 10^{14}]$ est la somme de 5 bicarrés, et que tout entier congru à 6, 21 ou 36 modulo 80 dans l'intervalle $[6\times 10^{12}\ ;\ 1.36\times 10^{23}]$ est la somme de 7 bicarrés. Nous indiquons également des résultats déduits de calculs portant sur les petites sommes de bicarrés. L'escalade de Dickson appliquée à ces résultats montre que tout entier de l'intervalle $[13793\ ;\ 10^{245}]$ est la somme de 16 bicarrés.

Abstract. We explain the algorithms that we have implemented to show that all integers congruent to 4 modulo 80 in the interval $[6\times 10^{12}\ ;\ 2.17\times 10^{14}]$ are sums of five fourth powers, and that all integers congruent to 6, 21 or 36 modulo 80 in the interval $[6\times 10^{12}\ ;\ 1.36\times 10^{23}]$ are sums of seven fourth powers. We also give some results related to small sums of biquadrates. Combining with the Dickson ascent method, we deduce that all integers in the interval $[13793\ ;\ 10^{245}]$ are sums of 16 biquadrates.

## 1. Introduction

Davenport [1] showed in 1939 that every sufficiently large integer is a sum of (at most) 16 biquadrates; Kempner [6], considering the integers $31\times 16^k$ had previously shown that the value 16 cannot be reduced. In the other direction, Thomas [8] showed in 1974 that every integer in the range $[13793\ ;\ 10^{80}]$ is a sum of 16 biquadrates. Although Davenport's method is effective, that is to say that it can lead to a numerical value for an integer $N_0$ beyond which every integer is a sum of 16 biquadrates, such a $N_0$ would be completely out of reach of numerical computation. In a forthcoming paper, Kawada, Wooley and the first-named author show

Manuscrit reçu le 31 janvier 2000.

that $10^{220}$ is an admissible value for $N_0$: it thus makes sense to pursue Thomas' computation. The main result of this paper is

**Theorem.** *Every positive integer in the interval* $[13793\ ;\ 10^{245}]$ *is the sum of sixteen fourth powers.*

The list of the 96 integers in $[1\ ;13792]$ which require at least 17 biquadrates is given in Proposition 7, at the end of the fifth section.

## 2. Scheme of the proof

In the sequel, we shall write that an integer $n$ is a $B_s$ (or is $B_s$) if $n$ is the sum of $s$ fourth powers. For $M > b \geq a \geq 0$, we shall say that $n$ belongs to $a \longleftrightarrow b$ modulo $M$ when the residue class of $n$ lies in the interval $[a, b]$.

Even if the general scheme of the proof is classical, we give in this section the main argument leading to our result.

The first step consists in the numerical verification that all integers in some finite arithmetic progression (we deal with residue 4 modulo 80) are $B_5$. We explain our algorithm in Section 3. Then, by a 11-fold application of the ascent argument (cf. Lemma 1), we deduce that all integers in the much larger interval $I = (5\,865\,530\,312\,564\ ;\ 10^{245}]$ and belonging to $4 \longleftrightarrow 15$ modulo 16 are $B_{16}$. The lower bound of $I$ is the largest integer found by our algorithm congruent to 4 modulo 80 which is not a $B_5$. A probabilistic argument, similar to that explained for the sums of four cubes (cf. [5]), would show that $5\,865\,530\,312\,564$ is most likely the last non-$B_5$ in this arithmetic progression.

By applying again the ascent method on some very short arithmetic progressions of $B_9$ (checked of course on a computer by a straightforward implementation), we show in Section 5 that any integer, not a multiple of 16, in the interval $(2.5 \times 10^5\ ;\ 10^{16})$ is a $B_{16}$. This section is completed by showing that any integer in $[13793\ ;\ 2.5 \times 10^5]$ is a $B_{16}$, and by giving the non-$B_{16}$ integers up to 13792.

In Section 6, we show how the remaining non-zero residue classes modulo 16 can be covered by a slight modification in the ascent application (cf. Lemma 2).

Now since 16 is a biquadrate, any integer $n \in [13793\ ;\ 16\times10^{245}]$ divisible by 16 will be a $B_{16}$ if $n/16$ is itself a $B_{16}$. We may repeat this argument until we obtain an integer $n/16^a$ which is either in the interval $[13793\ ;\ 10^{245}]$ and belonging to $1 \longleftrightarrow 15$ modulo 16, or lies in a short interval $[13793\ ;\ 2.5 \times 10^5]$ covered by the sequence of $B_{16}$.

We now give an ascent lemma, which is the key of our argument. This result is a slight variation of the standard greedy algorithm [2], and already appeared more or less in [4]. For any real number $x$, we denote by $\lceil x \rceil$ the smaller integer greater than or equal to $x$. For any non-empty sets $A$ and

$B$ of integers, we write $A+B$ the set of all the sums $m+n$ where $m \in A$ and $n \in B$.

**Lemma 1.** *Let $M \geq 1$ be an integer, and $a, b$ two residues modulo $M$. For $\ell_0$ and $L$ integer, we denote by $A$ the finite arithmetic progression consisting of the integers $n$ in $(\ell_0, L]$ congruent to $a$ modulo $M$, and $B$ an infinite sequence of biquadrates congruent to $b$ modulo $M$: $b_1^4 < b_2^4 < \cdots < b_s^4 < \cdots$. We assume that $T = \max_k(b_{k+1} - b_k)$ exists.*

*Then the sumset $A+B$ contains all integers congruent to $a+b$ modulo $M$ lying in the interval*

$$\Big(\ell_0 + b_1^4\,;\, L + \Big(\Big\lceil\Big(\frac{L-\ell_0}{4T}\Big)^{1/3}\Big\rceil - T\Big)^4\Big],$$

*which contains the interval*

$$\Big(\ell_0 + b_1^4\,;\, \ell_0 + \Big(\frac{L-\ell_0}{4T}\Big)^{4/3}\Big].$$

*Proof.* Let $A_k = A + b_k^4$. For any $k$ such that $b_k^4 - b_{k-1}^4 \leq L - \ell_0$, the set $A_{k-1} \cup A_k$ contains all the integers congruent to $a+b$ modulo $M$ in the interval $(\ell_0 + b_{k-1}^4, L + B_k^4]$. Since

$$b_{k-1} \geq b_k - T, \text{ for any } k \geq 2, \tag{1}$$

we have $b_k^4 - b_{k-1}^4 \leq 4Tb_k^3$, for any $k \geq 1$. Thus the set $\bigcup_{k=1}^s A_k$ contains all integers congruent to $a+b$ modulo $M$ in the interval $\big(\ell_0 + b_1^4, L + b_s^4\big)$, if $b_k \leq ((L-\ell_0)/4T)^{1/3}$ for any $k \leq s$.

By (1), the integer $b_s$ may be chosen greater or equal than

$$\lceil ((L-\ell_0)/4T)^{1/3} \rceil - T.$$

Lemma 1 is proved. □

The following lemma will enable us to use a modified ascent argument for $B_{14}$'s congruent to 1 modulo 16 by starting from $B_{13}$'s with non-zero residue class modulo 16. We have

**Lemma 2.** *Let $n$ be an integer congruent to* 1 *modulo* 16. *Then among any set of* 50 *consecutive integers,*
(i) *there exists an odd integer $u$ such that $m_u := (n-u^4)/16$ belongs to* $4 \longleftrightarrow 12$ *modulo* 16 *and* $m_u \equiv n \bmod 5$,
(ii) *there exists an odd integer $v$ such that $m_v := (n-v^4)/16$ belongs to* $4 \longleftrightarrow 12$ *modulo* 16 *and* $m_v \equiv n-1 \bmod 5$.

*Proof.* We first observe that the sequence $S := (s_{2q+1})_{q\geq 0}$ of the residue class modulo 16 of $(1-(2q+1)^4)/16$ is 32-periodic; its first 32 terms are

$$0, 11, 9, 10, 6, 13, 7, 4, 12, 15, 5, 14, 2, 1, 3, 8, 8, 3, 1, 2, 14, 5, 15, 12, 4, 7, 13, 6, 10, 9, 11, 0.$$

We now describe how we proceed if we require further the residue class modulo 5 of $m_u = (n - u^4)/16$ to be fixed (we have either $m \equiv n$ or $n - 1 \bmod 5$), we choose either $u \equiv 5 \bmod 10$ or $u \equiv 1 \bmod 10$.

We first look at the subsequence $S_0 := \{s_u \ : \ u \geq 0 \text{ and } u \equiv 5 \bmod 10\}$: $S_0$ is also 32-periodic and its first 32 terms are

$$9,4,2,3,15,6,0,13,5,8,14,7,11,10,12,1,1,12,10,11,7,14,8,5,13,0,\\ 6,15,3,2,4,9.$$

Let $k \geq 0$ such that $n = 1 + 16k$. In order to have $m_u$ in $4 \longleftrightarrow 12$ modulo 16 and $m_u \equiv n \bmod 5$, we may choose $u \equiv 5 \bmod 10$ such that $s_u + k$ belongs to $4 \longleftrightarrow 12$ modulo 16.

We easily observe that for each $k \mod 16$, the longest series of consecutive terms in $S_0 + k$ which are not in $4 \longleftrightarrow 12$ modulo 16 have at most four elements. Remember that a difference of five between the ranks of two elements in the sequence $S_0 + k$ corresponds to 25 in the whole sequence $S + k$, and thus corresponds to 50 in the set of integers. This proves (i).

We now examine the case where we need to get $u$ such that $n \equiv (n - v^4)/16 + 1 \bmod 5$. We thus take the subsequence $S_1 := \{s_v \ : \ v \geq 0 \text{ and } v \equiv 1 \bmod 10\}$: $S_1$ which is also 32-periodic and its first 32 terms are

$$0,13,5,8,14,7,11,10,12,1,1,12,10,11,7,14,8,5,13,0,6,15,3,2,4,9,9,\\ 4,2,3,15,6.$$

By considering again the shifted sequence $S_1 + k$ where $n = 1 + 16k$, we still realize that the longest series of consecutive elements of $S_1 + k$ which are not in $4 \longleftrightarrow 12$ modulo 16 is four terms long.

This ends the proof of Lemma 2. $\square$

Although the ascent method is rather efficient, we may largely improve the final bound if we jump by a single ascent from $B_5$'s to $B_7$'s with $B_2$'s instead of two ascents with $B_1$'s as was noticed in [4]. We shall use

**Lemma 3.** *Let $M \geq 1$ be an integer, and $a, b$ two residues modulo $M$. For $\ell_0$ and $L$ integers, we denote by $A$ the finite arithmetic progression consisting of the integer $n$ in $(\ell_0, L]$ congruent to $a$ modulo $M$, and by $B$ a set of integers $b_1 < b_2 < \cdots < b_s$ all congruent to $b$ modulo $M$ and such that $b_k - b_{k-1} \leq L - \ell_0$, for $2 \leq k \leq s$.*

*Then the sumset $A + B$ contains all integers congruent to $a + b$ modulo $M$ lying in the interval $(\ell_0 + b_1, L + b_s]$.*

*Proof.* This easily follows from the fact that for any $k$, $2 \leq k \leq s$, the set $A + b_{k-1} \cup A + b_k$ contains all the integers lying in the interval $(\ell_0 + b_{k-1}, L + b_k]$ and congruent to $a + b$ modulo $M$. $\square$

## 3. On the $B_5$'s congruent to 4 modulo 80

In this section, we explain how we obtain the following

**Proposition 1.** *Every integer lying in the interval*

$$(5\,865\,530\,312\,564\,;\,2.17\times 10^{14}]$$

*which is congruent to* 4 *modulo* 80 *is a* $B_5$.

Since the sequence of $B_3$'s has a zero density, the least number of summands necessary to represent a large arithmetic progression of integers is 4: it is a great problem to know whether the sequence of $B_4$'s has a positive density or not. On the contrary, a density argument and numerical evidence lead us to expect infinite arithmetic progressions of $B_5$'s.

Let us now explain our choice of the arithmetic progression 4 modulo 80.

Our goal is to find an arithmetic progression which is rich in sums of 5 biquadrates. For a fixed modulus $M$ and any residue $k$, this can be measured thanks to the number $\rho(k, M)$ which is the number of incongruent solutions of $k \equiv x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 \bmod M$. For an odd prime $p \equiv 3 \bmod 4$, since biquadrates coincide with squares modulo $p$, there is an almost uniform distribution of the $B_5$'s modulo $p$. For $p \equiv 1 \bmod 4$, the numbers $\rho(k,p)$ are badly distributed when $p$ is small and especially for $p = 5$. The number $\rho(k, 5)$ is maximal for $k = 4$. Indeed we have

| $k$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\rho(k,5)$ | 1025 | 20 | 160 | 640 | 1280 |

TABLE 1. *Number of representations modulo* 5

Biquadrates also are badly distributed modulo 16: every biquadrate is congruent to 0 or 1 modulo 16. Starting from an interval of $B_5$'s, we still have 11 biquadrates to add. Since every biquadrate is congruent to 0 or 1 modulo 16, in order to represent integer congruent to 15 modulo 16, we are led to select either $B_5$'s congruent to 4 modulo 16, or $B_5$'s congruent to 5 modulo 16. There have more sums of five fourth powers congruent to 4 modulo 16 than congruent to 5 modulo 16. Hence we have considered the arithmetic progression 4 modulo 80.

| $k$ | 4 | 5 |
|---|---|---|
| $\rho(k,16)$ | 163840 | 32768 |

TABLE 2. *Number of representations modulo* 16

In the following table, we describe the action of the ascent on the residue classes modulo 16 and modulo 5. In the right hand column, we indicate the value of $T$ when applying Lemma 1.

| | modulo 16 | modulo 5 | * $B_1$ mod 80 | maximal difference T |
|---|---|---|---|---|
| $B_{16}$ | $4 \longleftrightarrow 15$ | $0 \longleftrightarrow 4$ | 65, 0 , 1, 16 | 10 or 4 |
| $B_{15}$ | $4 \longleftrightarrow 14$ | $1 \longleftrightarrow 4$ | 65, 0 , 1, 16 | 10 or 4 |
| $B_{14}$ | $4 \longleftrightarrow 13$ | $1 \longleftrightarrow 3$ | 65, 0 , 1, 16 | 10 or 4 |
| $B_{13}$ | $4 \longleftrightarrow 12$ | $1 \longleftrightarrow 2$ | 65, 0 , 1, 16 | 10 or 4 |
| $B_{12}$ | $4 \longleftrightarrow 11$ | 1 | 1, 16 | 4 |
| $B_{11}$ | $4 \longleftrightarrow 10$ | 0 | 1, 16 | 4 |
| $B_{10}$ | $4 \longleftrightarrow 9$ | 4 | 1, 16 | 4 |
| $B_9$ | $4 \longleftrightarrow 8$ | 3 | 1, 16 | 4 |
| $B_8$ | $4 \longleftrightarrow 7$ | 2 | 1, 16 | 4 |
| $B_7$ | $4 \longleftrightarrow 6$ | 1 | 1, 16 | 4 |
| $B_6$ | $4 \longleftrightarrow 5$ | 0 | 1, 16 | 4 |
| $B_5$ | 4 | 4 | | |

* The residue class modulo 80 of the added $B_1$ is fixed among those given at each ascent step $B_j$ to $B_{j+1}$ according to the considered arithmetic progression modulo 80 of the $B_{j+1}$'s.

TABLE 3. *the 11-fold ascent*

A biquadrate is congruent to $0, 1, 16$ or $65$ modulo 80. Thus the only ordered representations of 4 modulo 80 as sums of five biquadrates are

$$4 \equiv 0 + 1 + 1 + 1 + 1 \equiv 1 + 1 + 1 + 16 + 65 \bmod 80$$

It is easy to see that the second representation type generates much more $B_5$'s than the first one (the ratio is 4). In our algorithm, we thus have first used the most productive representation, and the other one only when necessary.

Our algorithm may be described as follows:

- Compute all $B_1$'s up to a given bound $L$, divide them by 80, and rearrange their quotient in four different arrays $U_0$, $U_1$, $U_{16}$ and $U_{65}$ according to their remainder modulo 80.
- Calculate separately the $B_2$'s up to $L$ congruent to 1 modulo 80, and the ones congruent to 2 modulo 80 in the following way.
  - For each class 1 or 2 and each interval, a bits array is initialized to zero, and a two-loops routine on each pair $(U_j, U_k)$ – considering $(j, k) = (1, 1)$ for $B_2$'s congruent to 2, and $(j, k) = (0, 1)$ or $(16, 65)$ for $B_2$'s congruent to 1 – switches at each new $B_2$ the corresponding bit to one.
  - Then read the $B_2$'s identified by the location of the bits "one" in the bits array and arrange them in two different arrays according to their class modulo 80: array $V_1$ for the $B_2$'s congruent to 1 (mod 80) and array $V_2$ for the $B_2$'s congruent to 2 mod 80. Next interval is considered.

- In order to check that a fixed interval $I$ of integers congruent to 4 modulo 80 contains only $B_5$'s, we associate to each quotient modulo 80 a bit which is initialized to zero.
- A three loops routine on $U_1$, $V_1$ and $V_2$ sifts the interval $I$, by switching at each new $B_5$ the corresponding bit to one.
- When all the bits are to one, checking is stopped. Next interval is considered.

## 4. On the $B_7$'s congruent to $6, 21$ or $36$ modulo $80$

We deal in this section with $B_2$'s. We show

**Proposition 2.** *For each $k \in \{2, 17, 32\}$, the increasing sequence $\{b_j : j \geq 1\}$ of the $B_2$'s congruent to $k$ modulo 80 is such that there exists an element $b_s$ satisfying $b_s \geq 1.36 \times 10^{23}$ and $\max_{k \leq s}(b_k - b_{k-1}) \leq 2.11 \times 10^{14}$.*

A single application of Lemma 3 leads to the following

**Proposition 3.** *Every integer lying in the interval*

$$[5.87 \times 10^{12}\,;\, 1.36 \times 10^{23}]$$

*which is congruent to $4, 5$ or $6$ modulo 16 and to 1 modulo 5 is a $B_7$.*

We give in Table 4, for each step $j \geq 1$ of the ascent, the upper bound related to the sums of $5 + j$ fourth powers in the arithmetic progressions described in Table 1. We have denoted by $L_{5+j}$, $j \geq 1$, the decimal logarithms of these bounds. We give for comparison the bounds obtained when using two ascents with $B_1$'s from $B_5$'s to $B_7$'s. In the four last ascents from $B_{12}$'s to $B_{16}$'s, the bounds are calculated from Lemma 1 in the worst case $T = 10$.

| | two ascents with $B_1$'s | one single ascent with $B_2$'s |
|---|---|---|
| $L_{16}$ | 226.44 | 245.32 |
| $L_{15}$ | 171.43 | 185.59 |
| $L_{14}$ | 130.17 | 140.80 |
| $L_{13}$ | 99.23 | 107.20 |
| $L_{12}$ | 76.02 | 82.00 |
| $L_{11}$ | 58.22 | 62.70 |
| $L_{10}$ | 44.87 | 48.23 |
| $L_9$ | 34.85 | 37.38 |
| $L_8$ | 27.34 | 29.23 |
| $L_7$ | 21.71 | 23.13 |
| $L_6$ | 17.49 | |
| $L_5$ | 14.32 | 14.32 |

TABLE 4. *Upper bounds in the ascent*

Below we state precisely two useful consequences.

**Proposition 4.** *Every integer lying in the interval* $[5.87 \times 10^{12}\ ;\ 10^{107}]$ *which is congruent to* 1 *or* 2 *modulo* 5 *and to* 4, 5, 6, 7, 8, 9, 10, 11 *or* 12 *modulo* 16 *is a* $B_{13}$.

*Every integer lying in the interval* $[5.87 \times 10^{12}\ ;\ 10^{245}]$ *which is congruent to* 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 *or* 15 *modulo* 16 *is a* $B_{16}$.

It is not surprising to notice that the ascent with $B_2$'s is much better than two ascents with $B_1$'s. Indeed the numbers of $B_2$'s up to $x$ is asymptotically equivalent to $\frac{\Gamma(1/4)^2}{32\Gamma(3/2)}\sqrt{x}$, and the $B_2$'s appears to be more or less as well-distributed as a regular sequence of squares. Hence we may expect that one ascent with $B_2$'s increases the bound on $B_5$'s by a power 2, when two ascents with $B_1$'s gives only $\frac{4}{3}.\frac{4}{3} = \frac{16}{9}$. Unfortunately, the sequence of $B_2$'s is not so regular than expected, but this trick yet provides a significant gain on the bound for $B_7$'s, which leads to a major improvement on the bound for $B_{16}$'s.

We now present the calculations leading to Proposition 3. In each residue class $k \in \{2, 17, 32\}$ modulo 80, we simply have computed the values of $B_2$'s and noticed the highest difference.

In order to reach the upper bound $2 \times 10^{23}$, we proceed as follows. We easily compute the values of $B_2$ lying in an interval $[a, b[$ and put them in a large vector say $V$ (with at most $10^7$ numbers). We quickly sort this vector by ascending order and then compute all the differences between two consecutive $B_2$'s. As soon as a new highest difference is found, we record it as well as the two corresponding $B_2$'s.

To deal with $B_2$'s up to $10^{24}$, although these calculations have been performed on a DIGITAL workstation (64 bits) we need to use a special 128 bits structure. Since the number of $B_2$'s lying in an interval is decreasing, and in order to work for quickness with an interval of maximal length $L$, we were led to change several times the value of $L$ keeping in mind that the number of $B_2$'s for $V$ should be less than $10^7$. These calculations took about 20 hours CPU time in each residue class 2 or 32 and 40 hours in the residue class 17.

Table 5 below gives the largest differences between two consecutive $B_2$'s that occur in one of the arithmetic progressions 2, 17 or 32 modulo 80 in the range $[10^{23}\ ;\ 2 \times 10^{23}]$.

We observe that up to $1.36 \times 10^{23}$, in each residue class 2,17 or 32 modulo 80, the largest difference between two consecutive $B_2$'s is not greater than 210 565 453 462 800 which implies Proposition 3.

| $N$ | new largest difference at $N$ | | |
|---|---|---|---|
| | 2 | 17 | 32 |
| $1.043 \times 10^{23}$ | | 95 138 992 318 480 | |
| $1.089 \times 10^{23}$ | 194 899 909 381 840 | | |
| $1.135 \times 10^{23}$ | | | 203 592 000 262 640 |
| $1.157 \times 10^{23}$ | | 99 109 783 064 240 | |
| $1.161 \times 10^{23}$ | | 119 417 876 967 680 | |
| $1.170 \times 10^{23}$ | 205 382 223 918 000 | | |
| $1.205 \times 10^{23}$ | | | 210 565 453 462 800 |
| $1.368 \times 10^{23}$ | 233 263 552 681 200 | | |
| $1.484 \times 10^{23}$ | | | 240 353 667 777 520 |
| $1.680 \times 10^{23}$ | | 130 380 781 594 720 | |
| $1.747 \times 10^{23}$ | | | 252 139 980 148 720 |
| $1.756 \times 10^{23}$ | 238 326 205 393 520 | | |
| $1.806 \times 10^{23}$ | 242 236 651 101 680 | | |
| $1.827 \times 10^{23}$ | | | 276 110 546 369 280 |
| $1.917 \times 10^{23}$ | | 141 705 034 717 520 | |

TABLE 5. *Largest differences between consecutive $B_2$'s in the range* $[10^{23} ; 2 \times 10^{23}]$

## 5. Small values, exceptions

The last number $\ell_0$ congruent to 4 modulo 80 which is not a $B_5$ is too large and does not permit to check directly (even with a powerful computer) the whole interval $(0, \ell_0)$ to know whether a number is a $B_{16}$ or not.

Many computations on fourth powers have been performed. Those described by Thomas in [7, 8] lead to show that all integers in the interval $[13793 ; 10^{80}]$ are $B_{16}$ (see also [3] for comments), which is widely sufficient to our needs.

In order to make our result depending only on our algorithms, we chose to perform our own calculations which are closely adapted to our needs. We checked

**Proposition 5.** *All integers in the interval* $[13793 ; 2.5 \times 10^5]$ *are* $B_{16}$.
*All integers in the interval* $[2.5 \times 10^5 ; 5 \times 10^5]$ *congruent to* $1 \longleftrightarrow 8$ *modulo* 16 *are* $B_9$.

Starting from the $B_9$'s, we apply a 7-fold ascent using Lemma 1 with $T = 2$.

| | interval | modulo 16 | $B_1$ mod 16 | maximal difference T |
|---|---|---|---|---|
| $B_{16}$ | $[2.5 \times 10^5, 1.3 \times 10^{16}]$ | $1 \longleftrightarrow 15$ | 0, 1 | 2 |
| $B_{15}$ | $[2.5 \times 10^5, 1.0 \times 10^{13}]$ | $1 \longleftrightarrow 14$ | 0, 1 | 2 |
| $B_{14}$ | $[2.5 \times 10^5, 4.5 \times 10^{10}]$ | $1 \longleftrightarrow 13$ | 0, 1 | 2 |
| $B_{13}$ | $[2.5 \times 10^5, 7.9 \times 10^8]$ | $1 \longleftrightarrow 12$ | 0, 1 | 2 |
| $B_{12}$ | $[2.5 \times 10^5, 3.8 \times 10^7]$ | $1 \longleftrightarrow 11$ | 0, 1 | 2 |
| $B_{11}$ | $[2.5 \times 10^5, 4.1 \times 10^6]$ | $1 \longleftrightarrow 10$ | 0, 1 | 2 |
| $B_{10}$ | $[2.5 \times 10^5, 8.1 \times 10^5]$ | $1 \longleftrightarrow 9$ | 0, 1 | 2 |
| $B_9$ | $[2.5 \times 10^5, 5 \times 10^5]$ | $1 \longleftrightarrow 8$ | | |

TABLE 6. *The 7-fold ascent for small $B_{16}$'s*

We deduce

**Proposition 6.** *All integers not divisible by* 16 *in the interval* $[13793\ ;\ 10^{16}]$ *are* $B_{16}$.

We recall in the following the only known numbers which are not $B_{16}$.

**Proposition 7.** *The following* 96 *numbers*
47, 62, 63, 77, 78, 79, 127, 142, 143, 157, 158, 159, 207, 222, 223, 237, 238, 239
287, 302, 303, 317, 318, 319, 367, 382, 383, 397, 398, 399, 447, 462, 463, 477,
478, 479, 527, 542, 543, 557, 558, 559, 607, 622, 623, 687, 702, 703, 752, 767,
782, 783, 847, 862, 863, 927, 942, 943, 992, 1007, 1008, 1022, 1023, 1087,
1102, 1103, 1167, 1182, 1183, 1232, 1247, 1248, 1327, 1407, 1487, 1567,
1647, 1727, 1807, 2032, 2272, 2544, 3552, 3568, 3727, 3792, 3808, 4592,
4832, 6128, 6352, 6368, 7152, 8672, 10992, 13792
*are the only non-$B_{16}$ integers up to* 13792 *and satisfy*

- *the integers* $79 + 80k$, $k = 0, 1, \ldots, 6$ *are not* $B_{18}$*'s,*
- *the* 24 *integers*
  - $63 + 80k$, $k = 0, \ldots, 14$,
  - $78 + 80k$, $k = 0, \ldots, 6$,
  - $48 + 80k$, $k = 12, 15$,

  *are* $B_{18}$*'s but not* $B_{17}$*'s.*
- *the* 65 *integers*
  - $47 + 80k$, $k = 0, \ldots, 22, 46$,
  - $62 + 80k$, $k = 0, \ldots, 14$,
  - $77 + 80k$, $k = 0, \ldots, 6$,
  - $32 + 80k$, $k = 9, 12, 15, 25, 28, 44, 47, 57, 60, 79, 89, 108, 137, 172$,
  - $48 + 80k$, $k = 44, 47, 76, 79$,
  - $64 + 80k$, $k = 31$,

  *are* $B_{17}$*'s but not* $B_{16}$*'s.*

## 6. the remaining residues modulo 16 - End of the proof

We now look at the integers congruent to 1, 2 or 3 modulo 16.

Let $n$ be an integer congruent to 1 modulo 16 and to $1, 2$ or 3 modulo 5, lying in the interval $[13793\ ;\ 10^{L_{14}}]$. By Proposition 6, it suffices to consider the case $n \geq 10^{16}$.

By Lemma 2, we may write $n = 16m + u^4$ where $m$ belongs to $4 \longleftrightarrow 12$ modulo 16 and $u < \lfloor n^{1/4} \rfloor$. We assert that $u$ and $m$ may be chosen such that $10^{11} \leq m \leq 10^{107}$ and $m \equiv 1$ or 2 modulo 5.

Indeed let $u$ be the largest nonnegative integer less than $\lfloor n^{1/4} \rfloor$ such that

$$\begin{aligned} m &:= (n - u^4)/16 \in 4 \longleftrightarrow 12 \text{ modulo } 16, \\ u &\equiv \begin{cases} 5 \bmod 10 \text{ if } n \equiv 1 \text{ or } 2 \bmod 5, \\ 1 \bmod 10 \text{ if } n \equiv 3 \bmod 5. \end{cases} \end{aligned}$$

We plainly have $m \equiv 1$ or $2 \bmod 5$. Moreover we get

$$n - (\lfloor n^{1/4} \rfloor - 1)^4 \leq n - u^4 \leq n - (\lfloor n^{1/4} \rfloor - 50)^4,$$

then

$$n - (n^{1/4} - 1)^4 \leq n - u^4 \leq n - (n^{1/4} - 51)^4,$$

which gives the following bounds:

$$10^{11} \leq n^{3/4}/6 \leq m \leq 13n^{3/4} \leq 13 \times 10^{3L_{14}/4} \leq 10^{107},$$

by using $L_{14} = 140.8$ given in the third column of Table 4. By Proposition 4, we deduce that $m$ is a $B_{13}$. Thus $n = 2^4 m + u^4$ is a $B_{14}$.

It means that the upper bound $L_{14}$ is still available for the integers $n$ congruent to 1 modulo 16 to be $B_{14}$.

The last steps in the ascent in Tables 3 and 4 may be completed by

| | | modulo 16 | modulo 5 | maximal difference T |
|---|---|---|---|---|
| $B_{16}$ | $L_{16}$ | $1 \longleftrightarrow 3$ | $0 \longleftrightarrow 4$ | 10 |
| $B_{15}$ | $L_{15}$ | $1, 2$ | $1 \longleftrightarrow 4$ | 10 |
| $^*B_{14}$ | $L_{14}$ | 1 | $1 \longleftrightarrow 3$ | 50 |
| $B_{13}$ | $L_{13}$ | $4 \longleftrightarrow 12$ | $1 \longleftrightarrow 2$ | |

* means that the $B_{14}$'s are written as the sum of 16 times a $B_{13}$ and a $B_1$.

TABLE 7. *The ascent for the classes* $1, 2, 3$ *modulo* 16

We summarize the results proved up to now in the following

**Proposition 8.** *All integers in the interval* $[13793\ ;\ 10^{245}]$ *and belonging to* $1 \longleftrightarrow 15$ *modulo* 16 *are* $B_{16}$.

We end by considering the residue class 0 modulo 16. For that we use Proposition 8, which concerns the integers non divisible by 16, and also the first part of Proposition 5.

Let now $n$ be an integer congruent to 0 modulo 16 in the interval $[13793\ ;\ 16\times 10^{245}]$. We may write $n = 16^a m$ where either $m \in [13793\ ;\ 2.5\times 10^5]$, or $16 \nmid n$ and $m \in [13793\ ;\ 10^{245}]$. In either case, $m$ is a $B_{16}$, thus $n$ is also a $B_{16}$.

This ends the proof of the theorem.

## References

[1] H. DAVENPORT, *On Waring's problem for fourth powers.* Ann. of Math. **40** (1939), 731–747.

[2] L.E. DICKSON, *Recent progress on Waring's theorem and its generalizations.* Bull. Amer. Math. Soc. **39** (1933), 701-727.

[3] J-M. DESHOUILLERS, *Problème de Waring pour les bicarrés : le point en 1984.* Sém. Théor. Analyt. Nbres Paris, 1984-85, exp. 33.

[4] J-M. DESHOUILLERS, F. DRESS, *Numerical results for sums of five and seven biquadrates and consequences for sums of 19 biquadrates.* Math. Comp. **61**, 203 (1993), 195–207.

[5] J-M. DESHOUILLERS, F. HENNECART, B. LANDREAU, 7 373 170 279 850. Math. Comp. **69** (2000), 421-439.

[6] A. KEMPNER, *Bemerkungen zum Waringschen Problem.* Math. Ann. **72** (1912), 387–399.

[7] H.E. THOMAS, A numerical approach to Waring's problem for fourth powers. Ph.D., The University of Michigan, 1973.

[8] H.E. THOMAS, *Waring's problem for twenty-two biquadrates.* Trans. Amer. Math. Soc. **193** (1974), 427–430.

Jean-Marc DESHOUILLERS*, François HENNECART, Bernard LANDREAU
Laboratoire Théorie des nombres et algorithmique arithmétique
UMR 5465 CNRS
Université Bordeaux 1
351, cours de la Libération
33405 Talence Cedex
France
*E-mail* : {dezou,hennec,landreau}@math.u-bordeaux.fr

* Egalement membre de l'EA 2619, Université Victor Segalen Bordeaux 2