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Davenport-Hasse relations and an explicit Langlands correspondence, II: twisting conjectures

par Colin J. BUSHNELL et Guy HENNIART

To Jacques Martinet on his retirement

RÉSUMÉ. Soit F une extension finie de $\mathbb{Q}_p$. La correspondance de Langlands donne une bijection canonique entre l'ensemble $\mathcal{G}_F^0(n)$ des classes d'isomorphisme de représentations irréductibles de dimension n du groupe de Weil $\mathcal{W}_F$ de F, et l'ensemble $\mathcal{A}_F^0(n)$ des classes d'isomorphisme de représentations irréductibles supercuspidales de $\mathrm{GL}_n(F)$. Nous regardons le cas où n est une puissance de p, $n = p^m$. Dans un travail antérieur, nous avons construit une bijection $\boldsymbol{\pi}$ de $\mathcal{G}_F^0(p^m)$ sur $\mathcal{A}_F^0(p^m)$, grâce à la construction d'un changement de base modéré (non nécessairement galoisien). Si le changement de base satisfait certaines relations conjecturales, dites de Davenport-Hasse, et si l'on admet l'existence d'une correspondance de Langlands en degré p^m, alors cette correspondance n'est autre que $\boldsymbol{\pi}$. Dans ce papier, nous ne supposons pas *à priori* l'existence d'une correspondance de Langlands, mais nous voulons prouver directement, en supposant vérifiées les conjectures de Davenport-Hasse, que π est une telle correspondance. Nous réduisons le problème à des propriétés élémentaires des constantes locales $\varepsilon(\pi_1 \times \pi_2, s, \psi_F)$, pour $\pi_i \in \mathcal{A}_F^0(p^{m_i})$ (qui peuvent d'ailleurs se déduire de l'existence de la correspondance de Langlands et de propriétés analogues du côté galoisien). Au cours de cet article, nous obtenons de nouvelles propriétés inconditionnelles pour π, et décrivons complètement les propriétés de rationalité des fonctions L et des constantes locales de paires pour $\mathrm{GL}_n(F)$.

ABSTRACT. Let $F/\mathbb{Q}_p$ be a finite field extension. The Langlands correspondence gives a canonical bijection between the set $\mathcal{G}_F^0(n)$ of equivalence classes of irreducible n-dimensional representations of the Weil group $\mathcal{W}_F$ of F and the set $\mathcal{A}_F^0(n)$ of equivalence classes of irreducible supercuspidal representations of $\mathrm{GL}_n(F)$.

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This paper is concerned with the case where $n = p^m$. In earlier work, the authors constructed an explicit bijection $\boldsymbol{\pi} : \mathcal{G}^0_F(p^m) \to \mathcal{A}^0_F(p^m)$ using a non-Galois tame base change map. If this tame base change satisfies a certain conjectured *automorphic Davenport-Hasse relation,* and there exists a Langlands correspondence in p-power degree, then $\boldsymbol{\pi}$ is the Langlands correspondence. This paper is concerned with the problem of showing, without assuming *a priori* the existence of the Langlands correspondence, that (on the Davenport-Hasse conjecture) $\boldsymbol{\pi}$ preserves local constants of pairs, and so is a Langlands correspondence. The principal obstruction is the lack of knowledge of certain elementary properties of the local constant $\varepsilon(\pi_1 \times \pi_2, s, \psi_F)$, for $\pi_i \in \mathcal{A}^0_F(p^{m_i})$. We state these properties as conjectures (which are certainly true, as consequences of the existence of the Langlands correspondence and analogous properties of the Langlands-Deligne local constant) and show that they imply the desired result: $\boldsymbol{\pi}$ is a Langlands correspondence. In the process, we prove several new unconditional results concerning $\boldsymbol{\pi}$, and give a complete account of the rationality properties of L-functions and local constants of pairs for $\mathrm{GL}_n(F)$.

Introduction

Let p be a prime number. We fix an algebraic closure $\overline{\mathbb{Q}}_p/\mathbb{Q}_p$ of the p-adic number field $\mathbb{Q}_p$, and consider a finite field extension $F/\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$. We write $\mathcal{W}_F$ for the Weil group of $\overline{\mathbb{Q}}_p/F$. We use the standard notation $\mathfrak{o}_F$ for the discrete valuation ring in F, $\mathfrak{p}_F$ for the maximal ideal of $\mathfrak{o}_F$, $U^1_F = 1+\mathfrak{p}_F$ for the group of principal units, and v_F for the normalized valuation $F^\times \twoheadrightarrow \mathbb{Z}$.

The Langlands Conjecture for $\mathrm{GL}_n(F)$ has been proved, for all $n \geqslant 1$, in [16] and [20]. The methods of these two papers are different, but they both rely on constructions from [14], [15]. Their approach is global in nature, with a significant geometric component; they can therefore say nothing explicit about the local Langlands correspondence whose existence they establish. To get an explicit correspondence, once has to work in a local framework, using the classification of representations from [11]. This paper is concerned with the problem of obtaining a local proof of the existence of the correspondence in these terms. We deal only with a special case, but one which is particularly subtle and basic to the overall picture.

1. To specify this case, we take an integer $m \geqslant 0$ and write $\mathcal{G}^{\mathrm{wr}}_m(F)$ for the set of equivalence classes of irreducible continuous representations σ of $\mathcal{W}_F$ such that $\dim \sigma = p^m$ and which are *totally ramified* in the sense that

$\sigma \otimes \chi \not\cong \sigma$ for any unramified quasicharacter $\chi \neq 1$ of $\mathcal{W}_F$. There is a corresponding notion on the other side: we write $\mathcal{A}_m^{\mathrm{wr}}(F)$ for the set of equivalence classes of irreducible supercuspidal representations π of $\mathrm{GL}_{p^m}(F)$ such that $\chi\pi \not\cong \pi$ for any unramified quasicharacter $\chi \neq 1$ of $F^\times$. (Here and throughout $\chi\pi$ denotes the representation $g \mapsto \chi(\det g)\pi(g)$.) A special case of the Langlands Conjecture for $\mathrm{GL}_n(F)$ predicts the existence of a bijection

$$\mathcal{L}_m^F : \mathcal{G}_m^{\mathrm{wr}}(F) \longrightarrow \mathcal{A}_m^{\mathrm{wr}}(F)$$

which *preserves local constants of pairs:*

$$\varepsilon(\sigma_1 \otimes \sigma_2, s, \psi_F) = \varepsilon(\pi_1 \times \pi_2, s, \psi_F), \tag{1}$$

for $\sigma_i \in \mathcal{G}_{m_i}^{\mathrm{wr}}(F)$ and $\pi_i = \mathcal{L}(\sigma_i)$, $i = 1, 2$. Here, the first ε is the local constant of Langlands-Deligne [13] and the second that of Jacquet, Piatetskii-Shapiro and Shalika [22], [26], attached to a complex variable s and a non-trivial character ψ_F of the additive group of F. The family $\{\mathcal{L}_m^F\}$ has several other formal properties; together with (1), these specify it uniquely.

2. In [5], we constructed a bijection

$$\boldsymbol{\pi}_m^F : \mathcal{G}_m^{\mathrm{wr}}(F) \longrightarrow \mathcal{A}_m^{\mathrm{wr}}(F),$$

using the description of the elements of $\mathcal{A}_m^{\mathrm{wr}}(F)$ implied by [11] and group-theoretic constructions which are, at least in spirit, elementary and straightforward. The family $\{\boldsymbol{\pi}_m^F\}$ has many of the properties demanded of a Langlands correspondence (see especially [5, §2]). One also knows from [6] that, for $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$, we have

$$\mathcal{L}_m^F(\sigma) = \chi_\sigma \boldsymbol{\pi}_m^F(\sigma),$$

where χ_σ is an unramified character of $F^\times$ of finite order strictly dividing p^m. However, we do not know at this stage whether $\{\boldsymbol{\pi}_m^F\}$ enjoys the critical property of preserving local constants of pairs.

The construction of the correspondence $\boldsymbol{\pi}_m^F$ is based on that of a certain tame base change map. If this tame base change satisfies the *automorphic Davenport-Hasse relation* (recalled below as Conjecture A) then indeed $\boldsymbol{\pi}_m^F = \mathcal{L}_m^F$ for all m and F, [6].

3. In this paper, we are more concerned with how one could proceed without assuming *a priori* the existence of the Langlands correspondence. In particular, we consider the problem of showing directly that the correspondence $\{\boldsymbol{\pi}_m^F\}$ preserves local constants of pairs.

From now on, we make no use of the existence of the Langlands correspondence $\{\mathcal{L}_m^F\}$. Indeed, we shall not mention it again except in asides.

As we shall see, the principal obstruction to carrying out this programme is our present ignorance of the structure and significance of the local constant $\varepsilon(\pi_1 \times \pi_2, s, \psi_F)$. In this paper, we list as conjectures some basic properties of the function $(\pi_1, \pi_2) \mapsto \varepsilon(\pi_1 \times \pi_2, s, \psi_F)$ (the "twisting conjectures") and investigate their consequences. We show that they imply many cases of the Davenport-Hasse relation. If we assume the full Davenport-Hasse relation, they imply that the family $\{\boldsymbol{\pi}_m^F\}$ preserves local constants of pairs up to sign (if p is odd) or exactly (if $p = 2$). To eliminate the sign ambiguity, we have to assume a rather deeper conjecture (Conjecture B below) concerning the local constant. This conjecture implies that $\{\boldsymbol{\pi}_m^F\}$ has all the properties demanded of a Langlands correspondence.

4. We now give an abbreviated account of these conjectures. They are all suggested by analogous properties of the Langlands-Deligne local constant $\varepsilon(\sigma_1 \otimes \sigma_2, s, \psi_F)$, $\sigma_i \in \boldsymbol{\mathcal{G}}_{m_i}^{\mathrm{wr}}(F)$.

Temporarily let π_i be an irreducible smooth representation of $\mathrm{GL}_{n_i}(F)$, for $i = 1, 2$. We have

$$\varepsilon(\pi_1 \times \pi_2, s, \psi_F) = q^{-sf(\pi_1 \times \pi_2, \psi_F)}\, \varepsilon(\pi_1 \times \pi_2, 0, \psi_F),$$

where $f(\pi_1 \times \pi_2, \psi_F)$ is an integer and $q = q_F$ is the size of the residue field $\mathfrak{o}_F/\mathfrak{p}_F$ of F. An immediate consequence of the definition [22] of the local constant is that

$$\varepsilon(\chi\pi_1 \times \pi_2, s, \psi_F) = \chi(c)^{-1}\, \varepsilon(\pi_1 \times \pi_2, s, \psi_F)$$

for any unramified quasicharacter χ of $F^\times$ and any $c \in F^\times$ with $v_F(c) = -f(\pi_1 \times \pi_2, \psi_F)$. Our first conjecture is a refinement of this property.

Let K/F be a finite tame extension, and let

$$\begin{aligned} \boldsymbol{l}_{K/F} : \boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F) &\longrightarrow \boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(K), \\ \pi &\longmapsto \boldsymbol{l}_{K/F}(\pi) = \pi^K \end{aligned}$$

be the tame base change ("tame lifting") map defined in [5]. We write $\check{\pi}$ for the contragredient of π.

Twisting properties conjecture. *Let $\pi_i \in \boldsymbol{\mathcal{A}}_{m_i}^{\mathrm{wr}}(F)$, $i = 1, 2$, and suppose that $\pi_1 \not\cong \chi\check{\pi}_2$ for any tamely ramified quasicharacter χ of $F^\times$.*

(i) *There exists $c = c(\pi_1 \times \pi_2, \psi_F) \in F^\times$, uniquely determined modulo U_F^1, such that*

$$\varepsilon(\chi\pi_1 \times \pi_2, s, \psi_F) = \chi(c)^{-1}\varepsilon(\pi_1 \times \pi_2, s, \psi_F),$$

for any tamely ramified quasicharacter χ of $F^\times$.

(ii) *Let K/F be a finite tame extension, and put $\psi_K = \psi_F \circ Tr_{K/F}$. Suppose that $\pi_1^K \not\cong \chi\check{\pi}_2^K$ for any tamely ramified quasicharacter χ of $K^\times$. Then*

$$c(\pi_1^K \times \pi_2^K, \psi_K) \equiv c(\pi_1 \times \pi_2, \psi_F) \pmod{U_K^1}.$$

For the definitive statement of this conjecture, see §2 below.

Remark. This conjecture is certainly true, as a consequence of results in [8] and the existence of the Langlands correspondence. Direct proofs are known for many cases, but we do not consider the matter here.

5. We recall from [6] the *automorphic Davenport-Hasse relation.* We take an odd prime number $\ell \neq p$ and a totally ramified extension K/F of degree ℓ. We need the Adams operation $\pi \mapsto \pi^\ell$ on $\mathcal{A}_m^{\mathrm{wr}}(F)$ defined in [6, §3] or [5, §10], and also the unramified character $\Delta = \Delta_{\ell,m}$ of $F^\times$ defined by

$$\Delta(x) = \left(\frac{q}{\ell}\right)^{(p^m-1)v_F(x)},$$

where $\left(\frac{q}{\ell}\right)$ is the Legendre symbol. The Davenport-Hasse relation is then

Conjecture A. *Let $\pi_i \in \mathcal{A}_{m_i}^{\mathrm{wr}}(F)$, $i = 1,2$, and suppose that $\pi_1^K \not\cong \chi\check{\pi}_2^K$ for any unramified quasicharacter χ of $K^\times$. Then*

$$\varepsilon(\pi_1^K \times \pi_2^K, s, \psi_K) = q^{(\ell-1)f(\pi_1\times\pi_2,\psi_F)/2}\,\varepsilon(\Delta_{\ell,m_1}\pi_1^\ell \times \Delta_{\ell,m_2}\pi_2^\ell, \ell s, \psi_F^\ell).$$

This conjecture is known when m_1 or m_2 is 0 [6, Proposition 4.1]. In all other cases, we have $\Delta_{\ell,m_1}\Delta_{\ell,m_2} = 1$, so the relation then reduces to

$$\varepsilon(\pi_1^K \times \pi_2^K, s, \psi_K) = q^{(\ell-1)f(\pi_1\times\pi_2,\psi_F)/2}\,\varepsilon(\pi_1^\ell \times \pi_2^\ell, \ell s, \psi_F^\ell), \qquad (m_1m_2 \neq 0).$$

Remark. Conjecture A is a test of the "correctness" of the definition of $\boldsymbol{l}_{K/F}$. The Langlands correspondence $\{\mathcal{L}_m^F\}$ gives a map $\boldsymbol{l}'_{K/F} : \mathcal{A}_m^{\mathrm{wr}}(F) \to \mathcal{A}_m^{\mathrm{wr}}(K)$ corresponding to the restriction map $\mathcal{G}_m^{\mathrm{wr}}(F) \to \mathcal{G}_m^{\mathrm{wr}}(K)$; this satisfies the analogue of Conjecture A [6, Theorem 2.2].

6. The final entry in our list of conjectures is more overtly arithmetic in nature. For this, we need the normalized classical Gauss sum $G_F(x,\psi_F)$, $x \in F^\times$, defined in [18] and recalled in 3.3, 3.4 below.

Conjecture B. *Let $\pi_i \in \mathcal{A}_{m_i}^{\mathrm{wr}}(F)$, $i = 1,2$, and let ω_i denote the central quasicharacter of π_i. Suppose that $\pi_1 \not\cong \chi\check{\pi}_2$, for any tamely ramified quasicharacter χ of $F^\times$. Then*

$$\varepsilon(\pi_1 \times \pi_2, \tfrac12, \psi_F) \equiv \omega_1^{p^{m_2}}\omega_2^{p^{m_1}}(c)^{-1/p^{m_1+m_2}}\,G_F(c,\psi_F)^{p^{m_1+m_2}} \pmod{\mu_{p^\infty}},$$

where μ_{p^∞} denotes the group of all p-power roots of unity in $\mathbb{C}$ and

$$c = c(\pi_1 \times \pi_2, \psi_F).$$

Remark. In the case of m_1 or m_2 being zero, Conjecture B is proved in [5, 1.4]; the general case follows from [8] via the correspondence $\{\mathcal{L}_m^F\}$.

7. We give a summary of our results. In §1, we extend the domain of definition of the correspondence $\{\boldsymbol{\pi}_m^F\}$ to a larger set, using unramified automorphic induction [21] and the Langlands-Zelevinsky classification. The point is that this set is stable under the operations of base change (in the sense of [1]) in arbitrary degree and automorphic induction in p-power degree. We show that the extended correspondence commutes with such operations, up to twisting with unramified characters of finite p-power order. The results of §1 are completely unconditional, depending on no conjecture. Thus they have some independent interest, and they also form the first step in the proofs of our later results.

In §2, we give the definitive statement of the twisting properties conjecture. From this, we deduce in §3 some rationality properties of the numbers $\varepsilon(\pi_1 \times \pi_2, \frac{1}{2}, \psi_F)$. Here we employ a result (Theorem 3.2) describing the action of the group $\mathrm{Aut}(\mathbb{C})$ of all (not necessarily continuous) automorphisms of $\mathbb{C}$ on L-functions and local constants of pairs. This again depends on no conjecture, and is proved in §6.

These rationality properties have interesting consequences. Always assuming the twisting properties conjecture, we get:

Theorem 1. *Suppose that K/F is totally ramified of prime degree ℓ and that the normal closure of K/F has odd degree prime to p. Then Conjecture A holds for K/F.*

Further:

Theorem 2.

(a) *Conjecture B holds when $p = 2$.*

(b) *When p is odd, Conjecture B holds up to sign.*

8. Our main result, of which the following is only an approximate statement, is given in §5:

Theorem 3. *Assume Conjecture A and the twisting properties conjecture. Then:*

(a) *The correspondence $\{\boldsymbol{\pi}_m^F\}$ is compatible with tame base change in p-prime degree, with cyclic base change in arbitrary degree, and automorphic induction in p-power degree.*

(b) *If $p = 2$, the correspondence $\{\boldsymbol{\pi}_m^F\}$ preserves local constants of pairs,*

$$(*) \qquad \varepsilon(\sigma_1 \otimes \sigma_2, s, \psi_F) = \varepsilon(\pi_1 \times \pi_2, s, \psi_F),$$

for $\sigma_i \in \boldsymbol{\mathcal{G}}_{m_i}^{\mathrm{wr}}(F)$ and $\pi_i = \boldsymbol{\pi}(\sigma_i)$.

(c) *If p is odd, formula $(*)$ holds up to sign. It holds exactly if Conjecture B is true.*

Thus our conjectures imply the Langlands Conjecture in p-power dimension, via a proof in the spirit of [23] (the case $p^m = 2$) and [17] ($p^m = 3$). We also note that the properties listed in Theorem 3 determine the correspondence $\{\boldsymbol{\pi}^F_m\}$ uniquely, just as in [6].

1. Explicit wild correspondences

In this introductory section, we first recall from [5] the construction of the bijections

$$\boldsymbol{\pi}^F_m : \mathcal{G}^{\mathrm{wr}}_m(F) \longrightarrow \mathcal{A}^{\mathrm{wr}}_m(F).$$

There is a natural way to extend the definition of $\boldsymbol{\pi}^F_m$ to give a bijection, at the level of equivalence classes, between irreducible representations of $\mathcal{W}_F$ of dimension p^m and irreducible supercuspidal representations of $\mathrm{GL}_{p^m}(F)$, for each $F/\mathbb{Q}_p$ and $m \geqslant 0$. We extend the definition further, to a set which is stable under certain operations of base change, automorphic induction and tame lifting. The main point of the section is to show that this extended correspondence is compatible with these operations, *up to twisting by an unramified character of finite p-power order.*

All of the arguments and results of this section are completely unconditional. However, they also form an important first step in the proof of the more precise, but conditional, results below.

1.1. Let K/F be a finite, tamely ramified field extension. In [5], we constructed a canonical map

$$\boldsymbol{l}_{K/F} : \mathcal{A}^{\mathrm{wr}}_m(F) \longrightarrow \mathcal{A}^{\mathrm{wr}}_m(K).$$

The main properties of this "tame base change" map are listed in [5, §1]. We recall, in particular, that $\boldsymbol{l}_{K/F}$ is transitive in the field extension K/F, and natural with respect to isomorphisms of the extension K/F.

If K/F is a tame *cyclic* extension, base change in the sense of Arthur-Clozel [1] likewise gives us a map

$$\boldsymbol{b}_{K/F} : \mathcal{A}^{\mathrm{wr}}_m(F) \longrightarrow \mathcal{A}^{\mathrm{wr}}_m(K).$$

We have (see [5, 1.8]):

Proposition. (i) *Let K/F be a finite cyclic extension with $p \nmid [K{:}F]$. Then $\boldsymbol{b}_{K/F}(\pi) = \boldsymbol{l}_{K/F}(\pi)$, for all $\pi \in \mathcal{A}^{\mathrm{wr}}_m(F)$, $m \geqslant 0$.*

(ii) *If K/F is unramified of degree p and $\pi \in \mathcal{A}^{\mathrm{wr}}_m(F)$, there exists an unramified quasicharacter χ_π of $K^\times$, of finite order strictly dividing p^m, such that $\boldsymbol{b}_{K/F}(\pi) = \chi_\pi\, \boldsymbol{l}_{K/F}(\pi)$.*

The precise relation between $\boldsymbol{b}_{K/F}$ and $\boldsymbol{l}_{K/F}$, when K/F is unramified of degree p, remains unknown. This, however, will not trouble us.

1.2. We recall the definition of $\boldsymbol{\pi}_m^F$. For each $m \geqslant 0$, we define a subset $\mathcal{G}c_m^{\mathrm{wr}}(F)$ of $\mathcal{G}_m^{\mathrm{wr}}(F)$ as follows. A representation $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$ lies in $\mathcal{G}c_m^{\mathrm{wr}}(F)$ if there is a tower of fields $F = F_0 \subset F_1 \subset \cdots \subset F_m$, with each F_i/F_{i-1} cyclic and totally ramified of degree p, together with a quasicharacter χ of $F_m^\times$, such that $\sigma \cong \mathrm{Ind}_{F_m/F}(\chi)$. (Here and throughout we write $\mathrm{Ind}_{K/F}$ to mean $\mathrm{Ind}_{\mathcal{W}_K}^{\mathcal{W}_F}$, where K/F is a finite field extension.)

There is an analogous subset $\mathcal{A}c_m^{\mathrm{wr}}(F)$ of $\mathcal{A}_m^{\mathrm{wr}}(F)$: a representation $\pi \in \mathcal{A}_m^{\mathrm{wr}}(F)$ lies in $\mathcal{A}c_m^{\mathrm{wr}}(F)$ if there is a tower of fields $F = F_0 \subset F_1 \subset \cdots \subset F_m$, with each F_i/F_{i-1} cyclic and totally ramified of degree p, together with a quasicharacter χ of $F_m^\times$, such that

$$\pi \cong \boldsymbol{i}_{F_1/F}\boldsymbol{i}_{F_2/F_1}\cdots\boldsymbol{i}_{F_m/F_{m-1}}(\chi),$$

where $\boldsymbol{i}$ denotes the operation of *automorphic induction,* in the sense of [21].

We know from [19] that there is a canonical bijection

$${}^{c}\boldsymbol{\pi}_m^F : \mathcal{G}c_m^{\mathrm{wr}}(F) \longrightarrow \mathcal{A}c_m^{\mathrm{wr}}(F)$$

as follows. If $\sigma \in \mathcal{G}c_m^{\mathrm{wr}}(F)$ is given by the tower of fields $\{F_i\}$ and the quasicharacter χ as above, we set

$${}^{c}\boldsymbol{\pi}_m^F(\sigma) = \boldsymbol{i}_{F_1/F}\boldsymbol{i}_{F_2/F_1}\cdots\boldsymbol{i}_{F_m/F_{m-1}}(\chi).$$

Remark. The definition we have used for ${}^{c}\boldsymbol{\pi}$ is not apparently that of [19]. However, the two versions are equivalent, as follows from [10, 2.6].

1.3. We fix $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$; we view σ as a homomorphism $\mathcal{W}_F \to \mathrm{GL}_n(\mathbb{C})$, where $n = p^m$. Composing with the canonical projection, we get a projective representation $\bar\sigma : \mathcal{W}_F \to \mathrm{PGL}_n(\mathbb{C})$ *with finite image.* We may identify this image with $\mathrm{Gal}(E/F)$, for some finite Galois extension E/F. Let P denote the inverse image in $\mathcal{W}_F$ of some p-Sylow subgroup of the image of $\bar\sigma$, and let $K \subset E$ be the fixed field of P. Thus $p \nmid [K{:}F]$ and $P = \mathcal{W}_K$. The restriction $\sigma^K = \sigma \mid \mathcal{W}_K$ is effectively an irreducible representation of a central extension of a finite p-group and so $\sigma^K \in \mathcal{G}c_m^{\mathrm{wr}}(K)$. In this situation, the representation $\boldsymbol{\pi}_m^F(\sigma)$ is defined to be the unique element $\pi \in \mathcal{A}_m^{\mathrm{wr}}(F)$ such that:

$$\begin{aligned} \boldsymbol{l}_{K/F}(\pi) &= {}^{c}\boldsymbol{\pi}_m^K(\sigma^K), \quad \text{and} \\ \omega_\pi &= \det\sigma, \end{aligned} \tag{1.3.1}$$

where, as usual, ω_π denotes the central quasicharacter of π. For a full account of this definition, and of the properties of the maps $\boldsymbol{\pi}_m^F$, see [5, especially §2].

The tame field extension K/F constructed here is determined up to F-isomorphism; we refer to it as a *defect field* for σ. The *defect of* σ is the degree $[K{:}F]$.

It will be useful to record the behaviour of the defect field under various change of base field operations.

Proposition. *Let $\sigma \in \boldsymbol{\mathcal{G}}_m^{\mathrm{wr}}(F)$ have defect field E/F, and let K/F be tamely ramified of prime degree.*

(i) *If the field extensions K/F, E/F are linearly disjoint, then σ^K has defect field KE/K.*

(ii) *Otherwise, σ has a defect field E/F which contains K, and then E/K is a defect field for σ^K.*

Proof. The argument of [5, 2.7] adapts easily to the present case, so we omit the details. □

1.4. For an integer $n \geqslant 1$, we denote by $\boldsymbol{\mathcal{G}}_F^0(n)$ the set of equivalence classes of irreducible continuous representations of $\mathcal{W}_F$ of dimension n. Likewise, let $\boldsymbol{\mathcal{A}}_F^0(n)$ be the set of equivalence classes of irreducible supercuspidal representations of $\mathrm{GL}_n(F)$.

Let $\sigma \in \boldsymbol{\mathcal{G}}_F^0(p^m)$, and let $d \geqslant 1$ denote the number of unramified characters χ of $F^\times$ such that $\chi \otimes \sigma \cong \sigma$. Then $d = p^r$, where $0 \leqslant r \leqslant m$. If E/F is unramified of degree d, there is a representation $\tau \in \boldsymbol{\mathcal{G}}_{m-r}^{\mathrm{wr}}(E)$, uniquely determined up to the action of $\mathrm{Gal}(E/F)$, such that $\sigma = \mathrm{Ind}_{E/F}(\tau)$. We define

$$\boldsymbol{\pi}_m^F(\sigma) = \boldsymbol{i}_{E/F}\boldsymbol{\pi}_{m-r}^E(\tau). \tag{1.4.1}$$

Then, in support of a claim made in the introduction to [5], we have:

Proposition. *Let $\sigma \in \boldsymbol{\mathcal{G}}_F^0(p^m)$ and put $\pi = \boldsymbol{\pi}_m^F(\sigma)$.*

(i) *We have $\pi \in \boldsymbol{\mathcal{A}}_F^0(p^m)$, and $\boldsymbol{\pi}_m^F$ induces a bijection*

$$\boldsymbol{\pi}_m^F : \boldsymbol{\mathcal{G}}_F^0(p^m) \xrightarrow{\approx} \boldsymbol{\mathcal{A}}_F^0(p^m)$$

which is natural with respect to automorphisms of the base field F.

(ii) *We have $\boldsymbol{\pi}_m^F(\check{\sigma}) = \check{\pi}$ and $\omega_\pi = \det\sigma$. Moreover, for a quasicharacter χ of $F^\times$ and $\sigma \in \boldsymbol{\mathcal{G}}_F^0(p^m)$, we have*

$$\boldsymbol{\pi}_m^F(\chi \otimes \sigma) = \chi\boldsymbol{\pi}_m^F(\sigma).$$

Proof. Write $\sigma = \mathrm{Ind}_{E/F}(\tau)$, $\tau \in \boldsymbol{\mathcal{G}}_{m-r}^{\mathrm{wr}}(E)$ and E/F unramified of degree p^r as above. Set $\rho = \boldsymbol{\pi}_{m-r}^E(\tau)$. As γ ranges over $\mathrm{Gal}(E/F)$, the representations τ^γ are distinct. Since $\boldsymbol{\pi}_{m-r}^E$ is injective on $\boldsymbol{\mathcal{G}}_{m-r}^{\mathrm{wr}}(E)$ and natural with respect to automorphisms of E, we deduce that the representations $\rho^\gamma \in \boldsymbol{\mathcal{A}}_{m-r}^{\mathrm{wr}}(E)$ are also pairwise distinct. It follows that $\boldsymbol{i}_{E/F}(\rho)$ is supercuspidal (*cf.* [10, 2.6] and so $\boldsymbol{\pi}_m^F(\boldsymbol{\mathcal{G}}_F^0(p^m)) \subset \boldsymbol{\mathcal{A}}_F^0(p^m)$.

We can construct a map $\boldsymbol{\mathcal{A}}_F^0(p^m) \to \boldsymbol{\mathcal{G}}_F^0(p^m)$ as follows. For $\pi \in \boldsymbol{\mathcal{A}}_F^0(p^m)$, let d be the number of unramified quasicharacters χ of $F^\times$ such that $\chi\pi \cong \pi$. Then $d = p^r$, for some $r \leqslant m$. Let E/F be unramified of degree d. There is then $\rho \in \boldsymbol{\mathcal{A}}_{m-r}^{\mathrm{wr}}(E)$ such that $\pi = \boldsymbol{i}_{E/F}(\rho)$, and ρ is determined uniquely up to the action of $\mathrm{Gal}(E/F)$. We may write $\rho = \boldsymbol{\pi}^E_{m-r}(\tau)$, $\tau \in \boldsymbol{\mathcal{G}}^{\mathrm{wr}}_{m-r}(E)$. The representation $\sigma = \mathrm{Ind}_{E/F}(\tau)$ is irreducible, and $\boldsymbol{\pi}^F_m(\sigma) = \pi$, by definition. This process $\pi \mapsto \sigma$ is then inverse to $\boldsymbol{\pi}^F_m$, which is therefore bijective, as required.

The remaining assertions of the proposition are straightforward. $\square$

1.5. Now let $\boldsymbol{\mathcal{G}}_F(n)$ denote the set of equivalence classes of Φ-semisimple representations of the Weil-Deligne group of F. We define a subset $\boldsymbol{\mathcal{G}}_F((p))$ of $\bigcup_{n\geqslant 1} \boldsymbol{\mathcal{G}}_F(n)$ as follows: a representation ρ lies in this subset if, when viewed as a representation of $\mathcal{W}_F$, every composition factor of ρ has p-power dimension.

Similarly, let $\boldsymbol{\mathcal{A}}_F(n)$ denote the set of equivalence classes of irreducible smooth representations of $\mathrm{GL}_n(F)$. There is an analogous subset $\boldsymbol{\mathcal{A}}_F((p))$ of $\bigcup_{n\geqslant 1} \boldsymbol{\mathcal{A}}_F(n)$: a representation lies in this subset if its supercuspidal support consists of elements of $\boldsymbol{\mathcal{A}}^0_F(p^r)$, for various $r \geqslant 0$.

We briefly recall a classical idea [19], [25], [28]. Suppose given, for each $n \geqslant 1$, a bijection $\lambda^0_n : \boldsymbol{\mathcal{G}}^0_F(n) \to \boldsymbol{\mathcal{A}}^0_F(n)$, which takes determinants to central quasicharacters and is compatible with twisting by quasicharacters of $F^\times$. There is then a standard way of extending the family $\{\lambda^0_n\}$ to a family of bijections $\lambda_n : \boldsymbol{\mathcal{G}}_F(n) \to \boldsymbol{\mathcal{A}}_F(n)$, $n \geqslant 1$. This technique enables us to extend the family $\boldsymbol{\pi}^F_m$ to a family of bijections

$$\boldsymbol{\pi}^F = \boldsymbol{\pi}^F(n) : \boldsymbol{\mathcal{G}}_F((p)) \cap \boldsymbol{\mathcal{G}}_F(n) \xrightarrow{\approx} \boldsymbol{\mathcal{A}}_F((p)) \cap \boldsymbol{\mathcal{A}}_F(n).$$

Remark. Thus we have $\boldsymbol{\pi}^F(p^m) \mid \boldsymbol{\mathcal{G}}^{\mathrm{wr}}_m(F) = \boldsymbol{\pi}^F_m$. From now on, we omit the adornments F etc from the notation, at least when there is no fear of confusion.

We will only need to know this bijection explicitly in one case. We take $\sigma \in \boldsymbol{\mathcal{G}}_F((p)) \cap \boldsymbol{\mathcal{G}}_F(n)$, and we assume that σ is in fact a semisimple representation of $\mathcal{W}_F$. Thus $\sigma = \sigma_1 \oplus \cdots \oplus \sigma_t$, with $\sigma_i \in \boldsymbol{\mathcal{G}}^0_F(p^{m_i})$, for various integers $m_i \geqslant 0$. Set $\pi_i = \boldsymbol{\pi}^F_{m_i}(\sigma_i)$, and let π be the representation of $\mathrm{GL}_n(F)$ parabolically induced by $\pi_1 \otimes \cdots \otimes \pi_t$. One then gets easily:

Lemma. *Suppose that the representation π is irreducible. Then $\boldsymbol{\pi}^F(\sigma) = \pi$.*

If K/F is cyclic, then restriction induces a map $\boldsymbol{\mathcal{G}}_F((p)) \to \boldsymbol{\mathcal{G}}_K((p))$. If K/F is cyclic of degree p, induction gives a map $\boldsymbol{\mathcal{G}}_K((p)) \to \boldsymbol{\mathcal{G}}_F((p))$. We show that a similar phenomenon occurs on the other side.

Proposition. (i) *Let $\pi \in \boldsymbol{\mathcal{A}}_F((p))$, and let K/F be cyclic. Then $\boldsymbol{b}_{K/F}(\pi)$ lies in $\boldsymbol{\mathcal{A}}_K((p))$.*

(ii) *Let K/F be cyclic of degree p, and let $\pi \in \boldsymbol{\mathcal{A}}_K((p))$. Then $\boldsymbol{i}_{K/F}(\pi) \in \boldsymbol{\mathcal{A}}_F((p))$.*

Proof. In (i), it is enough to treat the case where K/F is of prime degree and, since base change respects supercuspidal support, we may take $\pi \in \mathcal{A}_F^0(p^m)$, for some m. If $\boldsymbol{b}_{K/F}(\pi)$ is supercuspidal, there is nothing to prove. Otherwise, $\boldsymbol{b}_{K/F}(\pi)$ is irreducibly parabolically induced from a representation of the form $\bigotimes_\gamma \rho^\gamma$, where ρ is supercuspidal and γ ranges over $\mathrm{Gal}(K/F)$ [10, 2.6]. Further, the factors ρ^γ are distinct. Thus $[K{:}F] = p$ and $\rho^\gamma \in \mathcal{A}_F^0(p^{m-1})$. It follows that $\boldsymbol{b}_{K/F}(\pi) \in \mathcal{A}_F((p))$, as required.

In (ii), we may again take $\pi \in \mathcal{A}_K^0(p^m)$. If $\boldsymbol{i}_{K/F}(\pi)$ is supercuspidal, there is nothing to prove. Otherwise, $\boldsymbol{i}_{K/F}(\pi)$ is irreducibly parabolically induced from a representation $\bigotimes_\chi \chi\tau$, where $\tau \in \mathcal{A}_F^0(p^m)$ and χ ranges over the group of characters of $F^\times$ trivial on norms from K (see [10, 2.6] again). This surely lies in $\mathcal{A}_F((p))$. ☐

1.6. We are now in a position to formulate and prove the main result of this section.

Theorem. (i) *Let $\sigma \in \mathcal{G}_F^0(p^m)$, and put $\pi = \boldsymbol{\pi}_m^F(\sigma)$. Let K/F be a finite cyclic extension. There is an unramified character χ_σ of $K^\times$, of finite order dividing p^m, such that*

$$\boldsymbol{b}_{K/F}(\pi) = \chi_\sigma \boldsymbol{\pi}^K(\sigma^K).$$

If $p \nmid [K{:}F]$, then $\chi_\sigma = 1$.

(ii) *Let $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$, and put $\pi = \boldsymbol{\pi}_m^F(\sigma)$. Let K/F be a finite tame extension. There is an unramified character χ_σ of $K^\times$, of finite order dividing p^m, such that*

$$\boldsymbol{l}_{K/F}(\pi) = \chi_\sigma \boldsymbol{\pi}^K(\sigma^K).$$

(iii) *Let K/F be cyclic of degree p, let $\tau \in \mathcal{G}_K^0(p^m)$, and put $\rho = \boldsymbol{\pi}_m^K(\tau)$. There is an unramified character χ_σ of $F^\times$, of finite order dividing p^{m+1}, such that*

$$\boldsymbol{i}_{K/F}(\rho) = \chi_\sigma \boldsymbol{\pi}^F(\mathrm{Ind}_{K/F}\,\tau).$$

The character χ_σ is trivial when K/F is unramified.

Proof. In both (i) and (ii), by transitivity it is enough to treat the case where K/F has prime degree ℓ.

Step 1. In (ii), if K/F is cyclic, the result is given by Proposition 1.1, [5, 2.3], and [6, 1.8.1]. We therefore prove (ii) under the assumption that K/F is not cyclic. Thus the prime $\ell = [K{:}F]$ is odd. Let L/F be the normal closure of K/F, and E/F the maximal unramified subextension of L/F. Thus L/E, E/F are cyclic and, by the first case, $\boldsymbol{l}_{L/F}(\pi) = \chi_1\boldsymbol{\pi}(\sigma^L)$, for χ_1 unramified. Thus, since L/K is unramified, there is an unramified character χ_2 of $K^\times$ such that $\chi_2\boldsymbol{\pi}(\sigma^K)$, $\boldsymbol{l}_{K/F}(\pi)$ have the same image under $\boldsymbol{l}_{L/K}$. By [5, Theorem 1.3], this means that $\boldsymbol{l}_{K/F}(\pi) = \chi_3\boldsymbol{\pi}(\sigma^K)$, with χ_3

unramified. The representations $\boldsymbol{l}_{K/F}(\pi)$, $\boldsymbol{\pi}(\sigma^K)$ have the same central quasicharacter $\det\sigma^K$, so $\chi_3^{p^m}=1$, as required.

Step 2. We now prove (i) in the case $\sigma\in\mathcal{G}_m^{\mathrm{wr}}(F)$. If $p\nmid[K:F]$, the result is given by [5, Th. 2.3(vi)]. If K/F is unramified of degree p, it is given by Step 1. We therefore need only consider the case where K/F is totally ramified of degree p. If we in fact have $\sigma\in\mathcal{G}\boldsymbol{c}_m^{\mathrm{wr}}(F)$, then $\boldsymbol{b}_{K/F}(\pi)=\boldsymbol{\pi}(\sigma^K)$ [19]. In general, let E/F be the normal closure of a defect field for σ. Then E/F is tame Galois, and $\sigma^E\in\mathcal{G}\boldsymbol{c}_m^{\mathrm{wr}}(E)$. The metacyclic base change operations $\boldsymbol{b}_{E/F}$, $\boldsymbol{b}_{KE/F}$ etc. are well-defined (*cf.* [4, 16.4, 16.6]) with the obvious transitivity properties. Thus, using the tame case (ii),

$$\boldsymbol{b}_{KE/K}\boldsymbol{b}_{K/F}(\pi)=\boldsymbol{b}_{KE/E}\boldsymbol{b}_{E/F}(\pi)=\chi_1\boldsymbol{b}_{KE/E}(\boldsymbol{\pi}(\sigma^E)),$$

with χ_1 unramified. Since $\sigma^E\in\mathcal{G}\boldsymbol{c}_m^{\mathrm{wr}}(E)$, we have

$$\boldsymbol{b}_{KE/E}(\boldsymbol{\pi}(\sigma^E))=\boldsymbol{\pi}(\sigma^{KE}),$$

and this, by the tame case again, is $\chi_2\boldsymbol{b}_{KE/K}(\boldsymbol{\pi}(\sigma^K))$, where χ_2 is unramified. In other words, there is an unramified character χ_3 of $K^\times$ such that $\boldsymbol{b}_{K/F}(\pi)$, $\chi_3\boldsymbol{\pi}(\sigma^K)$ have the same base change to KE/K. When σ^K is irreducible (that is, $\boldsymbol{\pi}(\sigma^K)$ is supercuspidal), it follows from [5, Theorem 1.3] that the representations $\boldsymbol{b}_{K/F}(\pi)$, $\chi_3\boldsymbol{\pi}(\sigma^K)$ differ by a tame character of $K^\times$. That is, $\boldsymbol{b}_{K/F}(\pi)=\alpha\boldsymbol{\pi}(\sigma^K)$, for some tame character α of $K^\times$. Comparing central quasicharacters, we get $\alpha^{p^m}=1$, whence α is unramified and the assertion follows.

We therefore assume that σ^K is not irreducible. Thus there is a representation $\tau\in\mathcal{G}_{m-1}^{\mathrm{wr}}(K)$ such that $\sigma^K=\bigoplus_\gamma\tau^\gamma$, with γ ranging over $\mathrm{Gal}(K/F)$ (and the τ^γ are pairwise inequivalent). Thus $\boldsymbol{\pi}(\sigma^K)$ is parabolically induced from $\bigotimes_\gamma\rho^\gamma$, where $\rho=\boldsymbol{\pi}(\tau)$. Similarly, by [10, 2.6], $\boldsymbol{b}_{K/F}(\pi)$ is parabolically induced by $\bigotimes_\gamma\rho_1^\gamma$, for some $\rho_1\in\mathcal{A}_{m-1}^{\mathrm{wr}}(K)$. By choosing ρ_1 properly within its $\mathrm{Gal}(K/F)$-orbit, we get $\boldsymbol{b}_{KE/K}(\rho_1)=\boldsymbol{b}_{KE/K}(\rho)$, and we are reduced to the supercuspidal case, which has already been settled.

Step 3. We now treat (i) for general $\sigma\in\mathcal{G}_F^0(p^m)$. Thus we have an unramified extension E/F of degree p^r, $r\geqslant 1$, and $\tau\in\mathcal{G}_{m-r}^{\mathrm{wr}}(E)$ such that $\sigma=\mathrm{Ind}_{E/F}(\tau)$. If we put $\rho=\boldsymbol{\pi}(\tau)$, then $\pi=\boldsymbol{i}_{E/F}(\rho)$. Suppose first that K/F is not unramified of degree p. Then $\sigma^K=\mathrm{Ind}_{KE/K}(\tau^{KE})$ and $\boldsymbol{b}_{K/F}(\pi)=\boldsymbol{i}_{KE/K}\boldsymbol{b}_{KE/E}(\rho)$. The result now follows from the case of Step 2 and the definition of $\boldsymbol{\pi}$. If K/F is unramified of degree p, the result is immediate.

Step 4. In (iii), the case of K/F unramified is straightforward. We therefore assume that K/F is totally ramified. We set $\sigma=\mathrm{Ind}_{K/F}(\tau)$, $\pi=\boldsymbol{i}_{K/F}(\rho)$. We assume first that σ is irreducible, whence π is supercuspidal,

$\pi \in \boldsymbol{\mathcal{A}}_F^0(p^m)$. We write $\pi = \boldsymbol{\pi}(\sigma_1)$. The representation $\boldsymbol{b}_{K/F}(\pi)$ is irreducibly parabolically induced from $\bigotimes_\gamma \rho^\gamma$, with distinct factors ρ^γ and γ ranging over $\mathrm{Gal}(K/F)$. By part (i), $\boldsymbol{\pi}(\sigma^K)$ is an unramified twist of this representation. Therefore some unramified twist of σ_1 contains τ on $\mathcal{W}_K$. Therefore $\sigma_1 = \chi\eta\sigma$, where χ is unramified and η is a character of $F^\times$ vanishing on norms from K. Since σ is induced from K, we have $\eta\sigma = \sigma$, and the result follows in this case.

This leaves the case where σ^K is not irreducible. We have $\tau = \tau^\gamma$ for every $\gamma \in \mathrm{Gal}(K/F)$. The representation ρ has the corresponding property and π is irreducibly parabolically induced from $\bigotimes_\chi \chi\pi_0$, for some $\pi_0 \in \boldsymbol{\mathcal{A}}_F^0(p^{m+1})$ with χ ranging over the group of characters of $F^\times$ trivial on norms from K. The representation π_0 satisfies $\boldsymbol{b}_{K/F}(\pi_0) = \rho$; by part (i), it is therefore of the form $\pi_0 = \boldsymbol{\pi}(\sigma_0)$, where $\sigma_0^K = \tau$. The assertion therefore follows in this case, with $\chi_\sigma = 1$. $\square$

2. Local constants under twisting

We fix a non-trivial continuous character ψ_F of the additive group of F. As a matter of notation, if E/F is a finite extension, we put $\psi_E = \psi_F \circ \mathrm{Tr}_{K/F}$.

In this section, we state some conjectural properties of the local constant $\varepsilon(\pi_1 \times \pi_2, s, \psi_F)$, for $\pi_i \in \boldsymbol{\mathcal{A}}_{m_i}^{\mathrm{wr}}(F)$. These form the definitive version of the twisting properties conjecture of the Introduction.

2.1. We first recall a simple consequence of the definition [22] of the local constant:

Lemma. *Let $\pi_i \in \boldsymbol{\mathcal{A}}_{m_i}^{\mathrm{wr}}(F)$, for $i = 1, 2$. We have*

$$\varepsilon(\chi\pi_1 \times \pi_2, s, \psi_F) = \chi(c)^{-1}\varepsilon(\pi_1 \times \pi_2, s, \psi_F),$$

for any unramified quasicharacter χ of $F^\times$ and any element $c \in F^\times$ such that $v_F(c) = -f(\pi_1 \times \pi_2, \psi_F)$.

We need to refine this result. For this, it will be convenient to divide into two cases.

Conjecture 1. *Let $\pi \in \boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F)$. There exists $c_{\pi\times\check{\pi}} \in F^\times$, well defined modulo U_F^1, such that*

$$\varepsilon(\chi\pi \times \check{\pi}, s, \psi_F) = \frac{\varepsilon(\chi, s, \psi_F)}{\varepsilon(1_F, s, \psi_F)}\, \chi(c_{\pi\times\check{\pi}})^{-1}\, \varepsilon(\pi \times \check{\pi}, s, \psi_F), \tag{2.1.1}$$

for all tame quasicharacters χ of $F^\times$.

Here and in the following 1_F denotes the trivial character of $F^\times$ (or of $\mathcal{W}_F$). We recall [7] that $\varepsilon(\pi \times \check{\pi}, \frac{1}{2}, \psi_F) = \omega_\pi(-1)^{p^m-1}$, while $f(\pi \times \check{\pi}, \psi_F)$ is given by [9, Theorem 6.5].

There is a complementary relation:

Conjecture 2. *For $i = 1, 2$, let $\pi_i \in \mathcal{A}^{\mathrm{wr}}_{m_i}(F)$. Assume that π_1 is not of the form $\chi\check{\pi}_2$ for any tame quasicharacter χ of $F^\times$. There is then an element $c_{\pi_1\times\pi_2} \in F^\times$, well-defined modulo U^1_F, such that*

$$\varepsilon(\chi\pi_1 \times \pi_2, s, \psi_F) = \chi(c_{\pi_1\times\pi_2})^{-1}\,\varepsilon(\pi_1 \times \pi_2, s, \psi_F), \tag{2.1.2}$$

for all tame quasicharacters χ of $F^\times$.

Note. In both statements the element c depends on the choice of ψ_F. When necessary we shall use the more precise notation $c(\pi_1 \times \pi_2, \psi_F)$.

In (2.1.2) we obviously have

$$c_{\chi\pi_1\times\pi_2} \equiv c_{\pi_1\times\pi_2} \pmod{U^1_F} \tag{2.1.3}$$

for all tame quasicharacters χ of $F^\times$. If on the contrary we have $\pi_1 = \chi\check{\pi}_2$ for a tame quasicharacter χ of $F^\times$, we *define*

$$c_{\pi_1\times\pi_2} = c_{\pi_1\times\check{\pi}_1}. \tag{2.1.4}$$

In both cases therefore, we have

$$c_{\chi\pi_1\times\pi_2} = c_{\pi_1\times\pi_2}, \text{ for } \chi \text{ tamely ramified.} \tag{2.1.5}$$

2.2. We now consider how the element $c_{\pi_1\times\pi_2}$ behaves under extension of the base field.

Conjecture 3. *Let K/F be a finite tame extension. For $i = 1, 2$ let $\pi_i \in \mathcal{A}^{\mathrm{wr}}_{m_i}(F)$, and let $\pi_i^K = \boldsymbol{l}_{K/F}(\pi_i)$. Then*

$$c(\pi_1^K \times \pi_2^K, \psi_K) \equiv c(\pi_1 \times \pi_2, \psi_F) \pmod{U^1_K}. \tag{2.2.1}$$

The identity (2.2.1) still holds if we replace, for example, π_1^K by $\chi\pi_1^K$, for an unramified quasicharacter χ of $K^\times$. Thus, if K/F is a *cyclic* tame extension (2.2.1) implies

$$c(\boldsymbol{b}_{K/F}\pi_1 \times \boldsymbol{b}_{K/F}\pi_2, \psi_K) \equiv c(\pi_1 \times \pi_2, \psi_F) \pmod{U^1_K}, \tag{2.2.2}$$

since $\boldsymbol{l}_{K/F}(\pi_i)$ and $\boldsymbol{b}_{K/F}(\pi_i)$ differ at most by an unramified character (Proposition 1.1).

Remark. Let $\sigma_i \in \mathcal{G}^{\mathrm{wr}}_{m_i}(F)$, $i = 1, 2$. The arguments of [8] then yield an element $c_{\sigma_1\otimes\sigma_2} \in F^\times/U^1_F$ with properties analogous to those above, relative to the local constant $\varepsilon(\sigma_1 \otimes \sigma_2, s, \psi_F)$. If $\pi_i = \mathcal{L}(\sigma_i)$, we then have $c_{\pi_1\times\pi_2} = c_{\sigma_1\otimes\sigma_2}$, since $\mathcal{L}$ preserves local constants of pairs. Thus (2.1.1), (2.1.2) follow from the Langlands Conjecture and [8, Corollaire 2.3, Théorème 1.3] respectively. The analogue of (2.2.1) for $c_{\sigma_1\otimes\sigma_2}$ follows from the defining property of c and the inductivity of ε; this implies (2.2.1) via the Langlands correspondence. Of course, the point for us is to obtain direct proofs of these results, without recourse to the existence of $\mathcal{L}^F_m$.

Known cases. A direct proof of Conjecture 1 is known. Likewise Conjecture 2 in the case $m_1 \neq m_2$. Conjecture 3 is known for $c(\pi \times \check{\pi}, \psi_F)$. (We do not prove these assertions here.)

2.3. If we consider the correspondences $\boldsymbol{\pi}^F_{m_i} : \mathcal{G}^{\mathrm{wr}}_{m_i}(F) \to \mathcal{A}^{\mathrm{wr}}_{m_i}(F)$ recalled in §1, we obtain:

Theorem. *For $i = 1, 2$, let $\sigma_i \in \mathcal{G}^{\mathrm{wr}}_{m_i}(F)$ and put $\pi_i = \boldsymbol{\pi}^F_{m_i}(\sigma_i)$. Then, on Conjectures 1–3, we have*

$$c_{\pi_1 \times \pi_2} \equiv c_{\sigma_1 \otimes \sigma_2} \pmod{U^1_F}.$$

Proof. If $\sigma_i \in \mathcal{G}c^{\mathrm{wr}}_{m_i}(F)$, $i = 1, 2$, then the assertion is immediate: the maps ${}^{c}\boldsymbol{\pi}$ preserve local constants of pairs [19]. The general case then follows from (2.2.1) and the definition of $\boldsymbol{\pi}^F_m$. □

2.4. For the moment, let π_i be an irreducible smooth representation of $\mathrm{GL}_{n_i}(F)$, for $i = 1, 2$. We consider briefly the variation of $\varepsilon(\pi_1 \times \pi_2, s, \psi_F)$ with the additive character ψ_F.

Let ψ'_F be some other non-trivial character of F; there is then a unique $\alpha \in F^\times$ such that $\psi'_F(x) = \psi_F(\alpha x)$, for $x \in F$. We have

(2.4.1) $$f(\pi_1 \times \pi_2, \psi'_F) = f(\pi_1 \times \pi_2, \psi_F) + n_1 n_2 v_F(\alpha).$$

Write ω_i for the central character of π_1 and put $\Omega = \omega_1^{n_2}\omega_2^{n_1}$. We have:

(2.4.2) $$\varepsilon(\pi_1 \times \pi_2, \tfrac{1}{2}, \psi'_F) = \Omega(\alpha)\, \varepsilon(\pi_1 \times \pi_2, \tfrac{1}{2}, \psi_F).$$

Both identities follow readily from the definitions in [22].

As a consequence of these identities, we see that all of the conjectures above are essentially independent of the choice of ψ_F: if they hold for one choice, then they hold for all.

3. Values of local constants for pairs

In this section, we investigate the arithmetic properties of the value of the local constant $\varepsilon(\pi_1 \times \pi_2, \frac{1}{2}, \psi_F)$, for $\pi_i \in \mathcal{A}^{\mathrm{wr}}_{m_i}(F)$. We prove, on the conjectures listed in §2, a slightly weakened version of Conjecture B (stated as Theorem 2 in the introduction). This depends on knowledge of the action of $\mathrm{Aut}(\mathbb{C})$ on the local constant. We give a general statement of the required result in 3.2, and prove it in §6.

3.1. We proved in [7] that

(3.1.1) $$\varepsilon(\pi_1 \times \check{\pi}_1, \tfrac{1}{2}, \psi_F) = \omega_{\pi_1}(-1)^{p^{m_1}-1}.$$

We shall next explore the value $\varepsilon(\pi_1 \times \pi_2, \frac{1}{2}, \psi_F)$ when π_2 is "distant from" $\check{\pi}_1$. This result depends on the conjectures listed in §2.

Proposition. *For $i = 1, 2$, let $\pi_i \in \mathcal{A}^{\mathrm{wr}}_{m_i}(F)$, and assume:*
(i) *ω_{π_i} has finite order, and*
(ii) *π_1 is not of the form $\chi\check{\pi}_2$ for any tame quasicharacter χ of $F^\times$.*
Then $\varepsilon(\pi_1 \times \pi_2, \frac{1}{2}, \psi_F)$ is a root of unity.

Proof. We first show that the analogous statement is true for Weil group representations. Indeed if, for $i = 1, 2$, $\sigma_i \in \mathcal{G}^{\mathrm{wr}}_{m_i}(F)$ has determinant of finite order, then $\sigma_1 \otimes \sigma_2$ is in fact a representation of a finite Galois group. If σ_1 is not of the form $\chi\check{\sigma}_2$ for any tame χ then $\sigma_1 \otimes \sigma_2$ has no tame component hence $\varepsilon(\sigma_1 \otimes \sigma_2, 0, \psi_F)$ is a real number times a root of unity [13, Appendix]. But $\varepsilon(\sigma_1 \otimes \sigma_2, \frac{1}{2}, \psi_F)$ is a complex number of modulus 1 and

$$\varepsilon(\sigma_1 \otimes \sigma_2, 0, \psi_F) = q^{f(\sigma_1 \otimes \sigma_2, \psi_F)/2}\, \varepsilon(\sigma_1 \otimes \sigma_2, \tfrac{1}{2}, \psi_F)$$

with an integral exponent $f(\sigma_1 \otimes \sigma_2, \psi_F)$. It follows that $\varepsilon(\sigma_1 \otimes \sigma_2, \frac{1}{2}, \psi_F)$ is indeed a root of unity.

We write $\pi_i = \boldsymbol{\pi}(\sigma_i)$, $\sigma_i \in \mathcal{G}^{\mathrm{wr}}_{m_i}(F)$. Suppose first that $\sigma_i \in \mathcal{G}\boldsymbol{c}^{\mathrm{wr}}_{m_i}(F)$, for $i = 1, 2$. We then have $\varepsilon(\pi_1 \times \pi_2, s, \psi_F) = \varepsilon(\sigma_1 \otimes \sigma_2, s, \psi_F)$, whence the result in this case.

In general, we can find a finite, tame, Galois extension E/F containing the defect fields of both σ_i. In particular, $\sigma_i^E \in \mathcal{G}\boldsymbol{c}^{\mathrm{wr}}_{m_i}(E)$, and $\boldsymbol{\pi}^E_{m_i}(\sigma_i^E) = \chi_i\, \boldsymbol{l}_{E/F}(\pi_i)$, for an unramified character χ_i of $E^\times$ of finite p-power order (Theorem 1.6). From the first case we have

$$\varepsilon(\boldsymbol{l}_{E/F}(\pi_1) \times \boldsymbol{l}_{E/F}(\pi_2), \tfrac{1}{2}, \psi_E) = \eta\, \varepsilon(\sigma_1^E \otimes \sigma_2^E, \tfrac{1}{2}, \psi_E),$$

where η is a root of unity of p-power order, whence

$$\varepsilon(\boldsymbol{l}_{E/F}(\pi_1) \times \boldsymbol{l}_{E/F}(\pi_2), \frac{1}{2}, \psi_E)$$

is a root of unity.

The conclusion of the last paragraph still holds if we replace $\boldsymbol{l}_{E/F}(\pi_i)$ by the Arthur-Clozel base change $\boldsymbol{b}_{E/F}(\pi_i)$, by Proposition 1.1.

Now let L/F be a finite cyclic extension and set $\pi_i^L = \boldsymbol{b}_{L/F}(\pi_i)$. We then have [1, I, Prop. 6.9]:

$$\lambda_{L/F}^{p^{m_1+m_2}}\, \varepsilon(\pi_1^L \times \pi_2^L, s, \psi_L) = \prod_{\chi} \varepsilon(\chi\pi_1 \times \pi_2, s, \psi_F) \tag{3.1.2}$$

where $\lambda_{L/F}$ is the Langlands constant

$$\lambda_{L/F} = \frac{\varepsilon(\mathrm{Ind}_{L/F}(1_L), s, \psi_F)}{\varepsilon(1_L, s, \psi_L)} \tag{3.1.3}$$

and χ runs through the group of characters of $F^\times$ trivial on $\mathrm{N}_{L/F}(L^\times)$. One knows from, for example, the calculations in [3, §10], that $\lambda_{L/F}$ is a root of unity when L/F is tame. The right hand side in the relation 3.1.2 is then, by (2.1.2), a root of unity times $\varepsilon(\pi_1 \times \pi_2, s, \psi_F)^{[L:F]}$. In the situation above, the extension E/F is of the form $E \supset L \supset F$, with L/F and E/L both tame and cyclic; in fact L/F is unramified. The result will therefore follow if we can prove that $\varepsilon(\pi_1^L \times \pi_2^L, \frac{1}{2}, \psi_L)$ is a root of unity. We next observe:

Lemma. *In the situation above, π_1^L is not of the form $\chi\check{\pi}_2^L$ for any tame quasicharacter χ of $L^\times$.*

Proof. This is only an issue in the case $m_1 = m_2$. The fact that π_1 is not a tame twist of $\check{\pi}_2$ is equivalent to the simple characters in the π_i not being conjugate in $\mathrm{GL}_{p^{m_1}}(F)$. This implies [5, 1.3] that the simple characters underlying the representations π_i^L are not conjugate in $\mathrm{GL}_{p^{m_1}}(L)$, and the Lemma follows. □

Because of this lemma, we can apply the above procedure twice to get the proposition. □

3.2. We can in fact be a little more precise than in 3.1, by taking into account the action of $\mathrm{Aut}(\mathbb{C})$ on local constants for pairs. We first state a completely general result on this matter (which does not depend on any conjectures).

We write $\boldsymbol{\mathcal{A}}_F(n)$ for the set of equivalence classes of irreducible smooth representations of $\mathrm{GL}_n(F)$. As explained in [6, §7], $\mathrm{Aut}(\mathbb{C})$ acts on $\boldsymbol{\mathcal{A}}_F(n)$; we write this action as $(\tau, \pi) \mapsto \tau\pi$, for $\tau \in \mathrm{Aut}(\mathbb{C})$, $\pi \in \boldsymbol{\mathcal{A}}_F(n)$. We extend $\tau \in \mathrm{Aut}(\mathbb{C})$ to an automorphism of the field $\mathbb{C}(q^{-s})$ of rational functions in q^{-s} by treating q^{-s} as an indeterminate on which τ acts trivially.

Theorem. *For $i = 1, 2$, let $\pi_i \in \boldsymbol{\mathcal{A}}_F(n_i)$. Then*

$$L(\tau\pi_1 \times \tau\pi_2, s + \tfrac{n_1+n_2}{2}) = \tau L(\pi_1 \times \pi_2, s + \tfrac{n_1+n_2}{2}),$$

$$\varepsilon(\tau\pi_1 \times \tau\pi_2, s + \tfrac{n_1+n_2}{2}, \tau\psi_F) = \left(\frac{\tau}{q_F}\right)^{a(\psi_F)n_1n_2} \tau\varepsilon(\pi_1 \times \pi_2, s + \tfrac{n_1+n_2}{2}, \psi_F),$$

where

$$\left(\frac{\tau}{q_F}\right) = \frac{\tau\sqrt{q_F}}{\sqrt{q_F}}$$

and

$$a(\psi_F) = min\{k \in \mathbb{Z} : \mathfrak{p}_F^k \subset Ker\psi_F\}.$$

The proof will be given in §6.

3.3. We are now ready to investigate Conjecture B. When some $m_i = 0$, the conjecture is proved in [5, Theorem 1.4], so we can assume throughout that $m_1 m_2 \neq 0$.

We write μ_{p^r} for the group of p^r-th roots of unity in $\mathbb{C}$ and $\mu_{p^\infty} = \bigcup_r \mu_{p^r}$.

We first recall that, when $p = 2$, the function G_F of [18] is identically 1.

Theorem. *Let $p = 2$. For $i = 1, 2$, let $\pi_i \in \mathcal{A}^{\mathrm{wr}}_{m_i}(F)$. Assume that π_1 is not of the form $\chi\check{\pi}_2$ for any tame quasicharacter χ of $F^\times$. Then, on the conjectures of §2, we have*

$$\varepsilon(\pi_1 \times \pi_2, \tfrac{1}{2}, \psi_F) \equiv \omega_{\pi_1}^{2^{m_2}} \omega_{\pi_2}^{2^{m_1}} (c_{\pi_1\times\pi_2})^{-1/2^{m_1+m_2}} \pmod{\mu_{2^\infty}}.$$

In particular, Conjecture B holds when $p = 2$.

Proof. Assume first that ω_{π_1} and ω_{π_2} are of finite 2-power order. Then the central type of π_i, $i = 1, 2$, factorizes through a finite 2-group hence is definable over $\mathbb{Q}(\mu_{2^r})$ for any sufficiently large integer r. The same is true of π_2. If $\tau \in \mathrm{Aut}(\mathbb{C})$ acts trivially on μ_{2^∞} it also acts trivially on $\sqrt{q}$ and ψ_F and we have

$$\begin{aligned}\varepsilon(\pi_1 \times \pi_2, \tfrac{1}{2}, \psi_F) &= \varepsilon(\tau\pi_1 \times \tau\pi_2, \tfrac{1}{2}, \tau\psi_F)\\ &= \tau\varepsilon(\pi_1 \times \pi_2, \tfrac{1}{2}, \psi_F)\end{aligned}$$

by Theorem 3.2. Since we know that $\varepsilon(\pi_1 \times \pi_2, \frac{1}{2}, \psi_F)$ is a root of unity (Proposition 3.1), it follows that it is a root of unity in some cyclotomic field $\mathbb{Q}(\mu_{2^r})$, hence a root of unity in μ_{2^∞}. In the general case, there are tame quasicharacters χ_i, $i = 1, 2$, such that $\pi_i = \chi_i \pi_i'$ with $\omega_{\pi_i'}$ of finite 2-power order. As $\omega_{\pi_i} = \chi_i^{2^{m_i}} \omega_{\pi_i'}$, the theorem follows from the special case and (2.1.2), noting that $c_{\pi_1\times\pi_2} = c_{\pi_1'\times\pi_2'}$. □

3.4. Suppose in this paragraph that p is odd. The map $G_F : F^\times \to \mathbb{C}^\times$ is defined as follows. Set $a = a(\psi_F)$, as in 3.2, and take $x \in F^\times$. There are two cases. First, we set

$$G_F(x, \psi_F) = 1 \quad \text{if} \quad v_F(x) \equiv a(\psi_F) \pmod 2.$$

Otherwise, we choose a prime element ϖ of $\mathfrak{o}_F$ and let b be the integer for which $x\varpi^{2b}\mathfrak{o}_F = \mathfrak{p}_F^{a-1}$. We then define

$$G_F(x, \psi_F) = q_F^{-1/2} \sum_{y \in \mathfrak{o}_F/\mathfrak{p}_F} \psi_F(x\varpi^{2b} y^2/2).$$

In particular,

$$G_F(x, \psi_F)^2 = \left(\frac{-1}{q}\right)^{v_F(x)+a(\psi_F)}. \tag{3.4.1}$$

Theorem. *Let p be odd. For $i = 1, 2$, let $\pi_i \in \mathcal{A}^{\mathrm{wr}}_{m_i}(F)$. Assume that π_1 is not of the form $\chi\check{\pi}_2$ for any tame quasicharacter χ of $F^\times$. Then, on the Conjectures of §2, we have*

$$\varepsilon(\pi_1 \times \pi_2, \tfrac{1}{2}, \psi_F)^2 \equiv \omega_{\pi_1}^{p^{m_2}} \omega_{\pi_2}^{p^{m_1}} (c_{\pi_1\times\pi_2})^{-2/p^{m_1+m_2}} G_F(c_{\pi_1\times\pi_2})^{2p^{m_1+m_2}} \pmod{\mu_{p^\infty}}.$$

That is, Conjecture B holds up to sign.

Proof. As above we reduce (using the twisting properties) to the case where the ω_{π_i} have finite p-power order. We can also assume that the m_i are non-zero.

We remark that if $\tau \in \mathrm{Aut}(\mathbb{C})$ acts trivially on μ_{p^∞} and $\sqrt{q}$ then it also acts trivially on $\varepsilon(\pi_1 \times \pi_2, \frac{1}{2}, \psi_F)$ (Theorem 3.2). More precisely if $\left(\frac{-1}{q}\right) = 1$ then if $\tau \in \mathrm{Aut}(\mathbb{C})$ acts trivially on μ_{p^∞}, it also fixes $\sqrt{q}$ and then $\varepsilon(\pi_1\times\pi_2, \frac{1}{2}, \psi_F)$ is a root of unity belonging to $\mathbb{Q}(\mu_{p^r})$ for large enough r. This implies $\pm\varepsilon(\pi_1 \times \pi_2, \frac{1}{2}, \psi_F) \in \mu_{p^\infty}$, and hence, arguing as in 3.3, we get the result.

Let now $\left(\frac{-1}{q}\right) = -1$. The exponent of q^{-s} in $\varepsilon(\pi_1 \times \pi_2, s, \psi_F)$ is $f = -v_F(c_{\pi_1\times\pi_2})$ and we deduce from Theorem 3.2 that if $\tau \in \mathrm{Aut}(\mathbb{C})$ acts trivially on μ_{p^∞} we have

$$\tau\varepsilon(\pi_1 \times \pi_2, \tfrac{1}{2}, \psi_F) = \left(\tfrac{\tau}{q}\right)^{f+a} \varepsilon(\pi_1 \times \pi_2, \tfrac{1}{2}, \psi_F).$$

But from (3.4.1) above we have

$$\tau G_F(x) = \left(\frac{\tau\sqrt{-1}}{\sqrt{-1}}\right)^{f+a} G_F(x),$$

for $x \in F$ with $v_F(x) = -f$. Since τ acts trivially on μ_{p^∞} we have $\left(\frac{\tau}{q}\right) = \tau\sqrt{-1}/\sqrt{-1}$; so $\varepsilon(\pi_1 \times \pi_2, \frac{1}{2}, \psi_F)$ and $G_F(c_{\pi_1\times\pi_2})^{p^{m_1+m_2}}$ have the same behaviour under τ. Their quotient belongs to $\mathbb{Q}(\mu_{p^r})$ for r sufficiently large and the square of the quotient belongs to μ_{p^∞}. □

3.5. We add some more consequences of Theorem 3.2, to be used in the next section.

Proposition. *For $i = 1, 2$, let $\pi_i \in \mathcal{A}^{\mathrm{wr}}_{m_i}(F)$. Then for any $\tau \in \mathrm{Aut}(\mathbb{C})$ we have*

$$c(\tau\pi_1 \times \tau\pi_2, \tau\psi_F) \equiv c(\pi_1 \times \pi_2, \psi_F) \pmod{U_F^1}.$$

This is an immediate consequence of Theorem 3.2 and the defining properties of $c(\pi_1 \times \pi_2, \psi_F)$.

Next, we need the Adams operation $\pi \mapsto \pi^\ell$ on $\mathcal{A}^{\mathrm{wr}}_m(F)$: see, for example, [6, §3.3] for the definition.

Corollary. *Let ℓ be a prime number, $\ell \neq p$. Then*

$$c(\pi_1^\ell \times \pi_2^\ell, \psi_F^\ell) \equiv c(\pi_1 \times \pi_2, \psi_F) \pmod{U_F^1}.$$

Proof. Let $\tau \in \mathrm{Aut}(\mathbb{C})$ have restriction $\zeta \mapsto \zeta^\ell$ to μ. Then $\tau\psi_F = \psi_F^\ell$ and by construction [5] π_i^ℓ differs from $\tau\pi_i$ by a tame quasicharacter. The corollary now follows from (2.1.3), (2.1.4) and the proposition. □

4. Davenport-Hasse relations

In this section, we prove some weak versions of Conjecture A, including that stated as Theorem 1 in the Introduction. Most results in this section depend on the conjectures of §2.

4.1. We let K/F be a totally ramified extension of odd prime degree $\ell \neq p$. As usual, we abbreviate $\pi^K = \boldsymbol{l}_{K/F}(\pi)$, for $\pi \in \mathcal{A}_m^{\mathrm{wr}}(F)$.

Lemma. *For $i = 1, 2$, let $\pi_i \in \mathcal{A}_m^{\mathrm{wr}}(F)$. Suppose that $\pi_1 \not\cong \chi\check{\pi}_2$ for any tamely ramified quasicharacter χ of $F^\times$. Then $\pi_1^\ell \not\cong \chi\check{\pi}_2^\ell$ (resp. $\pi_1^K \not\cong \chi\check{\pi}_2^K$) for any tamely ramified quasicharacter χ of $F^\times$ (resp. $K^\times$).*

Proof. The hypothesis means that the simple characters (in the sense of [11]) attached to π_1 and $\check{\pi}_2$ in $G = \mathrm{GL}_{p^m}(F)$ are not conjugate. It follows that the simple characters attached to π_1^ℓ and $\check{\pi}_2^\ell$ are not conjugate either, and hence that $\pi_1^\ell \not\cong \chi\check{\pi}_2^\ell$ for any tame quasicharacter χ of $F^\times$. Similarly by [5, Theorem 1.3] the simple characters attached to π_1^K and $\check{\pi}_2^K$ in $\mathrm{GL}_{p^{m_1}}(K)$ are not conjugate. Thus $\pi_1^K \not\cong \chi\check{\pi}_2^K$ for any tame quasicharacter χ of $K^\times$. □

4.2. Conjecture A can be proved by taking the values of the local constants at any given $s \in \mathbb{C}$, for example $s = \frac{1}{2}$, because of:

Proposition. *In the setting of Conjecture A we have*

$$f(\pi_1^\ell \times \pi_2^\ell, \psi_F^\ell) = f(\pi_1 \times \pi_2, \psi_F),$$
$$f(\pi_1^K \times \pi_2^K, \psi_K) = \ell f(\pi_1 \times \pi_2, \psi_F).$$

The absolute values of the two sides of formula $(*)$ in Conjecture A are equal.

Proof. Remembering that π_1^ℓ is not an unramified twist of $\check{\pi}_2^\ell$, the second formula follows from [5, Theorem 1.7]. To get the first formula, we apply the procedures of [9, §6]. It is immediate that a best common approximation to the simple characters θ_i underlying the π_i is also a best common approximation to the simple characters θ_i^ℓ. Since the conductor depends only on this best common approximation, the result follows. □

Remark. If we admit the conjectures listed in §2, the second formula of the proposition is immediate and the first a consequence of Theorem 3.2.

4.3. Let us assume now that $\pi_1 = \eta\check{\pi}_2$ for some tame quasicharacter η of $F^\times$; this relation determines η uniquely. We have $\pi_1^\ell = \eta^\ell\check{\pi}_2^\ell$ so the condition $\pi_1^\ell \neq \chi\check{\pi}_2^\ell$ for χ unramified means that η^ℓ is not unramified. We now prove Conjecture A for (π_1, π_2) in this situation.

Theorem. *For $i = 1, 2$, let $\pi_i \in \mathcal{A}_m^{\mathrm{wr}}(F)$ and assume that $\pi_2 = \eta\check{\pi}_1$ for some tame quasicharacter η of $F^\times$ such that η^ℓ is not unramified. Then (on the conjectures of §2) Conjecture A holds for π_1, π_2.*

Proof. We can assume $m > 0$ (since otherwise we know the conjecture is true [6, 4.1]). Also, the assertion is independent of the choice of additive character (2.4), so (to simplify calculations) we take $a(\psi_F) = 1$, in the notation of 3.2.

We have (by (2.1.1))

$$\begin{aligned}\varepsilon(\pi_1^\ell \times \pi_2^\ell, \ell s, \psi_F^\ell) &= \varepsilon(\pi_1^\ell \times \eta^\ell\check{\pi}_1^\ell, \ell s, \psi_F^\ell) = \varepsilon(\eta^\ell\pi_1^\ell \times \check{\pi}_1^\ell, \ell s, \psi_F^\ell)\\ &= \frac{\varepsilon(\eta^\ell, \ell s, \psi_F^\ell)}{\varepsilon(1_F, \ell s, \psi_F^\ell)}\,\eta^\ell(c(\pi_1 \times \check{\pi}_1, \psi_F))^{-1}\varepsilon(\pi_1^\ell \times \check{\pi}_1^\ell, \ell s, \psi_F^\ell).\end{aligned}$$

Taking values at $\frac{1}{2}$, we get

$$\begin{aligned}&\varepsilon(\pi_1^\ell \times \pi_2^\ell, \tfrac{\ell}{2}, \psi_F^\ell)\\ &\qquad = \mathfrak{g}(\eta^\ell)\eta^\ell(c(\pi_1 \times \check{\pi}_1, \psi_F))^{-1}\omega_{\pi_1^\ell}(-1)^{p^m-1}\,q^{(\ell-1)f(\pi_1\times\pi_2,\psi_F)/2},\end{aligned}$$

where

$$\mathfrak{g}(\eta^\ell) = \frac{1}{\sqrt{q}}\sum_{\mathfrak{o}_F/\mathfrak{p}_F} \eta^\ell(x)\psi_F^\ell(x).$$

This is obtained as follows. First, $\varepsilon(\eta^\ell, \ell s, \psi_F^\ell) = \mathfrak{g}(\eta^\ell)$ (and does not depend on s), while $\varepsilon(1_F, \ell/2, \psi_F^\ell)$ is equal to $q^{(1-\ell)/2}$. Next,

$$c(\pi_1^\ell \times \check{\pi}_1^\ell, \psi_F^\ell) = c(\pi_1 \times \check{\pi}_1, \psi_F)$$

by 3.5. Thirdly, $\varepsilon(\pi_1^\ell \times \check{\pi}_1^\ell, \frac{1}{2}, \psi_F^\ell) = \omega_{\pi_1^\ell}(-1)^{p^m-1}$ by [7, Théorème 1]. Finally,

$$f(\pi_1^\ell \times \check{\pi}_1^\ell, \psi_F) = f(\pi_1^\ell \times \pi_2^\ell, \psi_F) - 1$$

by [9, Theorem 6.5].

Similarly, with $\eta_K = \eta \circ \mathrm{N}_{K/F}$, we get

$$\begin{aligned}&\varepsilon(\pi_1^K \times \pi_2^K, \tfrac{1}{2}, \psi_K)\\ &\qquad = \frac{\varepsilon(\eta_K, \frac{1}{2}, \psi_K)}{\varepsilon(1_K, \frac{1}{2}, \psi_K)}\,\eta_K(c(\pi_1^K \times \pi_2^K, \psi_K))^{-1}\omega_{\pi_1^K}(-1)^{p^m-1}.\end{aligned}$$

Moreover,

$$c(\pi_1^K \times \pi_2^K, \psi_K) \equiv c(\pi_1 \times \pi_2, \psi_F) \pmod{U_F^1}$$

by (2.2.1) and $\omega_{\pi_1^K} = \omega_{\pi_1} \circ \mathrm{N}_{K/F}$. So we get

$$\varepsilon(\pi_1^K \times \pi_2^K, \tfrac{1}{2}, \psi_K) = \mathfrak{g}(\eta_K)\eta^\ell(c(\pi_1^K \times \pi_2^K, \psi_K))^{-1}\,\omega_{\pi_1^\ell}(-1)^{p^m-1},$$

where

$$\mathfrak{g}(\eta_K) = \frac{1}{\sqrt{q_K}} \sum_{\mathfrak{o}_K/\mathfrak{p}_K} \eta_K(x)\psi_K(x) = \mathfrak{g}(\eta^\ell).$$

The formula $(*)$ in Conjecture A therefore holds at $s = \frac{1}{2}$, then for all s by Proposition 4.2. □

Remark. Conjecture A holds when m_1 or m_2 is zero [6, 4.1]. In light of the foregoing therefore, when considering further cases of the conjecture we can assume that $m_1, m_2 > 0$ and π_1 is not of the form $\chi\check{\pi}_2$ for any tame quasicharacter χ of $F^\times$.

Remark. If $\pi_1 = \eta\check{\pi}_2$ with η^ℓ unramified, the conjecture does not hold, but it is easy to compute both sides using [7] and the twisting properties of §2.

4.4. As the next step in the proof of Theorem 1, we recall from (3.1.2):

Proposition. *Let K/F be a cyclic extension. For $i = 1, 2$ let $\pi_i \in \mathcal{A}_{m_i}^{\mathrm{wr}}(F)$ and put $\pi_i^K = \boldsymbol{b}_{K/F}(\pi_i)$. Then*

$$\lambda_{K/F}^{p^{m_1+m_2}}\,\varepsilon(\pi_1^K \times \pi_2^K, s, \psi_K) = \prod_\chi \varepsilon(\chi\pi_1 \times \pi_2, s, \psi_F),$$

where χ ranges through the characters of $F^\times$ trivial on $N_{K/F}(K^\times)$.

Here, $\lambda_{K/F}$ is the Langlands constant, as in (3.1.3). Combining this with (2.1.2), we get

Corollary. *Assume moreover that K/F is tame of degree ℓ and π_1 is not of the form $\chi\check{\pi}_2$ for any tame quasicharacter χ of $F^\times$. Then*

$$\lambda_{K/F}^{p^{m_1+m_2}}\,\varepsilon(\pi_1^K \times \pi_2^K, s, \psi_K) = \delta_{K/F}(c_{\pi_1\times\pi_2})^{-1}\varepsilon(\pi_1 \times \pi_2, s, \psi_F)^\ell,$$

where

$$\delta_{K/F} = \det \mathrm{Ind}_{K/F}(1_K) = \prod_\chi \chi,$$

with χ ranging through the characters of $F^\times$ trivial on $N_{K/F}(K^\times)$.

4.5. We can now deal with a special case of Theorem 1.

Proposition. *Let K/F be a cyclic, totally ramified extension of odd prime degree $\ell \neq p$. For $i = 1, 2$, let $\pi_i \in \mathcal{A}_{m_i}^{\mathrm{wr}}(F)$ and put $\pi_i^K = \boldsymbol{l}_{K/F}(\pi_i)$. Then (on the conjectures of §2) we have*

$$\varepsilon(\pi_1^K \times \pi_2^K, s, \psi_K) = \varepsilon(\pi_1 \times \pi_2, s, \psi_F)^\ell,$$

and Conjecture A is true for K/F.

Proof. As in 4.3 Remark 1, we can assume that $m_1, m_2 > 0$ and that $\pi_1 \not\cong \eta\check{\pi}_2$ for any tame quasicharacter η.

Since K/F has degree prime to p, we know that $\boldsymbol{l}_{K/F}(\pi_i) = \boldsymbol{b}_{K/F}(\pi_i)$ (Proposition 1.1). Also, since K/F is cyclic of degree ℓ, we have

$$q \equiv 1 \pmod{\ell}, \quad \lambda_{K/F} = \left(\frac{q}{\ell}\right) = 1, \quad \delta_{K/F} = 1_F,$$

and the first formula follows from Corollary 4.4.

To deduce Conjecture A, we follow the same lines as in [6, §2]. It is actually enough to prove it when $s = \frac{1}{2}$. By twisting π_1 and π_2 by tame quasicharacters (which does not affect the identity to be proved), we can in fact assume that π_1 and π_2 are defined over $\mathbb{Q}(\mu_{p^r})$ for a large enough integer r. Moreover, p^{m_1} and p^{m_2} have the same parity. Then

$$\varepsilon(\pi_1^\ell \times \pi_2^\ell, \tfrac{\ell}{2}, \psi_F^\ell) = q^{-(\ell-1)f(\pi_1\times\pi_2,\psi_F)/2}\varepsilon(\pi_1^\ell \times \pi_2^\ell, \tfrac{1}{2}, \psi_F^\ell)$$

and it is enough to prove that

$$\varepsilon(\pi_1^\ell \times \pi_2^\ell, \tfrac{1}{2}, \psi_F^\ell) = \varepsilon(\pi_1 \times \pi_2, \tfrac{1}{2}, \psi_F)^\ell.$$

Consider an automorphism τ of $\mathbb{C}$ whose restriction to roots of unity of order prime to ℓ is given by $\tau(\zeta) = \zeta^\ell$. Then

$$\left(\frac{\tau}{q}\right) = \left(\frac{q}{\ell}\right) = 1$$

and so

$$\begin{aligned}\varepsilon(\pi_1^\ell \times \pi_2^\ell, \tfrac{1}{2}, \psi_F^\ell) &= \varepsilon(\tau\pi_1 \times \tau\pi_2, \tfrac{1}{2}, \tau\psi_F) \\ &= \tau\varepsilon(\pi_1 \times \pi_2, \tfrac{1}{2}, \psi_F)\end{aligned}$$

by Theorem 3.2. Further,

$$\tau\varepsilon(\pi_1 \times \pi_2, \tfrac{1}{2}, \psi_F) = \varepsilon(\pi_1 \times \pi_2, \tfrac{1}{2}, \psi_F)^\ell,$$

since $\varepsilon(\pi_1 \times \pi_2, \frac{1}{2}, \psi_F)$ is a root of unity of order prime to ℓ. This proves Proposition 4.5. □

4.6. We prove Theorem 1 of the Introduction.

Theorem. *Let K/F be a totally ramified extension of odd prime degree $\ell \neq p$. For $i = 1, 2$ let $\pi_i \in \mathcal{A}^{\mathrm{wr}}_{m_i}(F)$ and put $\pi_i^K = \boldsymbol{l}_{K/F}(\pi_i)$. Assume that the Galois closure E/F of K/F is of odd degree prime to p. Then (on the conjectures of §2) we have*

$$\varepsilon(\pi_1^K \times \pi_2^K, s, \psi_K) = \varepsilon(\pi_1 \times \pi_2, s, \psi_F)^\ell,$$

and Conjecture A is true K/F.

Proof. In view of Proposition 4.5, we can assume that K/F is not cyclic, i.e., $\ell \nmid q-1$. Further, we can assume that $\pi_1 \not\cong \chi\check{\pi}_2$ for any tame quasicharacter χ of $F^\times$ (4.3).

Let E/F be the Galois closure of K/F and L/F the maximal unramified sub-extension of E/F. Then L/F is generated by the ℓ-th roots of unity and E/L is totally ramified and cyclic of degree ℓ. Put $d = [L{:}F]$. By assumption d is odd and prime to p.

By Proposition 4.5 applied to E/L we have

$$\varepsilon(\pi_1^E \times \pi_2^E, s, \psi_E) = \varepsilon(\pi_1^L \times \pi_2^L, s, \psi_F)^\ell.$$

By Corollary 4.4 applied to L/F we get

$$\varepsilon(\pi_1^L \times \pi_2^L, s, \psi_L) = \varepsilon(\pi_1 \times \pi_2, s, \psi_F)^d.$$

By Corollary 4.4 again but applied to E/K we get

$$\varepsilon(\pi_1^E \times \pi_2^E, s, \psi_E) = \varepsilon(\pi_1^K \times \pi_2^K, s, \psi_K)^d.$$

This shows that

$$\varepsilon(\pi_1^K \times \pi_2^K, s, \psi_K)^d = \varepsilon(\pi_1 \times \pi_2, s, \psi_F)^{\ell d}.$$

We can twist π_i by a tame quasicharacter to ensure that ω_{π_i} is of finite p-power order, $i = 1, 2$. This does not change the identity to be proved (by (2.1.2) and Corollary 3.5), but it does imply via 3.4 that $\varepsilon(\pi_1^K \times \pi_2^K, \frac{1}{2}, \psi_K)$ and $\varepsilon(\pi_1 \times \pi_2, \frac{1}{2}, \psi_F)$ are roots of unity in $\pm\mu_{p^\infty}$. Since d is odd and prime to p, it follows that

$$\varepsilon(\pi_1^K \times \pi_2^K, \tfrac{1}{2}, \psi_K) = \varepsilon(\pi_1 \times \pi_2, \tfrac{1}{2}, \psi_F)^\ell$$

and, by 4.2, equality holds for any s instead of $\frac{1}{2}$.

To finish the proof, we proceed as in 4.5, noting that d is the order of q mod ℓ and, since d is odd, q is a square mod ℓ. Thus

$$\left(\frac{\tau}{q}\right) = \left(\frac{q}{\ell}\right) = 1,$$

as required. □

5. Davenport-Hasse relations and the Langlands correspondence

5.1. In this section, we state and prove our main result, the exact version of Theorem 3 of the introduction. We use the extended correspondences $\boldsymbol{\pi}^F : \mathcal{G}_F((p)) \to \mathcal{A}_F((p))$ constructed in §1. Throughout, we assume the conjectures of §2. In that context, the theorem exhibits the consequences of the deeper conjectures A and B.

Theorem. *Assume the conjectures of §2 and Conjecture A. Then:*

(i) *Let $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$, and let K/F be a finite field extension of degree prime to p. Then*

$$\boldsymbol{l}_{K/F}\boldsymbol{\pi}_m^F(\sigma) = \boldsymbol{\pi}_m^K(\sigma^K).$$

(ii) *Let K/F be cyclic of degree p and let $\sigma \in \mathcal{G}_F^0(p^m)$. Then*

$$\boldsymbol{b}_{K/F}\boldsymbol{\pi}_m^F(\sigma) = \boldsymbol{\pi}_m^K(\sigma^K).$$

(iii) *Let K/F be cyclic of degree p and let $\tau \in \mathcal{G}_K^0(p^m)$. Then*

$$\boldsymbol{\pi}^F(\mathrm{Ind}_{K/F}(\tau)) = \boldsymbol{i}_{K/F}\boldsymbol{\pi}^K(\tau).$$

Let $\sigma_i \in \mathcal{G}_{m_i}^{\mathrm{wr}}(F)$ and put $\pi_i = \boldsymbol{\pi}^F(\sigma_i)$, $i = 1, 2$. Then:

(iv) *If σ_1 or σ_2 has odd defect, then*

$$\varepsilon(\sigma_1 \otimes \sigma_2, s, \psi_F) = \varepsilon(\pi_1 \times \pi_2, s, \psi_F)$$

while, in general,

$$\varepsilon(\sigma_1 \otimes \sigma_2, s, \psi_F)^2 = \varepsilon(\pi_1 \times \pi_2, s, \psi_F)^2.$$

(v) *If Conjecture B holds, then*

$$\varepsilon(\sigma_1 \otimes \sigma_2, s, \psi_F) = \varepsilon(\pi_1 \times \pi_2, s, \psi_F).$$

Remarks. 1. We observe first that, when $p = 2$, every $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$ has odd defect so (v) is contained in (iv) in this case. For p odd, (iv) implies (v) via Conjecture B and Theorem 2.3.

2. In the context of (iv), we know that $f(\sigma_1 \otimes \sigma_2, \psi_F) = f(\pi_1 \times \pi_2, \psi_F)$ (by [5, Theorem 2.3]). It is therefore enough to verify the ε-relations of (iv) at the point $s = \frac{1}{2}$.
3. In (iv), if we twist the σ_i with tame quasicharacters, the identities to be proven do not change, because of the twisting properties of local constants (and [5, Theorem 2.3]). Therefore, when convenient, we can assume that the $\det \sigma_i$ have finite p-power order.
4. Using (iv) and [8], we now see that Conjecture A implies the truth of Conjecture B in the case where $\boldsymbol{\pi}^{-1}(\pi_1)$ or $\boldsymbol{\pi}^{-1}(\pi_2)$ has odd defect.

Before proving the theorem, we observe that (i) can be extended slightly.

Corollary. *Let $\sigma \in \mathcal{G}_F^0(p^m)$ and let K/F be a finite cyclic extension. Then*

$$\boldsymbol{b}_{K/F}\boldsymbol{\pi}_m^F(\sigma) = \boldsymbol{\pi}_m^K(\sigma^K).$$

Proof. It is enough to treat the case where K/F has prime degree ℓ. If $\ell = p$, the assertion is given by part (ii) of the theorem, so take $\ell \neq p$. The result is then Theorem 1.6(i). $\square$

5.2. Let us consider (iv) first in the case where $\sigma_1 = \chi \otimes \check{\sigma}_2$ for a tame quasicharacter χ of $F^\times$. Then, by (2.1.1), we have

$$\varepsilon(\pi_1 \times \pi_2, s, \psi_F) = \frac{\varepsilon(\chi, s, \psi_F)}{\varepsilon(1_F, s, \psi_F)} \chi(c_{\pi_1 \times \check{\pi}_1})^{-1} \varepsilon(\pi_1 \times \check{\pi}_1, s, \psi_F)$$

and

$$\varepsilon(\sigma_1 \otimes \sigma_2, s, \psi_F) = \frac{\varepsilon(\chi, s, \psi_F)}{\varepsilon(1_F, s, \psi_F)} \chi(c_{\sigma_1 \otimes \check{\sigma}_1})^{-1} \varepsilon(\sigma_1 \otimes \check{\sigma}_1, s, \psi_F)$$

by the Galois analogue [8, Corollaire 2.3]. But by Theorem 2.3,

$$c_{\sigma_1 \otimes \check{\sigma}_1} \equiv c_{\pi_1 \times \pi_2} \pmod{U_F^1}$$

and by [7, Théorème 2, Théorème 1],

$$\begin{aligned} \varepsilon(\pi_1 \times \check{\pi}_1, \tfrac{1}{2}, \psi_F) &= \omega_{\pi_1}(-1)^{p^{m_1}-1}, \\ \varepsilon(\sigma_1 \otimes \check{\sigma}_1, \tfrac{1}{2}, \psi_F) &= \det \sigma_1(-1)^{p^{m_1}-1}. \end{aligned}$$

This proves (iv) of the theorem in the present case. □

Remark. In proving the equality of local constants in the remaining part of Theorem 5.1 we can henceforward assume that σ_1 is not of the form $\chi \otimes \check{\sigma}_2$ for any tame χ.

5.3. We prove Theorem 5.1 by induction. We first observe that all statements are easy when $m = m_1 = m_2 = 0$. Indeed, the identity in (v) holds, independent of Conjecture B, when either m_1 or m_2 is zero, by [5, Theorem 2.3(v)].

Inductive hypothesis. We fix an integer $m \geqslant 1$; we assume that (iv) holds whenever $m_1, m_2 < m$, and that the analogues of (i), (ii) hold for all $\sigma' \in \mathcal{G}^{\mathrm{wr}}_{m'}(F)$ (for (i)) or $\mathcal{G}^0_F(p^{m'})$ (for (ii)) and all $m' < m$. We also assume that the analogue of (iii) holds for $\tau' \in \mathcal{G}^0_K(p^{m'})$ when $m' + 1 < m$.
Step 1. We prove the ε-identities for $\sigma \in \mathcal{G}^{\mathrm{wr}}_m(F)$, $\sigma' \in \mathcal{G}^{\mathrm{wr}}_{m'}(F)$ with $m' < m$. We suppose first that $\sigma \in \mathcal{G}c^{\mathrm{wr}}_m(F)$. In particular, there is a cyclic extension E/F of degree p and $\tau \in \mathcal{G}c^{\mathrm{wr}}_{m-1}(E)$ such that $\sigma = \mathrm{Ind}_{E/F}(\tau)$. Further, $\boldsymbol{\pi}(\sigma) = \boldsymbol{i}_{E/F}\boldsymbol{\pi}(\tau)$, and both σ, τ have defect 1. We have (using [10, 2.6(c)] and omitting the obvious variables from the notation)

$$\varepsilon(\sigma \otimes \sigma') = \lambda_{E/F}^{p^{m+m'-1}} \varepsilon(\tau \otimes \sigma'^E) = \lambda_{E/F}^{p^{m+m'-1}} \varepsilon(\boldsymbol{\pi}(\tau) \times \boldsymbol{\pi}(\sigma')^E), \tag{5.3.1}$$

by induction, abbreviating $\boldsymbol{\pi}(\sigma')^E = \boldsymbol{b}_{E/F}\boldsymbol{\pi}(\sigma')$. The last term here is equal to $\varepsilon(\boldsymbol{i}_{E/F}\boldsymbol{\pi}(\tau) \times \boldsymbol{\pi}(\sigma')) = \varepsilon(\boldsymbol{\pi}(\sigma) \times \boldsymbol{\pi}(\sigma'))$, as required.

We now take a general $\sigma \in \mathcal{G}^{\mathrm{wr}}_m(F)$, and work by induction on the defect of σ. We choose a defect field E/F for σ, and a subfield K/F of E with

$[K{:}F] = \ell$ prime. Assume first that ℓ is odd. If K/F happens to be cyclic, we have

$$\varepsilon(\sigma^K \otimes \sigma'^K) = \lambda_{K/F}^{-p^{m+m'}} \prod_\eta \varepsilon(\eta\sigma \otimes \sigma'),$$

with η ranging over the characters of $F^\times$ which are trivial on norms from $K^\times$, and this expression reduces to $\lambda_{K/F}^{-p^{m+m'}} \varepsilon(\sigma \otimes \sigma')^\ell$. The defect of σ^K has the same parity as that of σ, and likewise for σ' (1.3), so by induction and Theorem 1.6(i),

$$\varepsilon(\sigma^K \otimes \sigma'^K) = \pm\varepsilon(\boldsymbol{b}_{K/F}(\pi) \times \boldsymbol{b}_{K/F}(\pi')),$$

where $\pi = \boldsymbol{\pi}(\sigma)$ etc. The minus sign can occur only if both σ and σ' have even defect.

On the other hand, by Corollary 4.4,

$$\varepsilon(\boldsymbol{b}_{K/F}(\pi) \times \boldsymbol{b}_{K/F}(\pi')) = \lambda_{K/F}^{-p^{m+m'}} \varepsilon(\pi \times \pi')^\ell.$$

Therefore

$$\varepsilon(\sigma \otimes \sigma')^\ell = \pm\varepsilon(\pi \times \pi')^\ell, \tag{5.3.2}$$

and the desired equality $\varepsilon(\sigma \otimes \sigma') = \pm\varepsilon(\pi \times \pi')$ now follows from Theorem 3.4.

We next assume that K/F is not cyclic. We can assume $m' \geqslant 1$, since otherwise the result is given by [5, Theorem 2.3]. Conjecture A and its Galois analogue [6, Theorem 2.2], together with induction, then give

$$\varepsilon(\sigma^\ell \otimes \sigma'^\ell, \ell s, \psi_F^\ell) = \pm\varepsilon(\pi^\ell \times \pi'^\ell, \ell s, \psi_F^\ell).$$

At this point, we assume (as we may — see 5.1 Remark 3) that $\det\sigma$, $\det\sigma'$ have finite p-power order. There is then $\tau \in \mathrm{Aut}(\mathbb{C})$ such that $\sigma^\ell = \tau\sigma$, $\pi^\ell = \tau\pi$ and similarly for σ', π'. Applying τ^{-1}, the desired result follows from Theorem 3.2.

This leaves us with the case $\ell = 2$. In particular, σ has even defect. We follow the argument for ℓ odd and K/F cyclic, paying no attention to the defect of σ': we come to the conclusion (corresponding to (5.3.2)) that $\varepsilon(\sigma \otimes \sigma')^2 = \pm\varepsilon(\pi \times \pi')^2$. Theorem 3.4 then yields $\varepsilon(\sigma \otimes \sigma') = \pm\varepsilon(\pi \times \pi')$, as desired. To eliminate the sign ambiguity when σ' has odd defect, we proceed by induction on the defect of σ'. This repeats again the first part of the argument with the rôles of σ and σ' interchanged.

So far, we have proved:

$$\varepsilon(\sigma \otimes \sigma', s, \psi_F) = \pm\varepsilon(\pi \times \pi', s, \psi_F),$$

for $\sigma \in \boldsymbol{\mathcal{G}}_m^{\mathrm{wr}}(F)$, $\sigma' \in \boldsymbol{\mathcal{G}}_{m'}^{\mathrm{wr}}(F)$ with $m' < m$, and $\pi = \boldsymbol{\pi}(\sigma)$, $\pi' = \boldsymbol{\pi}(\sigma')$. The minus sign can only occur when σ and σ' have even defect.

Step 2. The next step is to prove (ii) when K/F is unramified. Immediately, it is enough to treat the case $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$. We set $\pi = \boldsymbol{\pi}(\sigma)$ and, using [6, Theorem 6.1], we find $\pi' \in \mathcal{A}_{m'}^{\mathrm{wr}}(F)$, $m' < m$, such that $p \nmid f(\pi \times \pi', \psi_F)$. We put $\pi' = \boldsymbol{\pi}(\sigma')$ and $\pi^K = \boldsymbol{b}_{K/F}(\pi)$, etc. It follows (4.2) that $p \nmid f(\pi^K \times \pi'^K, \psi_K)$. On the other hand, $\pi^K = \chi\boldsymbol{\pi}(\sigma^K)$, for some unramified character χ of $K^\times$ of finite p-power order (1.6). Applying the ε-identities established in Step 1, we get

$$\varepsilon(\sigma^K \otimes \sigma'^K) = \pm\chi(\mathfrak{p}_K)^{f(\pi\times\pi',\psi_K)}\varepsilon(\pi^K \times \pi'^K),$$

(the minus sign only occurring when p is odd) while

$$\varepsilon(\sigma^K \otimes \sigma'^K) = \lambda_{K/F}^{-p^{m+m'}} \prod_\eta \varepsilon(\eta\sigma \otimes \sigma'),$$

$$\varepsilon(\pi^K \otimes \pi'^K) = \lambda_{K/F}^{-p^{m+m'}} \prod_\eta \varepsilon(\eta\pi \otimes \pi'),$$

with η ranging over the characters of $F^\times$ trivial on norms from $K^\times$. The right hand sides of these two equations are the same (up to sign if p is odd); we conclude that $\chi = 1$, as required.

Remark. It is now straightforward to show that the ε-identities hold for $\sigma \in \mathcal{G}_F^0(p^m)$, $\sigma' \in \mathcal{G}_F^0(p^{m'})$, $m' < m$.

Step 3. Now we prove (i) for $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$. The only case to be considered, because of 1.1 and 1.6, is that where K/F is of prime degree ℓ and not Galois. In particular, ℓ is odd. We use the same procedure as before: we set $\pi = \boldsymbol{\pi}(\sigma)$ and choose $\pi' = \boldsymbol{\pi}(\sigma') \in \mathcal{A}_{m'}^{\mathrm{wr}}(F)$, with $m' < m$ and $p \nmid f(\pi \times \pi', \psi_F)$. By 1.6, we have $\pi^K = \chi\boldsymbol{\pi}(\sigma^K)$, for some unramified character χ of finite p-power order. (Here we abbreviate $\pi^K = \boldsymbol{l}_{K/F}(\pi)$, etc.) By 4.2, we have $p \nmid f(\pi^K \times \pi'^K, \psi_K)$ and, by induction, $\pi'^K = \boldsymbol{\pi}(\sigma'^K)$. The result then follows from Conjecture A and 3.2.

Step 4. We next prove the ε-identities for $\sigma, \sigma' \in \mathcal{G}_F^0(p^m)$. Compatibility with unramified base change reduces us straightaway to the case $\sigma, \sigma' \in \mathcal{G}_m^{\mathrm{wr}}(F)$. If both σ and σ' lie in $\mathcal{G}c_m^{\mathrm{wr}}(F)$, the result is immediate. In general, (i) allows us to work by induction on the sum of the defects and use Conjecture A along with 3.2.

Step 5. We now turn to (ii), (iii), when K/F is ramified. (When K/F is unramified, (iii) is part of the definition and (ii) has been done in Step 2.) We start with (iii). We first take $\tau \in \mathcal{G}_{m-1}^{\mathrm{wr}}(K)$, put $\rho = \boldsymbol{\pi}(\tau)$, and show that

$$\boldsymbol{\pi}(\mathrm{Ind}_{K/F}(\tau)) = \boldsymbol{i}_{K/F}(\rho).$$

It will now be more convenient to use abbreviated notation like:

$$\rho_F = \boldsymbol{i}_{K/F}(\rho), \quad \pi^K = \boldsymbol{b}_{K/F}(\pi).$$

If $\operatorname{Ind}\tau$ is reducible, the result has been proved in the course of the proof of Theorem 1.6 (final remark in Step 4 of that proof). We therefore assume that $\operatorname{Ind}\tau$ is irreducible. Thus $\pi = \boldsymbol{i}_{K/F}(\rho)$ is supercuspidal. Assume first that it lies in $\mathcal{A}_m^{\mathrm{wr}}(F)$. We choose $\pi' = \boldsymbol{\pi}(\sigma') \in \mathcal{A}_{m'}^{\mathrm{wr}}(F)$ with $m' < m$ and $p \nmid f(\pi \times \pi', \psi_F)$. We know from 1.6 that $\pi = \chi\boldsymbol{\pi}(\operatorname{Ind}\tau)$, where χ is unramified of finite p-power order. Then

$$\begin{aligned}\varepsilon(\rho_F \times \pi') = \varepsilon(\chi\operatorname{Ind}(\tau)\otimes\sigma') &= \lambda_{K/F}^{p^{m+m'-1}}\varepsilon(\chi^K\tau\otimes\sigma'^K)\\ &= \lambda_{K/F}^{p^{m+m'-1}}\varepsilon(\chi^K\rho\times\pi'^K) = \varepsilon(\chi\rho_F\times\pi');\end{aligned}$$

it follows that $\chi(\mathfrak{p}_F)^{f(\rho_F\times\pi',\psi_F)} = 1$, whence χ is trivial as required.

Now we have the case where $\operatorname{Ind}\tau$ is irreducible, but induced from a degree p unramified extension E/F. The representations $\boldsymbol{\pi}(\operatorname{Ind}\tau)$, $\boldsymbol{\pi}(\tau)_F$ are distinguished by their base change to E. We have

$$\boldsymbol{\pi}(\operatorname{Ind}\tau)^E = \boldsymbol{\pi}(\operatorname{Ind}(\tau)^E) = \boldsymbol{\pi}(\operatorname{Ind}_{KE/E}(\tau^{KE})),$$

by the compatibility of $\boldsymbol{\pi}$ with unramified base change. On the other hand,

$$(\boldsymbol{\pi}(\tau)_F)^E = (\boldsymbol{\pi}(\tau)^{KE})_E,$$

by the compatibility of base change and automorphic induction [10, 2.6]. Since KE/K is unramified, this reduces to $\boldsymbol{\pi}(\tau^{KE})_E$. However, τ^{KE} is invariant under $\mathrm{Gal}(KE/E)$, so

$$\boldsymbol{i}_{KE/E}(\boldsymbol{\pi}(\tau^{KE}) = \boldsymbol{\pi}(\operatorname{Ind}_{KE/E}(\tau^{KE})),$$

by the first case. This gives the desired relation.

We now treat the general case of $\tau \in \mathcal{G}_K^0(p^{m-1})$; thus there is an unramified extension E/K of degree p^r say, and $\mu \in \mathcal{G}_{m-r-1}^{\mathrm{wr}}(E)$ such that $\tau = \mu_K$. Let L/F be unramified of degree p^r, so that $E = KL$ and E/L is totally ramified cyclic of degree p. Thus $\tau_F = \mu_F = (\mu_L)_F$. We have $\boldsymbol{\pi}^F(\tau_F) = \boldsymbol{\pi}^F((\mu_L)_F)$ which, by definition, is the same as $(\boldsymbol{\pi}^L(\mu_L))_F$. Since $\mu \in \mathcal{G}_{m-r-1}^{\mathrm{wr}}(E)$, we have (by the first case) $\boldsymbol{\pi}^L(\mu_L) = \boldsymbol{\pi}^E(\mu)_L$; it follows that

$$\boldsymbol{\pi}^F(\tau_F) = (\boldsymbol{\pi}^E(\mu)_L)_F = (\boldsymbol{\pi}^E(\mu)_K)_F$$

by the transitivity of automorphic induction. By definition, $\boldsymbol{\pi}^E(\mu)_K = \boldsymbol{\pi}^K(\mu_K) = \boldsymbol{\pi}^K(\tau)$, whence $\boldsymbol{\pi}^F(\tau_F) = \boldsymbol{\pi}^K(\tau)_F$, as required.

Step 6. We finally have to check (ii) in the case of K/F ramified. We reduce as usual to $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$. If σ^K is reducible, then σ is induced from $\mathcal{W}_K$, and we use (iii). We therefore assume σ^K is irreducible. If $\sigma^K \in \mathcal{G}_m^{\mathrm{wr}}(K)$, we can use the ε-identities as before. We therefore assume that σ^K is induced from an extension KE/K, with E/F unramified of degree p. This means that σ is induced from $\mathcal{W}_L$, where L/F is cyclic of degree p with $L \subset KE$.

Of course, L is neither K nor E. We put $\sigma = \mathrm{Ind}_{L/F}(\tau)$, $\tau \in \mathcal{G}^{\mathrm{wr}}_{m-1}(L)$. Thus

$$\sigma^K = \mathrm{Ind}_{L/F}(\tau)^K = \mathrm{Ind}_{KE/K}(\tau^{KE})$$

and

$$\boldsymbol{\pi}(\sigma^K) = \boldsymbol{\pi}(\mathrm{Ind}_{KE/K}(\tau^{KE})) = \boldsymbol{\pi}(\tau^{KE})_K,$$

whereas

$$\boldsymbol{\pi}(\sigma)^K = \boldsymbol{\pi}(\mathrm{Ind}_{L/F}(\tau))^K = (\boldsymbol{\pi}(\tau)_F)^K = (\boldsymbol{\pi}(\tau)^{KE})_K = \boldsymbol{\pi}(\tau^{KE})_K,$$

since KE/K is unramified. This completes the proof. □

6. Complex automorphisms

In this section, we prove Theorem 3.2. Our treatment is written as a continuation of [6, §7], where we discussed in detail the action of $\mathrm{Aut}(\mathbb{C})$ on smooth representations of groups like $\mathrm{GL}_n(F)$, and on various associated constructions.

6.1. For convenience, we start by recalling the identities to be proved. It will now be simpler to write just $\psi = \psi_F$ and put $a = a(\psi)$.

For $\pi_i \in \boldsymbol{\mathcal{A}}_{n_i}(F)$, $i = 1, 2$, and $\tau \in \mathrm{Aut}(\mathbb{C})$, we have to prove that

$$L(\tau\pi_1 \times \tau\pi_2, s + \tfrac{n_1+n_2}{2}) = \tau L(\pi_1 \times \pi_2, s + \tfrac{n_1+n_2}{2}),$$
$$\varepsilon(\tau\pi_1 \times \tau\pi_2, s + \tfrac{n_1+n_2}{2}, \tau\psi) = \tau\varepsilon(\pi_1 \times \pi_2, s + \tfrac{n_1+n_2}{2}, \psi) \left(\frac{\tau}{q_F}\right)^{an_1n_2},$$

where $\left(\frac{\tau}{q_F}\right) = \tau\sqrt{q_F}/\sqrt{q_F}$.

Note. In fact, what is important is the parity of n_1+n_2 rather than its exact value. Indeed, the L- and ε-factors are functions of q^{-s} and $\mathrm{Aut}(\mathbb{C})$ acts on the coefficients of these functions, fixing q^{-s}. It follows that we only need prove the identities above for $s + k/2$ replacing $s + (n_1 + n_2)/2$, where k is an integer with $k \equiv n_1 + n_2 \pmod 2$.

The method of proof is to return to the definitions of the L- and ε-factors in [22], and follow step by step the action of $\mathrm{Aut}(\mathbb{C})$. The functions $L(\pi_1 \times \pi_2, s)$ and $\varepsilon(\pi_1 \times \pi_2, s, \psi)$ are there initially defined for generic representations π_1, π_2, so the first step is to analyse the action of $\mathrm{Aut}(\mathbb{C})$ on generic representations.

6.2. We first recall some facts from [22]. We abbreviate $G_n = \mathrm{GL}_n(F)$ and write N_n for the subgroup of upper triangular unipotent matrices in G_n. On N_n we define a character $\theta = \theta_{n,\psi}$ by the formula

$$\theta(u_{ij}) = \psi\left(\sum_{i=1}^{n-1} u_{i,i+1}\right), \quad u = (u_{ij}) \in N_n.$$

An irreducible smooth representation (π, V) of G_n is called *generic* if there is a non-zero linear functional λ (called a *Whittaker functional*) on

V such that $\lambda(\pi(u)v) = \theta(u)\lambda(v)$ for $v \in V$ and $u \in N_n$. That condition does not, in fact, depend on ψ since, if ψ' is another continuous additive character of F, then the characters $\theta_{n,\psi}$, $\theta_{n,\psi'}$ of N_n are conjugate in the G_n-normalizer of N_n.

When (π, V) is generic, the Whittaker functional λ on V is unique up to multiplication by a non-zero scalar, and consequently (π, V) has a unique model as a space $\mathcal{W}(\pi;\psi)$ of locally constant functions $W : G_n \to \mathbb{C}$ satisfying

$$W(ug) = \theta(u)W(g), \quad g \in G_n,\ u \in N_n;$$

the group G_n acts on this space by right translation of functions. The space $\mathcal{W}(\pi;\psi)$ is called the Whittaker model of π. A G_n-isomorphism $V \to \mathcal{W}(\pi;\psi)$ is given by associating to $v \in V$ the function $g \mapsto \lambda(\pi(g)v)$. (We refer to [24] for these basic facts concerning Whittaker models and generic representations.)

Let $\pi \in \mathcal{A}_F(n)$ be generic and let λ be a Whittaker functional on the space V of π. If $\tau \in \mathrm{Aut}(\mathbb{C})$, we can form the representation $(\tau\pi, V)$ as in [6, §7]. Strictly speaking, $\tau\pi$ acts on the space $\mathbb{C} \otimes_\tau V$, which is an isomorphic copy of the underlying group of V, on which $\mathbb{C}$ acts via the automorphisms τ. We have

$$\tau \circ \lambda(\pi(u)v) = \tau\theta(u)\,\tau\lambda(v)) = \theta_{n,\tau\psi}(u)\,\tau\lambda(v),$$

for $v \in V$ and $u \in N_n$. Thus $\tau\lambda$ defines a $\tau\psi$-Whittaker functional on the space τV of $\tau\pi$. Consequently the map $f \mapsto \tau \circ f$ gives an additive isomorphism $\mathcal{W}(\pi;\psi) \cong \mathcal{W}(\tau\pi;\tau\psi)$ respecting the actions of G_n.

If $\pi \in \mathcal{A}_F(n)$ is generic, then $\check{\pi} \in \mathcal{A}_F(n)$ is also generic. In fact if we let w_n be the anti-diagonal matrix $(\delta_{i,n+1-i}) \in G_n$, then the map

$$W \longmapsto \widetilde{W} : g \mapsto W(w_n{}^t g^{-1})$$

gives a vector space isomorphism $\mathcal{W}(\pi;\psi) \cong \mathcal{W}(\check{\pi};\bar{\psi})$.

If $\tau \in \mathrm{Aut}(\mathbb{C})$ then $(\tau\pi)^\vee = \tau(\check{\pi})$ in $\mathcal{A}_F(n)$ and, for $W \in \mathcal{W}(\pi;\psi)$, we have

$$\widetilde{\tau W} = \tau\widetilde{W}.$$

6.3. We now recall the definitions of $L(\pi_1 \times \pi_2, s)$ and $\varepsilon(\pi_1 \times \pi_2, s, \psi)$ when the representations $\pi_i \in \mathcal{A}_F(n_i)$ are both generic. We may assume $n_2 \leqslant n_1$, as

$$\begin{aligned} L(\pi_1 \times \pi_2, s) &= L(\pi_2 \times \pi_1, s),\\ \varepsilon(\pi_1 \times \pi_2, s, \psi) &= \varepsilon(\pi_2 \times \pi_1, s, \psi). \end{aligned} \tag{6.3.1}$$

Indeed, if $n_1 \neq n_2$, this is part of the definition; in the case $n_1 = n_2$, the symmetry properties 6.3.1 are in [22, 2.12].

6.4. We start with the case $n_1 = n_2 = n$, and write $\mathcal{S}(F^n)$ for the space of Schwartz-Bruhat functions on F^n. On this space we have the Fourier transform

$$\begin{aligned} \mathcal{F}_\psi : \Phi &\longmapsto \hat{\Phi}, \\ \hat{\Phi}(y) &= \int_{F^n} \Phi(x)\, \psi(x \cdot {}^t y)\, dx, \end{aligned}$$

where the Haar measure dx on F^n is chosen to be self dual, so that $\hat{\hat{\Phi}}(x) = \Phi(-x)$, for $x \in F^n$ and $\Phi \in \mathcal{S}(F^n)$.

We put $\eta = (0, \dots, 0, 1) \in F^n$ and we fix an invariant measure $d\dot{g}$ on $N_n \backslash G_n$, such that the measure of any compact open set is rational.

For $W_1 \in \mathcal{W}(\pi_1; \psi)$, $W_2 \in \mathcal{W}(\pi_2; \bar{\psi})$, and $\Phi \in \mathcal{S}(F^n)$, we set

$$\Psi(s, W_1, W_2, \Phi) = \int_{N_n \backslash G_n} W_1(g) W_2(g) \Phi(\eta g) \, \| \det g \|^s \, d\dot{g},$$

where the absolute value denotes the normalized absolute value on F.

Then [22, Theorem 2.7] each such integral converges absolutely for $\mathrm{Re}(s)$ large enough and defines a rational function of q^{-s}, for which we use the same notation. When W_1, W_2, Φ vary, the rational functions $\Psi(s, W_1, W_2, \Phi)$ span a fractional ideal of $\mathbb{C}[q^s, q^{-s}]$ in $\mathbb{C}(q^{-s})$. That ideal has a unique generator of the form $P(q^{-s})^{-1}$, where $P(X) \in \mathbb{C}[X]$ satisfies $P(0) = 1$. By definition,

$$L(\pi_1 \times \pi_2, s) = P(q^{-s})^{-1},$$

whereas the ε-factor is determined by the functional equation

$$\frac{\Psi(1-s, \widetilde{W}_1, \widetilde{W}_2, \mathcal{F}_\psi(\Phi))}{L(\check{\pi}_1 \times \check{\pi}_2, 1-s)} = \omega_{\pi_2}(-1)^{n-1}\, \varepsilon(\pi_1 \times \pi_2, s, \psi)\, \frac{\Psi(s, W_1, W_2, \Phi)}{L(\pi_1 \times \pi_2, s)},$$

for any $W_1 \in \mathcal{W}(\pi_1; \psi)$, $W_2 \in \mathcal{W}(\pi_2; \bar{\psi})$, $\Phi \in \mathcal{S}(F^n)$.

6.5. Let $\tau \in \mathrm{Aut}(\mathbb{C})$; then $\Phi \mapsto \tau\Phi$ gives a group automorphism of $\mathcal{S}(F^n)$. However, this does not commute with Fourier transform: that is, in some circumstances we have $\tau\mathcal{F}_\psi\Phi \neq \mathcal{F}_{\tau\psi}(\tau\Phi)$. This is because the self-dual measure dx on $\mathcal{S}(F^n)$ need not be rational. Indeed, it is simple to check that dx is $\sqrt{q^{a(\psi)n}}$ times a Haar rational measure, so that

$$\tau\mathcal{F}_\psi(\Phi) = \left(\tfrac{\tau}{q}\right)^{an} \mathcal{F}_{\tau\psi}(\tau\Phi), \quad \Phi \in \mathcal{S}(F^n),$$

abbreviating $a = a(\psi)$. On the other hand, for $W_1 \in \mathcal{W}(\pi_1; \psi)$, $W_2 \in \mathcal{W}(\pi_2; \bar{\psi})$ and $\Phi \in \mathcal{S}(F^n)$, we have

$$\Psi(s, \tau W_1, \tau W_2, \tau\Phi) = \tau\Psi(s, W_1, W_2, \Phi),$$

because $d\dot{g}$ is a rational measure. This already shows that

$$L(\tau\pi_1 \times \tau\pi_2, s) = \tau L(\pi_1 \times \pi_2, s),$$
$$L(\tau\check{\pi}_1 \times \tau\check{\pi}_2, s) = \tau L(\check{\pi}_1 \times \check{\pi}_2, s),$$

which yields

$$L((\tau\pi_1)^\vee \times (\tau\pi_2)^\vee, 1-s) = \tau L(\check{\pi}_1 \times \check{\pi}_2, 1-s).$$

Of course, $\omega_{\tau\pi_2} = \tau\omega_{\pi_2}$ and, for $W_1 \in \mathcal{W}(\pi_1; \psi)$, $W_2 \in \mathcal{W}(\pi_2; \bar{\psi})$ and $\Phi \in \mathcal{S}(F^n)$,

$$\begin{aligned}\Psi(1-s, \widetilde{\tau W_1}, \widetilde{\tau W_2}, \mathcal{F}_{\tau\psi}(\tau\Phi)) &= \Psi(1-s, \tau\widetilde{W}_1, \tau\widetilde{W}_2, \mathcal{F}_{\tau\psi}(\tau\Phi)) \\ &= \left(\tfrac{\tau}{q}\right)^{an} \tau\Psi(1-s, \widetilde{W}_1, \widetilde{W}_2, \mathcal{F}_\psi(\Phi))\end{aligned}$$

and consequently the functional equation yields

$$\varepsilon(\tau\pi_1 \times \tau\pi_2, s, \tau\psi) = \left(\tfrac{\tau}{q}\right)^{an} \tau\varepsilon(\pi_1 \times \pi_2, s, \psi).$$

This proves Theorem 3.2 when $n_1 = n_2 = n$ and the π_i are generic.

6.6. Assume now that π_1, π_2 are generic and that $n_2 < n_1$.

Let j be an integer satisfying $0 \leqslant j \leqslant n_1 - n_2 - 1$, and put $k = n_1 - n_2 - 1 - j$.

Let $d\dot{g}$ be a rational invariant measure on $N_{n_1}\backslash G_{n_1}$. We let $\mathcal{S}(M_j)$ be the space of Schwartz-Bruhat functions on the space $M_j = \mathbb{M}(j \times n_2, F)$ of $j \times n_2$ matrices over F. On $\mathcal{S}(M_j)$ we then have the Fourier transform

$$\begin{aligned}\mathcal{F}_\psi : \Phi &\longmapsto \hat{\Phi}, \\ \hat{\Phi}(y) &= \int_{M_j} \Phi(x)\, \psi(\mathrm{tr}\,({}^t x \cdot y))\, dx,\end{aligned}$$

where the Haar measure dx on M_j is chosen to be self-dual: $\hat{\hat{\Phi}}(x) = \Phi(-x)$ for $x \in M_j$.

For j, k as above, $W_1 \in \mathcal{W}(\pi_1; \psi)$, $W_2 \in \mathcal{W}(\pi_2; \bar{\psi})$, we let

$$\begin{aligned}&\Psi(s, W_1, W_2; j) \\ &\quad = \int_{N_{n_2}\backslash G_{n_2}} \int_{M_j} W_1 \left[\begin{pmatrix} g & 0 & 0 \\ x & 1_j & 0 \\ 0 & 0 & 1_{k+1} \end{pmatrix}\right] W_2(g)\, \|\det g\|^{s - \frac{n_1 - n_2}{2}}\, dx d\dot{g}.\end{aligned}$$

(Here 1_j, for example, denotes the $j \times j$ identity matrix.) Then, [22, Theorem 2.7], the integral converges for $\mathrm{Re}(s)$ large enough and defines a rational function of q^{-s}. For j fixed, and when W_1 and W_2 vary, these rational functions span a fractional ideal of $\mathbb{C}[q^{-s}, q^s]$ in $\mathbb{C}(q^{-s})$ which is *independent of* j. That ideal has a unique generator of the form $P(q^{-s})^{-1}$, where

$P(X) \in \mathbb{C}[X]$ satisfies $P(0) = 1$, and

$$L(\pi_1 \times \pi_2, s) = P(q^{-s})^{-1}.$$

The ε-factor is determined by the functional equations (for any j, k as above)

$$\frac{\Psi(1-s, w_{1,2}\widetilde{W}_1, \widetilde{W}_2; k)}{L(\check{\pi}_1 \times \check{\pi}_2, 1-s)} = \omega_{\pi_2}(-1)^{n_1-1}\,\varepsilon(\pi_1 \times \pi_2, s, \psi)\,\frac{\Psi(s, W_1, W_2; j)}{L(\pi_1 \times \pi_2, s)},$$

for $W_1 \in \mathcal{W}(\pi_1; \psi)$, $W_2 \in \mathcal{W}(\pi_2; \bar{\psi})$, and where

$$\begin{aligned} w_{1,2} &= \begin{pmatrix} 1_{n_2} & 0 \\ 0 & w_{n_1-n_2} \end{pmatrix} \in \mathbb{M}_{n_1}(F), \\ w_{1,2}\widetilde{W}_1 : g &\longmapsto \widetilde{W}_1(gw_{1,2}). \end{aligned}$$

6.7. Let now $\tau \in \mathrm{Aut}(\mathbb{C})$ and fix the integers j, k as in 6.6. The self-dual Haar measure dx on M_j is $\sqrt{q^{ajn_2}}$ times a rational measure. Because, in the definition of $\Psi(s, W_1, W_2; j)$, the factor $\|\det g\|$ occurs with exponent $s - \frac{n_1-n_2}{2}$, the rationality property of such integrals is:

$$\Psi(s + \tfrac{n_1-n_2}{2}, \tau W_1, \tau W_2; j) = \left(\tfrac{\tau}{q}\right)^{ajn_2} \tau\Psi(s + \tfrac{n_1-n_2}{2}, W_1, W_2; j)$$

which implies

$$L(\tau\pi_1 \times \tau\pi_2, s + \tfrac{n_1-n_2}{2}) = \tau L(\pi_1 \times \pi_2, s + \tfrac{n_1-n_2}{2}) \tag{6.7.1}$$

and consequently also

$$L(\tau\check{\pi}_1 \times \tau\check{\pi}_2, 1 - (s + \tfrac{n_1-n_2}{2})) = \tau L(\check{\pi}_1 \times \check{\pi}_2, 1 - (s + \tfrac{n_1-n_2}{2})).$$

Again $\tau\omega_{\pi_2} = \omega_{\tau\pi_2}$ and

$$\begin{aligned} &\Psi(1 - (s + \tfrac{n_1-n_2}{2}), w_{1,2}\widetilde{\tau W}_1, \widetilde{\tau W}_2; k) \\ &\qquad = \Psi(1 - (s + \tfrac{n_1-n_2}{2}), \tau w_{1,2}\widetilde{W}_1, \tau\widetilde{W}_2; k) \\ &\qquad = \left(\tfrac{\tau}{q}\right)^{akn_2} \tau\Psi(1 - (s + \tfrac{n_1-n_2}{2}), w_{1,2}\widetilde{W}_1, \widetilde{W}_2; k) \end{aligned}$$

so that the functional equation yields

$$\varepsilon(\tau\pi_1 \times \tau\pi_2, s + \tfrac{n_1-n_2}{2}, \tau\psi) = \left(\tfrac{\tau}{q}\right)^{an_1n_2} \tau\varepsilon(\pi_1 \times \pi_2, s + \tfrac{n_1-n_2}{2}, \psi), \tag{6.7.2}$$

since $(j + k)n_2 = (n_1 - n_2 - 1)n_2 \equiv n_1n_2 \pmod 2$. This proves Theorem 3.2 when $n_2 < n_1$ and the π_i are generic.

6.8. We have now proved Theorem 3.2 in the case where the π_i are essentially tempered: this is more than enough for the applications in the rest of the paper. For completeness, we continue and prove it in the general case.

The definition of L- and ε-factors for general pairs uses the Langlands classification for GL_n (see [12]; for proofs, see [2], [27]). For our purposes, it is better to use the version of the Langlands classification which follows from the investigations of Zelevinsky [28]. Our reference will be Rodier's Bourbaki report [25] on the subject.

Let $\pi \in \boldsymbol{\mathcal{A}}_F(n)$. By Langlands' classification there is a parabolic subgroup P of $G_n = \mathrm{GL}_n(F)$, a tempered irreducible smooth representation ρ of the Levi quotient L of P, and a smooth homomorphism $\chi : L \to \mathbb{R}_+^\times$, positive with respect to P, such that π is the unique irreducible quotient of the representation of G_n induced (via P) from $\chi \otimes \rho$. The triple (P, ρ, χ) is unique up to conjugation in G_n.

When we try to investigate the action of $\mathrm{Aut}(\mathbb{C})$ on the Langlands classification, we run into two problems. The first is that parabolic induction, being normalized to make unitary representations go to unitary representations, is not $\mathrm{Aut}(\mathbb{C})$-equivariant. The second and more serious is that, for $\tau \in \mathrm{Aut}(\mathbb{C})$ and ρ tempered, the representation $\tau\rho$ is not necessarily tempered. Further, if $\chi : L \to \mathbb{R}_+^\times$ is as above, the quasicharacter $\tau\chi$ need not be real-valued, let alone positive with respect to P. We now clarify these matters.

6.9. We start with parabolic induction.

For $i = 1, 2$, let $\pi_i \in \boldsymbol{\mathcal{A}}_F(n_i)$. Put $n = n_1+n_2$ and let ρ be the representation of G_n parabolically induced by $\pi_1 \otimes \pi_2$. Then ρ is obtained by *un-normalized* induction from $\delta_P^{-1}(\pi_1 \otimes \pi_2)$, where δ_P^2 is the modulus character of the upper triangular parabolic subgroup P of G_n with diagonal blocks G_{n_1}, G_{n_2}. Consequently if $\tau \in \mathrm{Aut}(\mathbb{C})$, then $\tau\rho$ is obtained, by un-normalized induction, from $\tau\delta_P^{-1}(\tau\pi_1 \otimes \tau\pi_2)$: indeed, un-normalized induction is clearly $\mathrm{Aut}(\mathbb{C})$ equivariant.

Let $\epsilon : x \mapsto \left(\frac{\tau}{q}\right)^{v(x)}$, where $v = v_F$. Thus ϵ is a character of $F^\times$ such that $\epsilon^2 = 1$. For $(g_1, g_2) \in G_{n_1} \times G_{n_2}$, one calculates that

$$\tau\delta_P^{-1}(g_1, g_2) = \epsilon(\det g_1)^{n_2}\epsilon(\det g_2)^{n_1}\delta_P^{-1}(g_1, g_2).$$

Consequently, $\tau\rho$ is obtained, by parabolic induction, from the representation $\epsilon^{n_2}\tau\pi_1 \otimes \epsilon^{n_1}\tau\pi_2$.

If we use the notation $\pi_1 \circ \pi_2$ instead of ρ, we get

$$\epsilon^{-(n_1+n_2)}\tau(\pi_1 \circ \pi_2) = (\epsilon^{-n_1}\tau\pi_1) \circ (\epsilon^{-n_2}\tau\pi_2).$$

By transitivity of induction we get, for any positive integer r and $\pi_i \in \boldsymbol{\mathcal{A}}_F(n_i)$, $i = 1, \ldots, r$,

$$\epsilon^{-n}\tau(\pi_1 \circ \cdots \circ \pi_r) = (\epsilon^{-n_1}\tau\pi_1) \circ \cdots \circ (\epsilon^{-n_r}\tau\pi_r),$$

where $n = \sum_{i=1}^r n_i$.

6.10. Let us now investigate the action of $\mathrm{Aut}(\mathbb{C})$ on the Langlands classification, using [25]. We put $\boldsymbol{\mathcal{A}}_F = \bigcup_{n\geqslant 1} \boldsymbol{\mathcal{A}}_F(n)$, $\boldsymbol{\mathcal{A}}_F^0 = \bigcup_{n\geqslant 1} \boldsymbol{\mathcal{A}}_F^0(n)$. For $\pi \in \boldsymbol{\mathcal{A}}_F$ and $s \in \mathbb{C}$, we let $\pi(s)$ be the class of the representation $g \mapsto \|\det g\|^s\pi(g)$.

A *segment* of length $r \geqslant 1$ is an r-tuple $\Delta = (\rho, \rho(1), \ldots, \rho(r-1))$ where $\rho \in \boldsymbol{\mathcal{A}}_F^0$.

Let $\Delta = (\rho, \rho(1), \ldots, \rho(r-1))$ and $\Delta' = (\rho', \ldots, \rho'(r'-1))$ be two segments. We say that Δ *precedes* Δ' if $\rho' = \rho(m)$ for some integer m satisfying $1 \leqslant m \leqslant r-1$, $m + r' - 1 > r - 1$. (This is easily seen to be equivalent to the conditions in [25, 4.1]). The Langlands classification theorem, as completed by Bernstein and Zelevinsky, can then be stated as follows ([25, Théorème 3]):

1. For each segment $\Delta = (\rho, \ldots, \rho(r-1))$, the representation $\rho\circ\cdots\circ\rho(r-1)$ has a unique irreducible quotient. Its class is denoted $\ell(\Delta)$ and the map $\Delta \mapsto \ell(\Delta)$ is a bijection from the set of segments to the set of classes of essentially square-integrable elements in $\boldsymbol{\mathcal{A}}_F$.
2. Let $\Delta_1, \ldots, \Delta_r$ be segments. Assume that Δ_i does not precede Δ_j whenever $i < j$. Then the representation $\ell(\Delta_1) \circ \cdots \circ \ell(\Delta_r)$ has a unique irreducible quotient, the class of which is called $\ell(\Delta_1, \ldots, \Delta_r)$. If $\Delta'_1, \ldots, \Delta'_s$ are segments such that Δ'_i does not precede Δ_j when $i < j$, then $\ell(\Delta_1, \ldots, \Delta_r) = \ell(\Delta'_1, \ldots, \Delta'_s)$ if and only if the sequence $(\Delta'_1, \ldots, \Delta'_s)$ is a permutation of $(\Delta_1, \ldots, \Delta_r)$.
3. For $\pi \in \boldsymbol{\mathcal{A}}_F$, there exists a sequence $(\Delta_1, \ldots, \Delta_r)$ of segments such that Δ_i does not precede Δ_j when $i < j$ and $\pi = \ell(\Delta_1, \ldots, \Delta_r)$.

6.11. Let now $\tau \in \mathrm{Aut}(\mathbb{C})$. Let $\Delta = (\rho, \ldots, \rho(r-1))$ be a segment. Then clearly $\tau\ell(\Delta)$ is the class of the unique irreducible quotient of $\tau(\rho \circ \cdots \circ \rho(r-1))$. By 6.7, if $\rho \in \boldsymbol{\mathcal{A}}_F^0(t)$, then

$$\epsilon^{-rt}\tau(\rho \circ \cdots \circ \rho(r-1)) = \epsilon^{-t}\tau\rho \circ \cdots \circ \epsilon^{-t}\tau\rho(r-1),$$

so that $\epsilon^{-rt}\tau\ell(\Delta) = \ell(\epsilon^{-t}\tau\Delta)$, with the obvious notation

$$\tau\Delta = (\tau\rho, \ldots, \tau\rho(r-1)),$$

and $\chi\Delta = (\chi\rho, \ldots, \chi\rho(r-1))$ for any quasicharacter χ of $F^\times$.

Let s be a positive integer and, for $i = 1, \ldots, s$, let $\Delta_i = (\rho_i, \ldots, \rho_i(r_i - 1))$ be a segment, with $\rho_i \in \boldsymbol{\mathcal{A}}_F^0(t_i)$. Assume that, for $i < j$, Δ_i does not precede Δ_j. Then obviously $\tau\ell(\Delta_1, \ldots, \Delta_s)$ is the class of the unique irreducible quotient of $\tau(\ell(\Delta_1)\circ\cdots\circ\ell(\Delta_s))$. Putting $n=\sum_{i=1}^{s} r_i t_i$ we have that $\epsilon^{-n}\tau\ell(\Delta_1,\ldots,\Delta_s)$ is the class of the unique irreducible quotient of $\epsilon^{-r_1t_1}\tau\ell(\Delta_1)\circ\cdots\circ\epsilon^{-r_st_s}\tau\ell(\Delta_s)$, i.e., of $\ell(\epsilon^{-t_1}\tau\Delta_1)\circ\cdots\circ\ell(\epsilon^{-t_s}\tau\Delta_s)$.

If, for some $i<j$, $\epsilon^{-t_i}\tau\Delta_i$ precedes $\epsilon^{-t_j}\tau\Delta_j$ then necessarily $t_i=t_j$ and Δ_i precedes Δ_j which is impossible. Consequently

$$\epsilon^{-n}\tau\ell(\Delta_1,\ldots,\Delta_s)=\ell(\epsilon^{-t_1}\tau\Delta_1,\ldots,\epsilon^{-t_s}\tau\Delta_s).$$

6.12. We are now ready to prove Theorem 3.2 in general. Let $\pi\in\mathcal{A}_F(n)$, $\pi'\in\mathcal{A}_F(n')$. Write $\pi=\ell(\Delta_1,\ldots,\Delta_r)$, $\pi'=\ell(\Delta'_1,\ldots,\Delta'_{r'})$ in the Langlands classification as above. We put $\Delta_i=(\rho_i,\ldots,\rho_i(r_i-1))$, with $\rho_i\in\mathcal{A}_F^0(t_i)$, and $\Delta'_j=(\rho'_j,\ldots,\rho'_j(r'_j-1))$, with $\rho'_j\in\mathcal{A}_F^0(t'_j)$. Then by definition [22, §9]

$$L(\pi\times\pi',s)=\prod_{i=1}^{r}\prod_{j=1}^{r'}L(\ell(\Delta_i)\times\ell(\Delta'_j),s),$$

$$\varepsilon(\pi\times\pi',s,\psi)=\prod_{i=1}^{r}\prod_{j=1}^{r'}\varepsilon(\ell(\Delta_i)\times\ell(\Delta'_j),s,\psi).$$

(This definition is compatible with the cases where π and π' are generic.)

Let $\tau\in\mathrm{Aut}(\mathbb{C})$. We then have by 6.11

$$L(\tau\pi\times\tau\pi',s)=\prod_{i=1}^{r}\prod_{j=1}^{r'}L(\epsilon^{n-t_ir_i}\tau\ell(\Delta_i)\times\epsilon^{n'-t'_jr'_j}\tau\ell(\Delta'_j),s),$$

$$L(\tau\pi\times\tau\pi',s+\tfrac{n+n'}{2})=\prod_{i=1}^{r}\prod_{j=1}^{r'}L(\epsilon^{n+n'-t_ir_i-t'_jr'_j}\tau\ell(\Delta_i)\times\tau\ell(\Delta'_j),s+\tfrac{n+n'}{2}).$$

We know that for each i and j

$$L(\tau\ell(\Delta_i)\times\tau\ell(\Delta'_j),s+\tfrac{t_ir_i+t'_jr'_j}{2})=\tau L(\ell(\Delta_i)\times\ell(\Delta'_j),s+\tfrac{t_ir_i+t'_jr'_j}{2}).$$

Now let $\alpha\in\mathbb{C}^\times$ and $r\in\mathbb{Z}$; we have

$$\tau\left(1-\alpha q^{-s-r/2}\right)=\left(1-\left(\tfrac{\tau}{q}\right)^{r/2}\tau(\alpha)q^{-s-r/2}\right).$$

Consequently

$$\tau L(\ell(\Delta_i)\times\ell(\Delta'_j),s+\tfrac{n+n'}{2})=L(\epsilon^{n+n'-t_ir_i-t'_jr'_j}\tau\ell(\Delta_i)\times\tau\ell(\Delta'_j),s+\tfrac{n+n'}{2}).$$

Taking the product over i and j we finally get

$$L(\tau\pi\times\tau\pi',s+\tfrac{n+n'}{2})=\tau L(\pi\times\pi',s+\tfrac{n+n'}{2}),$$

as required for Theorem 3.2.

The proof for the ε-factor is analogous. By 6.11 we have

$$\varepsilon(\tau\pi \times \tau\pi', s, \tau\psi) = \prod_{i=1}^{r}\prod_{j=1}^{r'} \varepsilon(\epsilon^{n-t_ir_i}\tau\ell(\Delta_i) \times \epsilon^{n'-t'_jr'_j}\tau\ell(\Delta'_j), s, \tau\psi),$$

$$\varepsilon(\tau\pi \times \tau\pi', s + \tfrac{n+n'}{2}, \tau\psi) = \prod_{i=1}^{r}\prod_{j=1}^{r'} \varepsilon(\epsilon^{n+n'-t_ir_i-t'_jr'_j}\tau\ell(\Delta_i) \times \tau\ell(\Delta'_j), s + \tfrac{n+n'}{2}, \tau\psi),$$

and we know that for each i and j

$$\varepsilon(\tau\ell(\Delta_i) \times \tau\ell(\Delta'_j), s + \tfrac{t_ir_i+t'_jr'_j}{2}, \tau\psi) = \left(\tfrac{\tau}{q}\right)^{at_ir_i.t'_jr'_j} \tau\varepsilon(\ell(\Delta_i) \times \ell(\Delta'_j), s + \tfrac{t_ir_i+t'_jr'_j}{2}, \psi),$$

which implies

$$\varepsilon(\epsilon^{n+n'-t_ir_i-t'_jr'_j}\tau\ell(\Delta_i) \times \tau\ell(\Delta'_j), s + \tfrac{n+n'}{2}, \tau\psi) = \left(\tfrac{\tau}{q}\right)^{at_ir_i.t'_jr'_j} \tau\varepsilon(\ell(\Delta_i) \times \ell(\Delta'_j), s + \tfrac{n+n'}{2}, \psi).$$

Taking the product over i and j we get

$$\begin{aligned}\varepsilon(\tau\pi \times \tau\pi', s + \tfrac{n+n'}{2}, \tau\psi) &= \left(\tfrac{\tau}{q}\right)^{a\sum_{i,j} t_ir_i.t'_jr'_j} \tau\varepsilon(\pi \times \pi', s + \tfrac{n+n'}{2}, \psi) \\ &= \left(\tfrac{\tau}{q}\right)^{ann'} \tau\varepsilon(\pi \times \pi', s + \tfrac{n+n'}{2}, \psi),\end{aligned}$$

as required. □

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