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# The mean values of logarithms of algebraic integers

par ARTŪRAS DUBICKAS

RÉSUMÉ. Soit $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ l'ensemble des conjugués d'un entier algébrique $\alpha$ de degré $d$, n'étant pas une racine de l'unité. Dans cet article on propose de minorer

$$M_p(\alpha) = \sqrt[p]{\frac{1}{d}\sum_{i=1}^{d} |\log|\alpha_i||^p}$$

où $p > 1$.

ABSTRACT. Let $\alpha$ be an algebraic integer of degree $d$ with conjugates $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$. In the paper we give a lower bound for the mean value

$$M_p(\alpha) = \sqrt[p]{\frac{1}{d}\sum_{i=1}^{d} |\log|\alpha_i||^p}$$

when $\alpha$ is not a root of unity and $p > 1$.

## 1. INTRODUCTION.

Let $\alpha$ be an algebraic number of degree $d \geq 2$ with

$P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0 = a_d(x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_d)$

as its minimal polynomial over $\mathbb{Z}$ and $a_d$ positive. Following Mahler, the Mahler measure of $\alpha$ is defined by

$$M(\alpha) = a_d \prod_{i=1}^{d} \max(1, |\alpha_i|).$$

The house of an algebraic number is the maximum of the modulus of its conjugates:

$$\overline{|\alpha|} = \max\{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_d|\}.$$

Put also

$$d(\alpha) = \max\left\{\overline{|\alpha|}, \overline{|\alpha^{-1}|}\right\} = \max\{|\alpha_1|, \ldots, |\alpha_d|, 1/|\alpha_1|, \ldots, 1/|\alpha_d|\}$$

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for the "symmetric deviation" of conjugates from the unit circle. Denote for $p > 0$

$$M_p(\alpha) = \sqrt[p]{\frac{1}{d}\sum_{i=1}^{d} \big|\log|\alpha_i|\big|^p}.$$

Our main concern here is the lower bound for this mean value when $\alpha$ is an algebraic integer ($a_d = 1$) which is not a root of unity.

In 1933, D.H. Lehmer [8] asked whether it is true that for every positive $\varepsilon$ there exists an algebraic number $\alpha$ for which $1 < M(\alpha) < 1 + \varepsilon$. In its strong form Lehmer's problem has been reformulated as whether it is true that if $\alpha$ is not a root unity then $M(\alpha) \geq \alpha_0 = 1.1762808...$ where $\alpha_0$ is the root of the polynomial

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$

In 1971, C.J. Smyth [16] proved that if $\alpha$ is a non–reciprocal algebraic integer then $M(\alpha) \geq \theta = 1.32471...$ where $\theta$ is the real root of the polynomial $x^3 - x - 1$. This result reduces Lehmer's problem to the case of reciprocal algebraic integers (those with minimal polynomial satisfying the identity $P(x) \equiv x^d P(1/x)$). P.E. Blanksby and H.L.Montgomery [2] used Fourier analysis to prove that $M(\alpha) > 1 + 1/52d \log(6d)$. In 1978, C.L. Stewart [18] proved that $M(\alpha) > 1 + 1/10^4 d \log d$. Although this result is weaker than the previous one, the method used has become very important and led to further improvements. Recently M. Mignotte and M. Waldschmidt [12] obtained Stewart's result via the interpolation determinant.

In 1979, E. Dobrowolski [4] obtained a remarkable improvement of these results showing that for each $\varepsilon > 0$, there exists an effective $d(\varepsilon)$ such that for $d > d(\varepsilon)$

$$M(\alpha) > 1 + (1-\varepsilon)\left(\frac{\log\log d}{\log d}\right)^3. \tag{1.1}$$

D.C. Cantor and E.G. Straus [3] in 1982 introduced the interpolation determinant to simplify Dobrowolski's proof and to replace the constant $1-\varepsilon$ by $2-\varepsilon$. Finally, R. Louboutin [9] was able to improve this constant to $9/4 - \varepsilon$. M. Meyer [11] obtained Louboutin's result using a version of Siegel's lemma due to Bombieri and Vaaler. Recently P. Voutier [19] showed that inequality (1) holds for all $d \geq 2$ with the weaker constant $1/4$ instead of $1 - \varepsilon$.

In 1965, A. Schinzel and H. Zassenhaus [13] conjectured that there exists an absolute positive constant $\gamma$ such that $\overline{|\alpha|} > 1 + \gamma/d$ whenever $\alpha$ is not a root of unity. The best known result on this problem is due to the author [5]:

we have

$$\overline{|\alpha|} > 1 + \left(\frac{64}{\pi^2} - \varepsilon\right)\frac{1}{d}\left(\frac{\log\log d}{\log d}\right)^3 \tag{1.2}$$

where $d > d_1(\varepsilon)$. In fact, both inequalities (1), (2) and the respective conjectures can be considered in terms of the lower bound for $M_p(\alpha)$. Indeed, notice that

$$M_1(\alpha) = \frac{2\log M(\alpha) - \log|a_0|}{d}.$$

Therefore, for $|a_0| \geq 2$,

$$M_1(\alpha) = \frac{2\log|a_0| - \log|a_0|}{d} \geq \frac{\log 2}{d}.$$

If $|a_0| = 1$, then

$$M_1(\alpha) = \frac{2\log M(\alpha)}{d}.$$

Louboutin's result can be written as follows

$$dM_1(\alpha) > \left(\frac{9}{2} - \varepsilon\right)\left(\frac{\log\log d}{\log d}\right)^3. \tag{1.3}$$

Taking $p = \infty$, we can write the inequality (2) in the following form

$$dM_\infty(\alpha) = d\log d(\alpha) \geq d\log\overline{|\alpha|} > \left(\frac{64}{\pi^2} - \varepsilon\right)\left(\frac{\log\log d}{\log d}\right)^3. \tag{1.4}$$

The function $p \to M_p(\alpha)$ is nondecreasing. Hence the inequality $dM_p(\alpha) \geq c_p$ where $1 < p < \infty$ and $c_p > 0$ lies between the conjecture of Lehmer $p = 1$ and the "symmetric" form of the conjecture of Schinzel and Zassenhaus $p = \infty$ (see also [1] for a problem which lies between these two conjectures). We have noticed above that the conjectural value for $c_1$ is $2\log\alpha_0$. It would be of interest to find out whether it is true that $d(\alpha) \geq \sqrt[d]{2}$. The equality holds for the polynomial $x^d - 2$. We conjecture that the answer to the above question is affirmative, so that $c_\infty = \log 2$. In this paper, we take up the interpolation determinant again (see [3],[5],[9],[10], [19]) and fill the gap between inequalities (3) and (4) (Theorem 2). One can also consider the mean value of conjugates of an algebraic integer

$$m_p(\alpha) = \sqrt[p]{\frac{1}{d}\sum_{i=1}^{d}|\alpha_i|^p}$$

and the mean value of the differences

$$t_p(\alpha) = \sqrt[p]{\frac{2}{d(d-1)}\sum_{i\leq j}|\alpha_i - \alpha_j|^p}.$$

The lower bound for $m_1(\alpha)$ where $\alpha$ is a totally positive integer was considered by I. Schur [14], C.L. Siegel [15], C.J. Smyth [17]. In 1988, M. Langevin [7] solved Favard's problem proving that $t_\infty(\alpha) := \max_{i,j} |\alpha_i - \alpha_j| > 2 - \varepsilon$ for an algebraic integer of a sufficiently large degree. The author [6] proved that $t_2(\alpha) > \sqrt[4]{e} - \varepsilon$. The problem of finding an upper bound for $t_{-\infty}(\alpha) := 1/\min_{i \neq j} |\alpha_i - \alpha_j|$ is known as a separation problem. In this article, we apply the lower bound for $M_2(\alpha)$ to estimate $m_p(\alpha)$ from below (Theorem 3).

## 2. Statement of the results.

The notations are the following. Let $G(x)$ be a real valued function in $[0;1]$ such that $G(0) = 1$, $G(1) = 0$. Let also the derivative of $G(x)$ be continuous and negative in the interval $(0;1)$. Put

$$I = \int_0^1 G(x)dx, \tag{2.5}$$

$$J = \int_0^1 \Big(G(x)\Big)^2 dx, \tag{2.6}$$

$$L = \int_0^1 \Big(G'(x)\Big)^2 dx. \tag{2.7}$$

Put also for brevity

$$\delta(d) = \left(\frac{\log\log d}{\log d}\right)^3.$$

Let $\alpha$ be a reciprocal algebraic integer, i.e. $d = 2m$, $m \in \mathrm{N}$, $\alpha_{2m} = 1/\alpha_1$, $\alpha_{2m-1} = 1/\alpha_2, \ldots, \alpha_{m+1} = 1/\alpha_m$ where $|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_m| \geq 1$. Suppose also that $\alpha$ is not a root of unity. With these hypotheses, our main result is the following:

**Theorem 1.** *For every $\varepsilon > 0$ there exists $d_0(\varepsilon)$ such that we have*

$$\sum_{j=1}^{d/2} \Big(I - \frac{2j}{d} J\Big) \log|\alpha_j| > \frac{1-\varepsilon}{L} \delta(d) \tag{2.8}$$

*whenever $d > d_0(\varepsilon)$.*

The constant $d_0(\varepsilon)$ and the constants $d_1(\varepsilon)$,$d_2(\varepsilon)$,$d_3(\varepsilon)$, $d_4$, $d_5(p)$ used below are effective. Taking $G(x) = (1-x)^2$, we get $I = 1/3$, $J = 1/5$, $L = 4/3$. Hence the following inequality holds:

**Corollary 1.** *For every $\varepsilon > 0$ there exists $d_1(\varepsilon)$ such that*

$$\sum_{j=1}^{d/2}\Big(1-\frac{6j}{5d}\Big)\log|\alpha_j| > \Big(\frac{9}{4}-\varepsilon\Big)\delta(d)$$

*whenever $d > d_1(\varepsilon)$.*

This inequality obviously implies Louboutin's result. On the other hand, taking $G(x) = 1-\sin(\pi x/2)$, we have $I = 1-2/\pi$, $J = 3/2-4/\pi$, $L = \pi^2/8$. Hence

$$\sum_{j=1}^{d/2}\Big(1-\frac{2}{\pi}-\Big(3-\frac{8}{\pi}\Big)\frac{j}{d}\Big)\log|\alpha_j| > \Big(\frac{8}{\pi^2}-\varepsilon\Big)\delta(d)\,.$$

We can replace in the inequality above $\log|\alpha_j|$ by $|\alpha_j|-1$, and so Theorem 1 yields the following Corollary.

**Corollary 2.** *For every $\varepsilon > 0$ there exists $d_2(\varepsilon)$ such that for $d > d_2(\varepsilon)$ we have*

$$\sum_{j=1}^{d/2}\tau_j|\alpha_j| > 1+\Big(\frac{64}{\pi^2}-\varepsilon\Big)\frac{\delta(d)}{d},$$

*where*

$$\tau_j = \Big(1-\frac{2}{\pi}\Big)\frac{8}{d}-\Big(3-\frac{8}{\pi}\Big)\frac{8j-4}{d^2}.$$

Corollary 2 implies the inequality (2), since $\sum_{j=1}^{d/2}\tau_j = 1$. The following theorem fills the gap between (3) and (4).

**Theorem 2.** *Let $1 < p < \infty$ and $\varepsilon > 0$. Then there is $d_3(\varepsilon)$ such that for $d > d_3(\varepsilon)$ we have*

$$dM_p(\alpha) > \Big(b_p-\varepsilon\Big)\delta(d),$$

*where the constant $b_p$ is given by*

$$b_p = \frac{2}{L}\left(\frac{(2p-1)J}{(p-1)\big(I^{(2p-1)/(p-1)}-(I-J)^{(2p-1)/(p-1)}\big)}\right)^{1-1/p}. \tag{2.9}$$

We are not solving the problem of computing the maximum in (9) for a fixed $p$ from the interval $(1;\infty)$. However, notice that if $G(x) = (1-x)^{1.7}$ and $p = 2$ then by (5)-(7) and (9) we get $b_2 > 6.2679$.

**Corollary 3.** *There is $d_4 > 0$ such that for $d > d_4$ we have*

$$dM_2(\alpha) > 6.2679\delta(d).$$

**Theorem 3.** *If $\alpha$ is an algebraic integer which is not a root of unity, then for every $p > 0$ there exists $d_5(p)$ such that for $d > d_5(p)$ we have*

$$\Big(m_p(\alpha)\Big)^p > 1 + 19.64\Big(p\,\delta(d)/d\Big)^2.$$

*In particular,*

$$m_1(\alpha) = \frac{|\alpha_1| + \cdots + |\alpha_d|}{d} > 1 + 19.64\Big(\frac{\delta(d)}{d}\Big)^2.$$

*Proof of Theorem 1.* Let $f(x)$ be a continuous non-negative function in $[0;1]$ such that $\int_0^1 f(x)dx = 1$, and let $G(x) = \int_x^1 f(y)dy$. Put

$$\begin{aligned} s &= \left[\frac{L}{2}\Big(\frac{\log d}{\log\log d}\Big)^2\right], \\ k_0 &= \left[\frac{s^2 \log s}{\log d}\right], \\ k_r &= \left[s\,f\Big(\frac{r}{s}\Big)\right], \quad 1 \le r \le s. \end{aligned}$$

Define

$$\begin{aligned} h_0(z) &= h(z) = \Big(1, z, z^2, \dots, z^{N-1}\Big)^t, \\ h_k(z) &= \frac{z^k}{k!}\frac{d^k h(z)}{d^k z} = \left(0, \dots, \binom{N-2}{k} z^{N-2}, \binom{N-1}{k} z^{N-1}\right)^t. \end{aligned}$$

Consider the determinant

$$D = det\Big|\Big|h_{u_r}\big(\alpha_j^{p_r}\big)\Big|\Big|,$$

where the matrix consists of $N = \big(k_0 + k_1 + \cdots + k_s\big)d$ columns, $u_r = 0, 1, \dots, k_r - 1$, $j = 1, 2, \dots, d$. Here $p_r$ is the $r$-th prime number $\big(p_0 = 1, p_1 = 2, p_2 = 3, \dots\big)$. Recall that $\alpha$ is reciprocal and $\alpha_{2m} = 1/\alpha_1, \dots, \alpha_{m+1} = 1/\alpha_m$. Then see $\big([3], [5], [9], [10], [19]\big)$ the determinant $D$ is given by

$$D = \pm \prod \Big(\alpha_i^{p_u} - \alpha_j^{p_v}\Big)^{k_u k_v} \Big(\alpha_i^{-p_u} - \alpha_j^{-p_v}\Big)^{k_u k_v} \prod \Big(\alpha_i^{p_u} - \alpha_j^{-p_v}\Big)^{k_u k_v}$$

where the first product is taken over $i, j = 1, 2, \dots, m$ and $0 \le u \le v \le s$ (if $u = v$, then $i < j$). The second product is taken over all $i, j = 1, 2, \dots, m$; $u, v = 0, 1, 2, \dots, s$. Let us denote these products by $P_1$ and $P_2$ respectively.

We first consider $P_1$. We have:

$$\begin{aligned}
P_1 &= \pm \prod \left(\alpha_i^{p_u} - \alpha_j^{p_v}\right)^{2k_u k_v} \prod \alpha_i^{-p_u k_u k_v} \alpha_j^{-p_v k_u k_v} \\
&= \pm \prod \left(\alpha_i^{p_u} - \alpha_j^{p_v}\right)^{2k_u k_v} \prod_{i,j;u<v} \alpha_i^{-p_u k_u k_v} \prod_{i,j;u<v} \alpha_j^{-p_v k_u k_v} \\
&\quad \times \prod_{i<j;u} \left(\alpha_i \alpha_j\right)^{-p_u k_u^2} \\
&= \pm M(\alpha)^{-m\left(\sum_{u<v} p_u k_u k_v + \sum_{u>v} p_u k_u k_v\right) - (m-1)\sum p_u k_u^2} \\
&\quad \times \prod \left(\alpha_i^{p_u} - \alpha_j^{p_v}\right)^{2k_u k_v} \\
&= \pm \prod \left(\alpha_i^{p_u} - \alpha_j^{p_v}\right)^{2k_u k_v} M(\alpha)^{-m \sum p_u k_u \sum k_v + \sum p_u k_u^2}.
\end{aligned}$$

Next, we have for the product $P_2$

$$\begin{aligned}
P_2 &= \prod \left(1 - \alpha_i^{-p_u} \alpha_j^{-p_v}\right)^{k_u k_v} \prod \alpha_i^{p_u k_u k_v} \\
&= \prod \left(1 - \alpha_i^{-p_u} \alpha_j^{-p_v}\right)^{k_u k_v} M(\alpha)^{m \sum p_u k_u \sum k_v}
\end{aligned}$$

Combining these results we find

$$D = \pm \prod \left(\alpha_i^{p_u} - \alpha_j^{p_v}\right)^{2k_u k_v} \prod \left(1 - \alpha_i^{-p_u} \alpha_j^{-p_v}\right)^{k_u k_v} M(\alpha)^{\sum p_u k_u^2}.$$

Now from each term $\alpha_i^{p_u} - \alpha_j^{p_v}$ in the first product we take

1. $\alpha_i^{p_u}$, if $u = v$, $i < j$;
2. $\alpha_j^{p_v}$, if $u < v$, $j \leq i$;
3. $\alpha_i^{p_u} \alpha_j^{p_v - p_u}$, if $u < v$, $i < j$.

This is the key point of our argument. Write the determinant $D$ as follows

$$\begin{aligned}
D = \; \pm \; & \prod \alpha_i^{2p_u k_u^2} \left(1 - \left(\alpha_j/\alpha_i\right)^{p_u}\right)^{2k_u^2} \prod \alpha_j^{2p_v k_u k_v} \left(\alpha_i^{p_u} \alpha_j^{-p_v} - 1\right)^{2k_u k_v} \\
\times \; & \prod \alpha_i^{2p_u k_u k_v} \alpha_j^{2(p_v - p_u) k_u k_v} \left(\alpha_j^{p_u - p_v} - \left(\alpha_j/\alpha_i\right)^{p_u}\right)^{2k_u k_v} \\
\times \; & \prod \left(1 - \alpha_i^{-p_u} \alpha_j^{-p_v}\right)^{k_u k_v} M(\alpha)^{\sum p_u k_u^2}
\end{aligned}$$

Denote $y_1 = \alpha_2/\alpha_1$, $y_2 = \alpha_3/\alpha_2, \ldots, y_{m-1} = \alpha_m/\alpha_{m-1}$, $y_m = 1/\alpha_m$. Then $D$ can be expressed in the form

$$\begin{aligned} D &= \pm M(\alpha)^{\;p_u k_u^2} \prod \alpha_i^{2p_u k_u^2} \prod \alpha_j^{2p_v k_u k_v} \prod \alpha_i^{2p_u k_u k_v} \alpha_j^{2(p_v - p_u)k_u k_v} \\ &\quad \times p(y_1, y_2, \ldots, y_m) \\ &= \prod_{j=1}^{m} \alpha_j^{s_j} \times p(y_1, y_2, \ldots, y_m) \end{aligned}$$

where $p(y_1, \ldots, y_m)$ is a polynomial in $y_1, y_2, \ldots, y_m$. The power $s_j$ is given by

$$\begin{aligned} s_j &= \sum p_u k_u^2 + 2(m-j)\sum p_u k_u^2 + 2(m-j+1)\sum_{u<v} p_v k_u k_v \\ &\quad + 2(m-j)\sum_{u<v} p_u k_u k_v + 2(j-1)\sum_{u<v}(p_v - p_u)k_u k_v \\ &= (2m-2j+1)\sum p_u k_u^2 + 2m\sum_{u<v} p_v k_u k_v + (2m-4j+2)\sum_{u<v} p_u k_u k_v \\ &= (2m-2j+1)\sum p_u k_u^2 + 2m\sum_{u<v} p_v k_u k_v \\ &\quad + (2m-4j+2)\Big(\sum p_u k_u \sum k_v - \sum p_u k_u^2 - \sum_{v<u} p_u k_u k_v\Big) \\ &= (d-4j+2)\sum p_u k_u \sum k_v + (4j-2)\sum_{v<u} p_u k_u k_v + (2j-1)\sum p_u k_u^2 \\ &= (d-4j+2)\sum p_u k_u \sum k_v + (4j-2)\sum_{v\le u} p_u k_u k_v - (2j-1)\sum p_u k_u^2 \end{aligned}$$

Using the maximum modulus principle and the inequalities $|y_j| \le 1$, $j = 1, 2, \ldots, m$, we have

$$\Big|p(y_1, y_2, \ldots, y_m)\Big| \le \Big|p(y_1^0, y_2^0, \ldots, y_m^0)\Big|,$$

where $|y_1^0| = |y_2^0| = \cdots = |y_m^0| = 1$. Now by Hadamard's inequality we find (see [5])

$$\log|D| \le \frac{1}{2} d \log\Big(d\sum_{v=0}^{s} k_v\Big) \sum_{v=0}^{s} k_v^2 + \sum_{j=1}^{d/2} s_j \log|\alpha_j|.$$

On the other hand (see [9]),

$$\log|D| \ge k_0 d \sum_{v=1}^{s} k_v \log p_v.$$

For $d$ tending to infinity the following asymptotic formulas hold:

$$\sum_{v=1}^{s} k_v \log p_v \sim \sum_{v} s\, f\left(\frac{v}{s}\right) \log v \sim s^2 \log s \int_0^1 f(x)dx \sim s^2 \log s \sim$$
$$\sim \frac{L^2}{2} \frac{(\log d)^4}{(\log\log d)^3},$$
$$k_0 \sim \frac{L^2}{2}\left(\frac{\log d}{\log\log d}\right)^3,$$
$$\sum_{v=0}^{s} k_v^2 \sim k_0^2 + \sum_{v=1}^{s} s^2 f^2\left(\frac{v}{s}\right) \sim k_0^2 + s^3 \int_0^1 f^2(x)dx \sim \frac{3}{8} L^4 \left(\frac{\log d}{\log\log d}\right)^6.$$

Similarly,

$$s_j \sim (d-4j)\, s^5 \log s \int_0^1 f(x) x\, dx + 4j\, s^5 \log s \int_0^1 f(x) x \left(\int_0^x f(y)dy\right) dx.$$

Since

$$\int_0^1 f(x)xdx = -\int_0^1 G'(x)xdx = \int_0^1 G(x)dx = I$$

and

$$\int_0^1 f(x)x\left(\int_0^x f(y)dy\right)dx = \int_0^1 f(x)x\big(1-G(x)\big)dx$$
$$= I - \int_0^1 f(x)x\, G(x)dx = I + \int_0^1 G'(x)\, G(x)\, x\, dx$$
$$= I + \frac{1}{2}\int_0^1 \left(G^2(x)\right)' x\, dx$$
$$= I - \frac{1}{2}\int_0^1 G^2(x)dx = I - \frac{1}{2}J,$$

we have

$$s_j \sim s^5 \log s\Big((d-4j)I + 4j(I - \frac{12}{J})\Big)$$
$$\sim \big(dI - 2jJ\big)\frac{L^5}{16}\frac{(\log d)^{10}}{(\log\log d)^9}.$$

For a sufficiently large $d$ we have

$$\begin{aligned}
&\sum_{j=1}^{d/2} \big(dI - 2jJ\big) \log|\alpha_j| \\
> \;& (1-\varepsilon)\frac{16(\log\log d)^9}{L^5(\log d)^{10}}\left(\frac{dL^4(\log d)^7}{4(\log\log d)^6} - \frac{3dL^4(\log d)^7}{16(\log\log d)^6}\right) \\
= \;& (1-\varepsilon)\frac{d}{L}\left(\frac{\log\log d}{\log d}\right)^3.
\end{aligned}$$

This inequality implies (8). □

*Proof of Theorem 2.* If $\alpha$ is not reciprocal, then by Smyth's result [16] $dM_1(\alpha) \geq 2\log\theta$, and the theorem follows from $M_p(\alpha) \geq M_1(\alpha)$. Let $\alpha$ be reciprocal. Then by (8) and by Hölder's inequality we have

$$\begin{aligned}
1 - \frac{\varepsilon}{L}\delta(d) &< \sum_{j=1}^{d/2}\Big(I - \frac{2j}{d}J\Big)\log|\alpha_j| \\
&\leq \left(\sum_{j=1}^{d/2}\big(\log|\alpha_j|\big)^p\right)^{1/p}\left(\sum_{j=1}^{d/2}\Big(I - \frac{2j}{d}J\Big)^q\right)^{1/q}
\end{aligned}$$

where $1/p + 1/q = 1$.

Note first that for a reciprocal $\alpha$

$$\left(\sum_{j=1}^{d/2}\Big(\log|\alpha_j|\Big)^p\right)^{1/p} = (d/2)^{1/p}M_p(\alpha).$$

For $d$ tending to infinity we have

$$\begin{aligned}
\sum_{j=1}^{d/2}\Big(I - \frac{2j}{d}J\Big)^q &\sim \frac{d}{2}\int_0^1 \Big(I - Jx\Big)^q dx \\
&= \frac{d\big(I^{q+1} - (I-J)^{q+1}\big)}{2J(q+1)}.
\end{aligned}$$

Hence

$$1 - \frac{\varepsilon_1}{L}\delta(d) < \frac{d}{2}M_p(\alpha)\left(\frac{I^{q+1} - (I-J)^{q+1}}{J(q+1)}\right)^{1/q},$$

and Theorem 2, where the constant $b_p$ is given by (9), follows. □

*Proof of Theorem 3.* We have

$$\begin{aligned}\big(m_p(\alpha)\big)^p &= \frac{1}{d}\sum_{i=1}^{d}|\alpha_i|^p \\ &= \frac{1}{d}\sum_{i=1}^{d}\exp\left(p\log|\alpha_i|\right) \\ &= \frac{1}{d}\sum_{i=1}^{d}\sum_{j=0}^{\infty}\frac{(p\log|\alpha_i|)^j}{j!} \\ &= \frac{1}{d}\sum_{j=0}^{\infty}\frac{p^j}{j!}\sum_{i=1}^{d}\Big(\log|\alpha_i|\Big)^j.\end{aligned}$$

If $\alpha$ is reciprocal, then the inner sum equals $d\big(M_j(\alpha)\big)^j$ for even $j$ and zero for odd $j$. Hence

$$\big(m_p(\alpha)\big)^p = 1+\sum_{k=1}^{\infty}\frac{p^{2k}}{(2k)!}\Big(M_{2k}(\alpha)\Big)^{2k} > 1+\frac{p^2}{2}\Big(M_2(\alpha)\Big)^2.$$

Utilizing Corollary 3 we have

$$\Big(M_2(\alpha)\Big)^2 > 39.28\left(\frac{\delta(d)}{d}\right)^2,$$

if $d$ is large enough and the statement of Theorem 3 follows.

Suppose now that $\alpha$ is not reciprocal. If

$$|a_0| = \prod_{i=1}^{d}|\alpha_i| \geq 2$$

then

$$\begin{aligned}\big(m_p(\alpha)\big)^p &= \frac{|\alpha_1|^p+\cdots+|\alpha_d|^p}{d} \geq \prod_{i=1}^{d}|\alpha_i|^{p/d} \geq 2^{p/d} > 1+\frac{p\log 2}{d} \\ &> 1+19.64\left(\frac{p\delta(d)}{d}\right)^2\end{aligned}$$

for $d > d_5(p)$. Hence it is sufficient to consider the case when $|a_0| = 1$. Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be the conjugates of $\alpha$ lying strictly outside the unit circle. Put

$$\Lambda = \prod_{i=1}^{r}|\alpha_i|.$$

Then

$$\begin{aligned}\big(m_p(\alpha)\big)^p &= \frac{|\alpha_1|^p+\cdots+|\alpha_d|^p}{d}\\ &\geq \frac{r}{d}\big(|\alpha_1|\ldots|\alpha_r|\big)^{p/r}+\frac{d-r}{d}\big(|\alpha_{r+1}|\ldots|\alpha_d|\big)^{p/(d-r)}\\ &= \frac{r}{d}\Lambda^{p/r}+\frac{d-r}{d}\Lambda^{-p/(d-r)}.\end{aligned}$$

We shall show now that the last expression is greater than

$$1+\frac{(\log\theta)^2}{2}\Big(\frac{p}{d}\Big)^2$$

where $\theta=1.32471\ldots.$ Indeed, if

$$h(\Lambda)=\frac{r}{d}\Lambda^{p/r}+\frac{d-r}{d}a\Lambda^{-p/(d-r)}$$

then

$$\begin{aligned}h'(\Lambda) &= \frac{p}{d}\Lambda^{p/r-1}-\frac{p}{d}\Lambda^{-p/(d-r)-1}\\ &= \frac{p}{\Lambda d}\Big(\Lambda^{p/r}-\Lambda^{-p/(d-r)}\Big).\end{aligned}$$

Therefore, the function $h(\Lambda)$ is increasing in the interval $(1;\infty)$ and by Smyth's theorem

$$h(\Lambda)\geq h(\theta)=\frac{r}{d}\theta^{p/r}+\frac{d-r}{d}\theta^{-p/(d-r)}.$$

Put for brevity $p=zd$ and $r=yd$. We are going to prove that

$$g(z)=y\theta^{z/y}+(1-y)\theta^{-z/(1-y)}-1-\frac{(\log\theta)^2}{2}z^2>0$$

for $z>0$ and $0<y<1$. Indeed, $g(0)=0$ and

$$\begin{aligned}g'(z) &= \theta^{z/y}\log\theta-\theta^{-z/(1-y)}\log\theta-(\log\theta)^2z\\ &> \theta^{z/y}\log\theta-\log\theta-(\log\theta)^2z\\ &> \Big(1+\frac{z\log\theta}{y}\Big)\log\theta-\log\theta-(\log\theta)^2z\\ &= z\Big(\frac{1}{y}-1\Big)(\log\theta)^2>0.\end{aligned}$$

Therefore, with our hypotheses

$$\big(m_p(\alpha)\big)^p>1+\frac{(\log\theta)^2}{2}\Big(\frac{p}{d}\Big)^2>1+19.64\Big(\frac{p\delta(d)}{d}\Big)^2$$

for $d>d_5(p)$. This completes the proof of Theorem 3. □

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