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**Non-vanishing of n -th derivatives of twisted
elliptic L -functions in the critical point**

par JACEK POMYKAŁA

RÉSUMÉ. On note $L^{(n)}(s, E)$ la dérivée n -ième de la série L de Hasse-Weil associée à une courbe elliptique modulaire E définie sur \mathbb{Q} . On évalue dans cet article le nombre de tordues E_d , $d \leq D$, de la courbe elliptique E telles que $L^{(n)}(1, E_d) \neq 0$.

ABSTRACT. Let E be a modular elliptic curve over \mathbb{Q} and $L^{(n)}(s, E)$ denote the n -th derivative of its Hasse-Weil L -series. We estimate the number of twisted elliptic curves E_d , $d \leq D$ such that $L^{(n)}(1, E_d) \neq 0$.

1. Introduction

Let $E = E/\mathbb{Q}$ be a modular elliptic curve over \mathbb{Q} with conductor N defined by the Weierstrass equation $y^2 = f(x)$ and let \mathbb{D} be defined as follows:

$$\mathbb{D} = \{d - \text{square-free} : 0 < d \equiv -\nu^2 \pmod{4N} \text{ for some } \nu \text{ prime to } 4N\}.$$

For any $d \in \mathbb{D}$ we consider the twisted elliptic curve E_d given by the equation $-dy^2 = w(x)$. We denote by $L(s, E)$ and $L(s, E_d)$ the Hasse-Weil L -functions associated to the curves E and E_d , respectively.

The celebrated Birch and Swinnerton-Dyer conjecture (see [B-S]) asserts that the rank of the group of rational points of E/\mathbb{Q} is equal to the vanishing order of the associated Hasse-Weil function $L(s, E)$ at $s = 1$. Kolyvagin [Ko] has proved that

E/\mathbb{Q} has rank equal to zero if:

- 1) $L(E, 1) \neq 0$,
- 2) There exists $d \in \mathbb{D}$ such that $L(s, E_d)$ has a simple zero at $s = 1$.

E/\mathbb{Q} has rank equal to one if:

- 1') $L(s, E)$ has a simple zero at $s = 1$.
- 2') There exists $d \in \mathbb{D}$ such that $L(1, E_d) \neq 0$.

The condition 2') is true according to Waldspurger's theorem (see [Wa]).

The condition 2) has been proved to hold for infinitely many $d \in \mathbb{D}$ (see [B-F-H], [M-M]).

Iwaniec [Iw] has also proved a quantitative result on this condition. Let

$$N(D) = \#\{d \in \mathbb{D}, d \leq D : L'(1, E_d) \neq 0\}.$$

He has obtained the estimate

$$N(D) \gg D^{\frac{2}{3}-\varepsilon}$$

with arbitrary $\varepsilon > 0$.

The above exponent is improved to $1-\varepsilon$ in [P-P]. Here we will generalize this result to the n -th derivative of $L(s, E_d)$ where n is an arbitrary non-negative integer.

In this connection let $w = w(E)$ be the sign in the functional equation (see (1)). We define

$$N_n(D) = \#\{d \in \mathbb{D}, d \leq D : L^{(n)}(1, E_d) \neq 0\}.$$

We will prove

THEOREM 1. *Let ε be an arbitrary positive real number and n be a fixed non-negative integer. Then we have as D tends to infinity*

$$N_n(D) \gg (n + |w - 1|)D^{1-\varepsilon}.$$

where the constant implied in the symbol \gg depends on n and ε .

Our result is based on a recent large sieve type estimates over fundamental discriminants obtained by Heath-Brown [HB] (see Theorem 3 of [P-P]) and the method applied by Iwaniec in [Iw].

2. Outline of the proofs

For $\text{Re } s > 1$ the corresponding L -functions are given by

$$L(s, E) = \sum_{k=1}^{\infty} a_k k^{-s}, \quad L(s, E_d) = \sum_{k=1}^{\infty} \chi_d(k) a_k k^{-s},$$

where $\chi_d(\cdot) = \left(\frac{\cdot-d}{4N}\right)$ (the Kronecker symbol) is a real character to modulus d prime to $4N$. They have the analytic prolongation to the whole complex plane, where they satisfy the equations

$$\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L(s, E) = w \left(\frac{\sqrt{N}}{2\pi}\right)^{2-s} \Gamma(2-s) L(E, 2-s), \quad (1)$$

$$\left(\frac{d\sqrt{N}}{2\pi}\right)^s \Gamma(s) L(s, E_d) = w_d \left(\frac{d\sqrt{N}}{2\pi}\right)^{2-s} \Gamma(2-s) L(2-s, E_d), \quad (2)$$

where w and $w_d = \pm 1$ are suitable constants depending on the reduction of E at the primes dividing N , which satisfy the equality

$$w_d = w \chi_d(-N). \quad (3)$$

From (1) and (2) it follows that

$$w = (-1)^{\text{ord}_{s=1} L(s, E)}, \quad w_d = (-1)^{\text{ord}_{s=1} L(s, E_d)}$$

Theorem 1 is a consequence of the following two theorems.

THEOREM 2. *For any integer $n \geq 0$ we have the asymptotical equality*

$$\sum_{d \leq D}^* L^{(n)}(1, E_d) F\left(\frac{d}{Y}\right) \sim c_n L(1) \int F(t) dt D(\ln D)^n, \quad \text{as } D \rightarrow \infty,$$

where F is a smooth function compactly supported in \mathbb{R}^+ with positive mean-value and the star (*) above means that the summation is restricted to $d \in \mathbb{D}$. The (nonzero) constant $L(1)$ is described in [Iw], while

$$c_n = \begin{cases} \frac{1}{2}(1-w)c & \text{if } n = 0 \\ w(-2)^{n-1}c & \text{if } n \geq 1 \end{cases} \quad (4)$$

$$c = \frac{3}{\pi^2 N} \prod_{p|4N} \left(1 - \frac{1}{p^2}\right)^{-1} \times \quad (5)$$

$$\times \#\{d \pmod{4N} : d \equiv -\nu^2 \pmod{4N}, (\nu, 4N) = 1\}.$$

The proof follows the idea exploited in [Iw] (cf. also [M-M] and [P-S]). We postpone it to sect.3.

Along the same lines as Theorem 3 of [P-P] we obtain

THEOREM 3. *Let $\varepsilon > 0$ and n be a fixed non-negative integer. Then we have*

$$\sum_{d \leq D}^* |L^{(n)}(1, E_d)|^2 \ll_{\varepsilon} D^{1+\varepsilon}$$

Now from Theorem 2 and Theorem 3 we obtain by an application of the Cauchy-Schwarz inequality that

$$(n+|w-1|)D(\ln D)^n \ll \sum_{d \leq D}^* |L^{(n)}(1, E_d)| \leq \left(\sum_{d \leq D}^* |L^{(n)}(1, E_d)|^2 \right)^{\frac{1}{2}} N_n(D)^{\frac{1}{2}}$$

hence

$$N_n(D) \gg (n+|w-1|)D^{1-2\varepsilon}$$

and Theorem 1 follows.

3. Proof of Theorem 2

For $n \geq 0$ we introduce the approximate functions

$$A_n(X, \chi_d) = \sum_{m=1}^{\infty} a_m \frac{\chi_d(m)}{m} V_n\left(\frac{2\pi m}{X}\right) \quad (6)$$

with

$$V_n(X) = \frac{1}{2\pi i} \int_{(3/4)} \frac{\Gamma(s)}{s^n} X^{-s} ds, \quad (7)$$

where we integrate over the line $\operatorname{Re} s = \frac{3}{4}$.

By the functional equation (2) we have

$$L(1+s, E_d) = w_d \left(\frac{d\sqrt{N}}{2\pi} \right)^{-2s} L(1-s, E_d) \frac{\Gamma(1-s)}{\Gamma(1+s)}.$$

Therefore

$$\begin{aligned} \frac{1}{2\pi i} \int_{(-3/4)} L(1+s, E_d) \frac{\Gamma(s)}{s^n} \left(\frac{X}{2\pi} \right)^s ds \\ &= \frac{w_d}{2\pi i} \int_{(-3/4)} L(1-s, E_d) \frac{\Gamma(1-s)}{\Gamma(1+s)} \frac{\Gamma(s)}{s^n} \left(\frac{2\pi X}{d^2 N} \right)^s ds \\ &= -\frac{w_d}{2\pi i} \int_{(-3/4)} L(1-s, E_d) \frac{\Gamma(-s)}{s^n} \left(\frac{2\pi X}{d^2 N} \right)^s ds \\ &= (-1)^{n+1} \frac{w_d}{2\pi i} \int_{(3/4)} L(1+s, E_d) \frac{\Gamma(s)}{s^n} \left(\frac{2\pi}{d^2 N/X} \right)^{-s} ds \\ &= (-1)^{n+1} w_d A_n \left(\frac{d^2 N}{X}, \chi_d \right). \end{aligned}$$

Hence defining

$$G_n(s, d, X) = L(1+s, E_d) \frac{\Gamma(s)}{s^n} \left(\frac{X}{2\pi} \right)^s$$

we obtain by the Cauchy theorem

$$\begin{aligned} \operatorname{res}_{s=0} G_n(s, d, X) &= \frac{1}{2\pi i} \int_{(3/4)} G_n(s, d, X) ds - \frac{1}{2\pi i} \int_{(-3/4)} G_n(s, d, X) ds \\ &= A_n(X, \chi_d) + (-1)^n w_d A_n \left(\frac{d^2 N}{X}, \chi_d \right). \end{aligned}$$

By the definition of \mathbb{D} we have $w_d = w\chi_d(-N) = -w$ for any $d \in \mathbb{D}$. Hence letting $X = d\sqrt{N}$ we obtain

$$\operatorname{res}_{s=0} G_n(s, d, d\sqrt{N}) = \begin{cases} 2A_n(d\sqrt{N}, \chi_d) & \text{if } n \equiv \frac{w+1}{2} \pmod{2} \\ 0 & \text{if } n \equiv \frac{w-1}{2} \pmod{2}. \end{cases} \quad (8)$$

On the other hand the residuum of G_n may be expressed in terms of the derivatives $L^{(k)}(s, E_d)$, $k = 0, 1, 2, \dots$, by means of the Laurent expansions

$$\Gamma(s) = s^{-1} + \gamma_0 + \gamma_1 s + \dots,$$

$$\left(\frac{X}{2\pi}\right)^s = \sum_{k=0}^{\infty} b_k s^k,$$

where $-\gamma_0$ is the Euler constant and $b_k = b_k(X) = \frac{1}{k!} (\ln \frac{X}{2\pi})^k$.

Namely we prove

Lemma. For any $n \geq 0$ we have

$$\begin{aligned} \operatorname{res}_{s=0} G_n(s, d, X) &= b_0 \frac{L^{(n)}(1, E_d)}{n!} \\ &+ \sum_{l=0}^{n-1} \left(b_{l+1} + \sum_{m=0}^l \gamma_m b_{l-m} \right) \frac{L^{(n-1-l)}(1, E_d)}{(n-1-l)!}. \end{aligned}$$

Proof. We have

$$\begin{aligned} G_n(s, d, X) &= s^{-n} \left(\sum_{k=0}^{\infty} \frac{L^{(k)}(1, E_d)}{k!} s^k \right) \left(s^{-1} + \sum_{k=0}^{\infty} \gamma_k s^k \right) \left(\sum_{k=0}^{\infty} b_k s^k \right) \\ &= s^{-n} \left(\sum_{k=0}^{\infty} \frac{L^{(k)}(1, E_d)}{k!} s^k \right) \left(\sum_{k=0}^{\infty} b_k s^{k-1} + \sum_{k=0}^{\infty} f_k s^k \right), \end{aligned}$$

where

$$f_k = \sum_{l=0}^k b_l \gamma_{k-l}.$$

The expression in the second bracket is equal to

$$\frac{b_0}{s} + \sum_{k=0}^{\infty} b_{k+1} s^k + \sum_{k=0}^{\infty} f_k s^k = \frac{b_0}{s} + \sum_{k=0}^{\infty} (f_k + b_{k+1}) s^k.$$

Hence

$$G_n(s, d, X) = s^{-n} \left(\frac{b_0}{s} \sum_{k=0}^{\infty} \frac{L^{(k)}(1, E_d)}{k!} s^k + \sum_{k=0}^{\infty} c_k s^k \right),$$

where

$$c_k = \sum_{l=0}^k (f_l + b_{l+1}) \frac{L^{(k-l)}(1, E_d)}{(k-l)!}.$$

Therefore

$$\begin{aligned} \operatorname{res}_{s=0} G_n(s, d, X) &= b_0 \frac{L^{(n)}(1, E_d)}{n!} + c_{n-1} \\ &= b_0 \frac{L^{(n)}(1, E_d)}{n!} + \sum_{l=0}^{n-1} (b_{l+1} + f_l) \frac{L^{(n-1-l)}(1, E_d)}{(n-1-l)!}, \end{aligned}$$

as required.

Since $b_l(X) = \frac{1}{l!} \left(\ln \frac{X}{2\pi} \right)^l$ we obtain asymptotically

$$\operatorname{res}_{s=0} G_n(s, d, X) \sim \sum_{l=0}^n b_l(X) \frac{L^{(n-l)}(1, E_d)}{(n-l)!}, \quad \text{as } X \rightarrow \infty. \quad (9)$$

By (8) and the Lemma we obtain the formula

$$\begin{aligned} \sum_{l=0}^n b_l(d\sqrt{N}) \frac{L^{(n-l)}(1, E_d)}{(n-l)!} &\sim \\ &\sim \begin{cases} 2A_n(d\sqrt{N}, \chi_d) & \text{if } n \equiv \frac{w+1}{2} \pmod{2} \\ 0 & \text{if } n \equiv \frac{w-1}{2} \pmod{2}, \end{cases} \quad (10) \end{aligned}$$

as $d \rightarrow \infty$. Next we sum both sides of the above equality multiplied by the weighted function $F\left(\frac{d}{D}\right)$ over the numbers $d \in \mathbb{D}$. The contribution of the right-hand side of (10) is evaluated on the basis of the results on square-free sieve obtained in § 7-§ 9 of [Iw]. Precisely we have

$$\sum_{d \in \mathbb{D}}^* 2A_n(d\sqrt{N}, \chi_d) F\left(\frac{d}{D}\right) \sim cD \int F(t) B_n(tD\sqrt{N}) dt, \quad (11)$$

where

$$B_n(X) = \frac{1}{2\pi i} \int_{(3/4)} L(s+1) \frac{\Gamma(s)}{s^n} \left(\frac{X}{2\pi} \right)^s ds,$$

the constant c is defined by (5) and the series $L(s)$ is defined in §9 of [Iw].

In order to find the asymptotical behaviour of the right-hand side of (11) we denote

$$\bar{G}_n(s, X) = L(s+1) \frac{\Gamma(s)}{s^n} \left(\frac{X}{2\pi}\right)^s$$

and apply the contour integration to obtain

$$\begin{aligned} B_n(X) &= \frac{1}{2\pi i} \int_{(3/4)} \bar{G}_n(s, X) ds \\ &= \operatorname{res}_{s=0} \bar{G}_n(s, X) + \frac{1}{2\pi i} \int_{(-1/4)} \bar{G}_n(s, X) ds \\ &= \operatorname{res}_{s=0} \bar{G}_n(s, X) + O(X^{-1/4}). \end{aligned}$$

Applying the asymptotical equality (9) with G_n replaced by \bar{G}_n we obtain

$$B_n(X) \sim \sum_{l=0}^n b_l(X) \frac{L^{(n-l)}(1)}{(n-l)!} \sim b_n(X) L(1) \sim \frac{1}{n!} (\ln X)^n L(1), \quad \text{as } X \rightarrow \infty.$$

Hence the right-hand side of (11) is asymptotically equal to

$$L(1) \frac{c_0}{n!} D \int F(t) \ln^n(tD\sqrt{N}) dt \sim L(1) \frac{c_0}{n!} \int F(t) dt D(\ln D)^n.$$

Therefore in view of (10) we obtain the asymptotic equality

$$\begin{aligned} &\sum_{d \leq D} \sum_{l=0}^n b_l(d\sqrt{N}) \frac{L^{(n-l)}(1, E_d)}{(n-l)!} F\left(\frac{d}{D}\right) \sim \\ &\sim \begin{cases} \frac{c}{n!} L(1) \int F(t) dt D(\ln D)^n & \text{if } n \equiv \frac{w+1}{2} \pmod{2} \\ 0 & \text{if } n \equiv \frac{w-1}{2} \pmod{2}. \end{cases} \end{aligned}$$

Hence we obtain immediately that the constant c_0 in Theorem 2 is equal to $(\frac{1-w}{2})c$, where c is the constant defined by (5).

Assuming that

$$\sum_{d \in \mathbb{D}} L^{(k)}(1, E_d) F\left(\frac{d}{D}\right) \sim c_k L(1) \int F(t) dt D(\ln D)^k$$

where c_k are some constants we see that they have to satisfy the equality

$$\sum_{k=0}^n \binom{n}{k} c_k = \begin{cases} c & \text{if } n \equiv \frac{w+1}{2} \pmod{2} \\ 0 & \text{if } n \equiv \frac{w-1}{2} \pmod{2}. \end{cases} \quad (12)$$

To complete the proof of Theorem 2 it remains to prove that (12) holds with the constants c_k defined by (4). Indeed we have

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} c_k &= \left(\frac{1-w}{2}\right) c + \sum_{k=1}^n \binom{n}{k} c_k \\ &= c \left(\frac{1-w}{2} - w \left(\frac{-1}{2} \sum_{k=1}^n \binom{n}{k} (-2)^k \right) \right) \\ &= c \left(\frac{1-w}{2} - \frac{w}{2} ((-1)^n - 1) \right) \\ &= \begin{cases} c & \text{if } n \equiv \frac{w+1}{2} \pmod{2} \\ 0 & \text{if } n \equiv \frac{w-1}{2} \pmod{2} \end{cases} \end{aligned}$$

as it is claimed. This completes the proof of Theorem 2.

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