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## A new lower bound for the football pool problem for 7 matches

par LAURENT HABSIEGER<sup>1</sup>

RÉSUMÉ. Notons  $K_3(7, 1)$  le cardinal minimal d'un code ternaire de longueur 7 et de rayon de recouvrement un. Dans un précédent article, nous avons amélioré la minoration  $K_3(7, 1) \geq 147$  en montrant que  $K_3(7, 1) \geq 150$ . Dans cette note, nous prouvons que  $K_3(7, 1) \geq 153$ .

ABSTRACT. Let  $K_3(7, 1)$  denote the minimum cardinality of a ternary code of length 7 and covering radius one. In a previous paper, we improved on the lower bound  $K_3(7, 1) \geq 147$  by showing that  $K_3(7, 1) \geq 150$ . In this note, we prove that  $K_3(7, 1) \geq 153$ .

### 1. Introduction

Let  $\mathbb{F}_3$  be the finite field with three elements and  $n$  be some positive integer. Define the Hamming distance between two elements  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  of  $\mathbb{F}_3^n$  by  $d(x, y) = |\{i \in \{1, \dots, n\} : x_i \neq y_i\}|$ . For  $x \in \mathbb{F}_3^n$  and  $r \in \mathbb{N}$ , the sphere of center  $x$  and radius  $r$  is denoted  $S_r(x)$  and is defined by  $S_r(x) = \{y \in \mathbb{F}_3^n : d(x, y) = r\}$ .

A covering code with covering radius one is a subset  $C$  of  $\mathbb{F}_3^n$  such that the following condition holds:

$$(1) \quad \forall x \in \mathbb{F}_3^n, \exists y \in C : d(x, y) \leq 1.$$

Let  $A$  denote the characteristic function of  $C$ . For  $i \in \mathbb{Z}$  let, as usual, the function  $A_i$  be defined by  $A_i(x) = |C \cap S_i(x)|$ . More generally, for any function  $F$  defined on  $\mathbb{F}_3^n$ , we define the function  $F_i$  by

$$F_i(x) = \sum_{y \in S_i(x)} F(y),$$

as in [4]. Then the covering condition (1) becomes

$$\forall x \in \mathbb{F}_3^n, (A_0 + A_1)(x) \geq 1.$$

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The problem of determining  $K_3(n, 1)$ , the minimal cardinality of  $C$ , has been widely studied [2] and is known as the “football pool problem”. Summing the covering conditions over  $\mathbb{F}_3^n$  leads to the sphere covering bound  $(2n + 1)K_3(n, 1) \geq 3^n$ . For instance this gives  $K_3(7, 1) \geq 146$ . Chen and Honkala [1] found the lower bound 147 and in [3] we gave the lower bound 150. The purpose of this paper is to prove the following Theorem.

**THEOREM 1.**  $K_3(7, 1) \geq 153$ .

## 2. Proof of Theorem 1

Let us consider the case  $n = 7$ . Let  $C$  be a covering code with covering radius one and suppose that  $|C| \leq 152$ . Up to adding some points to  $C$ , we may assume that  $|C| = 152$ . Let  $xy$  denotes the concatenation of  $x$  and  $y$ . For  $x \in \mathbb{F}_3^3$ , let us put

$$N(x) = \sum_{y \in \mathbb{F}_3^4} A(xy) = |\{y \in \mathbb{F}_3^4 : xy \in C\}|,$$

so that  $\sum_{i=0}^3 N_i(x) = |C| = 152$ . We proved in [3] that the following condition holds:

$$(2) \quad \forall x \in \mathbb{F}_3^3, \quad 9N_0(x) + N_1(x) \geq 81.$$

By summing this inequality over  $S_2(x)$  and  $S_3(x)$ , we obtain the additional conditions

$$(3) \quad \forall x \in \mathbb{F}_3^3, \quad 4N_1(x) + 11N_2(x) + 3N_3(x) \geq 972,$$

$$(4) \quad \forall x \in \mathbb{F}_3^3, \quad 2N_2(x) + 12N_3(x) \geq 648.$$

The linear combination  $43(2) + 5(3) + 4(4)$  now gives the inequality  $63|C| + 324N(x) \geq 10935$ , which implies the lower bound  $N(x) \geq 5$ .

Let us put  $D(x) = N(x) - 5$ , so that  $\sum_{i=0}^3 D_i(x) = 152 - 5 \times 27 = 17$ . The condition (2) becomes

$$\forall x \in \mathbb{F}_3^3, \quad 9D(x) + D_1(x) \geq 6,$$

which is equivalent to the new condition

$$(5) \quad \forall x \in \mathbb{F}_3^3, \quad 6D(x) + D_1(x) \geq 6.$$

We shall need the following lemma.

LEMMA 2. Let  $u$  be an element of  $\mathbb{F}_3^3$  such that  $d(0, u) = 3$ . Then the following property holds

$$(6) \quad \forall x \in \mathbb{F}_3^3, D(x - u) + D(x) + D(x + u) \geq 1.$$

*Proof.* Up to isometry, we may assume that  $u = 111$ . Let us define the function  $\phi$  on  $\mathbb{F}_3^3$  by  $\phi(x) = D(x - u) + D(x) + D(x + u)$ . Let us add the condition (5) for  $x - u$ ,  $x$  and  $x + u$ . We obtain  $6\phi(x) + \phi_1(x) \geq 18$ . Since we know that

$$\phi_1(x) \leq \sum_{x \in \mathbb{F}_3^3} D(x) = 17,$$

we have  $6\phi(x) \geq 1$  and the lemma follows.  $\square$

Since  $\sum_{i=0}^3 D_i(x) = 17 < 2 \times 9$ , there exists some  $a \in \mathbb{F}_3^3$  such that  $D(a0) + D(a1) + D(a2) \leq 1$ , say  $a = 00$ . If  $D(000) = D(001) = D(002) = 0$ , it follows from condition (5) that  $3 \times 6 \leq \sum_{i=0}^3 D_i(x) = 17$ , which is impossible. Therefore, we may assume that, up to translation,  $D(000) = D(001) = 0$  and  $D(002) = 1$ . The condition (5) and Lemma 2 then give us the following inequalities:

$$(7) \quad D(100) + D(200) + D(010) + D(020) \geq 5,$$

$$(8) \quad D(101) + D(201) + D(011) + D(021) \geq 5,$$

$$(9) \quad D(111) + D(222) \geq 1,$$

By using (5) for  $x = 222$ , (7-9) and the evaluation  $D(002) = 1$ , we get

$$\begin{aligned} 18 \leq \sum_{x \in S_1(000) \cup S_1(001) \cup S_1(222)} D(x) + D(111) + 7D(222) \\ \leq \sum_{x \in \mathbb{F}_3^3} D(x) + 6D(222) = 17 + 6D(222), \end{aligned}$$

which shows that  $D(222) \geq 1$ . By symmetry we also have  $D(112)$ ,

$D(122), D(212) \geq 1$ , which implies the inequality

$$(10) \quad D(222) + D(112) + D(122) + D(212) \geq 4.$$

We take the following linear combination of inequalities: the inequalities (5) with  $x \in \{110, 111, 120, 121, 210, 211, 220, 221\}$ , plus seven times the inequalities (7-8) and (10) plus nine times the equation  $D(002) = 1$ . In this way, we obtain

$$155 \leq 9 \left( \sum_{x \in \mathbb{F}_3^3 \setminus \{012, 022, 102, 202\}} D(x) \right) \leq 9 \times 17 = 153,$$

which is impossible. This completes the proof of Theorem 1.

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