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Some infinite products with interesting continued fraction expansions

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RÉSUMÉ. We display several infinite products with interesting continued fraction expansions. Specifically, for various small values of k — necessarily excluding $k = 3$ since that case cannot occur, we display infinite products in the field of formal power series whose truncations yield their every k-th convergent.

1. Introduction

Let $0 < \lambda_0 < \lambda_1 < \dots$ be a sequence of rational integers, and let $F(x)$ be the infinite product

(1) $$F(x) = \prod_{h=0}^{\infty}(1 + x^{-\lambda_h})$$

with continued fraction expansion $[a_0(x)\,, a_1(x)\,, \dots]$. In [1] it is shown that for $n = 0,\ 1,\ 2,\ \dots$ each truncation

(2) $$F_n(x) = \prod_{h=0}^{n-1}(1 + x^{-\lambda_h})$$

is a convergent p_m/q_m of $F(x)$ if, and only if,

(3) $$\lambda_{n+1} > 2(\lambda_0 + \dots + \lambda_n)\,.$$

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Moreover, then the degree of the subsequent partial quotient a_{m+1} is given by

$$\deg a_{m+1} = \lambda_{n+1} - 2(\lambda_0 + \cdots + \lambda_n). \tag{4}$$

Furthermore, one sees that the truncations provide precisely every second convergent if, and only if,

$$2\lambda_n \mid \lambda_{n+1} \tag{5}$$

for $n = 0, 1, 2, \ldots$. In fact, given (5), the partial quotients of $F(x)$ can readily be listed explicitly and are of rapidly increasing degree. Henceforth, we only consider products for which (3) is satisfied. So their every truncation yields a convergent.

Here we will be concerned with classes of infinite products whose continued fraction expansions have the property that those convergents arising from the truncations of the product are again separated by some few convergents. In this context we say that two consecutive truncations p_m/q_m and p_{m+k}/q_{m+k} of our product are k*-apart* if, as suggested by the notation, there are exactly $k-1$ intermediate convergents. Our aim is to use the above mentioned 2-apart examples to construct additional products with truncations a fixed bounded distance apart. We shall explicitly give examples with regular gaps of size 4, 5, 6 and 8. We also illustrate how the matrix interpretation of continued fractions can be nicely employed to cleanly generate the partial quotients of the products we study.

2. Terminology

We use the terminology of [1] and [2]. Namely, given a field $\mathbb{K}$, we let $\mathbb{L} = \mathbb{K}((x^{-1}))$ denote the field of formal Laurent series in x^{-1} over $\mathbb{K}$. Then each $F \in \mathbb{L}$ is of shape

$$\sum_{h=-d}^{\infty} a_h x^{-h}, \qquad a_{-d} \neq 0, \quad d \in \mathbb{Z}$$

and we say that the degree of F is $\deg F = d$. The integral part of F, denoted $\lfloor F \rfloor$, is the polynomial

$$\sum_{h=-d}^{0} a_h x^{-h}$$

in $\mathbb{K}[x]$. A series $F \in \mathbb{L}$ has a unique continued fraction expansion denoted by

$$[c_0\,, c_1\,, c_2\,, \ldots]$$

where the partial quotients c_h are polynomials in x, and have degree at least 1 for $h \geq 1$. From the general theory of continued fractions we have the fundamental correspondence whereby for $h = 0, 1, 2, \ldots$

$$p_h/q_h = [c_0\,, c_1\,, \cdots, c_h]$$

if, and only if,

$$\begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_h & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_h & p_{h-1} \\ q_h & q_{h-1} \end{pmatrix}$$

defining the convergents p_h/q_h by products of certain matrices. Further, on setting

$$R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we observe that for $c \in \mathbb{Z}$,

$$R^c J = \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} = J L^c\,.$$

Then we may write formally that,

$$\begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots = R^{c_0} L^{c_1} R^{c_2} \cdots,$$

whence if $\longleftrightarrow$ denotes the correspondence between matrix products and continued fractions we have

$$[c_0\,, c_1\,, c_2\,, \ldots] \longleftrightarrow R^{c_0} L^{c_1} R^{c_2} \ldots.$$

The correspondence between continued fractions and the so called $R-L$ sequences is particularly convenient in studying the multiplication or division of a continued fraction by a rational quantity. This is illustrated in detail in [1]. We assume the relevant terminology here.

We also recall from [2] that a rational function $p(x)/q(x)$ is a convergent p_h/q_h to F if, and only if,

$$\deg\left(F - \frac{p}{q}\right) < -2\deg q \tag{6}$$

and that the degree of the next partial quotient is then given by

$$\deg c_{m+1} = -\deg\left(F - \frac{p}{q}\right) - 2\deg q. \tag{7}$$

In the sequel we suppose $\mathbb{K}$ to be a field of characteristic zero.

3. The Examples Revealed

As mentioned above, we already know a class of infinite products such that the truncations are always 2–apart; namely

$$F(x;\lambda)=\prod_{h=0}^{\infty}(1+x^{-\lambda_h}), \tag{8}$$

where $\lambda=(\lambda_0,\lambda_1,\dots)$ satisfies the conditions

$$\lambda_{n+1}>2(\lambda_0+\cdots+\lambda_n) \quad \text{and} \quad 2\lambda_n \mid \lambda_{n+1} \tag{9}$$

for all $n\geq 0$. It transpires that we can use such known cases to construct a slue of new examples with the property we want. In particular if

$$G(x)=(1+x^{-r})F(x;\lambda) \tag{10}$$

then the truncations of G (although in general no longer 2–apart) will still be of bounded distance apart:

THEOREM 1. *Let*

$$G(x)=(1+x^{-r})\prod_{i=0}^{\infty}(1+x^{-\lambda_i})$$

where

$$\lambda_i>2(r+\lambda_0+\cdots+\lambda_i-1) \quad \textit{and} \quad 2\lambda_i \mid \lambda_{i+1} \quad \textit{for all } i\geq 0.$$

Then for all $n=0,\,1,\,2,\,\dots$, the successive truncations of $G(x)$

$$\frac{p_m}{q_m}=(1+x^{-r})\prod_{i=0}^{n-1}(1+x^{-\lambda_i}), \qquad \frac{p_{m+k}}{q_{m+k}}=(1+x^{-r})\prod_{i=0}^{n}(1+x^{-\lambda_i})$$

are convergents of G at most $(2r+2)$–apart.

Proof. By criterion (3) it follows that the stated truncations certainly are convergents and further that

$$\deg c_{m+1}=\lambda_n-2r-2\sum_{i=0}^{n-1}\lambda_i.$$

Moreover the condition $2\lambda_i \,\big|\, \lambda_{i+1}$ leads to a, so to speak, gratuitous intermediate convergent

$$\frac{p_{m+j}}{q_{m+j}} = \frac{x^{\lambda_n - r - \sum_{i=0}^{n-1}\lambda_i}(x^r+1)}{(x^{\lambda_n}-1)/\prod_{i=0}^{n-1}(x^{\lambda_i}+1)}$$

with

$$\deg c_{m+j+1} = 2\lambda_n - 2\deg q_{m+j}\,,$$

as may be readily checked with an appeal to (6). Between them these two partial quotients soak up most of the available degree and (as the remaining c_{m+i} are of degree at least one) we obtain

$$\begin{aligned} k-2 &\le \sum_{i=1}^{k} \deg c_{m+i} - \deg c_{m+1} - \deg c_{m+j+1} \\ &= 2\big(\deg q_{m+j} - (\lambda_n - r - \sum_{i=0}^{n-1}\lambda_i)\big) \le 2r. \end{aligned}$$

Note: we see at once the the truncations are 2–apart if and only if

$$(x^r+1) \,\big|\, (x^{\lambda_n}-1)/\prod_{i=0}^{n-1}(x^{\lambda_i}+1)$$

and that otherwise (since by degree considerations $j \ne 1$ and $j \ne k-1$) we must have separation $k \ge 4$. We shall see later that there are in fact no 3–apart examples. Since one can envisage cases were all the remaining quotients are indeed linear the bound $2r+2$ seems reasonably precise; we shall see cases where this bound is sharp when $r = 1$, 2 or 3.

Not surprisingly, the phenomenon of one product inheriting the boundedness property from another is rather more general:

THEOREM 2. *If $F(x)$ and $G(x)$ are infinite products (whose truncations are all convergents) which differ only by a factor consisting of a rational function*

$$G(x) = \frac{f(x)}{g(x)}F(x) \qquad f(x), g(x) \in \mathbb{K}[x] \setminus \{0\},$$

then the truncations of $G(x)$ are of bounded distance apart if, and only if, the truncations of $F(x)$ are of bounded distance apart.

The result is immediate from the following (admittedly rough) bound:

LEMMA 1. *Suppose that p_i/q_i and p_{i+k}/q_{i+k} are convergents to a Laurent series $F(x)$ such that*

$$\frac{p'_m}{q'_m} = \frac{f(x)}{g(x)}\frac{p_i}{q_i}, \qquad \text{and} \qquad \frac{p'_{m+t}}{q'_{m+t}} = \frac{f(x)}{g(x)}\frac{p_{i+k}}{q_{i+k}}$$

are convergents to

$$G(x) = \frac{f(x)}{g(x)}F(x) \qquad f(x), g(x) \in \mathbb{K}[x] \setminus \{0\}.$$

Then

$$t \le (k+1)\deg fg + k.$$

Proof. The proof employs a similar degree counting argument:

We suppose that there are $M+1$ convergents p'_{u+j}/q'_{u+j} $(0 \le j \le M)$ to $G(x)$ with

$$\deg q_i - \deg f \le \deg q'_{u+j} \le \deg q_{i+k} + \deg g.$$

Further suppose that there are N convergents $p_{i+n(j)}/q_{i+n(j)}$ $(1 \le j \le N)$ to $F(x)$ with $0 \le n(j) < k$ and $\deg c_{i+n(j)+1} > \deg fg$. Now since

$$\deg\left(G - \frac{f}{g}\frac{p_{i+n(j)}}{q_{i+n(j)}}\right) = -2\deg gq_{i+n(j)} - \left(\deg c_{i+n(j)+1} - \deg fg\right)$$

we see from criterion (6) that

$$\frac{f}{g}\frac{p_{i+n(j)}}{q_{i+n(j)}} = \frac{p'_{u+s(j)}}{q'_{u+s(j)}}$$

all are convergents to G with

$$\deg c'_{u+s(j)+1} \ge \deg c_{i+n(j)+1} - \deg fg\,,$$

and plainly

$$\deg q_i - \deg f \le \deg q_{u+s(j)} < \deg q_{i+k} + \deg g\,,$$

so that $0 \le s(j) \le M$. In particular, as

$$\sum_{j=1}^{N} \deg c'_{u+s(j)+1} \ge \sum_{\substack{j=1 \\ \deg c_{i+j} > \deg fg}}^{k} \deg c_{i+j} - N\deg fg \ge$$

$$\ge \deg q_{i+k} - \deg q_i - k\deg fg\,,$$

these convergents monopolise most of the available degree. Hence, as the remaining convergents are at worst linear we see that

$$
\begin{aligned}
M - N \le \sum_{j=1}^{M} \deg c'_{u+j} - \sum_{j=1}^{N} \deg c'_{u+s(j)+1} \le \\
\le \big(\deg q_{i+k} - \deg q_i + \deg fg\big) - \big(\deg q_{i+k} - \deg q_i - k \deg fg\big) = \\
= (k+1) \deg fg.
\end{aligned}
$$

The bound follows at once since trivially $N \le k$ and $t \le M$.

Notice that we made no prior assumption here about the nature of $F(x)$; for example we might consider products

$$F(x) = \frac{f(x)}{g(x)} \prod_{i=0}^{\infty} \left(1 + \frac{f_i}{g_i}\right)$$

with

$$\deg g_n / f_n > 2 \sum_{i=0}^{n-1} \deg g_i + \deg fg$$

to ensure that the 'truncations'

$$\frac{f}{g} \prod_{i=0}^{n} \left(1 + \frac{f_i}{g_i}\right)$$

are convergents (needless to say we shall do no such thing here).

4. The Transduction Process Justified

We saw in Theorem 1 that examples with the desired property could be obtained by multiplying a product known to have bounded gaps (for example the 2–apart products) by a rational function. In this section we show how the matrix formulation alluded to in §2 can be used to generate the new quotients from the old.

We suppose that

$$G(x) = \frac{f(x)}{g(x)} F(x) \qquad f(x), g(x) \in \mathbb{K}[x] \setminus \{0\}$$

where

$$F(x) = [c_0\,, c_1\,, c_2\,, \ldots], \qquad G(x) = [b_0\,, b_1\,, b_2\,, \ldots]$$

so that in terms of the matrix equivalence

$$R^{b_0}L^{b_1}R^{b_2}\cdots=\begin{pmatrix} f & 0\\ 0 & g\end{pmatrix}R^{c_0}L^{c_1}R^{c_2}\cdots.$$

Hence we may obtain the b_i's from the c_i's by passing the multiplying matrix through the R^c's and L^c's term by term. At each stage this amounts to multiplying each R^c or L^c on the left by a matrix of determinant fg of the form

$$\begin{pmatrix} A & 0\\ C & B\end{pmatrix}\quad\text{or}\quad\begin{pmatrix} C & B\\ A & 0\end{pmatrix}=J\begin{pmatrix} A & 0\\ C & B\end{pmatrix},\qquad J=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}$$

where $\deg C<\deg A$. By applying Euclid's algorithm to the rows of the resulting matrix (that is to say by writing

$$\begin{pmatrix} \alpha & \beta\\ \gamma & \delta\end{pmatrix}=L^d\begin{pmatrix} \alpha & \beta\\ \gamma-d\alpha & \delta-d\beta\end{pmatrix},\quad\text{or}\quad R^{d'}\begin{pmatrix} \alpha-d'\gamma & \beta-d'\delta\\ \gamma & \delta\end{pmatrix}$$

where $d=\lfloor\delta/\beta\rfloor$ or $d'=\lfloor\beta/\delta\rfloor$ and so on) we output a successsion of $R^{d'}$'s and L^d's until we are left once again with a matrix

$$\begin{pmatrix} A' & 0\\ C' & B'\end{pmatrix}\quad\text{or}\quad\begin{pmatrix} C' & B'\\ A' & 0\end{pmatrix}\quad\text{with }\deg C'<\deg A'.$$

In the special case that all the c_i's $i\geq 1$ have $\deg c_i>\deg fg$ (as will be the case in all of our examples) the procedure can be described fairly concretely:

As $\deg c_0$ may well be zero the first transduction can be atypical. Denoting by p_i/q_i the convergents to $c_0f/g=[d_0\,,d_1\,,\ldots,d_r]$ where $c_0f=\alpha(x)p_r$ and $g=\alpha(x)q_r$ we see that

$$\begin{aligned}\begin{pmatrix} f & 0\\ 0 & g\end{pmatrix}R^{c_0}&=\begin{pmatrix} p_r & p_{r-1}\\ q_r & q_{r-1}\end{pmatrix}\begin{pmatrix} (-1)^{r-1}q_{r-1}f & \alpha\\ (-1)^n q_r f & 0\end{pmatrix}\\ &=R^{d_0}L^{d_1}R^{d_2}\cdots L^{d_n}J\begin{pmatrix} A & 0\\ C & B\end{pmatrix}\qquad\text{if } n \text{ is even}\\ &\qquad\text{or}\quad R^{d_0}L^{d_1}R^{d_2}\cdots R^{d_n}\begin{pmatrix} A & 0\\ C & B\end{pmatrix}\qquad\text{if } n\text{ is odd}\end{aligned}$$

where $\deg C<\deg A$.

The general transition through an L^c is particularly simple:

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} L^{c_i} = L^{\lfloor \frac{Bc_i+C}{A} \rfloor} \begin{pmatrix} A & 0 \\ C' & B \end{pmatrix}$$

$$J \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} L^{c_i} = R^{\lfloor \frac{Bc_i+C}{A} \rfloor} J \begin{pmatrix} A & 0 \\ C' & B \end{pmatrix}$$

with $\deg C' = \deg \left(Bc_i + C - A \left\lfloor \frac{Bc_i+C}{A} \right\rfloor\right) < \deg A$.

The general passage through an R^c is a little more complicated and breaks down most naturally into two parts:

We first let p_i/q_i denote the convergents to $A/C = [e_0\,, e_1\,, \ldots, e_t]$ where $A = \mu(x)p_t$ and $C = \mu(x)q_t$ and observe that

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} R^{c_i} = \begin{pmatrix} p_t & p_{t-1} \\ q_t & q_{t-1} \end{pmatrix} \begin{pmatrix} \mu & u \\ 0 & v \end{pmatrix}$$

where $u = \mu c_i + (-1)^t B p_{t-1}$ and $v = (-1)^{t-1} B p_t$; of course there is the convention that

$$\begin{pmatrix} p_{-1} & p_{-2} \\ q_{-1} & q_{-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so that if $C = 0$ we take $t = -1$ and do nothing. Hence if we let p'_i/q'_i denote the convergents to $u/v = [h_0\,, h_1\,, \ldots, h_s]$ with $u = \lambda(x)p'_s$ and $v = \lambda(x)q'_s$ we obtain

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} R^{c_i} = \begin{pmatrix} p_t & p_{t-1} \\ q_t & q_{t-1} \end{pmatrix} \begin{pmatrix} p'_s & p'_{s-1} \\ q'_s & q'_{s-1} \end{pmatrix} \begin{pmatrix} (-1)^{s-1}\mu q'_{s-1} & \lambda \\ (-1)^s \mu q'_s & 0 \end{pmatrix}.$$

Thus

$$\begin{aligned} \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} R^{c_i} &= R^{e_0} L^{e_1} \cdots R^{h_{s-1}} L^{h_s} J \begin{pmatrix} A' & 0 \\ C' & B' \end{pmatrix} && \text{if } s+t \text{ is even} \\ &= R^{e_0} L^{e_1} \cdots L^{h_{s-1}} R^{h_s} \begin{pmatrix} A' & 0 \\ C' & B' \end{pmatrix} && \text{if } s+t \text{ is odd} \end{aligned}$$

and

$$\begin{aligned} J \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} R^{c_i} &= L^{e_0} R^{e_1} \cdots L^{h_{s-1}} R^{h_s} \begin{pmatrix} A' & 0 \\ C' & B' \end{pmatrix} && \text{if } s+t \text{ is even} \\ &= L^{e_0} R^{e_1} \cdots R^{h_{s-1}} L^{h_s} J \begin{pmatrix} A' & 0 \\ C' & B' \end{pmatrix} && \text{if } s+t \text{ is odd} \end{aligned}$$

where $\deg C' < \deg A'$.

So as we pass through successive R's and L's the R's of the output are always followed by L's and vice versa. Moreover due to the condition $\deg c_i > \deg fg = \deg AB$ we see that (with the possible exception of the very first output d_0) all our outputs are of degree greater then or equal to one. Hence in this case we are indeed obtaining (term by term) the legitimate expansion $b_0, b_1, \ldots$ of $G(x)$. Since our output is never less then our input we can (by feeding the output back in as input) use this approach on $F(x)$ satisfying a functional equation of the form:

$$F(x) = \frac{f(x)}{g(x)} F(x^\lambda) \qquad \lambda > \deg fg.$$

We shall begin the next section with just such an example.

Of course the described process still functions if we allow the possibility of input quotients with $\deg c_i = \deg fg$. In that case the output is still sustainable but may well now contain rogue constant terms — although techniques do exist for individually removing such illegalities relying on the observations that

$$\begin{aligned}[][\ldots, a(x), 1, b(x), c(x), d(x), \ldots] = \\ = [\ldots, a(x)+1, -b(x)-1, -c(x), -d(x), \ldots]\end{aligned}$$

and

$$\begin{aligned}k[a(x), k, b(x), c(x), d(x), \ldots] = \\ = [ka(x), 1, kb(x), k^{-1}c(x), kd(x), \ldots].\end{aligned}$$

Input containing terms c_i with $\deg c_i < \deg(fg)$ is more problematical; not only is the above program no longer quite suitable but there genuinely seems no way of preventing the possibility of R's followed R's or L's by L's so that (with the danger of wholescale back-tracking through cancellation) we no longer have any right to assume that the output we produce from a finite number of transductions bears much relation to the final expansion.

Note that with the exception of the two possibly large quotients $\lfloor \frac{Bc_i + C}{A} \rfloor$ and $h_0 = \lfloor u/v \rfloor$ the remaining quotients produced in passing through an $L^{c_i} R^{c_{i+1}}$ are (by considering the degree of the denominators C and v) of limited degree; a phenomenon that we have already encountered in the case described in Theorem 1. In fact since the sum of their degrees is at most $2 \deg fg$ we see that each $L^{c_i} R^{c_{i+1}}$ produces at most $2 \deg fg + 2$ of output. More careful analysis of common factors could no doubt be used to recover the bound of Theorem 1 in this way. When we are interested in truncations we can (by considering the potentially large degree of the subsequent partial quotient) use these observations to locate their position in the output; if the truncation p_n/q_n is (as indicated) the $(n+1)$-st convergent to $F(x)$ then when n is even the expansion for the truncation convergent fp_n/gq_n to $G(x)$ corresponds simply to the output generated by passing through $R^{c_0}L^{c_1}\cdots R^{c_n}$. When n is odd we need in addition the $e_0, e_1, \ldots, e_t$ produced (when $C \neq 0$) in partially passing through $R^{c_{n+1}}$. Of course the former situation is preferable and indeed when $F(x)$ is our familiar 2–apart product we can arrange for the truncations to be exactly the 'even convergents' (at the expense of a tolerably illegal and in some ways preferable expansion as we shall see below).

Although the cases are somewhat limited in form we are faced with potentially infinitely many different possibilities for the transition matrix $\begin{pmatrix} A & 0 \\ C & B \end{pmatrix}$. We would hope that the transduction process produces, if not a finite set, then at least only matrices of a similar enough shape that we can avoid simply performing transitions ad infinitum. In the next section we see this technique in action.

5. The Matrices in Action

As mentioned earlier, it was shown in [2] that any infinite product of the form

$$F(x;\lambda) = \prod_{i=0}^{\infty}(1 + x^{-\lambda_i})$$

where $\lambda = (\lambda_0, \lambda_1, \ldots)$ satisfies

$$\lambda_{i+1} > 2(\lambda_0 + \lambda_1 + \cdots + \lambda_i) \quad \text{and} \quad 2\lambda_i \mid \lambda_{i+1} \quad \text{for all } i \geq 0,$$

has truncations 2–apart with expansion

$$F(x;\lambda) = [b_0(x;\lambda)\,, b_1(x;\lambda)\,, b_2(x;\lambda)\,, \ldots]$$

where

$$b_0(x;\lambda) = 1, \qquad b_1(x;\lambda) = x^{\lambda_0}, \qquad b_2(x;\lambda) = -\frac{(x^{\lambda_1-\lambda_0}+1)}{(x^{\lambda_0}+1)},$$

and

$$b_{2n+1}(x;\lambda) = -x^{\lambda_0+\cdots+\lambda_{n-1}} \prod_{i=0}^{n-1}(x^{\lambda_i}+1),$$

$$b_{2n+2}(x;\lambda) = -\left(x^{\lambda_n-(\lambda_0+\cdots+\lambda_{n-1})}(x^{\lambda_{n+1}-2\lambda_n}-1)\right) \Big/ \prod_{i=0}^{n}(x^{\lambda_i}+1).$$

We begin this section by using the method of transduction described above to give an alternative proof of this fact:

From the functional equation

$$F(x;\lambda) = (1+x^{-\lambda_0})F(x;\lambda'), \qquad \lambda' = (\lambda_1,\lambda_2,\ldots)$$

we see that

$$R^{b_0(x;\lambda)}L^{b_1(x;\lambda)}R^{b_2(x;\lambda)}\cdots = \\ = \begin{pmatrix} x^{\lambda_0}+1 & 0 \\ 0 & x^{\lambda_0} \end{pmatrix} R^{b_0(x;\lambda')}L^{b_1(x;\lambda')}R^{b_2(x;\lambda')}\cdots.$$

The trivial observation $b_0(x;\lambda') = 1$ entails that the initial transition yields

$$\begin{pmatrix} x^{\lambda_0}+1 & 0 \\ 0 & x^{\lambda_0} \end{pmatrix} R = RL^{x^{\lambda_0}} \begin{pmatrix} x^{\lambda_0}+1 & 1 \\ -x^{\lambda_0}(x^{\lambda_0}+1) & 0 \end{pmatrix}.$$

Now since $\lambda_i \mid \lambda_{i+1}$ we see that $F(x;\lambda')$ is a polynomial in x^{λ_1} and so possesses partial quotients with $\deg b_i(x;\lambda') \geq \lambda_1 > 2\lambda_0$ for all $i \geq 1$. Hence from our previous comments we are allowed to read off the $b_i(x;\lambda)$ directly from our output as it is generated term by term. Thus we obtain $b_0(x;\lambda) = 1, b_1(x;\lambda) = x^{\lambda_0}$ and we proceed to the next transition armed with the knowledge that $b_1(x;\lambda') = x^{\lambda_1}$ and the assumption that $2\lambda_0 \mid \lambda_1$:

$$\begin{pmatrix} x^{\lambda_0}+1 & 1 \\ -x^{\lambda_0}(x^{\lambda_0}+1) & 0 \end{pmatrix} L^{x^{\lambda_1}} = R^{-(x^{\lambda_1-\lambda_0}+1)/(x^{\lambda_0}+1)} \begin{pmatrix} 1 & 1 \\ -x^{\lambda_0}(x^{\lambda_0}+1) & 0 \end{pmatrix}.$$

Therefore $b_2(x;\lambda) = -(x^{\lambda_1-\lambda_0}+1)/(x^{\lambda_0}+1)$ and at the next stage

$$\begin{pmatrix} 1 & 1 \\ -x^{\lambda_0}(x^{\lambda_0}+1) & 0 \end{pmatrix} R^{-(x^{\lambda_2-\lambda_1}+1)/(x^{\lambda_1}+1)} \\ = L^{-x^{\lambda_0}(x^{\lambda_0}+1)} R^{-x^{\lambda_1-\lambda_0}(x^{\lambda_2-2\lambda_1}-1)/(x^{\lambda_1}+1)(x^{\lambda_0}+1)} \begin{pmatrix} 1 & 0 \\ 0 & x^{\lambda_0}(x^{\lambda_0}+1) \end{pmatrix}.$$

Hence $b_3(x;\lambda)$ and $b_4(x;\lambda)$ are of the stated form.

From this point onwards the transition matrix moves through the R's and L's unchanged. In particular under the assumption (as we have just seen for $N = 1$) that $x^{\lambda_0}(x^{\lambda_0}+1)$ divides $b_{2N+2}(x;\lambda')$ we obtain at the N-th subsequent stage

$$\begin{pmatrix}1 & 0\\ 0 & x^{\lambda_0}(x^{\lambda_0}+1)\end{pmatrix} L^{b_{2N+1}(x;\lambda')} = \\ = L^{x^{\lambda_0}(x^{\lambda_0}+1)b_{2N+1}(x;\lambda')}\begin{pmatrix}1 & 0\\ 0 & x^{\lambda_0}(x^{\lambda_0}+1)\end{pmatrix}$$

and

$$\begin{pmatrix}1 & 0\\ 0 & x^{\lambda_0}(x^{\lambda_0}+1)\end{pmatrix} R^{b_{2N+2}(x;\lambda')} = \\ = R^{b_{2N+2}(x;\lambda')/x^{\lambda_0}(x^{\lambda_0}+1)}\begin{pmatrix}1 & 0\\ 0 & x^{\lambda_0}(x^{\lambda_0}+1)\end{pmatrix}.$$

A simple induction argument leads at once to the relations

$$b_{2N+3}(x;\lambda) = x^{\lambda_0}(x^{\lambda_0}+1)b_{2N+1}(x;\lambda')$$
$$b_{2N+4}(x;\lambda) = b_{2N+2}(x;\lambda')/x^{\lambda_0}(x^{\lambda_0}+1)$$

and the stated formulae for $b_{2N+1}(x;\lambda)$ and $b_{2N+2}(x;\lambda)$. Our previous comments or simple degree considerations show us that the truncations are always 2–apart.

From now on we will assume such an expansion which (in the manner of [2]) it will prove convenient to write in the slightly illegal form

$$F(x;\lambda) = [1\,,\, x^{\lambda_0}-1\,,\, 1\,,\, c_3\,,\, c_4\,,\, c_5\,,\, \ldots]$$

with

$$c_{2n+2} = x^{\lambda_0+\cdots+\lambda_{n-1}}\prod_{i=0}^{n-1}(x^{\lambda_0}+1)$$

$$c_{2n+1} = x^{\lambda_{n-1}-(\lambda_0+\cdots+\lambda_{n-2})}(x^{\lambda_n-2\lambda_{n-1}}-1)\Big/\prod_{i=0}^{n-1}(x^{\lambda_i}+1).$$

We use this to generate the quotients of

$$G(x) = [a_0\,,\, a_1\,,\, a_2\,,\, \ldots]$$

where

$$G(x)=(1+x^{-r})\prod_{i=0}^{\infty}(1+x^{-\lambda_i}),\qquad (r,\lambda_0)=1 \tag{11}$$

with

$$\lambda_i>2(r+\lambda_0+\cdots+\lambda_{i-1})\quad\text{and}\quad 2\lambda_i\mid\lambda_{i+1}\quad\text{for all } i\geq 0\,. \tag{12}$$

From the functional equation

$$G(x)=(1+x^{-r})F(x;\lambda)$$

we obtain the matrix correspondence

$$R^{a_0}L^{a_1}R^{a_2}\cdots=\begin{pmatrix}x^r+1 & 0\\ 0 & x^r\end{pmatrix}RL^{x^{\lambda_0}-1}RL^{c_3}R^{c_4}\cdots$$

where the c_i all are polynomials in x^{λ_0} and hence of degree $\deg c_i\geq\lambda_0>2r$. In particular (from our earlier analysis) it is clear that (once we have safely passed through the initial $RL^{x^{\lambda_0}-1}R$) we are allowed to read off the a_i from the output, transition by transition. Of course in general we cannot expect our transition matrix to remain constant as occurred in the above example; however we can hope that it changes in a fairly predictable manner.

THEOREM 3. *Let*

$$G(x)=(1+x^{-1})\prod_{i=0}^{\infty}(1+x^{-\lambda_i}),\qquad(\lambda_0,2)=1$$

with λ_i satisfying (12). *Then the truncations of G are eventually 4-apart (the initial gap is 3) with partial quotients a_i given by*

$$\begin{aligned}a_{4n+1}&=\tfrac{1}{16}\big(c_{2n+1}(x)+xc_{2n+1}(-1)\big)\;/\;x(x+1)\\ a_{4n+2}&=-2^{n+3}\,(\lambda_0/\lambda_n)\,(x+1)\\ a_{4n+3}&=-\left(\lambda_n/2^{n+1}\lambda_0\right)^2 xc_{2n+2}(x)\;/\;(x+1)\\ a_{4n+4}&=2^{n+3}\,(\lambda_0/\lambda_n)\,(x+1)\end{aligned}$$

for $n \geq 1$. The first 5 irregular partial quotients are given implicitly below.

Proof. We first note the general matrix transition formulae

$$\begin{pmatrix} Ax(x+1) & 0 \\ Bx & D \end{pmatrix} L^{c_{2n+1}} =$$
$$= L^{\frac{D}{A}(c_{2n+1}(x)+xc_{2n+1}(-1))/x(x+1)} \begin{pmatrix} Ax(x+1) & 0 \\ (B - Dc_{2n+1}(-1))x & D \end{pmatrix},$$

$$\begin{pmatrix} Ax(x+1) & 0 \\ Bx & D \end{pmatrix} R^{c_{2n+2}} =$$
$$= R^{\frac{A}{B}(x+1)} L^{-\frac{B^2}{AD}xc_{2n+2}/(x+1)} R^{-\frac{A}{B}(x+1)} \begin{pmatrix} Ax(x+1) & 0 \\ Bx & D \end{pmatrix}.$$

Passing the multiplier through the first three terms gives us the first five partial quotients $a_0, a_1, \ldots, a_4$:

$$\begin{pmatrix} x+1 & 0 \\ 0 & x \end{pmatrix} RL^{x^{\lambda_0}-1} R =$$
$$= RL^x R^{-(x^{\lambda_0-1}-1)/(x+1)} L^{-\frac{1}{4}(2x+1)} R^{4(2x+1)} \begin{pmatrix} 4x(x+1) & 0 \\ -\frac{1}{2}x & \frac{1}{4} \end{pmatrix}.$$

Now applying the above formulae with $A = 4$, $D = \frac{1}{4}$, $B_0 = -\frac{1}{2}$ we see that passage through the n-th block $L^{c_{2n+1}} R^{c_{2n+2}}$ takes the form

$$\begin{pmatrix} 4x(x+1) & 0 \\ B_{n-1}x & \frac{1}{4} \end{pmatrix} L^{c_{2n+1}} R^{c_{2n+2}} =$$
$$L^{(c_{2n+1}(x)+xc_{2n+1}(-1))/16x(x+1)} R^{4(x+1)/B_n} \times$$
$$\times L^{-B_n^2 xc_{2n+2}(x)/(x+1)} R^{-4(x+1)/B_n} \begin{pmatrix} 4x(x+1) & 0 \\ B_n x & \frac{1}{4} \end{pmatrix}$$

with $B_n = B_{n-1} - \frac{1}{4}c_{2n+1}(-1) = -\left(\frac{1}{2}\right)^{n+1}(\lambda_n/\lambda_0)$. Hence the alleged partial quotients. Moreover by previous comments or by simple degree considerations we see that the truncations coincide exactly with the finite continued fraction expansions produced by these transitions and hence are all (bar the first) four–apart.

THEOREM 4. *Let*

$$G(x)=(1+x^{-2})\prod_{i=0}^{\infty}(1+x^{-\lambda_i}), \qquad (\lambda_0,2)=1$$

with λ_i *satisfying* (12). *Then*

(*i*) *if* $4 \mid \lambda_1$ *then the truncations are eventually* 2*–apart (initial gap* 5*) with partial quotients*

$$a_{2n+6}=\tfrac{1}{16}x^2(x^2+1)c_{2n+2}\,, \qquad a_{2n+7}=16c_{2n+3}/x^2(x^2+1)\,,$$

for $n\ge 1$.

(*ii*) *if* $4\nmid\lambda_1$ *then the truncations are eventually* 6*–apart (initial gap* 5*) with partial quotients*

$$\begin{aligned}
a_{6n+7} &= 16\{c_{2n+3}(x)+(\lambda_{n+1}-2\lambda_n/2^n\lambda_1)\,x^2(\varepsilon x-1)\}\;/\;x^2(x^2+1)\\
a_{6n+8} &= -\tfrac{1}{16}2^n\,(\lambda_1/\lambda_{n+1})\,(\varepsilon x+1)\\
a_{6n+9} &= -8\left(\tfrac{1}{2}\right)^n(\lambda_{n+1}/\lambda_1)\,(\varepsilon x-1)\\
a_{6n+10} &= \tfrac{1}{4}(x^2c_{2n+4}(x))/(x^2+1)\\
a_{6n+11} &= 8\left(\tfrac{1}{2}\right)^n(\lambda_{n+1}/\lambda_1)\,(\varepsilon x-1)\\
a_{6n+12} &= \tfrac{1}{16}2^n\,(\lambda_1/\lambda_{n+1})\,(\varepsilon x+1)
\end{aligned}$$

for $n\ge 1$ *where*

$$\varepsilon=\begin{cases}1 & \textit{if } \lambda_0\equiv 3 \pmod 4\,,\\ -1 & \textit{if } \lambda_0\equiv 1 \pmod 4\,.\end{cases}$$

The first 8 *or* 13 *partial quotients can be discovered below.*

Proof. In both cases the initial transitions produce the first 7 quotients

$$\begin{pmatrix}x^2+1 & 0\\ 0 & x^2\end{pmatrix}RL^{x^{\lambda_0}-1}R=RL^{x^2}R^{-x\left(x^{\lambda_0-3}-\varepsilon\right)/(x^2+1)}\times$$
$$\times\,L^{-\varepsilon x+1}R^{-\frac{1}{2}\varepsilon x-\frac{3}{4}}L^{-8\varepsilon x+12}R^{-\frac{1}{16}(\varepsilon x+1)}\begin{pmatrix}-\frac{1}{4}x^2(x^2+1) & 0\\ 4x^2(1-\varepsilon x) & -4\end{pmatrix}.$$

In the first case when $4\mid\lambda_1$ we need only one further atypical transition

$$\begin{pmatrix}-\frac{1}{4}x^2(x^2+1) & 0\\ 4x^2(1-\varepsilon x) & -4\end{pmatrix}L^{c_3}=$$
$$=L^{-16\{x^2(1-\varepsilon x)-c_3(x)\}/x^2(x^2+1)}\begin{pmatrix}-\frac{1}{4}x^2(x^2+1) & 0\\ 0 & -4\end{pmatrix}$$

after which the transition matrix remains unchanged with

$$\begin{pmatrix} Ax^2(x^2+1) & 0 \\ 0 & B \end{pmatrix} R^{c_{2n+2}} = R^{Ax^2(x^2+1)c_{2n+2}/B} \begin{pmatrix} Ax^2(x^2+1) & 0 \\ 0 & B \end{pmatrix}$$
$$\begin{pmatrix} Ax^2(x^2+1) & 0 \\ 0 & B \end{pmatrix} L^{c_{2n+3}} = L^{Bc_{2n+3}/Ax^2(x^2+1)} \begin{pmatrix} Ax^2(x^2+1) & 0 \\ 0 & B \end{pmatrix}$$

for $n \geq 1$. The stated quotients are then immediate.

When $4 \nmid \lambda_1$ we require two further irregular transitions

$$\begin{pmatrix} -\frac{1}{4}x^2(x^2+1) & 0 \\ 4x^2(1-\varepsilon x) & -4 \end{pmatrix} L^{c_3} R^{c_4} =$$
$$= L^{16c_3(x)/x^2(x^2+1)} R^{\frac{1}{16}(\varepsilon x+1)} L^{-8(1-\varepsilon x)} \times$$
$$\times R^{\frac{1}{4}\{x^2 c_4(x)-(\varepsilon x+1)\}/(x^2+1)} L^{8(\varepsilon x-1)} R^{\frac{1}{16}(\varepsilon x+1)} \begin{pmatrix} -\frac{1}{4}x^2(x^2+1) & 0 \\ 4x^2(\varepsilon x-1) & -4 \end{pmatrix}$$

before the process settles down with

$$\begin{pmatrix} Ax^2(x^2+1) & 0 \\ B_{n-1}x^2(\varepsilon x-1) & C \end{pmatrix} L^{c_{2n+1}} =$$
$$L^{\{Cc_{2n+1}-(B_n-B_{n-1})x^2(\varepsilon x-1)\}/Ax^2(x^2+1)} \begin{pmatrix} Ax^2(x^2+1) & 0 \\ B_n x^2(\varepsilon x-1) & C \end{pmatrix},$$

$$\begin{pmatrix} Ax^2(x^2+1) & 0 \\ Bx^2(\varepsilon x-1) & C \end{pmatrix} R^{c_{2n+2}} = R^{\frac{A}{B}(\varepsilon x+1)} L^{\frac{B}{2A}(\varepsilon x-1)} \times$$
$$\times R^{4Ax^2 c_{2n+2}/C(x^2+1)} L^{-\frac{B}{2A}(\varepsilon x-1)} R^{-\frac{A}{B}(\varepsilon x+1)} \begin{pmatrix} Ax^2(x^2+1) & 0 \\ Bx^2(\varepsilon x-1) & C \end{pmatrix}$$

for $n \geq 2$, where

$$B_n = B_{n-1} - C2^{-(n-1)}(\lambda_n - 2\lambda_{n-1})/\lambda_1 .$$

The given partial quotients then follow fairly easily.

THEOREM 5. *Let*

$$G(x) = (1+x^{-3}) \prod_{i=0}^{\infty} (1+x^{-\lambda_i}), \qquad 2 \mid \lambda_0$$

with λ_i satisfying (12).

(*i*) *If* $(3,\lambda_i)=1$ *for all* i *then the truncations of* G *all are* 5*–apart with partial quotients*

$$a_{5n+2}=\begin{cases} (-1)^{n+1}\dfrac{c_{2n+1}(x)}{x^3(x+1)} & \text{if } \lambda_{n-1}\not\equiv\lambda_n \pmod 6\\ -\dfrac{\{x^4(x+1)+(-1)^n c_{2n+1}(x)\}}{x^3(x^3+1)} & \text{if } \lambda_{n-1}\equiv 2, \lambda_n\equiv 2 \pmod 6\\ \dfrac{(x^4(x+1)+(-1)^{n+1}c_{2n+1}(x))}{x^3(x^3+1)} & \text{if } \lambda_{n-1}\equiv 4, \lambda_n\equiv 4 \pmod 6\end{cases}$$

$$a_{5n+3}=\begin{cases} x & \text{if } \lambda_n\equiv 2 \pmod 6\\ -(x^2-x+1) & \text{if } \lambda_n\equiv 4 \pmod 6\end{cases}$$

$$a_{5n+4}=\begin{cases} x-1 & \text{if } \lambda_n\equiv 2 \pmod 6\\ \dfrac{(1+(-1)^n x^3 c_{2n+2})(x+1)}{(x^2-x+1)} & \text{if } \lambda_n\equiv 4 \pmod 6\end{cases}$$

$$a_{5n+5}=\begin{cases} -\dfrac{(1+(-1)^n x^3 c_{2n+2})(x+1)}{(x^2-x+1)} & \text{if } \lambda_n\equiv 2 \pmod 6\\ -(x-1) & \text{if } \lambda_n\equiv 4 \pmod 6\end{cases}$$

$$a_{5n+6}=\begin{cases} x^2-x+1 & \text{if } \lambda_n\equiv 2 \pmod 6\\ -x & \text{if } \lambda_n\equiv 4 \pmod 6\end{cases}$$

for $n\ge 1$. *The first seven atypical partial quotients can be found below.*

(*ii*) *If* $3\mid\lambda_N$ *for some* $N\ge 1$ *(where we take* N *to be minimal) then the truncations of* $G(x)$ *are eventually (after the* N*'th truncation)* 2*–apart with partial quotients*

$$a_{5N+2n+1}=\begin{cases} x^3(x^3+1)c_{2(N+n)}(x) & \text{if } N \text{ is odd}\\ -x^3(x^3+1)c_{2(N+n)}(x) & \text{if } N \text{ is even}\end{cases}$$

$$a_{5N+2n+2}=\begin{cases} c_{2(N+n)+1}(x)/x^3(x^3+1) & \text{if } N \text{ is odd}\\ -c_{2(N+n)+1}(x)/x^3(x^3+1) & \text{if } N \text{ is even}\end{cases}$$

for $n\ge 1$. *The first* $(5N+2)$ *quotients are as in case* (*i*), *and* a_{5N+2} *can be found below.*

Proof. If $3\not|\,\lambda_i$ for any i we experience four basic transition matrices:

$$A=\begin{pmatrix} x^3(x^3+1) & 0\\ x^3(x^2-1) & 1\end{pmatrix},\qquad B=\begin{pmatrix} x^3(x^3+1) & 0\\ -x^3(x+1) & 1\end{pmatrix},$$

$$C=\begin{pmatrix} -x^3(x^2-1) & 1\\ -x^3(x^3+1) & 0\end{pmatrix},\qquad D=\begin{pmatrix} x^3(x+1) & 1\\ -x^3(x^3+1) & 0\end{pmatrix}.$$

After the initial transitions

$$\begin{pmatrix} x^3+1 & 0 \\ 0 & x^3 \end{pmatrix} RL^{x^{\lambda_0}-1}R =$$

$$RL^{x^3}R^{-x\left(\frac{x^{\lambda_0-4}-1}{x^3+1}\right)}L^{-(x^2-x+1)}R^{(x+1)^2}L^{-(x-1)}R^{-x}A \quad \text{if } \lambda_0 \equiv 4 \pmod 6$$

$$RL^{x^3}R^{-x^2\left(\frac{x^{\lambda_0-5}+1}{x^3+1}\right)}L^{x}R^{(x-1)}L^{-(x+1)^2}R^{(x^2-x+1)}B \quad \text{if } \lambda_0 \equiv 2 \pmod 6$$

we have the following identities:

If n is odd

$$\begin{array}{ll}
AL^{c_{2n+1}}=L^{c_{2n+1}/x^3(x^3+1)}A & \text{if } \lambda_{n-1}\equiv 4, \quad \lambda_n \equiv 2 \pmod 6 \\
AL^{c_{2n+1}}=L^{\{c_{2n+1}+x^4(x+1)\}/x^3(x^3+1)}B & \text{if } \lambda_{n-1}\equiv 4, \quad \lambda_n \equiv 4 \pmod 6 \\
BL^{c_{2n+1}}=L^{\{c_{2n+1}-x^4(x+1)\}/x^3(x^3+1)}A & \text{if } \lambda_{n-1}\equiv 2, \quad \lambda_n \equiv 2 \pmod 6 \\
BL^{c_{2n+1}}=L^{c_{2n+1}/x^3(x^3+1)}B & \text{if } \lambda_{n-1}\equiv 2, \quad \lambda_n \equiv 4 \pmod 6
\end{array}$$

and

$$\begin{aligned}
AR^{c_{2n+2}} &= R^{x}L^{(x-1)}R^{(x+1)(x^3c_{2n+2}-1)/(x^2-x+1)}L^{(x^2-x+1)}D \\
BR^{c_{2n+2}} &= R^{-(x^2-x+1)}L^{-(x+1)(x^3c_{2n+2}-1)/(x^2-x+1)}R^{-(x-1)}L^{-x}C.
\end{aligned}$$

If n is even

$$\begin{array}{ll}
DL^{c_{2n+1}}=R^{-c_{2n+1}/x^3(x^3+1)}D & \text{if } \lambda_{n-1}\equiv 2, \quad \lambda_n \equiv 4 \pmod 6 \\
DL^{c_{2n+1}}=R^{-\{c_{2n+1}+x^4(x+1)\}/x^3(x^3+1)}C & \text{if } \lambda_{n-1}\equiv 2, \quad \lambda_n \equiv 2 \pmod 6 \\
CL^{c_{2n+1}}=R^{-\{c_{2n+1}-x^4(x+1)\}/x^3(x^3+1)}D & \text{if } \lambda_{n-1}\equiv 4, \quad \lambda_n \equiv 4 \pmod 6 \\
CL^{c_{2n+1}}=R^{-c_{2n+1}/x^3(x^3+1)}C & \text{if } \lambda_{n-1}\equiv 4, \quad \lambda_n \equiv 2 \pmod 6
\end{array}$$

and

$$\begin{aligned}
CR^{c_{2n+2}} &= L^{x}R^{(x-1)}L^{-(x+1)(x^3c_{2n+2}+1)/(x^2-x+1)}R^{(x^2-x+1)}B \\
DR^{c_{2n+2}} &= L^{-(x^2-x+1)}L^{(x+1)(x^3c_{2n+2}+1)/(x^2-x+1)}L^{-(x-1)}R^{-x}A.
\end{aligned}$$

So each time we pass through a block $L^{c_{2n+1}}R^{c_{2n+2}}$ we output 5 partial quotients and if n is even (respectively odd) either an A (respectively C) if $\lambda_n \equiv 4 \pmod 6$ or a B (respectively D) if $\lambda_n \equiv 2 \pmod 6$. The stated quotients can be read off and then degree considerations confirm that the truncations do indeed occur as every fifth convergent.

Now if $3 \mid \lambda_N$ for some $N \geq 1$ (assumed minimal) then the process remains the same until we hit $L^{c_{2N+1}}$. In particular the $a_0, a_1, \ldots, a_{5N+1}$ are generated as above. Writing

$$E = \begin{pmatrix} x^3(x^3+1) & 0 \\ 0 & 1 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 1 \\ -x^3(x^3+1) & 0 \end{pmatrix}$$

we see that

$$\begin{aligned} AL^{c_{2N+1}} &= L^{\{c_{2N+1}+x^3(x^2-1)\}/x^3(x^3+1)}E \\ &\qquad \text{if } N \text{ is odd and } \lambda_{N-1} \equiv 4 \pmod 6, \\ BL^{c_{2N+1}} &= L^{\{c_{2N+1}-x^3(x+1)\}/x^3(x^3+1)}E \\ &\qquad \text{if } N \text{ is odd and } \lambda_{N-1} \equiv 2 \pmod 6, \\ CL^{c_{2N+1}} &= R^{-\{c_{2N+1}-x^3(x^2-1)\}/x^3(x^3+1)}F \\ &\qquad \text{if } N \text{ is even and } \lambda_{N-1} \equiv 4 \pmod 6, \\ DL^{c_{2N+1}} &= R^{-\{c_{2N+1}+x^3(x+1)\}/x^3(x^3+1)}F \\ &\qquad \text{if } N \text{ is even and } \lambda_{N-1} \equiv 2 \pmod 6, \end{aligned}$$

after which the transition matrix remains unchanged with

$$\begin{aligned} ER^{c_{2n+2}} &= R^{x^3(x^3+1)c_{2n+2}}E & FR^{c_{2n+2}} &= L^{-x^3(x^3+1)c_{2n+2}}F \\ EL^{c_{2n+3}} &= L^{c_{2n+3}/x^3(x^3+1)}E & FL^{c_{2n+3}} &= R^{-c_{2n+3}/x^3(x^3+1)}F \end{aligned}$$

for $n \geq N$. The stated partial quotients are then evident and degree considerations or otherwise show that the truncations are indeed 2–apart as claimed.

THEOREM 6. *Let*

$$G(x) = (1+x^{-3})\prod_{i=0}^{\infty}(1+x^{-\lambda_i}), \qquad 2 \nmid \lambda_0, \quad 6 \mid \lambda_1$$

with λ_i satisfying (12). *The the truncations of G are eventually 4–apart (the initial gap is 6) with partial quotients*

$$\begin{aligned} a_{4n+5} &= \left(2^{n-1}3\lambda_0/\lambda_n\right) B^{-2}(x+1) \\ a_{4n+6} &= \left(\lambda_n/2^{n-1}3\lambda_0\right)^2 B^2x^3(x^2-x+1)c_{2n+2}(x)/(x+1) \\ a_{4n+7} &= -\left(2^{n-1}3\lambda_0/\lambda_n\right) B^{-2}(x+1) \\ a_{4k+8} &= -B^2\{c_{2n+3}(x) + \frac{1}{3}x^3(x^2-x+1)c_{2n+3}(-1)\}/x^3(x^3+1) \end{aligned}$$

for $n \geq 1$ *where*

$$B = \begin{cases} \frac{1}{4} & \text{if } \lambda_0 \equiv 1 \pmod 6 \\ 4 & \text{if } \lambda_0 \equiv 5 \pmod 6). \end{cases}$$

The first nine partial quotients $a_0, \ldots, a_8$ *can be found below.*

Proof. If λ_0 is odd and $6 \mid \lambda_1$ then we have the following transition relations:

$$\begin{pmatrix} Ax^3(x^2-x+1) & B \\ Cx^3(x^3+1) & 0 \end{pmatrix} R^{c_{2n+2}} =$$

$$L^{\frac{C}{A}(x+1)} R^{-\frac{A^2}{BC}(x^3(x^2-x+1)c_{2n+2})/(x+1)} L^{-\frac{C}{A}(x+1)} \begin{pmatrix} Ax^3(x^2-x+1) & B \\ Cx^3(x^3+1) & 0 \end{pmatrix}$$

for $n \geq 1$ and

$$\begin{pmatrix} Ax^3(x^2-x+1) & B \\ Cx^3(x^3+1) & 0 \end{pmatrix} L^{c_{2n+1}} =$$

$$R^{\frac{B}{C}(c_{2n+1}(x)+\frac{1}{3}x^3(x^2-x+1)c_{2n+1}(-1))/x^3(x^3+1)} \begin{pmatrix} Dx^3(x^2-x+1) & B \\ Cx^3(x^3+1) & 0 \end{pmatrix}$$

for $n \geq 2$ with $D = A - \frac{1}{3}Bc_{2n+1}(-1)$.

Here we have to endure some rather painful initial transitions:

$$\begin{pmatrix} x^3+1 & 0 \\ 0 & x^3 \end{pmatrix} RL^{x^{\lambda_0}-1} RL^{c_3}$$

$$= RL^{x^3} R^{-x\left(\frac{x^{\lambda_0-4}+1}{x^3+1}\right)} L^{(x^2+x+1)} R^{\frac{1}{2}x-\frac{3}{4}} L^{8x-4} R^{\frac{1}{16}(x+1)} L^{-16x} \times$$

$$\times R^{-\frac{1}{16}\{c_3(x)-x^3(x^2+1)-\beta_0 x^3(x^2-x+1)\}/x^3(x^3+1)} \begin{pmatrix} \frac{1}{4}\beta_0 x^3(x^2-x+1) & \frac{1}{4} \\ -4x^3(x^3+1) & 0 \end{pmatrix}$$

if $\lambda_0 \equiv 1 \pmod 6$ and

$$= RL^{x^3} R^{-x^2\left(\frac{x^{\lambda_0-5}-1}{x^3+1}\right)} L^{-x} R^{(x+1)} L^{\frac{1}{2}x-\frac{1}{4}} R^{8x-12} L^{\frac{1}{16}(x^2+x+1)} \times$$

$$\times R^{-16\{c_3(x)+x^3(x-1)-\beta_0 x^3(x^2-x+1)\}/x^3(x^3+1)} \begin{pmatrix} 4\beta_0 x^3(x^2-x+1) & 4 \\ -\frac{1}{4}x^3(x^3+1) & 0 \end{pmatrix}$$

if $\lambda_0 \equiv 5 \pmod 6$, where $\beta_0 = -\frac{1}{3}(\lambda_1/\lambda_0)$.

Hence after passing through the k-th block $R^{c_{2k+2}} L^{c_{2k+3}}$ we output 4 partial quotients and a transition matrix of the form

$$\begin{pmatrix} \beta_k B x^3(x^2-x+1) & B \\ -\frac{1}{B}x^3(x^3+1) & 0 \end{pmatrix},$$

where $\beta_k = \beta_{k-1} - \frac{1}{3}c_{2k+3}(-1) = -\lambda_{k+1}/2^k 3\lambda_0$. This makes the theorem evident.

Notice that we have covered here all cases of (11) with $r = 1$ or 2 and a number of cases when $r = 3$. The remaining cases with $r = 3$ become rather too unwieldy to handle by locating the quotients in this way (when $2 \not| \lambda_0$ and $3 \not| \lambda_i$, not only do we have to force through multiplication by the two polynomials $(x+1)$ and $(x^2 - x + 1)$ but, as we have seen before, the multiple factor $(x+1)^2$ in $(1+x^{-3})(1+x^{-\lambda_0})$ spawns unwieldy constants). These missing cases will be given in §6 and are found by considering convergents, as opposed to evaluating partial quotients.

6. The Convergents Considered

Given the convergents p_n/q_n and p_{n+h}/q_{n+h} and the intermediate partial quotients $a_{n+1}, \dots, a_{n+h}$ it is fairly easy to retrieve the missing convergents p_{n+1}/q_{n+1}, $\dots$, p_{n+h-1}/q_{n+h-1} in the following manner:

We define

$$S(r,m) = p_{r+m}q_r - q_{r+m}p_r. \tag{13}$$

and observe that $S(r,m)$ can be straightforwardly written in terms of a_{r+2}, $\dots$, a_{r+m} by means of the easily proved iteration relation

$$S(r,m) = a_{r+m}S(r,m-1) + S(r,m-2)$$

where, as is well known,

$$S(r,0) = 0 \qquad S(r,1) = (-1)^r.$$

It is not hard to then show (by induction on r) that the intermediate p_{n+r}'s and q_{n+r}'s, $1 \le r \le h-1$ are given by;

$$\begin{aligned} p_{n+r} &= \big(p_{n+h}S(n,r) + p_n S(n+r, h-r)\big)/S(n,h) \\ q_{n+r} &= \big(q_{n+h}S(n,r) + q_n S(n+r, h-r)\big)/S(n,h). \end{aligned} \tag{14}$$

If the reader should object that knowing p_n/q_n and p_{n+h}/q_{n+h} can only determine p_n, q_n and p_{n+h}, q_{n+h} up to a constant we observe that to obtain p_{n+r}/q_{n+r} we need only determine the ratio of the leading coefficients of p_{n+r} and p_n and this is simply the product of the leading coefficients of the $a_{n+1}, \dots, a_{n+h}$.

In fact not only must the missing convergents p_{n+r}/q_{n+r} be of a special form but conversely given polynomials A and B with $\deg AB < \deg S(n,h)$ such that

$$\frac{p}{q} = \frac{p_{n+h}A + p_nB}{q_{n+h}A + q_nB}$$

satisfies

$$\deg q < \deg(q_{n+h}A + q_nB) - \frac{1}{2}\deg(S(n,h)AB)\,,$$

then p/q is an intermediate convergent p_{n+i}/q_{n+i} for some $0 \le i \le h$. This fact is an easy consequence of criterion (6) since

$$\begin{aligned}\deg\left(F - \frac{p}{q}\right) &= \deg\left(\frac{BS(n,h)}{q_{n+h}(q_{n+h}A + q_nB)} + F - \frac{p_{n+h}}{q_{n+h}}\right)\\ &= \deg ABS(n,h) - 2\deg(q_{n+h}A + q_nB)\,,\end{aligned}$$

where clearly

$$\deg S(n,h) = \sum_{i=2}^{h} \deg a_{n+i} = \deg q_{n+h} - \deg q_n - \deg a_{n+1}\,.$$

Hence in order to find the $(n+i)$-th missing convergent it is necessary and sufficient to look for polynomials A, B (sharing no superfluous common factors) with

$$\deg A = \sum_{j=2}^{i} \deg a_{n+j}, \qquad \deg AB < \deg S(n,h)$$

and

$$S(n,h) \mid p_{n+h}A + p_nB, \qquad S(n,h) \mid q_{n+h}A + q_nB$$

giving

$$\frac{p_{n+i}}{q_{n+i}} = \frac{p_{n+h}A + p_nB}{q_{n+h}A + q_nB}$$

with

$$\deg a_{n+i+1} = \deg S(n,h) - \deg AB.$$

The limited form of the intermediate convergents allowed by (14) enables us to determine quite precisely when we can have very small gaps :

THEOREM 7. *Let*

$$G(x)=\prod_{i=0}^{\infty}(1+b_ix^{-\lambda_i}),\qquad b_i\in\mathbb{K}\setminus\{0\}$$

with the $\lambda_i\in\mathbb{N}$ *satisfying*

$$\lambda_{i+1}>2(\lambda_0+\cdots+\lambda_i)\quad\textit{for all}\quad i\geq 0\,.$$

Then the truncation convergents

$$\frac{p_m}{q_m}=\prod_{i=0}^{n-1}(1+b_ix^{-\lambda_i}),\qquad \frac{p_{m+h}}{q_{m+h}}=\prod_{i=0}^{n}(1+b_ix^{-\lambda_i})$$

are

(*i*) 2*–apart if and only if*

$$\prod_{i=0}^{n-1}(x^{\lambda_i}+b_i)\mid(x^{\lambda_n}-b_n)$$

with intermediate convergent

$$\frac{p_{m+1}}{q_{m+1}}=\frac{x^{\lambda_n-\sum_{i=0}^{n-1}\lambda_i}}{(x^{\lambda_n}-b_n)/\prod_{i=0}^{n-1}(x^{\lambda_i}+b_i)}\,.$$

(*ii*) 3*–apart only in the special case*

$$\frac{p_1}{q_1}=\left(1+\frac{1}{X}\right),\qquad \frac{p_4}{q_4}=\left(1+\frac{1}{X}\right)\left(1+\frac{c}{X^\lambda}\right)\quad c\neq(-1)^\lambda,$$

where $X=b_0^{-1}x^{\lambda_0}$.

We immediately obtain the following:

COROLLARY 1. *The infinite product*

$$G(x)=\prod_{i=0}^{\infty}(1+x^{-\lambda_i}),\qquad \lambda_i>2(\lambda_0+\cdots+\lambda_{i-1})\tag{15}$$

has truncations which are eventually 2*–apart (that is, from the* $(N-1)$*-st truncation onwards) if and only if* (*i*) $(x^{\lambda_i}+1, x^{\lambda_j}+1)=1$ *for all* $i,j \le N-1,\ i \ne j$,

(*ii*) $2\lambda_i \mid \lambda_N$ *for* $i \le N-1$,

(*iii*) $2\lambda_i \mid \lambda_{i+1}$ *for all* $i \ge N$.

There are no infinite products of the form (15) *whose truncations (except possibly the first) are ever three–apart.*

Proof. We suppose that μ and ρ are constants such that

$$p_m = \rho \prod_{i=0}^{n-1}(x^{\lambda_i}+b_i) \qquad p_{m+h} = \mu \prod_{i=0}^{n}(x^{\lambda_i}+b_i)$$
$$q_m = \rho x^{\sum_{i=0}^{n-1}\lambda_i} \qquad q_{m+h} = \mu x^{\sum_{i=0}^{n}\lambda_i}$$

so that

$$S(m,h) = \mu\rho b_n \prod_{i=0}^{n-1} x^{\lambda_i}(x^{\lambda_i}+b_i).$$

Now if $h=2$ we see from (14) that the middle convergents are necessarily given by

$$p_{m+1} = \frac{\mu(x^{\lambda_n}+b_n)-\rho}{(-1)^m\mu\rho b_n\prod_{i=0}^{n-1}x^{\lambda_i}}$$
$$q_{m+1} = \frac{\mu x^{\lambda_n}-\rho}{(-1)^m\mu\rho b_n\prod_{i=0}^{n-1}(x^{\lambda_i}+b_i)}.$$

For the first of these to be a polynomial we require that $\mu b_n = \rho$; substituting this in the second gives the stated criterion. Conversely if such a condition is satisfied then by appealing to our previous comments or to criterion (6) we see that the given fraction is an intermediate convergent and moreover by degree considerations that it is the only one.

Now if $h=3$ the equations (13) and (14) produce

$$p_{m+1} = \frac{\mu(x^{\lambda_n}+b_n)-\rho a_{m+3}}{(-1)^m\mu\rho\prod_{i=0}^{n-1}x^{\lambda_i}} \qquad q_{m+1} = \frac{\mu x^{\lambda_n}-\rho a_{m+3}}{(-1)^m\mu\rho\prod_{i=0}^{n-1}(x^{\lambda_i}+b_i)}$$
$$p_{m+2} = \frac{\mu(x^{\lambda_n}+b_n)a_{m+2}+\rho}{(-1)^m\mu\rho\prod_{i=0}^{n-1}x^{\lambda_i}} \qquad q_{m+2} = \frac{\mu x^{\lambda_n}a_{m+2}+\rho}{(-1)^m\mu\rho\prod_{i=0}^{n-1}(x^{\lambda_i}+b_i)}$$

and

$$(16) \qquad a_{m+3}a_{m+2}+1 = (-1)^m S(m,3) = (-1)^m\mu\rho b_n \prod_{i=0}^{n-1} x^{\lambda_i}(x^{\lambda_i}+b_i).$$

From the equations for p_{m+1}, p_{m+2} we see that

$$\begin{aligned}\mu b_n - \rho a_{m+3} &= A(x)x^{\sum_{i=0}^{n-1}\lambda_i}\\ \mu b_n a_{m+2} + \rho &= B(x)x^{\sum_{i=0}^{n-1}\lambda_i}\end{aligned}$$

for some polynomials $A(x)$ and $B(x)$. However since

$$\deg a_{m+2} + \deg a_{n+3} = 2\sum_{i=0}^{n-1}\lambda_i, \qquad \deg a_{m+2},\ \deg a_{m+3} \geq 1$$

we see that A, B are both in fact non-zero constants. Hence substituting these expressions for a_{m+2} and a_{m+3} back into (16) we obtain an expression of the form

$$x^{\sum_{i=0}^{n-1}\lambda_i} - \left(\frac{\mu b_n B + \rho A}{AB}\right) = (-1)^{m+1}\frac{(\mu\rho b_n)^2}{AB}\prod_{i=0}^{n-1}(x^{\lambda_i} + b_i)\,.$$

In view of the rapid increase of the $\lambda_i > 2\sum_{j=0}^{i-1}\lambda_j$, comparison of the number of terms on the right shows that such a relation is only possible if $n = 1$. Finally substituting for a_{m+3} in q_{m+1} gives

$$q_2 = \frac{\mu(x^{\lambda_1} - b_1) + Ax^{\lambda_0}}{-\mu\rho b_1(x^{\lambda_0} + b_0)}$$

from which one easily deduces that $\lambda_0 \mid \lambda_1$. An obvious change of variable reduces this to the case $(1 + X^{-1})(1 + cX^{-\lambda})$ which as we have seen is 2–apart if $c = (-1)^\lambda$, and when $c \neq (-1)^\lambda$ is easily checked to be 3–apart with expansion:

$$[1\,, X\,, -\frac{(X^{\lambda-1} - (-1)^{\lambda-1})}{c(X+1)}\,, \frac{b}{(1-b)}X + \frac{b}{(1-b)^2}\,, \frac{(1-b)^3}{b^2}X - \frac{(1-b)^2}{b}]$$

where $b = (-1)^\lambda c$.

Our examples have featured constant gaps between the truncations; lest the reader be misled into believing that this is generally so we give the following nice counter-example:

THEOREM 8. *Let*

$$G(x) = (1 + x^{-5}) \prod_{i=0}^{\infty} (1 + x^{-\lambda_i})$$

with the λ_i satisfying (12) *and*

$$\lambda_i \equiv 6 \cdot 2^i \pmod{10} \quad \textit{for all } i \geq 0 .$$

Then the truncation convergents

$$\frac{p_N}{q_N} = (1 + x^{-5}) \prod_{i=0}^{m} (1 + x^{-\lambda_i}), \qquad \frac{p_{N+M}}{q_{N+M}} = (1 + x^{-5}) \prod_{i=0}^{m+1} (1 + x^{-\lambda_i})$$

are 5*–apart when $m \equiv 3 \pmod 4$,* 6*–apart when $m \equiv 2 \pmod 4$,* 7*–apart when $m \equiv 0 \pmod 4$ and* 8*–apart when $m \equiv 1 \pmod 4$ with intermediate convergents:*

$$\frac{p_{N+j}}{q_{N+j}} = \frac{A_j(x) x^{\lambda_{m+1} - \sum_{i=0}^{m} \lambda_i - 5} + B_j(x)(x+1)\prod_{i=0}^{m}(x^{\lambda_i}+1)}{\left(A_j(x)\frac{(x^{\lambda_{m+1}}-1)}{(x+1)\prod_{i=0}^{m}(x^{\lambda_i}+1)} + B_j(x) x^{\sum_{i=0}^{m}\lambda_i + 5}\right) \Big/ \Phi_{10}(x)}$$

$$\frac{p_{N+k}}{q_{N+k}} = \left(\Phi_{10}(x) x^{\lambda_{m+1} - \sum_{i=0}^{m}\lambda_i - 5}\right) \Big/ \frac{(x^{\lambda_{m+1}} - 1)}{(x+1)\prod_{i=0}^{m}(x^{\lambda_i}+1)}$$

$$\frac{p_{N+k+h}}{q_{N+k+h}} = \frac{A'_h(x)(x+1)\prod_{i=0}^{m+1}(x^{\lambda_i}+1) + B'_h(x) x^{\lambda_{m+1} - \sum_{i=0}^{m}\lambda_i - 5}}{\left(A'_h(x) x^{\sum_{i=0}^{m+1}\lambda_i + 5} + B'_h(x)\frac{(x^{\lambda_{m+1}}-1)}{(x+1)\prod_{i=0}^{m}(x^{\lambda_i}+1)}\right) \Big/ \Phi_{10}(x)}$$

for $1 \leq j \leq k-1$ and $1 \leq h \leq M-k-1$ where $\Phi_{10}(x) = (x^5+1)/(x+1)$ and

	$m \equiv 0$	$m \equiv 1$	$m \equiv 2$	$m \equiv 3$	$\pmod 4$
(A_1, B_1)	$(1, u)$	$(1, -x^2)$	$(1, x^3)$	$(1, -1)$	
(A_2, B_2)	$(x, 1)$	(v, w)	(w, v)		
(A_3, B_3)		$(x^3, 1)$	$(-x^2, 1)$		
(A'_1, B'_1)	$(1, x^3)$	$(1, -x^2)$	$(-1, 1)$	$(1, x)$	
(A'_2, B'_2)	(w, v)	(v, w)		$(u, 1)$	
(A'_3, B'_3)	$(-x^2, 1)$	$(x^3, 1)$			

with $u(x) = 1 - x + x^2 - x^3$, $v(x) = x^2 - x + 1$, and $w(x) = 1 - x$.

Proof. The above expressions may appear formidable but the process for obtaining them is fairly simple. Previous remarks guarantee us the stated convergent p_{N+k}/q_{N+k}. Hence to find the convergents between p_N/q_N and p_{N+k}/q_{N+k} it is enough to find polynomials A_i, B_i with $\deg A_iB_i \le 3$ such that the stated denominators are polynomials. Thus starting with $A_1 = 1$ we find successive A_j, B_j with $\deg A_j = 4 - \deg B_{j-1}$ until we have filled in the gap (that is, $\sum(4 - \deg A_jB_j) = 4$). Similarly for the A'_h, B'_h. Of course we have made things here simple by choosing the λ_i such that for a root ζ of $\Phi_{10}(x)$ we have

$$\frac{(\zeta^{\lambda_{m+1}}-1)}{(\zeta+1)\prod_{i=0}^{m}(\zeta^{\lambda_i}+1)} = -1, \quad \zeta^{5+\sum_{i=0}^{m}\lambda_i} = \begin{cases} \zeta & \text{if } m \equiv 0 \pmod 4 \\ \zeta^3 & \text{if } m \equiv 1 \pmod 4 \\ -\zeta^2 & \text{if } m \equiv 2 \pmod 4 \\ -1 & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

Note that it is not hard to see that a judicious replacement of a subsequence of the λ_i by $\lambda_m \equiv 6\cdot 2^{m+2} \pmod{10}$ can even inject an appearance of utter randomness in these different spacings.

As promised, we conclude with the remaining $r = 3$ cases:

THEOREM 9. *Let*

$$G(x) = (1+x^{-3})\prod_{i=0}^{\infty}(1+x^{-\lambda_i}), \qquad (\lambda_0, 6) = 1$$

with λ_i satisfying (12).

Then the truncation convergents

$$\frac{p_N}{q_N} = (1+x^{-3})\prod_{i=0}^{n-1}(1+x^{-\lambda_i}), \qquad \frac{p_{N+M}}{q_{N+M}} = (1+x^{-3})\prod_{i=0}^{n}(1+x^{-\lambda_i})$$

are

(*i*) 8*–apart if* $3\nmid\lambda_n$ *with intermediate convergents:*

$$\frac{p_{N+j}}{q_{N+j}} = \frac{A_j(x)x^{\lambda_n-\sum_{i=0}^{n-1}\lambda_i-3} + B_j(x)\prod_{i=0}^{n-1}(x^{\lambda_i}+1)}{\left(A_j(x)\frac{(x^{\lambda_n}-1)}{\prod_{i=0}^{n-1}(x^{\lambda_i}+1)} + B_j(x)x^{\sum_{i=0}^{n-1}\lambda_i+3}\right)\Big/(x^3+1)}$$

$$\frac{p_{N+4}}{q_{N+4}} = (x^3+1)x^{\lambda_n-\sum_{i=0}^{n-1}\lambda_i-3}\Bigg/\frac{(x^{\lambda_n}-1)}{\prod_{i=0}^{n-1}(x^{\lambda_i}+1)}$$

$$\frac{p_{N+4+h}}{q_{N+4+h}} = \frac{A'_h(x)\prod_{i=0}^{n}(x^{\lambda_i}+1) + B'_h(x)x^{\lambda_n-\sum_{i=0}^{n-1}\lambda_i-3}}{\left(A'_h(x)x^{\sum_{i=0}^{n}\lambda_i+3} + B'_h(x)\frac{(x^{\lambda_n}-1)}{\prod_{i=0}^{n-1}(x^{\lambda_i}+1)}\right)\Big/(x^3+1)}$$

for $h, j = 1, 2, 3$ *with*

	$\lambda_n \equiv 2 \pmod 6$	$\lambda_n \equiv 4 \pmod 6$
(A_1, B_1)	$(1,\quad a(x^2 - x + 1) - \varepsilon x)$	$(1,\quad e(x^2 - x + 1) - \varepsilon(x - 1))$
(A_2, B_2)	$(\varepsilon a x + c,\quad bx + a)$	$(ex + d,\quad -\varepsilon dx + f)$
(A_3, B_3)	$(b(x^2 - x + 1) + k(x - 1),\quad \varepsilon k)$	$(d(x^2 - x + 1) - kx,\quad -\varepsilon k)$
(A_1', B_1')	$(k\varepsilon,\quad a(x^2 - x + 1) - kx)$	$(k,\quad f(x^2 - x + 1) - k\varepsilon(x - 1))$
(A_2', B_2')	$(ax + b,\quad cx + \varepsilon a)$	$(fx - d\varepsilon,\quad dx + e)$
(A_3', B_3')	$(c(x^2 - x + 1) + (x - 1),\quad \varepsilon)$	$(d(x^2 - x + 1) + \varepsilon x,\quad 1)$

where ε *and* k *are the constants*

$$k = -2\left(\frac{1}{2}\right)^n\left(\frac{\lambda_n}{\lambda_0}\right), \qquad \varepsilon = \begin{cases} (-1)^n & \text{if } \lambda_0 \equiv 1 \pmod 6 \\ (-1)^{n-1} & \text{if } \lambda_0 \equiv 5 \pmod 6 \end{cases}$$

and $a, \ldots, f$ *are constants given by* $a = -\frac{1}{3}(k + \varepsilon)$, $b = -\frac{1}{3}(\varepsilon - 2k)$, $c = -\frac{1}{3}(\varepsilon k - 2)$, $d = -\frac{1}{3}(k - \varepsilon)$, $e = -\frac{1}{3}(2\varepsilon + k)$ *and* $f = -\frac{1}{3}(2k\varepsilon + 1)$.

(ii) *are* 4*–apart if* $3 \mid \lambda_n$ *with intermediate convergents*

$$\frac{p_{N+1}}{q_{N+1}} = \frac{x^{\lambda_n - 3 - \sum_{i=0}^{n-1}\lambda_i} + \frac{2}{3}\left(\frac{1}{2}\right)^n\left(\frac{\lambda_n}{\lambda_1}\right)(x^2 - x + 1)\prod_{i=0}^{n-1}(x^{\lambda_i} + 1)}{\left(\frac{(x^{\lambda_n} - 1)}{(x^2 - x + 1)\prod_{i=0}^{n-1}(x^{\lambda_i} + 1)} + \frac{2}{3}\left(\frac{1}{2}\right)^n\left(\frac{\lambda_n}{\lambda_0}\right)x^{3 + \sum_{i=0}^{n-1}\lambda_i}\right) \Big/ (x + 1)}$$

$$\frac{p_{N+2}}{q_{N+2}} = \left(x^{\lambda_n - 3 - \sum_{i=0}^{n-1}\lambda_i}(x + 1)\right) \Big/ \frac{(x^{\lambda_n} - 1)}{(x^2 - x + 1)\prod_{i=0}^{n-1}(x^{\lambda_i} + 1)}$$

$$\frac{p_{N+3}}{q_{N+3}} = \frac{(x^2 - x + 1)\prod_{i=0}^{n}(x^{\lambda_i} + 1) + \frac{3}{2}\cdot 2^n\left(\frac{\lambda_0}{\lambda_n}\right)x^{\lambda_n - 3 - \sum_{i=0}^{n-1}\lambda_i}}{\left(x^{3 + \sum_{i=0}^{n} x^{\lambda_i}} + \frac{3}{2}\cdot 2^n\left(\frac{\lambda_0}{\lambda_n}\right)\frac{(x^{\lambda_n} - 1)}{(x^2 - x + 1)\prod_{i=0}^{n-1}(x^{\lambda_i} + 1)}\right) \Big/ (x + 1)}$$

Proof. As before, the proof amounts to tediously finding A_i, B_i such that the denominators are polynomial and so that the various degrees add up.

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