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Analytic and arithmetic theory of semigroups with divisor theory.

par ALFRED GEROLDINGER AND JERZY KACZOROWSKI

1. Introduction

In this paper we develop the analytic theory of semigroups with divisor theory having finite divisor class group such that each class contains approximately the same number of elements with norms $\leq x$. The theory of algebraic numbers yields many interesting examples of such semigroups. As in the classical case our treatment heavily depends on L-functions associated with the characters of the divisor class group. The theory of such L-functions is in many ways similar to the classical one with (at least) one serious exception in which $L(1,\chi)=0$ can hold for certain characters χ. If this occurs, we call the semigroup to be of type β ; otherwise the semigroup is said to be of type α, cf. 3) Definition 3.

Our first aim is to describe up to arithmetical isomorphism the structure of semigroups under consideration, cf. 6) Theorems 1, 2, 3 and 4. We do this by using structural G-mappings, cf. 5) Definition 4, which are counterparts to explicit formulae in the classical theory. It turns out that semigroups of types α and β have quite different structures.

Next we consider some quantitative questions. We derive asymptotic formulae for counting functions of elements having a factorization of length k, cf. 12) Theorem 5, and of elements having at most k factorizations of distinct lengths, cf. 12) Theorem 6. Moreover, we show that in some sense most sets of lengths have a typical form, cf. 12) Theorem 7. In all cases results for semigroups of type α and β differ significantly.

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2. Preliminaries on semigroups with divisor theory

Throughout this paper, a semigroup is a commutative, multiplicative semigroup with identity, usually denoted by 1; if not stated otherwise, we always assume that the cancellation law holds. If S is a semigroup, then $S^\times$ denotes its group of invertible elements and $\mathcal{Q}(S)$ is a quotient group of S. A semigroup S is called reduced, if $S^\times = \{1\}$; obviously $S/S^\times$ is a reduced semigroup. For a set P let $\mathcal{F}(P)$ be the free abelian semigroup with uniquely determined basis P. Furthermore we use the standard notions of divisibility theory in a semigroup as described in [Gi, §6].

Definition 1. *A divisor theory for a semigroup S is a semigroup homomorphism $\partial : S \to \mathcal{F}(P)$ from S into a free abelian semigroup $\mathcal{F}(P)$ with the following properties:*

(D1) *If $a, b \in S$ and $\partial a \mid \partial b$ in $\mathcal{F}(P)$, then $a|b$ in S.*
(D2) *For every $\alpha \in \mathcal{F}(P)$ there exist $a_1, \dots, a_n \in S$ with $\alpha = gcd\{\partial a_1, \dots, \partial a_n\}$.*

The quotient group $Cl(S) = \mathcal{Q}(\mathcal{F}(P))/\mathcal{Q}(\partial(S))$ is called divisor class group of S; we write $Cl(S)$ additively and for $\alpha \in \mathcal{F}(P)$ we denote by $[\alpha] \in Cl(S)$ the divisor class containing α.

For a recent paper on semigroups with divisor theory see [HK 1]. We now list some of the main properties of semigroups with divisor theory which will be used in the sequel:

Properties of semigroups with divisor theory:

1) Obviously condition (D2) in the definition is equivalent to

(D2)' $\forall p \in P \ \exists a_1, \dots, a_n \in S$ with $p = \gcd\{\partial a_1, \dots, \partial a_n\}$.

2) A semigroup S has a divisor theory if and only if the reduced semigroup $S/S^\times$ has a divisor theory. (cf. [HK 1, Bem. 2])

3) If $\partial_i : S \to D_i \quad (i = 1, 2)$ are divisor theories, then there exists a unique semigroup isomorphism $\phi : D_1 \to D_2$ such that $\partial_2 = \phi \circ \partial_1$ (cf. [HK 2, Korollar zu Satz 2]).

4) Let $\partial_i : S_i \to \mathcal{F}(P_i) \quad (i = 1, 2)$ be semigroups with divisor theories. Then the following conditions are equivalent:

i) $S_1/S_1^\times \simeq S_1/S_2^\times$

ii) there is a group isomorphism $\phi : Cl(S_1) \to Cl(S_2)$ such that

$$\operatorname{card}(P_1 \cap g) = \operatorname{card}(P_2 \cap \phi(g))$$

for every $g \in Cl(S_1)$ (cf. [HK 1, Satz 3]).

5) If a semigroup S admits a semigroup homomorphism $\partial : S \to D$ from S into a free abelian semigroup D satisfying (D1), then S has a divisor theory; cf. [HK 1, Bem. 5].

6) Semigroups with divisor theory and Krull semigroups (as introduced in [Ch]) are equivalent concepts [Ge-HK, Th. 1]. Using this equivalence, Krause proved that the multiplicative semigroup of an integral domain R has a divisor theory if and only if R is a Krull domain ([Kr, Prop.]).

Our first aim is to derive a characterization of reduced semigroups with divisor theory, which allows us to give a unified description of semigroups with divisor theory, which are quite different at first sight.

Let R be a Dedekind domain, $P = P(R)$ the set of non-zero prime ideals, $\mathfrak{I}(R)$ the multiplicative semigroup of non-zero ideals, $\mathcal{H}(R) \subseteq \mathfrak{I}(R)$ the subsemigroup of principal ideals and $R^\bullet = R \setminus \{0\}$ the multiplicative semigroup of R. Then

$$R^\bullet / R^{\bullet\times} \simeq \mathcal{H}(R), \qquad \mathfrak{I}(R) \simeq \mathcal{F}(P)$$

and

$$\partial_R : \mathcal{H}(R) \hookrightarrow \mathfrak{I}(R) \xrightarrow{\sim} \mathcal{F}(P)$$

is a divisor theory of $\mathcal{H}(R)$ with divisor class group being the usual ideal class group $Cl(R)$.

Let Γ_0 be a semigroup (for which we do not assume the cancellation law), $\Gamma < \Gamma_0^\times$ a subgroup and $\pi : R^\bullet \to \Gamma_0$ a semigroup homomorphism. Then $\widetilde{R}_{\Gamma,\pi} = \pi^{-1}(\Gamma)$ is a subsemigroup of $R^\bullet$ and we have

i) $\widetilde{R}_{\Gamma,\pi}^\times = \widetilde{R}_{\Gamma,\pi} \cap R^{\bullet\times}$

ii) $\widetilde{R}_{\Gamma,\pi} / \widetilde{R}_{\Gamma,\pi}^\times \simeq \{aR \mid \pi(a) \in \Gamma\} = R_{\Gamma,\pi} \subseteq \mathcal{H}(R)$.

PROPOSITION 1. *For a reduced semigroup S the following conditions are equivalent:*

i) *S has a divisor theory.*

ii) *There exist a Dedekind domain R and Γ, π as above such that $S \simeq R_{\Gamma,\pi}$.*

Proof. **i)** $\Rightarrow$ **ii)**. If S has a divisor theory, then by [G-HK, Th.4] there exist a Dedekind domain R and a subset $P_0 \subseteq P(R)$ such that

$$S \simeq H = \{\alpha \in \mathcal{H}(R) \mid \partial_R(\alpha) \in \mathcal{F}(P_0)\}\,.$$

Obviously $H = R_{\Gamma,\pi}$ with $\pi : R \to \mathcal{H}(R) \xrightarrow{\partial_R} \mathcal{F}(P) \hookrightarrow \mathcal{Q}(\mathcal{F}(P)) = \Gamma_0$ and $\Gamma = \mathcal{Q}(\mathcal{F}(P_0))$.

ii) $\Rightarrow$ **i)**. Since $\partial_R \mid R_{\Gamma,\pi} : R_{\Gamma,\pi} \to \mathcal{F}(P)$ satisfies (D1), $R_{\Gamma,\pi}$ has a divisor theory. $\square$

We conclude this section with some examples of semigroups with divisor theory.

Examples. Let R be a Dedekind domain.

1) By a cycle f of R we mean a formal product

$$f = f_0\sigma_1 \ldots \sigma_r$$

where $f_0 \in \mathfrak{J}(R)$ is an ideal of R, $r \geq 0$ and $\sigma_1, \ldots, \sigma_r : R \to \mathbb{R}$ are ring monomorphisms. Two elements $\alpha, \beta \in R$ are called congruent modulo $f, \alpha \equiv \beta \mod f$, if

$$\alpha \equiv \beta \mod f_0 \text{ and sign } \sigma_i(\alpha) = \text{ sign } \sigma_i(\beta) \text{ for } 1 \leq i \leq r\,.$$

Obviously the set of congruence classes R/f is a multiplicative semigroup. Let $\pi : R \to R/f$ the canonical epimorphism and $\Gamma < (R/f)^\times$ a subgroup. Then $R_{\Gamma,\pi}$ is a generalized ray class semigroup in the sense of [HK2, Def. 5] and $\partial_R | R_{\Gamma,\pi} : R_{\Gamma,\pi} \to \mathcal{F}(\{p \in P \mid p \nmid f_0\})$ is a divisor theory for $R_{\Gamma,\pi}$ [HK2, Satz 7].

2) Let $m \in \mathbb{N}_+$.

i) We set $\Gamma_0 = (\mathbb{Z}/m\mathbb{Z}, +), \Gamma < \Gamma_0$ a subgroup, $\pi : \mathcal{F}(P) \to (\mathbb{N}, +)$ a completely additive function and $\bar{\pi} : R \to \mathcal{F}(P) \xrightarrow{\pi} (\mathbb{N}, +) \xrightarrow{\rho} \Gamma_0$, where ρ denotes the canonical epimorphism. Then $R_{\Gamma,\bar{\pi}} = \{aR \mid \pi(aR) + m\mathbb{Z} \in \Gamma\}$.

ii) Setting $\Gamma_0 = (\mathbb{Z}/m\mathbb{Z}, \cdot), \Gamma < \Gamma_0^\times$ and $\pi : \mathcal{F}(P) \to (\mathbb{N}_+, \cdot)$ we obtain a multiplicative analogon to **i)**.

3) Let K be a quotient field of $R, K' \subseteq K$ a subfield such that K/K' is finite algebraic, $\pi = \mathcal{N}_{K/K'} : R \to K' = \Gamma_0$ the norm function and $\Gamma = F \subseteq$

$(K' \setminus \{0\}, \cdot)$ a multiplicative subgroup. Then $R_{\Gamma,\pi} = \{aR \mid \mathcal{N}_{K/K'}(a) \in F\}$ (cf. [Le]).

3. Arithmetical semigroups

We start with the definition of an arithmetical semigroup, which is the central notion in this paper. Most results will be derived in this setting.

DEFINITION 2. *A semigroup S with divisor theory $\partial : S \to D$ is called an arithmetical semigroup, if the following conditions are satisfied:*

1) *$Cl(S)$ is finite.*
2) *There exists a norm function $\|\cdot\| : D \to \mathbb{N}_+$ such that*
   i) *$\|a\| > 1$ for every $a \in D \setminus \{1\}$.*
   ii) *$\|ab\| = \|a\|\|b\|$ for all $a, b \in D$.*
3) **Condition $\tilde{\mathbf{A}}$** *: There exist two real numbers $\alpha > 0$ and $\theta < 1$ such that for every class $g \in Cl(S)$ we have*

$$\sum_{\substack{a \in g \\ \|a\| \le x}} \left(1 - \frac{\| a \|}{x}\right) = \frac{\alpha}{2h} x + O(x^{\theta}) \text{ as } x \to \infty .$$

Let S be an arithmetical semigroup. We write $\widehat{Cl(S)}$ for the dual group of $Cl(S)$ and let χ_0 denote the trivial character. For $\chi \in \widehat{Cl(S)}$ we define the L-function $L(s, \chi)$ as

$$L(s, \chi) = \sum_{a \in D} \chi([a]) \|a\|^{-s} ,$$

$s = \sigma + it$ being a complex variable.

By a non-trivial theory of L-functions we mean that for all $\chi \in \widehat{Cl(S)}$, $L(s, \chi)$ converges absolutely for $\sigma > 1$, and allows analytic continuation to a half-plane $\sigma > \theta$ with $\theta < 1$; all $L(s, \chi)$ should be holomorphic on this half-plane except $L(s, \chi_0)$, which has a simple pole at $s = 1$. Condition $\tilde{A}$ implies (by partial summation) that an arithmetical semigroup has a non-trivial theory of L-functions.

All examples of arithmetical semigroups, which are discussed below, satisfy the following

Condition A*: There exist two real numbers $\alpha > 0$ and $\theta < 1$ such that for every class $g \in Cl(S)$ we have

$$\sum_{\substack{a \in g \\ \|a\| \leq x}} 1 = \frac{\alpha}{h} x + O(x^{\theta}) \quad \text{as} x \to \infty .$$

By partial summation Condition A^* implies Condition $\widetilde{A}$, but it is not obvious whether or not A^* and $\widetilde{A}$ are equivalent. However, if

$$\#\{a \in D \mid \| a \| = n\} = O(n^{\epsilon}) \text{ for every positive } \epsilon,$$

then $\widetilde{A}$ and A^* are indeed equivalent, which follows from Perron's formula applied to $L(s, \chi)$.

Examples. Let R be the ring of integers of an algebraic number field K. Then R is a Dedekind domain with finite ideal class group and the absolute norm $\mathcal{N}_{K/Q} : \mathfrak{I}(R) \to \mathbb{N}_+$ with $\mathcal{N}_{K/Q}(I) = (R : I)$ is a norm function in the sense of Definition 2. We continue the examples given in the previous section.

1) Let f be a cycle, $\Gamma_0 = R/f, \pi : R \to R/f$ the canonical epimorphism and let $[1] \in (R/f)^{\times}$ denote the unit element. Then $R_{\{[1]\},\pi} = \{aR \mid a \equiv 1 \mod f\}, Cl(R_{\{[1]\},\pi})$, the group of f-ideal classes, is finite, Axiom A^* is satisfied and $L(1, \chi) \neq 0$ for every $\chi_0 \neq \chi \in \widehat{Cl(R_{\{[1]\},\pi})}$ (cf. [La; VI Th. 1 and Th. 3, VIII Th. 8]).

For an arbitrary subgroup $\Gamma < \Gamma_0^{\times}$ there is an epimorphism λ :

$$Cl(R_{\{[1]\},\pi}) \to Cl(R_{\Gamma,\pi})$$

and thus A^* holds for $R_{\Gamma,\pi}$ and $L(1, \chi) \neq 0$ for every $\chi_0 \neq \chi \in \widehat{Cl(R_{\Gamma,\pi})}$.

Note, that for a non-principal order $R' \subseteq R$ with conductor f_0 and for a subgroup $\Gamma < (R'/f_0)^{\times} < (R/f_0)^{\times}$, $\{a \in R' \mid a + f \in \Gamma\} = \tilde{R}_{\Gamma,\pi}$.

2) For $m \in \mathbb{N}_+$ let $\Gamma < (\mathbb{Z}/m\mathbb{Z}, +)$ be a subgroup and $\pi : \mathcal{F}(P) \to (\mathbb{N}, +)$ a completely additive prime-independent arithmetical function with $\gcd(m, \pi(p)) = 1$ for any $p \in P$. Then $R_{\Gamma,\bar{\pi}} = \{aR \mid \pi(aR) + m\mathbb{Z} \in \Gamma\}$ and we assert that $\partial_R \mid R_{\Gamma,\bar{\pi}} : R_{\Gamma,\bar{\pi}} \to \mathcal{F}(P)$ is a divisor theory.

For this it is sufficient to show that for every $p \in P$ there are $\alpha_1, \ldots, \alpha_n$ in $R_{\Gamma,\bar{\pi}}$ such that $p = \gcd\{\partial_R(\alpha_1), \ldots, \partial_R(\alpha_n)\}$. Let $p \in P$; we choose two further primes $p_1, p_2 \in [p] \in Cl(R)$.

For $i \in \{1,2\}$ let $\mathfrak{a}_i = (p p_i^{\mathrm{ord}([p])-1}) p_i^{\mathrm{ord}([p])(m-1)}$; then $\mathfrak{a}_i$ is principal, $\pi(\mathfrak{a}_i) + m\mathbb{Z} = m \ \mathrm{ord}([p])\pi(p) + m\mathbb{Z} = 0 \in \Gamma$ and $\gcd(\mathfrak{a}_1, \mathfrak{a}_2) = p$.

Since $\mathcal{Q}(\partial_R R_{\Gamma,\bar{\pi}}) < \mathcal{Q}(\partial_R \mathcal{H}(R)) < \mathcal{Q}(\mathcal{F}(P))$ there is a canonical epimorphism

$$\lambda : Cl(R_{\Gamma,\bar{\pi}}) = {}^{\mathcal{Q}(\mathcal{F}(P))}\!\big/_{\mathcal{Q}(\partial_R R_{\Gamma,\bar{\pi}})} \to {}^{\mathcal{Q}(\mathcal{F}(P))}\!\big/_{\mathcal{Q}(\partial_R \mathcal{H}(R))} = Cl(R)\,;$$

further

$$\mathrm{Ker}(\lambda) = {}^{\mathcal{Q}(\partial_R \mathcal{H}(R))}\!\big/_{\mathcal{Q}(\partial_R R_{\Gamma,\bar{\pi}})} \simeq {}^{\mathbb{Z}/m\mathbb{Z}}\!\big/_{\Gamma}$$

and thus $Cl(R_{\Gamma,\bar{\pi}})$ is finite.

In general $R_{\Gamma,\bar{\pi}}$ does not satisfy $\tilde{A}$. But if $Cl(R)$ is trivial, $(\mathbb{Z}/m\mathbb{Z} : \Gamma) = 2$ and $\pi(p) = 1$, then $h = 2$ and $R_{\Gamma,\bar{\pi}}$ has two L-functions: $L(s, \chi_0) = \zeta_K(s)$ and $L(s, \chi_1) = \zeta_K(2s)/\zeta_K(s)$; ζ_K being the Dedekind zeta function of K. Hence, under the Riemann Hypothesis for this function, $R_{\Gamma,\bar{\pi}}$ satisfies A^*. Observe that in this case we have $L(1, \chi_1) = 0$ (compare [F-S], example III).

Motivated by the above examples we conclude this section with the following definition.

DEFINITION 3. *Let S be an arithmetical semigroup. We say that S is of type α, if $L(1,\chi) \neq 0$ for all $\chi \in \widehat{Cl(S)}$ with $\chi \neq \chi_0$. Otherwise we say that S is of type β.*

4. L-functions

In this section we point out some basic facts concerning L-functions associated with an arithmetical semigroup S. One can expect that such L-functions share many properties of zeta-functions used in the theory of numbers. This is really the case. Arguing as in the classical case (cf. e.g. [T1], [T2], [P]) we state the following facts. Proofs are omitted.

For $\sigma > 1$ and every $\chi \in \widehat{Cl(S)}$

$$L(s, \chi) = \prod_{p \in P} \left(1 - \chi([p]) \parallel p \parallel^{-s}\right)^{-1}$$

and writing $\tau = |t| + 20$

$$|L(\sigma + it, \chi)| \ll_\epsilon \tau^2 \quad \text{for } \sigma > \theta + \epsilon, \epsilon > 0\,.$$

Denoting by $\rho = \beta + i\gamma$ a generic zero of $L(s,\chi)$ we have in case $\rho \neq 1$

$$\beta \leq 1 - \frac{A}{\log(|\gamma| + 20)}\,,$$

($A = A(S)$ is positive) and for fixed $\theta_1, \theta_2, \quad \theta < \theta_1 < \theta_2 < 1$

$$\frac{L'}{L}(s,\chi) = \sum_{\substack{\beta \geq \theta_1 \\ |\gamma - t| \leq 1}} \frac{1}{s-\rho} + O(\log \tau) \tag{4.1}$$

for $|t| \geq 1, \sigma > \theta_2$. Moreover, putting for $t > 0$, $\chi \in \widehat{Cl(S)}$

$$N(t,\chi) = \#\{\rho = \beta + i\gamma \mid \beta \geq \theta_1, |\gamma| \leq t, L(\rho,\chi) = 0\}\,,$$

we have

$$N(t+1,\chi) - N(t,\chi) \ll \log \tau \quad (t > 0)\,. \tag{4.2}$$

For $s = \sigma + it$,

$$\sigma > 1 - \frac{A}{4\log \tau}, \quad |t| \geq 1,$$

we have

(4.3)

$$|L(s,\chi)| \ll \log \tau, \quad |\log L(s,\chi)| \ll \log\log \tau, \text{ and } |\frac{L'}{L}(s,\chi)| \ll \log^2 \tau\,.$$

We denote by $\mu_\chi : [\theta_1, \infty) \to \mathbb{R}$ the Lindelöf function associated with $L(s,\chi)$:

$$\mu_\chi(\sigma) = \limsup_{|t| \to \infty} \frac{\log^+ |L(\sigma + it, \chi)|}{\log \tau}. \tag{4.4}$$

Obviously $\mu_\chi(\sigma) = 0$ for $\sigma \geq 1$ and $\mu_\chi(\theta_1) \leq 2$. Moreover μ_χ is continuous, non-increasing and convex downwards.

5. Structural G-mappings

Let G be a finite, abelian group and let $\mathbb{N}_1 = \mathbb{N}_+ \setminus \{1\}$. Let for $\mathfrak{d} < 1$,

$$H_\mathfrak{d} = \{\rho \in \mathbb{C} \mid \mathfrak{d} < Re\ \rho < 1\}\,.$$

We now introduce structural G-mappings. In the analytic theory of arithmetical semigroups they play the role of explicit formulae.

DEFINITION 4. *We say that* $a : \mathbb{N}_1 \times G \to \mathbb{N}$ *is a structural G-mapping of density* $\kappa > 0$ *when there exists a real number* $\mathfrak{d} \in (0,1)$ *and a function* $\nu : H_{\mathfrak{d}} \times G \to \mathbb{C}$ *satisfying the following conditions:*

i) *for every* $\chi \in \hat{G}$ *and* $\rho \in H_{\mathfrak{d}}$ *the number*

$$m(\rho,\chi) = \sum_{g\in G} \chi(g)\nu(\rho,g) \tag{5.1}$$

belongs to $\mathbb{N}$;

ii) *m has discrete support and*

$$\sum_{t\le \mathcal{I}m\,\rho\le t+1} m(\rho,\chi) = O(\log\tau) \tag{5.2}$$

for every $t \in \mathbb{R}$ $\quad (\tau = |t| + 20)$;

iii) *for every* $x \ge 1, T \in \mathbb{R}$ *and* $g \in G$ *we have*

$$\sum_{n\le x} a(n,g)\log n = \kappa x - \sum_{\substack{\rho\in H_{\mathfrak{d}}\\ |\gamma - T|\le 1}} \frac{\nu(\rho,\chi)}{\rho} x^{\rho} + E(x,T,g) \tag{5.3}$$

where

$$E(x,T,g) = O_{T,g}(x) \tag{5.4}$$

as $x \to \infty$;

iv) *there exist constants*

$$0 < \lambda < 1/2\pi, \quad A_1 > 0, \quad \mathfrak{d} < \mathfrak{d}' < 1, \quad \alpha_0 > 0$$

such that uniformly for

$$\sigma > \mathfrak{d}', \quad |t - \frac{T}{2\pi}| \le \lambda, \quad 0 < \alpha < \alpha_0$$

and for every $g \in G$ *we have*

$$|\hat{f}_{\alpha,\sigma}(t)| \le A_1 \frac{\log\tau}{\tau} \tag{5.5}$$

where $\hat{f}_{\alpha,\sigma}$ denotes the Fourier transform of the function

$$f_{\alpha,\sigma}(u) = f_{\alpha,\sigma;g,T}(u) = \begin{cases} \frac{E(e^u,T,g)e^{-\sigma u}}{\Gamma(\alpha u+1)} & \text{for } u \geq 0\,. \\ 0 & \text{for } u < 0\,. \end{cases} \tag{5.6}$$

Γ being the classical Euler's gamma function.

DEFINITION 5. *We say that a structural G-mapping is quasi-Riemannian, when $\nu \equiv 0$.*

For a quasi-Riemannian structural G-mapping conditions **i)** and **ii)** in Definition 4 can be neglected; (5.3) and (5.4) read as follows

$$\sum_{n \leq x} a(n,g) \log n = \kappa x + E(x,g)\,, \tag{5.3'}$$

$$E(x,g) = O_g(x)\,. \tag{5.4 '}$$

It is evident that a quasi-Riemannian structural G-mapping of density κ can be regarded as a family of quasi-Riemannian structural C_1-mappings (C_1 - the trivial group) of densities κ labeled by elements of $G : \{a_g\}_{g\in G}$. Formally every a_g is defined on the set $\mathbb{N}_1 \times C_1$; to simplify the notation we consider structural C_1-mappings as functions on $\mathbb{N}_1$. Instead of "structural C_1-mapping" we say simply "structural mapping".

PROPOSITION 2. *The function $a : \mathbb{N}_1 \to \mathbb{N}$ is a quasi-Riemannian structural mapping of density κ if and only if for $x \geq 1$ we have*

$$\sum_{n \leq x} a(n,g) \log n = \kappa x + E(x)\,,$$

where

$$E(x) = O(x^\theta) \quad \text{as } x \to \infty \tag{5.7}$$

for certain $\theta < 1$ and, moreover, for $\sigma > \theta$

$$\hat{\varphi_\sigma}(t) = O(\frac{\log \tau}{\tau}), \tag{5.8}$$

where

$$\varphi_\sigma(t) = \begin{cases} E(e^u)e^{-\sigma u} & \textit{for } u \geq 0, \\ 0 & \textit{for } u < 0\,. \end{cases} \tag{5.9}$$

Proof. Let $a : \mathbb{N}_1 \to \mathbb{N}$ be quasi-Riemannian and structural and let $f_{\alpha,\sigma}$ be defined by (5.6); we remark here that in the case under consideration E does not depend on T and g. Let us fix $\theta \in (\vartheta', 1)$. For $N \geq 2$, $0 < \alpha < \alpha_0$ and $\sigma > \vartheta'$ we have, using the inverse Fourier transform and (5.5)

$$\begin{aligned} F_{\alpha,\sigma}(N) := &\frac{1}{\log(1+\frac{1}{N})}\int_{\log N}^{\log(N+1)} f_{\alpha,\sigma}(u)du \\ &= \int_{-\infty}^{+\infty} \hat{f}_{\alpha,\sigma}(t)\frac{(N+1)^{2\pi i t} - N^{2\pi i t}}{2\pi i t\log(1+\frac{1}{N})}dt \\ &\ll \int_{-N}^{N} \frac{\log\tau}{\tau}dt + N\int_{N}^{\infty}\frac{\log t}{t^2}dt \ll \log^2 N\,. \end{aligned}$$

For $\log N \leq u < v \leq \log(N+1)$ we have

$$\begin{aligned} |f_{\alpha,\sigma}(u) - f_{\alpha,\sigma}(v)| = |\sum_{n\leq N} a(n)\log n\Big(\frac{e^{-\sigma u}}{\Gamma(\alpha u+1)} - \frac{e^{-\sigma v}}{\Gamma(\alpha v+1)}\Big) \\ - \kappa\Big(\frac{e^{(1-\sigma)u}}{\Gamma(\alpha u+1)} - \frac{e^{(1-\sigma)v}}{\Gamma(\alpha v+1)}\Big) | \ll \sigma N^{-\sigma} \ll 1\,. \end{aligned}$$

Hence for $\log N \leq u < \log(N+1)$

$$f_{\alpha,\sigma}(u) = F_{\alpha,\sigma}(N) + O(1) \ll \log^2 N \ll u^2$$

and therefore uniformly for $\sigma > \vartheta', 0 < \alpha < \alpha_0$ we have

$$E(e^u) \ll u^2 e^{\sigma u}\Gamma(\alpha u+1) \quad \text{as } u \to \infty\,.$$

Making $\alpha \to 0+$ and putting $\sigma = \sigma_0 = \frac{1}{2}(\vartheta' + \theta)$ we obtain:

$$E(e^u) \ll u^2 e^{\sigma_0 u} \ll e^{\theta u} \quad \text{as } u \to \infty\,;$$

(5.7) therefore follows. From this inequality we get

$$\int_{-\infty}^{+\infty} |\varphi_\sigma(t)|dt \leq C_0 \quad \text{uniformly for } \sigma > \theta.$$

Hence, by Lebesgue bounded convergence theorem

$$\lim_{\alpha\to 0+} \hat{f}_{\alpha,\sigma}(t) = \hat{\varphi_\sigma}(t) \text{ for all } t \in \mathbb{R}.$$

Now it is obvious that (5.8) follows from (5.5).

6. Structural theorems for arithmetical semigroups

Our goal now is to describe the structure of arithmetical semigroups up to arithmetical isomorphism.

DEFINITION 6. *Arithmetical semigroups S_1 and S_2 with divisor theories*

$$\partial_i : S_i \to D_i, \quad i = 1,2,$$

and norm functions

$$\|\cdot\|_i : S_i \to \mathbb{N}_+, \quad i = 1,2$$

are arithmetically isomorphic (or briefly a-isomorphic), if there exists an isomorphism $\varphi : S_1/S_1^\times \to S_2/S_2^\times$ such that $\|\bar\partial_1(\alpha)\|_1 = \|\bar\partial_2 \circ \varphi(\alpha)\|_2$ for every $\alpha \in S_1/S_1^\times$.

(Here $\bar\partial_i : S_i/S_i^\times \to D_i$ denotes the homomorphism induced by ∂_i, compare **2.** §2).

Suppose S_1 and S_2 are a-isomorphic. Then, according to **3.** §2 there exists a unique semigroup isomorphism $\phi : D_1 \to D_2$ such that $\bar\partial_2 \circ \varphi = \phi \circ \bar\partial_1$. Hence the diagram

$$\begin{array}{ccccc} S_1/S_1^\times & \xrightarrow{\bar\partial_1} & D_1 & \searrow^{\|\cdot\|_1} & \\ \varphi\downarrow & & \phi\downarrow & & \mathbb{N}_+ \\ S_2/S_2^\times & \xrightarrow{\bar\partial_2} & D_2 & \nearrow_{\|\cdot\|_2} & \end{array}$$

commutes and we have

PROPOSITION 3. *Arithmetical semigroups $(S_i, \partial_i, \|\cdot\|_i)$, $i = 1,2$, are a-isomorphic if and only if there exists a group isomorphism $\psi : Cl(S_1) \to Cl(S_2)$ such that for every pair $(n,g) \in \mathbb{N}_1 \times Cl(S_1)$,*

$$\#\{p \in g \cap P_1 \mid \|p\|_1 = n\} = \#\{p \in \psi(g) \cap P_2 \mid \|p\|_2 = n\}.$$

Hence the structure of an arithmetical semigroup is completely determined by the numbers $\#\{p \in P \cap g \mid \|p\| = n\}, (n,g) \in \mathbb{N}_1 \times Cl(S)$.

THEOREM 1. *Let S be an arithmetical semigroup of type α with the divisor class group G of order h. Then the function $a : \mathbb{N}_1 \times G \to \mathbb{N}$ defined by*

$$a(n,g) = \#\{p \in g \cap P \mid \|p\| = n\} \tag{6.1}$$

is a structural G-mapping of density $\kappa = 1/h$.

THEOREM 2. *For every finite abelian group G of order h and every structural G-mapping a of density $\kappa = 1/h$ there exists exactly one (up to a-isomorphism) arithmetical semigroup S of type α having the divisor class group isomorphic to G and such that $\#\{p \in g \cap P \mid \|p\| = n\} = a(n,g)$ for each pair $(n,g) \in \mathbb{N}_1 \times G$.*

THEOREM 3. *Let S be an arithmetical semigroup of type β with the divisor class group G of order h and let $\chi_1 \in \hat{G}$ be such that $L(1,\chi_1) = 0$. Then $h = 2h_1$, where $h_1 = \#\ker \chi_1$.*

Moreover, for each $g \notin \ker \chi_1$, the function

$$a_g : \mathbb{N}_1 \to \mathbb{N}$$

defined by

$$a_g(n) = \#\{p \in g \cap P \mid \|p\| = n\}$$

is a structural, quasi-Riemannian mapping of density $1/h_1$. For $g \in \ker \chi_1$ we have

$$\#\{p \in g \cap P \mid \|p\| \le x\} = O(x^\theta),\ \theta < 1,\ \text{as } x \to \infty. \tag{6.2}$$

THEOREM 4. *Let G be a finite abelian group of order $h = 2h_1$ and let H be a subgroup of order h_1. Let, moreover, $\{a_g\}_{g \notin H}$ be a family of quasi-Riemannian structural mappings of densities $1/h_1$ and let $\{b_g\}_{g \in H}$ be a family of functions $b_g : \mathbb{N}_1 \to \mathbb{N}$ satisfying $\sum_{n \le x} b_g(n) = O(x^\theta), \theta < 1$, as $x \to \infty$. Then there exists exactly one (up to a-isomorphism) arithmetical semigroup S of type β having the divisor class group isomorphic to G and such that for all $n \in \mathbb{N}_1$,*

$$\#\{p \in g \cap P \mid \|p\| = n\} = a_g(n) \text{ for } g \notin H \tag{6.3}$$

and

$$\#\{p \in g \cap P \mid \|p\| = n\} = b_g(n) \text{ for } g \in H\,. \tag{6.4}$$

Moreover, there exists $\chi_1 \in \hat{G}$ *satisfying* $L(1,\chi_1) = 0$ *and* $\ker \chi_1 = H$.

7. Proof of Theorem 1

Let S be an arithmetical semigroup of type α and let θ be the exponent appearing in the condition $\tilde{A}$. Without loss of generality we assume that $\theta \in (1/2, 1)$. Let us take any $\mathfrak{d} \in (\theta, 1)$ and define $\nu : H_{\mathfrak{d}} \times G \to \mathbb{C}$ as follows

$$\nu(\rho, g) = \frac{1}{h} \sum_{\chi \in \hat{G}} \overline{\chi(g)} m(g, \chi);$$

$m(\rho,\chi)$ being the multiplicity of the zero of $L(s,\chi)$ at $s = \rho$.

We check if a and ν satisfy conditions **i)** – **iv)** from Definition 4.

Condition **i)** is immediate and **ii)** follows from (4.2). Applying the classical Perron formula, shifting the line of integration to the left and using (4.3) we get

$$\sum_{n \le x} a(n,g)\left(1 - \frac{n}{x}\right) \log n = \frac{1}{2h} x + O\left(x e^{-C_0 \sqrt{\log x}}\right).$$

Hence

$$E(x,T,g) \ll x + \sum_{\substack{\rho \in H_{\mathfrak{d}} \\ |\gamma - T| \le 1}} \frac{|\nu(\rho,g)|}{|\rho|} x + \sum_{n \le 2x} a(n,g)\left(1 - \frac{n}{2x}\right) \log n$$
$$\ll x$$

and **iii)** follows.

By partial summation we have using (5.3)

$$-\frac{L'}{L}(s,\chi) = f_0(s,\chi) + \frac{\epsilon(\chi)s}{s-1} - s \sum_{g \in G} \chi(g) \sum_{\substack{\rho \in H_{\mathfrak{d}} \\ |\gamma - T| \le 1}} \frac{\nu(\rho,g)}{\rho(s-\rho)} \tag{7.1}$$
$$+ s \sum_{g \in G} \chi(g) \int_1^\infty \frac{E(\xi,T,g)}{\xi^{s+1}} d\xi,$$

Where

$$f_0(s,\chi) = \sum_{p \in P} \sum_{m \ge 2} \frac{\chi(m[p])}{||p||^{ms}} \tag{7.2}$$

is holomorphic and bounded for $\sigma > \mathfrak{d}$, and

$$\epsilon(\chi) = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{otherwise}. \end{cases}$$

We write

$$H(s,T,g) = \int_1^\infty \frac{E(\xi,T,g)}{\xi^{s+1}} d\xi \,. \tag{7.3}$$

Using (7.1), (7.2), (4.1) and (4.2) we see that $H(s,T,g)$ is holomorphic for $|t-T| \le 1, \sigma > \mathfrak{d}$ and for $\sigma > \mathfrak{d}_1, |t-T| \le \lambda_1$ with $\mathfrak{d} < \mathfrak{d}_1 < 1, 0 < \lambda_1 < 1$ we have

$$H(s,T,g) = O\left(\frac{\log \tau}{\tau}\right). \tag{7.4}$$

Let for $\alpha > 0$

$$H_\alpha(s,T,g) = \int_0^\infty \frac{E(e^u,T,g)}{\Gamma(\alpha u + 1)} e^{-su} du \,.$$

Then, using the Henkel's integral [S-Z]

$$\frac{1}{\Gamma(\alpha u + 1)} = \frac{1}{2\pi i} \int_l e^w w^{-\alpha u - 1} dw$$

taken round the contour consisting of the half-line $(-\infty, -1/e)$ on the lower side of the real axis, the circumference $C(0,1/e)$ and the half-line $(-1/e, -\infty)$ on the upper side of the real axis, we get

$$H_\alpha(s,T,g) = \frac{1}{2\pi i} \int_l e^w H(s + \alpha \log w, T, g) \frac{dw}{w} \,. \tag{7.5}$$

Let now $0 < \lambda_2 < \lambda_1, \mathfrak{d}_1 < \mathfrak{d}_2 < 1$ and $\alpha_0 = \min(\mathfrak{d}_2 - \mathfrak{d}_1, \frac{1}{\pi}(\lambda_1 - \lambda_2))$. Then for $0 < \alpha < \alpha_0, w \in l, \mathrm{Re}\, s = \sigma \ge \mathfrak{d}_2, |t-T| \le \chi_2$ we have

$$\mathrm{Re}(s + \alpha \log w) \ge \mathfrak{d}_2 + \alpha \log |w| \ge \mathfrak{d}_2 - \alpha_0 \ge \mathfrak{d}_1$$

$$|\, \mathrm{Im}(s + \alpha \log w) - T \,| = |\, t - T + \alpha \arg w \,| \le |t-T| + \alpha_0 \pi \le \lambda_1 \,.$$

Hence, using (7.4) and (7.5)

$$\begin{aligned} |H_\alpha(s,T,g)| &\ll \int_l e^{\mathrm{Re}\, w} |H(s + \alpha \log w, T, g)| \frac{|dw|}{|w|} \\ &\ll \frac{\log \tau}{\tau} \,. \end{aligned}$$

Since $H_\alpha(s,T,g) = \hat{f}_{\alpha,\sigma;g,T}(\frac{t}{2\pi})$ condition **iv)** is satisfied with $\lambda = {}^{\lambda_2}/_{(2\pi)}$ and $\mathfrak{d}' = \mathfrak{d}_2$. $\square$

8. Proof of Theorem 2

Let a be a structural G-mapping of density $\kappa = 1/h$. We construct an arithmetical semigroup S as follows. Let $\{P_g(n)\}_{(n,g)\in\mathbb{N}_1\times G}$ denote a family of sets satisfying:

1) $\#P_g(n) = a(n,g)$ for all $(n,g)\in\mathbb{N}_1\times G$,

2) $P_g(n)\cap P_{g'}(n') = \emptyset$ for $(n,g)\neq(n',g')$.

We write

$$P_g = \bigcup_{n\in\mathbb{N}_1} P_g(n), \quad P = \bigcup_{g\in G} P_g\,.$$

Let $D = \mathcal{F}(P)$. For $a\in D$ and $g\in G$ we write as usual

$$\Omega_g(a) = \sum_{\substack{p^k|a,p^{k+1}\nmid a\\ p\in P_g}} k \tag{8.1}$$

and put

$$S = \{a\in D|\sum_{g\in G}\Omega_g(a)g = 0\}\,.$$

S is a subsemigroup of D and the injection $\partial : S\to D$ is a divisor theory. It can easily be seen that $Cl(S)\simeq G$.

We prove now that S with a suitably defined norm-function is an arithmetical semigroup satisfying requirements of Theorem 2.

We define the function $||\cdot||$ on P as follows: $||p|| = n$ when $p\in\bigcup_{g\in G} P_g(n)$. Next, we extend it to D multiplicatively.

Using (5.3) and (5.4) we have

$$\sum_{\substack{||p||\le x\\ p\in P}} 1 = \sum_{g\in G}\sum_{\sqrt{x}\le n\le x} a(n,g) + 0(x^{1/2})$$

$$\ll \frac{1}{\log x}\sum_{n\in g}\sum_{n\le x} a(n,g)\log n + x^{1/2} \ll x/\log x\,.$$

Hence

$$\sum_{p\in P}\frac{1}{\|p\|^2}\ll 1,\ \sum_{\substack{\|p\|\le x\\ p\in P}}\frac{1}{\|p\|}\ll \log\log x$$

and consequently

$$\sum_{\substack{\|a\|\le x\\ a\in D}}\frac{1}{\|a\|}\le \prod_{\substack{\|p\|\le x\\ p\in P}}\left(1-\frac{1}{\|p\|}\right)^{-1}=\exp\Big(\sum_{\substack{\|p\|\le x\\ p\in P}}\frac{1}{\|p\|}+O(1)\Big)=(\log x)^{O(1)}\,.$$

Hence, by partial summation, all $L(s,\chi),\chi\in\hat{G}$ converge absolutely for $\sigma>1$. Therefore

$$L(s,\chi)=\prod_{p\in P}\left(1-\frac{\chi([p])}{\|p\|^s}\right)^{-1}\qquad\text{for } \sigma>1\,.$$

Taking logarithm, derivating and using (5.3) we arrive at (7.1). But this time (7.1) is valid for $\sigma>1$ only. We want to continue analytically L'/L to the left; for this we need more information about $H(s,T,g)$ (defined by (7.3)).

Let us consider

$$H_\alpha(\sigma+it,T,g)=\hat{f}_{\alpha,\sigma;g,T}(t/2\pi)\,.$$

For $\alpha>0$ this is an entire function and according to (5.5).

$$|H_\alpha(s,T,g)|=O\left(\frac{\log\tau}{\tau}\right)$$

uniformly for $\mathfrak{d}'\le\sigma\le 2,|t-T|\le 2\pi\lambda,0<\alpha<\alpha_0$.

From the classical theorem of Stielties-Osgood [S-Z, page 119] there exist a sequence $\alpha_n\to 0,0<\alpha_n<\alpha_0$ and a function $H_0(s,T,g)$ holomorphic for

$$\mathfrak{d}'<\sigma\le 2,\quad |t-T|\le 2\pi\lambda \tag{8.2}$$

such that $\lim\limits_{n\to\infty}H_{\alpha_n}(s,T,g)=H_0(s,T,g)$ uniformly for $s=\sigma+it$ satisfying (8.2).

Let l be the same contour as in the proof of Theorem 1. For $3/2 \leq \sigma \leq 2, |t-T| \leq 2\pi\lambda, w \in l$ we have $\mathrm{Re}(s+\alpha_n \log w) \geq 3/2 - \alpha_n \geq 5/4$ for sufficiently large n. Hence

$$|H(s+\alpha_n \log w, T, g) - H(s,T,g)| \leq \alpha_n |\log w| \max_{z \in [s, s+\alpha_n \log w]} |H'(z)|$$

$$\ll \alpha_n |\log w| \int_1^\infty \xi^{-5/4} \log \xi d\xi \ll \alpha_n |\log w|$$

and consequently

$$|H_{\alpha_n}(s,T,g) - H(s,T,g)| \ll \int_l e^{\mathrm{Re} w} |H(s+\alpha_n \log w, T, g) - H(s,T,g)| \frac{|dw|}{|w|}$$

$$\ll \alpha_n \int_l e^{\mathrm{Re} w} \frac{|\log w|}{|w|} |dw| \ll \alpha_n \to 0$$

as $n \to \infty$. This means that $H = H_0$ and H is holomorphic for $s = \sigma + it, \sigma > \vartheta', |t-T| \leq 2\pi\lambda$. Moreover, for such s we have

$$|H(s,T,g)| = \lim_{n\to\infty} |H_{\alpha_n}(s,T,g)| \ll \frac{\log \tau}{\tau}.$$

Hence, according to (7.1), L'/L has analytic continuation to a meromorphic function on $\sigma > \vartheta'$ and in this region we have

$$\frac{L'}{L}(s,\chi) = -\frac{\epsilon(\chi)}{s-1} + \sum_{\substack{\beta > \vartheta \\ |\gamma - t| \leq 1}} \frac{m(\rho,\chi)}{s-\rho} + O(\log \tau).$$

Thus L'/L has simple poles at $s = \rho$ with residues $m(\rho,\chi)$ and (in case $\chi = \chi_0$) a simple pole at $s = 1$ with residue -1.

For $\sigma > 1$ we have

$$L(s,\chi) = L(2,\chi) \exp\Big\{\int_2^s \frac{L'}{L}(z,\chi) dz\Big\};$$

the path of integration consists of two line segments joining 2 with $2+it$ and $\sigma + it$. This gives analytic continuation of $L(s,\chi)$ to a meromorphic function on the half-plane $\sigma > \vartheta'$. $L(s,\chi)$ has zeros at $s = \rho$ with multiplicities $m(\rho,\chi)$; $L(s,\chi_0)$ has a simple pole at $s = 1$.

For $\mathfrak{d}' < \sigma < 2, |t| \geq 1$ we have

$$|L(s,\chi)| \ll \exp\{\mathrm{Re} \int_{2+it}^{\sigma+it} \frac{L'}{L}(z,\chi)dz\}$$

$$= \exp\{\mathrm{Re} \sum_{\substack{\rho \in H_{\mathfrak{d}} \\ |\gamma - t| \leq 1}} \int_{2+it}^{\sigma+it} \frac{m(\rho,\chi)}{z-\rho} dz + O(\log\tau)\}$$

$$= \prod_{\substack{\rho \in H_{\mathfrak{d}} \\ |\gamma - t| \leq 1}} \left|\frac{s+it-\rho}{2+it-\rho}\right|^{m(\rho,\chi)} \tau^{O(1)} = \tau^{O(1)}.$$

Hence $L(s,\chi)$ is of finite order for $\sigma > \theta$. The Lindelöf function μ_χ is therefore well-defined (see (4.4)) and, as usual, it is non-increasing, continuous and vanishes for $\sigma \geq 1$. Let $\mathfrak{d}'' \in (\mathfrak{d}', 1)$ be such that $\mu_\chi(\mathfrak{d}'') = 1/2$ for all $\chi \in \hat{G}$. Using Perron's formula and shifting the line of integration to $\sigma = \mathfrak{d}''$, we get

$$\sum_{\substack{\|a\| \leq x \\ a \in g}} (1 - \frac{\|a\|}{x}) = \frac{\alpha}{2h} x + O(x^{\mathfrak{d}''}),$$

where $\alpha = \underset{s=1}{\mathrm{Re}}\, s\, L(s,\chi_0)$. Condition $\tilde{A}$ therefore follows.

Since $L'/L(s,\chi), \chi \neq \chi_0$ are regular at $s = 1$, we have $L(1,\chi) \neq 0 \quad (\chi \neq \chi_0)$. Hence S is an arithmetical semigroup of type α. Using Proposition 3 we see that S is uniquely determined up to a-isomorphism; Theorem 2 therefore follows. □

9. Proof of Theorem 3

Let us consider the function

$$Z(s) = \prod_{\chi \in \hat{G}} L(s,\chi) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \sigma > 1.$$

The local factor corresponding to $p \in g \cap P$ equals

$$\prod_{\chi \in \hat{G}} (1 - \frac{\chi(g)}{\|p\|^s})^{-1} = (1 - \frac{1}{\|p\|^{s \operatorname{ord} g}})^{-\frac{h}{\operatorname{ord} g}}.$$

Hence the coefficients a_n are non-negative. Since the zero of $L(s,\chi_1)$ at $s = 1$ cancels the pole of $L(s,\chi_0)$, $Z(s)$ is holomorphic at $s = 1$. From Landau's theorem it follows that the series $\sum_{n=1}^{\infty} a_n n^{-s}$ converges absolutely for $\sigma > \theta_0$ with $\theta_0 < 1$. Hence $Z(s)$ is equals to its Euler product and thus $L(s,\chi) \neq 0$ for $\sigma > \theta_0$, $(s,\chi) \neq (1,\chi)$. In particular we see that there exists exactly one character χ with $L(1,\chi) = 0$. It is necessarily real and thus $h = 2h_1 = 2\#\ker\chi_1$.

Using the Borel-Caratheodory and Hadamard three circle theorem we see that $\mu_\chi(\sigma) = 0$ for $\sigma \geq \theta_0$. Applying Perron's formula we obtain for $\theta \in (\theta_0, 1)$

$$\sum_{g\in G} \chi(g) \sum_{n\leq x} a_g(n) \log n = \epsilon_1(\chi)x + O(x^\theta),$$

where

$$\epsilon_1(\chi) = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ -1 & \text{if } \chi = \chi_1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\sum_{n\leq x} a_g(n)\log n = \frac{\epsilon_g}{h_1}x + E_g(x)$, with $E_g(x) = O(x^\theta)$ as $x \to \infty$ and

$$\epsilon_g = \begin{cases} 1 & \text{for } g \notin \ker\chi_1, \\ 0 & \text{for } g \in \ker\chi_1. \end{cases}$$

This proves (6.2). Moreover, using the Borel-Caratheodory theorem once more we see that $|L'/L(s,\chi)| \ll \log\tau$ for $|s-1| \geq 1, \sigma > \theta$. Hence writing

$$\varphi_{\sigma,g}(u) = \begin{cases} E_g(e^u)e^{-\sigma u} & \text{for } u \geq 0, \\ 0 & \text{for } u < 0, \end{cases}$$

we get $(s = \sigma + it)$:

$$\hat{\varphi_\sigma}\left(\frac{t}{2\pi}\right) = \int_1^\infty \frac{E_g(u)}{u^{s+1}}du$$

$$= \frac{1}{s}\frac{1}{h}\sum_{\chi\in\hat{G}} \overline{\chi(g)}\left\{-\frac{L'}{L}(s,\chi) - f_0(s,\chi) - \frac{\epsilon_1(\chi)s}{s-1}\right\} = O\left(\frac{\log\tau}{\tau}\right).$$

From Proposition 2 it follows now that a_g is a structural, quasi-Riemannian mapping of density $1/h_1$ and the result follows. □

10. Proof of Theorem 4

By an obvious modification of the proof of theorem 2 we construct an arithmetical semigroup S satisfying (6.3) and (6.4). It remains to prove that it is of type β. Let $\tilde{\chi_1}$ be the non-trivial character of G/H. Then $\chi_1 = \tilde{\chi_1} \circ \emptyset$, where $\emptyset : G \to G/H$ is the canonical surjection is a real character of G. Of course $\ker \chi_1 = H$ and by (5.7) we have ($\kappa = 1/h_1$)

$$\frac{L'}{L}(s, \chi_1) = \sum_{g \notin H} \int_2^\infty \sum_{n \leq \xi} a_g(n) \log n \frac{d\xi}{\xi^{\sigma+1}} + O(1) = \frac{1}{\sigma - 1} + O(1).$$

Hence $L(1, \chi_1) = 0$ and the result follows. □

11. Corollaries to structural theorems

COROLLARY 1. *Let G be a finite abelian group of order h, $h = 2h_1$, and χ_1 a non-trivial real character of G. If the Generalized Riemann Hypothesis (G.R.H.) for Dirichlet's L-function (mod h_1^2) holds, then there exist infinitely many non a-isomorphic semigroups of type β with divisor class group isomorphic to G and such that $L(1, \chi_1) = 0$.*

Proof. Under the G.R.H. the function

$$a(n) = \begin{cases} \varphi(h_1) & \text{if } n \text{ is prime, } n \equiv 1 \pmod{h_1^2}, \\ 0 & \text{otherwise,} \end{cases}$$

is a quasi-Riemannian, structural mapping of density $1/h_1$. Now it suffices to apply Theorem 4 with $a_g = a$ for $g \notin \ker \chi_1$ and

$$b_g(n) = \begin{cases} k & \text{for } n = 2, \\ 0 & \text{otherwise,} \end{cases}$$

for $g \in \ker \chi_1$; $k = 1, 2, \ldots$. □

COROLLARY 2. *Let G be a finite abelian group. Then there exist infinitely many non a-isomorphic arithmetical semigroups of type α with divisor class group isomorphic to G.*

Proof. It is evident that if $\phi : G' \to G$ is a group epimorphism and $a : \mathbb{N}_1 \times G' \to \mathbb{N}$ a structural G'-mapping of density κ, then $a_\phi : \mathbb{N}_1 \times G \to \mathbb{N}$ defined by

$$a_\phi(n, g) = \sum_{g' \in \phi^{-1}(\{g\})} a(n, g')$$

is a structural G-mapping of density $\kappa \# \ker \phi$. Let us observe now that for every finite abelian group G there exist a natural number q and a group epimorphism $\phi : (\mathbb{Z}/q\mathbb{Z})^\times \to G$. Moreover, the function $a : \mathbb{N}_1 \times (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{N}$ defined by

$$a(n,g) = \begin{cases} 1 & \text{when } n \text{ is prime, } n + q\mathbb{Z} = g, \\ 0 & \text{otherwise,} \end{cases}$$

is a structural $(\mathbb{Z}/q\mathbb{Z})^\times$-mapping of density $1/\varphi(q)$. Hence a_ϕ is structural of density $1/h, h = \#G$. Together with a_ϕ, the functions $(k = 1, 2, \dots)$

$$a_\phi^{(k)}(n,g) = \begin{cases} a_\phi(n,g) & \text{for } n \geq 3, \\ a_\phi(2,g) + k & \text{for } n = 2, \end{cases}$$

are structural of densities $1/h$. We apply now theorem 2 and observe that the resulting semigroups are non a-isomorphic. $\square$

12. Quantitative aspects of factorization in arithmetical semigroups

Throughout this chapter, let S be a semigroup with divisor theory $\partial : S \to \mathcal{F}(P)$, finite divisor class group G and $G_0 = \{g \in G \mid g \cap P \neq \emptyset\}$. Every element $a \in S \setminus S^\times$ has a factorization $a = u_1 \dots u_k$ into irreducible elements $u_1, \dots, u_k \in S$, where k is called length of the factorization. Since in general S is not factorial, several factorization properties were introduced to describe the non-uniqueness of factorization.

If S is an arithmetical semigroup and $Z \subseteq \partial S$ a subset defined by a factorization property, one can study the asymptotic behaviour of the corresponding counting function

$$Z(x) = \#\{\alpha \in Z \mid ||\alpha|| \leq x\}.$$

These investigations were started by W. Narkiewicz (in the case where S is the multiplicative semigroup of the ring of integers in an algebraic number field), and among others the following sets have been considered, for every $k \in \mathbb{N}_+$,

$$\begin{aligned} M'_k &= \{\partial a \mid k \in L(a)\}, \\ M_k &= \{\partial a \mid \min L(a) \leq k\}, \\ G_k &= \{\partial a \mid \#L(a) \leq k\}, \\ F_k &= \{\partial a \mid a \text{ has at most } k \text{ distinct factorization}\}; \end{aligned}$$

here $L(a)$ denotes the set of lengths of possible factorizations of a. Concerning the contribution of various authors to the study of the asymptotic behaviour of these and other counting functions, see ([Ge3], [HK3], [HK-M]) and the literature cited there.

We deal with M_k', M_k, G_k and a set being defined later. In this chapter we introduce the necessary notions to state the results; the proofs will be carried out in the following chapters.

Let $\mathfrak{L}(S) = \{L(a) \mid a \in S \setminus S^\times\}$ denote the system of sets of lengths and for $L \in \mathfrak{L}(S)$ let

$$\Delta(L) = \{s - r \mid r < s; r, s \in L;\ t \notin L \text{ for } r < t < s\}$$

the set of distances and

$$\Delta(S) = \bigcup_{L \in \mathfrak{L}(S)} \Delta(L).$$

We recall briefly the notion of block semigroups. Let $H \subseteq G$ be a non-empty subset and $\mathcal{F}(H)$ the free abelian semigroup with basis H, i.e. every element $S \in \mathcal{F}(H)$ has the form

$$S = \prod_{g \in H} g^{v_g(S)} \quad \text{with } v_g(S) \in \mathbb{N}\,,$$

and we call

$$\sigma(S) = \sum_{g \in H} v_g(S) \in \mathbb{N}$$

the size of S.

The subsemigroup $\mathcal{B}(H) = \{S \in \mathcal{F}(H) \mid \sum_{g \in H} v_g(S) g = 0 \in G\} \subseteq \mathcal{F}(H)$ is called block semigroup over H and the elements of $\mathcal{B}(H)$ are called blocks. Block semigroups were introduced in [N] and it was first observed in [HK1] that they admit a divisor theory.
Let the block homomorphism $\beta : S \to \mathcal{B}(G)$ be defined by $\beta(a) = 1$, if $a \in S^\times$ and by

$$\beta(a) = [p_1] \ldots [p_m]$$

if $\partial a = p_1 \ldots p_m$.

For $a \in S$ and $g \in G_0$ we set $v_g(a) = v_g(\beta(a))$ and we write $\mathfrak{L}(H)$ and $\Delta(H)$ instead of $\mathfrak{L}(\mathcal{B}(H))$ and $\Delta(\mathcal{B}(H))$. The importance of block semigroups for the investigation of non-unique factorizations in semigroups with divisor theory lies in the following fundamental correspondence between S and $\mathcal{B}(G_0)$ (cf. [Ge1, Proposition 1]):

i) an element $a \in S$ is irreducible if and only if $\beta(a)$ is irreducible in $\mathcal{B}(G_0)$.

ii) $L(a) = L(\beta(a))$.

iii) $\Delta(S) = \Delta(G_0)$ and $\mathfrak{L}(S) = \mathfrak{L}(G_0)$.

If Z is either M_k', M_k, or G_k, we proceed as follows: we show that Z is a finite union of sets of the form

$$\Omega(H, \mathrm{s}) = \{\partial a \mid a \in S \setminus S^\times \text{ with } v_g(a) = s_g \text{ for every } g \in G_0 \setminus H\},$$

where $H \subseteq G_0$ is a subset and $\mathrm{s} \in \mathbb{N}^{G_0 \setminus H}$. This turns out to be a purely combinatorial problem and the remaining analytical task is to study $\Omega(H, \mathrm{s})(x)$, which will be tackled in chapter 14.

In order to state our result on $M_k(x)$, we define, for a non-empty subset H of G, Davenport's constant

$$D(H) = \max\{\sigma(B) \mid B \in \mathcal{B}(H) \text{ irreducible }\} \in \mathbb{N},$$

which has turned out to be an important invariant in the qualitative as well as in the quantitative study of non-unique factorization. The value of $D(G)$, however, is known just for special types of groups i.e. p-groups and groups with maximal p-rank $r \leq 2$ (cf. [M]).

Theorem 5. *Let S be an arithmetical semigroup with divisor class group G and $k \in \mathbb{N}_+$. Then*

$$M_k^{(\prime)}(x) = \frac{x}{\log x} V^{(\prime)}(\log\log x) + O\Big(\frac{x(\log\log x)^{b_0^{(\prime)}}}{(\log x)^2}\Big),$$

where $V^{(\prime)} \in \mathbb{C}[X]$ has positive leading coefficient and $b_0^{(\prime)} \in \mathbb{N}$ with $0 \leq b_0^{(\prime)} \leq \deg V^{(\prime)} + 1$.

If S is of type α, then $\deg V^{(\prime)} = kD(G) - 1$.
If S is of type β, then $\deg V^{(\prime)} = kD(G \setminus \ker \chi_1) - 1$.

Next we consider the sets G_k for $k \in \mathbb{N}_+$. By definition the following conditions are equivalent:

i) $\Delta(S) = \emptyset$.

ii) $\#L = 1$ for every $L \in \mathfrak{L}(S)$.

iii) $G_k = G_1 = \partial S$ for every $k \in \mathbb{N}_+$.

It can be seen easily, that, if $\Delta(S) \neq \emptyset$, then $\partial S \supsetneqq G_k$ for any $k \in \mathbb{N}_+$. In the next proposition we list all cases with $\Delta(S) = \emptyset$, and since it will be of interest in a moment, all cases with $\#\Delta(S) = 1$.

PROPOSITION 4. *Let G be a finite abelian group.*

1) **i)** $\Delta(G) = \emptyset$ *if and only if* $G \in \{C_1, C_2\}$.

ii) $\Delta(G) = \{1\}$ *if and only if* $G \in \{C_3, C_3^2, C_2^2\}$.

iii) *In all other cases* $\#\Delta(G) \geq 2$ *and* $1 = \min \Delta(G)$.

2) *Let $K < G$ be a subgroup of index 2 and $G_0 \subseteq G$ a subset such that $G \setminus K \subseteq G_0$.*

i) $\Delta(G_0) = \emptyset$ *if and only if* $G \in \{C_2, C_2^2\}$ *and* $G_0 \subseteq \{0\} \cup (G \setminus K)$.

ii) $\Delta(G_0) = \{1\}$ *if and only if* $G = G_0 = C_2^2$.

iii) $\Delta(G_0) = \{2\}$ *if and only if* $G \in \{C_4, C_2^3\}$ *and* $G_0 \subseteq \{0\} \cup (G \setminus K)$.

iv) $\Delta(G_0) = \{2, 4\}$ *if and only if* $G = C_6$ *and* $G_0 \subseteq \{0\} \cup (G \setminus K)$.

v) *In all other cases* $\#\Delta(G_0) \geq 2$ *and* $1 = \min \Delta(G_0)$.

Proof. **1)** If $G \in \{C_1, C_2\}$, then $\Delta(G) = \emptyset$. If $\#G \geq 3$, then $1 = \min \Delta(G)$ (see [Ge2, Theorem 1], for example). Because $\max \Delta(G) \leq D(G) - 2$ ([Ge1, Prop.3]), it follows $\Delta(G) = \{1\}$ for $G \in \{C_3, C_2^2\}$.

To show that $\Delta(C_3^2) = \{1\}$, take irreducible blocks $A_1, \ldots, A_i, B_1, \ldots, B_k \in \mathcal{B}(C_3^2)$ with $A_1 \ldots A_i = B_1 \ldots B_k, i < k$ and $j \notin L(A_1 \ldots A_i)$ for $i < j < k$. Then it has to be verified that $k = i + 1$. We proceed by induction on i. For $i = 2$ this follows from the special form of irreducible blocks of size 4 and 5.
The induction step is similar to that of Prop. 3 in [Ge1].

If $G \notin \{C_1, C_2, C_3, C_3^2, C_2^2\}$, then either C_2^3 or C_3^3 are subgroups of G, or $\exp(G) \geq 4$. In each case it can be seen easily that $\#\Delta(G) \geq 2$.

2) If G_0 and G are as in **i)** - **iv)**, then $\Delta(G_0)$ has the asserted form. If $G \in \{C_2^3, C_4, C_6\}$ and $G_0 \setminus \{0\} \supsetneqq (G \setminus K)$, then $1 = \min \Delta(G_0)$.

Now assume $G \notin \{C_2, C_2^2, C_2^3, C_4, C_6\}$ and choose $k \in \mathbb{N}_+$ maximal such that $G \simeq G_{2k} \oplus H$ and $K \simeq C_k \oplus H$. We construct irreducible blocks $B_1, B_2 \in \mathcal{B}(G \setminus K)$, such that $B_1 B_2$ has a factorization into three irreducible blocks.

a) $k \geq 4$. For $g \in C_{2k}$ with $\mathrm{ord}(g) = 2k$ let
$B_1 = (-3g, 0)^2(5g, 0)(g, 0)$ and $B_2 = (g, 0)^{2k}$.

b) $k \geq 2$ and $\#H \geq 2$. For $g \in C_{2k}$ with $\mathrm{ord}(g) = 2k$ and $0 \neq h \in H$ let
$B_1 = (-g, -h)(-g, h)(-g, 0)(3g, 0)$ and $B_2 = (g, 0)^{2k}$.

c) $k = 1$. If H contains an element g of order 4, then choose
$B_1 = B_2 = (1, g)^2(1, 2g)(1, 0)$.
If H contains three linear independent elements e_1, e_2, e_3 of order 2, then choose
$B_1 = (1, e_1)(1, e_2)(1, e_1 + e_2)(1, 0)$ and
$B_2 = (1, e_2 + e_3)(1, e_1 + e_3)(1, e_1 + e_2)(1, 0)$.

It remains to verify that for $G \notin \{C_2, C_2^2, C_2^3, C_4, C_6\}$ $\#\Delta(G_0) \geq 2$. Just as before, this can be seen by obvious examples. □

For a non-empty subset H of G we set

$$\mu(H) = \max\{\#H' \mid H' \subseteq H \text{ with } \Delta(H') = \emptyset\}.$$

In the following chapter we derive upper and lower bounds for the invariant $\mu(G)$ and determine its precise value for some special types of abelian groups.
Finally for $H \subseteq G_0$ with $\Delta(H) = \emptyset$ and $k \in \mathbb{N}_+$ let

$$S(G_0, H, k) = \{\mathrm{s} \in \mathbb{N}^{G_0 \setminus H} \mid \emptyset \neq F(H, \mathrm{s}) \subseteq G_k\};$$

we set $S(G_0, G_0, k) = \emptyset$; if $\#H = \mu(G_0)$, then obviously $S(G_0, H, k)$ is finite.

THEOREM 6. *Let S be an arithmetical semigroup with divisor class group G, $\Delta(S) \neq \emptyset$ and $k \in \mathbb{N}_+$. Then*

$$G_k(x) = \frac{x}{(\log x)^{1-\lambda_0}} V(\log\log x) + O\Big(\frac{x(\log\log x)^{b_0}}{(\log x)^{1-\lambda_0+\gamma}}\Big),$$

where $V \in \mathbb{C}[X]$ has positive leading coefficient, $b_0 \in \mathbb{N}$ with $0 \leq b_0 \leq \deg V + 1$.
If S is of type α, then $\lambda_0 = \frac{\mu(G)}{h}$,

$$\gamma = \begin{cases} 1 & h = 1, \\ \frac{1}{h}\big(1 - \cos\frac{2\pi}{h}\big) & \textit{otherwise}, \end{cases}$$

and

$$\deg V = \max\Big\{ \sum_{g\in G\setminus H} s_g \Big| s \in S(G,H,k),$$
$$H \subseteq G \text{ with } \Delta(H)=\emptyset \text{ and } \#H = \mu(G)\Big\}.$$

If S is of type β, then $\lambda_0 = \frac{2}{h}\mu(G\setminus \ker\chi_1), \gamma = \min\{1, \frac{2}{h}(1-\cos\frac{2\pi}{h})\}$ *and*

$$\deg V \geq \max\Big\{ \sum_{g\in (G\setminus \ker\chi_1)\setminus H} s_g \Big| s \in S(G\setminus\ker\chi_1, H, k),$$
$$H \subseteq G\setminus\ker\chi_1 \text{ with } \Delta(H)=\emptyset \text{ and } \#H = \mu(G\setminus\ker\chi_1)\Big\}.$$

In [Ge1, Satz 1] it was shown that in the semigroup S sets of lengths have a well-defined structure. Indeed, it was proved that there exists a constant $M(S)$ such that for every $L \in \mathfrak{L}(S)$

$$\begin{array}{llll} L = & & & \\ \{x_1,\dots,x_\alpha, & y,\; y+\delta_1, & \dots, y+\delta_l, & y+d, \\ & y+\delta_1+d, & \dots, y+\delta_l+d, & y+2d, \\ & \dots & \dots\quad\dots & \dots \\ & y+\delta_1+(k-1)d, & \dots, y+\delta_l+(k-1)d, & y+kd,\; z_1,\dots,z_\beta\} \end{array}$$

with $x_1 < \dots < z_\beta, l \geq 0, \alpha \leq M(S), \beta \leq M(S)$ and $d \leq D(G)-2$.

There are examples which show that α, β and l can be strictly positive. However, most $L \in \mathfrak{L}(S)$ have a typical form which is (at least if S is of type α) as simple as possible. To make this precise, we define the following sets

$$T = \{\partial(a)\,|L(a) = \{x_1,\dots,x_\alpha,y,y+1,\dots,y+k,z_1,\dots,z_\beta\} \text{ with}$$
$$x_1 < \dots < z_\beta; \alpha,\beta \leq M(S) \text{ and } \Delta(L(a)) \subseteq \{1,2\}\},$$

$T_\delta = \{\partial(a) \mid \Delta(L(a)) \subseteq \{\delta\}\}$ with $\delta \in \{1,2\}$, and we show:

THEOREM 7. *Let S be an arithmetical semigroup with $\#\Delta(S) \geq 2$.*

1) *If S is of type α, then*

$$T_1(x) = \partial S(x) + O\Big(\frac{x}{(\log x)^{1/h}}\Big).$$

2) *If S is of type β, G its divisor class group and G_0 the set of classes containing prime divisors, then we have:*

If $G = C_6$ and $G_0 \subseteq \{0\} \cup (G \setminus \ker \chi_1)$, then

$$T_2(x) = \partial S(x) + O\Big(\frac{x(\log\log x)^\delta}{(\log x)^{2/h}}\Big)\,;$$

otherwise

$$T(x) = \partial S(x) + O\Big(\frac{x(\log\log x)^\delta}{(\log x)^{2/h}}\Big)\,.$$

In both cases $\delta \in \mathbb{N}$.

13. Some results on $\mu(G)$

The following properties are easy consequences from the definition.

PROPOSITION 5. *Let G be a finite abelian group.*

1) *$\mu(C_1) = 1$, and if $\#G \geq 2$, then*

$$2 \leq \mu(G) \leq \frac{\#G - 2^r}{2} + r + 1\,,$$

where r denotes the 2-rank of G.

2) *If $G = \bigoplus_{i=1}^{r} G_i$, then $\mu(G) \geq 1 + \sum_{i=1}^{r}(\mu(G_i) - 1)$.*

Proof. 1) Let $\#G \geq 2$; then for every $0 \neq g \in G$ $\Delta(\{0, g\}) = \emptyset$ and thus $\mu(G) \geq 2$.
Let $G_0 = \{a_1, \dots, a_k, b_1, \dots, b_l\} \subseteq G$ with $\Delta(G_0) = \emptyset, \#G_0 = \mu(G)$, ord $(a_i) > 2$ and ord$(b_j) \leq 2$. Since for every $a \in G$ with ord$(a) \geq 3$ $\Delta(\{a, -a\}) \neq \emptyset$, it follows $k \leq \frac{\#G - 2^r}{2}$. Further we have $l \leq \mu(C_2^r)$. Because $\mu(C_2^r) = r + 1$ (this can be seen directly or by the next Theorem) we are done.

2) Let $i \in \{1, \dots, r\}, H_i \subseteq G_i$ a subset with $0 \notin H_i$, $\Delta(H_i) = \emptyset$ and $\#H_i = \mu(G_i) - 1$; further let $\pi_i : G_i \to \bigoplus_{j=1}^{r} G_j$ be the canonical embedding.

Then $\Delta(\{0\} \cup \bigcup_{i=1}^{r} \pi_i(H_i)) = \emptyset$, which implies the assertion. □

The subsequent results rest upon a characterization due to L. Skula of subsets H of a finite abelian group G with $\Delta(H) = \emptyset$. For this we define the cross number

$$k(B) = \sum_{g \in G} \frac{v_g(B)}{\operatorname{ord}(g)}$$

of a block $B \in \mathcal{B}(G)$.

PROPOSITION 6. *For a subset H of a finite abelian group G the following conditions are equivalent:*

1) $\Delta(H) = \emptyset$.

2) $k(B) = 1$ *for every irreducible block* $B \in \mathcal{B}(H)$.

Proof. [Sk, Theorem 3.1] □

PROPOSITION 7. *For $n = p_1 \dots p_r p_{r+1}^{k_{r+1}} \dots p_s^{k_s}$ with distinct primes $p_1, \dots, p_r$ and $k_{r+1} \geq 2, \dots, k_s \geq 2$ we have*

$$1 + [\frac{3r}{2}] + \sum_{i=r+1}^{s} k_i \leq \mu(C_n) \leq d(n)\,,$$

where $d(n)$ denotes the number of divisors of n.

Proof. The right inequality is due to J. Sliwa [Sl, Theorem 3].
For the left inequality it suffices to show that

1) $\mu(C_{p_1 p_2}) \geq 4$

2) $\mu(C_{p^k}) \geq k + 1$;

then Proposition 5.2 implies the assertion. **1)** is true, since for $g \in G$ with $\operatorname{ord}(g) = p_1 p_2$, $\Delta(\{0, g, p_1 g, p_2 g\}) = \emptyset$; and **2)** holds, since for $g \in G$ with $\operatorname{ord}(g) = p^k$, $\Delta(\{0, g, pg, \dots, p^{k-1} g\}) = \emptyset$. □

Next we consider elementary p-groups. For $p \in \mathbb{P}$ and $r \in \mathbb{N}_+$, C_p^r may be viewed as an r-dimensional vector space over the finite field F_p.

LEMMA 1. *Let $p \in \mathbb{P}, r \in \mathbb{N}_+$ and $H \subset C_p^r$ with $\Delta(H) = \emptyset$ and $\#H = \mu(C_p^r)$.*

1) *For an irreducible block $B \in \mathcal{B}(H)$ we have $k(B) = 1$ if and only if $\sigma(B) = p$.*

2) *H contains a basis* $e_1, \dots, e_r$ *of* C_p^r.

3) *If* $a \in H$ *with* $a = \sum_{i=1}^{r} a_i e_i, 0 \le a_i \le p-1$, *then* $\sum_{i=1}^{r} (p - a_i) = p - 1$.

4) *If* $a, a' \in H$ *with* $a \neq a'$ *and* $a = \sum_{i=1}^{r} a_i e_i, a' = \sum_{i=1}^{r} a_i' e_i, 0 \le a_i, a_i' \le p-1$, *then* $a_i \neq a_i'$ *for every* $a_i \neq 0$.

Proof. **1)** is obvious.

2) Let $\{e_1, \dots, e_r\} \subseteq H$ be a basis of the subgroup $\bar{H}$ generated by H. Then $1 \le k \le r$, and since by Proposition 5 $k < r$ implies $\mu(\bar{H}) < \mu(C_p^r)$, it follows $k = r$.

3) Since $B = a \prod_{i=1}^{r} e_i^{p-a_i}$ is an irreducible block in $\mathcal{B}(H)$ with $\sigma(B) = 1 + \sum_{i=1}^{r} (p - a_i)$, **1)** implies the assertion.

4) Assume to the contrary, that $a_i = a_i' \neq 0$ for some $i \in \{1, \dots, r\}$. Then there are $n_j \in \mathbb{N}$ such that

$$B = a^{p-1} a' \prod_{\substack{j=1 \\ j \neq i}}^{r} e_j^{n_j} \in \mathcal{B}(H)$$

is irreducible, but $a \neq a'$ implies $\sum_{j \neq i} n_j > 0$, and thus $\sigma(B) = p + \sum_{j \neq i} n_j > p$, a contradiction. □

THEOREM 8. *For* $p \in P$ *and* $r \in \mathbb{N}_+$

$$1 + \left(r - 2\left[\frac{r}{2}\right]\right) + p\left[\frac{r}{2}\right] \le \mu(C_p^r) \le 1 + \left[p\frac{r}{2}\right].$$

Proof. **1)** For the left inequality it suffices to show that $\mu(C_p^2) \ge p + 1$, since then it follows

$$\mu(C_p^r) = \mu\left((C_p^2)^{[\frac{r}{2}]} \oplus C_p^{r-2[\frac{r}{2}]}\right) \ge 1 + p \cdot \left[\frac{r}{2}\right] + \left(r - 2\left[\frac{r}{2}\right]\right) \cdot 1.$$

For a basis e_1, e_2 of C_p^2 we consider $H = \{0, i\, e_1 + (1-i)e_2 \mid 0 \le i \le p-1\}$. Let

$$B = \prod_{i=0}^{p-1} (i\, e_1 + (1 - i)e_2)^{n_i} \in \mathcal{B}(H)$$

be irreducible. Then

$$\sum_{i\equiv 0}^{p-1} n_i i \equiv 0 \mod p \quad \text{and}$$

$$\sum_{i=0}^{p-1} n_i(1-i) \equiv 0 \mod p\,.$$

Therefore $\sum\limits_{i=0}^{p-1} n_i \equiv 0 \mod p$, and because $\sigma(B) = \sum\limits_{i=0}^{p-1} n_i \le D(C_p^2) = 2p-1$, it follows $\sigma(B) = p$.

2) To show the right inequality, we take a subset $H \subseteq C_p^r$ with $\Delta(H) = \emptyset$ and $\#H = \mu(C_p^r)$. There is a basis $e_1, \dots, e_r$ of C_p^r such that

$$H = \{0, e_1, \dots, e_r, a^{(1)}, \dots, a^{(\kappa)}\}$$

with $a^{(i)} = \sum\limits_{j=1}^{r} a_j^{(i)} e_j$ and $0 \le a_j^{(i)} \le p-1$.

Let $\mathcal{D}$ be the set of all subsets $D \subseteq \{1, \dots, r\}$ with $\#D \le r-2$, For $1 \le i \le \kappa$, we set

$$D(a^{(i)}) = \{1 \le j \le r \mid a_j^{(i)} = 0\},$$

and for $D \in \mathcal{D}$ let

$$N(D) = \#\{1 \le i \le \kappa \mid D(a^{(i)}) = D\}.$$

Since $\mu(C_p) = 2$, it follows $D(a^{(i)}) \in \mathcal{D}$, for $1 \le i \le \kappa$, and therefore

$$\sum_{D\in\mathcal{D}} N(D) = \kappa.$$

By Lemma 1.4 we have, for every $j \in \{1, \dots, r\}$

$$\sum_{\substack{D\in\mathcal{D}\\ j\notin D}} N(D) \le p-2$$

and hence

$$\sum_{j=1}^{r} \sum_{\substack{D\in\mathcal{D}\\ j\notin D}} N(D) \le r(p-2).$$

On the other hand, since $\#D \le r-2$ for every $D \in \mathcal{D}$, it follows

$$2\sum_{D\in\mathcal{D}} N(D) \le \sum_{j=1}^{r} \sum_{\substack{D\in\mathcal{D}\\ j\notin D}} N(D).$$

Putting it all together we obtain

$$\mu(C_p^r) = 1+r+\kappa = 1+r+\sum_{D\in\mathcal{D}} N(D) \le 1+r+\frac{1}{2}r(p-2) = 1+\frac{1}{2}rp\,.$$

□

14. The function $\Omega(H,\mathrm{s})(x)$

Let S be an arithmetical semigroup with divisor theory $\partial : S \to \mathcal{F}(P)$, divisor class group G and $G_0 = \{g \in G \mid g \cap P \neq \emptyset\}$. Further let $H \subseteq G_0$ be a subset and $\mathrm{s} = (s_g)_{g\in G_0\setminus H} \in \mathbb{N}^{G_0\setminus H}$. It can easily be seen that $\Omega(H,\mathrm{s})$ is not empty if and only if the following two conditions hold:

i) $\sum_{g\in G_0\setminus H} s_g = 0$ implies $H \neq \emptyset$.

ii) $\sum_{g\in G_0\setminus H} s_g g$ is contained in the subgroup generated by $H \cup \{0\}$.

If S is of type α, then $G_0 = G$; otherwise $G \setminus \ker\chi_1 \subseteq G_0$. In the extremal case when $b_g = 0$ for all $g \in \ker\chi_1$ (see Theorem 4), we have $G \setminus \ker\chi_1 = G_0$.

Having at our disposal the non-trivial theory of L-functions we can establish asymptotics for $\Omega(H,\mathrm{s})(x)$ as x rends to infinity.

PROPOSITION 8. *With all notations as above and $\Omega(H,\mathrm{s}) \neq \emptyset$ we have*

$$\Omega(H,\mathrm{s})(x) = \frac{x}{(\log x)^{1-\lambda_0}} V(\log\log x) + O\Big(\frac{x(\log\log x)^{b_0}}{(\log x)^{1-\lambda_0+\gamma}}\Big)$$

where $V \in \mathbb{C}[X]$ has positive leading coefficient and $b_0 \in \mathbb{N}$ with $0 \le b_0 \le \deg V + 1$.
If S is of type α, then $\lambda_0 = \frac{\#H}{h}$,

$$\deg V = \begin{cases} \sum_{g\in G\setminus H} s_g & H \neq \emptyset, \\ \sum_{g\in G\setminus H} s_g - 1 & H = \emptyset, \end{cases}$$

and

$$\gamma = \begin{cases} 1 & h = 1 \text{ or } H = \emptyset, \\ \frac{1}{h}\left(1 - \cos\frac{2\pi}{h}\right) & \text{otherwise.} \end{cases}$$

If S is of type β, then $\lambda_0 = \frac{2}{h}\#\big((G \setminus \ker\chi_1) \cap H\big)$,

$$\deg V = \begin{cases} \sum_{g \in G_0 \setminus (H \cup \ker\chi_1)} s_g & H \not\subseteq \ker\chi_1, \\ \sum_{g \in G_0 \setminus (H \cup \ker\chi_1)} s_g - 1 & H \subseteq \ker\chi_1, \end{cases}$$

and

$$\gamma = \begin{cases} 1 & H = \emptyset, \\ \min\left\{1, \frac{2}{h}(1 - \cos\frac{2\pi}{h})\right\} & \text{otherwise.} \end{cases}$$

Proof. If S is of type α, then the proof is essentially the same as in [Ka]. If S is of type β, then the proof differs in some technical details; required changes are, however easy to perform. □

15. Proof of Theorem 5

Let S be an arithmetical semigroup with divisor class group $G, G_0 \subseteq G$ the set of classes containing prime divisors and $k \geq 1$.

Since $\mathcal{B}(G_0)$ has only finitely many irreducible elements, the sets

$$\begin{aligned} U_k(G_0) &= \{B \in \mathcal{B}(G_0) \mid \min L(B) \leq k\} \quad \text{and} \\ U_k'(G_0) &= \{B \in \mathcal{B}(G_0) \mid k \in L(B)\} \end{aligned}$$

are finite. Thus

$$M_k^{(\prime)} = \dot{\bigcup_{B \in U_k^{(\prime)}(G_0)}} \Omega\big(\emptyset, (v_g(B))_{g \in G_0}\big),$$

and Proposition 8 implies Theorem 5 with

$$\deg V^{(\prime)} = \begin{cases} \max\{\sigma(B) \mid B \in U_k^{(\prime)}(G)\} - 1, & \text{if } S \text{ is of type } \alpha. \\ \max\{\sum\limits_{g \in G \setminus \ker\chi_1} v_g(B) \mid B \in U_k^{(\prime)}(G_0)\} - 1, & \text{if } S \text{ is of type } \beta. \end{cases}$$

So it remains to verify the following Lemma.

LEMMA 2. **1)** *For every non-empty subset* $H \subseteq G$

$$kD(H) = \max\{\sigma(B) \mid B \in U_k^{(l)}(H)\}.$$

2) *For every non-empty subset* $H \subseteq G$ *and every proper subgroup* K *of* G *with* $G \setminus K \quad \subseteq H$ *we have*

$$\max\{\sum_{g \in G \setminus K} v_g(B) \mid B \in U_k^{(l)}(H)\} = \max\{\sigma(B) \mid B \in U_k^{(l)}(G \setminus K)\}.$$

Proof.

1) Let $H \subseteq G$. Since for $B \in U_k(H)$, $\sigma(B) \leq (\min L(B))D(H) \leq kD(H)$, and since $U_k^l(H) \subseteq U_k(H)$, it follows

$$\max\{\sigma(B) \mid B \in U_k^l(H)\} \leq \max\{\sigma(B) \mid B \in U_k(H)\} \leq kD(H).$$

On the other hand let $B \in \mathcal{B}(H)$ irreducible with $\sigma(B) = D(H)$. Then $B^k \in U_k^l(H)$ and $\sigma(B^k) = kD(H)$, which implies

$$kD(H) \leq \max\{\sigma(B) \mid B \in U_k^l(H)\}.$$

2) Obviously $\max\{\sum\limits_{g \in G \setminus K} v_g(B) \mid B \in U_k(H)\} \geq$

$\max\{\sum\limits_{g \in G \setminus K} v_g(B) \mid B \in U_k^l(H)\} \geq \max\{\sigma(B) \mid B \in U_k^l(G \setminus K)\} \overset{(1)}{=}$
$\max\{\sigma(B) \mid B \in U_k(G \setminus K)\}$.

So for $B = \prod\limits_{i=1}^{u} a_i \prod\limits_{j=1}^{v} b_j \in U_k(H)$ with $a_i \in G \setminus K$ and $b_j \in K \cap H$ we have to find a $B' \in U_k(G \setminus K)$ with $\sigma(B') \geq u$.

We consider a factorization of $B = \prod\limits_{j=1}^{l} C_j$ into irreducible blocks C_j with $l = \min L(B) \leq k$, and we set

$$C_j = \prod_{i=1}^{u_j} a_i^{(j)} \prod_{i=1}^{v_j} b_k^{(j)}.$$

Let $u_j \geq 1$. Because $-\sum_{i=1}^{u_j} a_i^{(j)} = \sum_{i=1}^{v_j} b_i^{(j)} \in K$ and $a_{u_j}^{(j)} \in G \setminus K$ it follows that $-\sum\limits_{i=1}^{u_j-1} a_i^{(j)} = a_0^{(j)} \in G \setminus K$, and therefore

$$C_j' = a_0^{(j)} \prod_{i=1}^{u_j-1} a_i^{(j)}$$

is an irreducible block in $\mathcal{B}(G \setminus K)$.
If $u_j = 0$, we set $C'_j = 1 \in \mathcal{B}(G \setminus K)$. Finally $B' = \prod_{j=1}^{l} C'_j \in \mathcal{B}(G \setminus K)$ with $\min L(B') \le l \le k$ and $\sigma(B') = u$. $\square$

16. Proof of Theorem 6

PROPOSITION 9. *Let S be a semigroup with divisor theory, finite divisor class group $G, G_0 = \{g \in G \mid g$ contains a prime divisor$\}$, $\Delta(G_0) \ne \emptyset$ and $k \in \mathbb{N}_+$. Then there exists some $z \in \mathbb{N}_+$ such that*

$$G_k = \bigcup_{\substack{H \\ \Delta(H)=\emptyset}} \bigcup_{\mathbf{s} \in S(G_0,H,k)} \Omega(H,\mathbf{s}) = \bigcup_{\substack{H \\ \Delta(H)=\emptyset}} \bigcup_{\substack{\mathbf{s} \in S(G_0,H,k)\cap \\ \{0,\dots,z\}^{G_0\setminus H}}} \Omega(H,\mathbf{s}) .$$

Proof. **1)** For a subset $H \subseteq G_0$ with $\Delta(H) \ne \emptyset$ fix a block $C = C(H) \in \mathcal{B}(H)$ with $\#L(C) > k$; then $\#L(CC') > k$ for every $C' \in \mathcal{B}(H)$. For every $g \in G_0$ we define

$$w_g = \max\{v_g(C(H)) \mid H \subseteq G_0 \text{ with } \Delta(H) \ne \emptyset\} .$$

These numbers have the following property: if for a block $B \in \mathcal{B}(G_0)$ $\Delta(\{g \in G_0 \mid v_g(B) \ge w_g\}) \ne \emptyset$, then $\#L(B) > k$.

2) For $H \subseteq G_0$ with $\Delta(H) = \emptyset$ let $T(H)$ the set of all $\mathbf{t} \in \mathbb{N}^{G_0\setminus H}$ such that

$$\Omega(H,\mathbf{t}) \cap \{\partial a \mid \#L(a) > k\} \ne \emptyset .$$

By [C-P, Theorem 9.18] $T(H)$ has only finitely many minimal elements (with respect to the usual order in $\mathbb{N}^{G_0\setminus H}$), say $\mathbf{t}^{H,1}, \dots, \mathbf{t}^{H,\pi_H}$. We choose $a^{H,j} \in S$ with $v_g(a^{H,j}) = t_g^{H,j}$ for every $g \in G_0 \setminus H$ and with $\#L(a^{H,j}) > k$.
For every $g \in G_0$ we define

$$u_g = \max\{v_g(a^{H,j}) \mid 1 \le j \le \pi_H, H \subseteq G_0 \quad \text{with } \Delta(H) = \emptyset\} .$$

By construction we get, for every $H \subseteq G_0$ with $\Delta(H) = \emptyset$ and every $\mathbf{t} \in T(H)$,

$$G_k \cap \{\partial a \mid v_g(a) \ge u_g \ \forall g \in H, \ \ v_g(a) = t_g \ \forall g \in G_0 \setminus H\} = \emptyset .$$

3) Let $z = \max\{u_g, w_g \mid g \in G_0\}$. Then

$$\bigcup_{H,\Delta(H)=\emptyset} \bigcup_{\mathbf{s}\in S(G_0,H,k)} \Omega(H,\mathbf{s}) \subseteq G_k =$$

$$\bigcup_{H\subseteq G_0} \{\partial a | \#L(a) \le k,\ v_g(a) \ge z\ \forall g \in H,\ v_g(a) < z\ \forall g \in G_0 \setminus H\} \overset{(1)}{=}$$

$$\bigcup_{H,\Delta(H)=\emptyset} \{\dots\} \overset{(2)}{\subseteq}$$

$$\bigcup_{H,\Delta(H)=\emptyset} \bigcup_{\substack{\mathbf{s}\in S(G_0,H,k)\cap \\ \{0,\dots,z-1\}^{G_0\setminus H}}} \Omega(H,\mathbf{s})\,. \quad \square$$

Proposition 8 and 9 imply Theorem 6 except for the assertion on $\deg V$, if S is of type β. In this case we obtain

$$\deg V = \max\Big\{ \sum_{g\in(G\setminus\ker\chi_1)\setminus H} s_g \mid \mathbf{s} \in S(G_0,H,k), H \subseteq G_0$$
$$\text{with } \Delta(H) = \emptyset \text{ and} \#(H \cap (G \setminus \ker\chi_1)) = \mu(G \setminus \ker\chi_1)\Big\}.$$

So it remains to prove the following Lemma.

LEMMA 3. $\deg V \ge \max\Big\{\sum_{g\in(G\setminus\ker\chi_1)\setminus H'} s'_g \mid \mathbf{s}' \in S(G\setminus\ker\chi_1, H', k),$ $H' \subseteq G\setminus\ker\chi_1$ *with* $\Delta(H') = \emptyset$ *and* $\#H' = \mu(G\setminus\ker\chi_1)\Big\}$.

Proof. Let $H' \subseteq G\setminus\ker\chi_1$ with $\Delta(H') = \emptyset, \#H' = \mu(G\setminus\ker\chi_1)$ and $\mathbf{s}' \in S(G\setminus\ker\chi_1, H', k)$. For the canonical projection $\pi : \mathbb{N}^{G_0\setminus H'} \to \mathbb{N}^{(G\setminus\ker\chi_1)\setminus H'}$ we have $S(G\setminus\ker\chi_1, H', k) \subseteq \pi(S(G_0, H', k))$. Thus there exists an $\mathbf{s} \in S(G_0, H', k)$ with $\pi(\mathbf{s}) = \mathbf{s}'$ and the assertion follows. $\square$

17. Proof of Theorem 7

Let S be an arithmetical semigroup of type α with $\#\Delta(S) \ge 2$. Then [Ge2, Theorem 1] implies

$$\partial S \setminus T_1 \subseteq \bigcup_{g\in G\setminus\{0\}} \Omega(G\setminus\{g\}, 0)\,.$$

So the assertion follows by Proposition 8.

If S is an arithmetical semigroup of type β with $\#\Delta(S) \geq 2$, then Proposition 8 together with the following Proposition 10 imply the result.

From now on till the end of this chapter, let S be a semigroup with divisor theory, G its divisor class group, $K < G$ a subgroup of index 2 such that $G \setminus K \subseteq G_0 = \{g \in G \mid g$ contains a prime divisor$\}$ and $\#\Delta(G_0) \geq 2$.

By [Gel, Prop. 6] there exist a block $B_1 \in \mathcal{B}(G_0)$ and a constant $M(G_0)$ such that for every $B \in \mathcal{B}(G_0)$

$$(*) \qquad L = L(B_1 B) = \{x_1, \ldots, x_\alpha, y, y+d, \ldots, y+kd, z_1, \ldots, z_\beta\},$$

with $x_1 < \cdots < z_\beta$, $d = \min \Delta(G_0)$, α, $\beta \leq M(G_0)$ and $\Delta(L) \subseteq \{1, \ldots, D(G_0) - 2\}$.

If $G = C_6$ and $G_0 \subseteq \{0\} \cup (G \setminus K)$, then by Proposition 4,

$2 = \min \Delta(G_0) = \min \Delta(G \setminus K)$ and $D(G \setminus K) = 6 \geq 4$. In all other cases Proposition 4 implies that $1 = \min \Delta(G_0) = \min \Delta(G \setminus K)$ and $\#\Delta(G \setminus K) \geq 2$; since by Proposition 3 in [Gel] $\max\Delta(G \setminus K) \leq D(G \setminus K) - 2$, it follows $D(G \setminus K) \geq 4$.

Because in both cases $\min \Delta(G_0) = \min \Delta(G \setminus K)$ we can choose $B_1 \in \mathcal{B}(G \setminus K)$ (as can be seen by the construction used in the proof of Prop. 6 in [Gel]).

PROPOSITION 10. *There exists a block $B_0 \in \mathcal{B}(G \setminus K)$ such that, for every $B \in \mathcal{B}(G_0)$, $L = L(BB_0)$ has the form (*) with:*

$x_1 < \cdots < z_\beta$, $d = \min \Delta(G_0), \alpha, \beta \leq M(G_0)$ *and* $\Delta(L) \subseteq \{1, 2\}$.

The proof of Proposition 10 will be carried out in a series of lemmata.

LEMMA 4. *There exists a block $B_2 \in \mathcal{B}(G \setminus K)$ such that for every $B \in \mathcal{B}(G_0)$*

$$L(B_2 B) = \{u_1, \ldots, u_r, z_0, \ldots, z_{M(G_0)}\}$$

with $u_1 < \cdots < z_{M(G_0)}$ and $|z_{i+1} - z_i| \leq 2$ for $0 \leq i \leq M(G_0) - 1$.

Proof. Since $D(G \setminus K) \geq 4$, there exists an irreducible block $C = \prod_{i=1}^4 g_i \in \mathcal{B}(G \setminus K)$ of size 4. Because $C' = \prod_{i=1}^4 (-g_i) \in \mathcal{B}(G \setminus K)$ we obtain $A = CC' = \prod_{i=1}^4 (g_i)(-g_i) \in \mathcal{B}(G \setminus K)$. Now it can be easily verified that $B_2 = A^{M(G_0)+1}$ has the required property. $\square$

LEMMA 5. *Let* $B \in \mathcal{B}(G_0)$ *with* $v_g(B) \geq D(G_0) + 1$, *for every* $g \in G \setminus K$, *and with* $\min L(B) = r \in \mathbb{N}_+$. *Then one of the following two conditions holds:*

a) *B has a factorization of the form $B = C_1 \dots C_r$ with irreducible C_j such that $L(C_1C_2) \cap \{3,4\} \neq \emptyset$.*

b) *For every $g \in G \setminus K$, B has a factorization of the form*

$$B = (-g)(g) \prod_{j=2}^{r} C_j.$$

Proof. Suppose that **a)** does not hold and choose an element $g \in G \setminus K$. Then we set

$$u = \min\{\sigma(C_1) \mid B \text{ has a factorization } B = C_1 \dots C_r \\ \text{with irreducible } C_j \text{ and } v_g(C_1) > 0\},$$

and we have to prove that $u = 2$. Assume to the contrary, that $u \geq 3$. Let $B = C_1 \dots C_r$ and $C_1 = g \prod_{i=2}^{u} a_i$. Then at least one element of $a_2, \dots, a_u$ is in $G \setminus K$. Without restriction let $a_2 \in G \setminus K$. If $a_3 \in K$, we put $s = 3$; if $a_3 \in G \setminus K$, we assume $a_4 \in G \setminus K$ and put $s = 4$. So $b_1 = \sum_{j=2}^{s} a_j \in G \setminus K$.

Since $v_{b_1}(C_2 \dots C_r) = v_{b_1}(C_1 \dots C_r) - v_{b_1}(C_1) \geq v_{b_1}(B) - D(G_0) \geq 1$, we may assume that $C_2 = \prod_{j=1}^{v} b_j$. We define $C_1' = g \cdot b_1 \prod_{j=s+1}^{u} a_j$ and $C_2' = \prod_{i=2}^{s} a_i \prod_{j=2}^{v} b_j$. C_1' and C_2' are blocks with $C_1'C_2' = C_1C_2$. C_1' is irreducible and C_2' is a product of three irreducible blocks at most. Since **a)** does not hold, C_2' is irreducible. But then we obtain $B = C_1'C_2'C_3 \dots C_r$ with $\sigma(C_1') < u$, a contradiction to the minimality of u. □

LEMMA 6. *Let* $B \in \mathcal{B}(G_0)$ *with* $v_g(B) \geq D(G_0) + 7$, *for every* $g \in G \setminus K$, *and with* $\min L(B) = r \in \mathbb{N}_+$. *Then condition* **a)** *of Lemma 5 holds.*

Proof. Let $CC' = \prod_{i=1}^{4} (g_i)(-g_i)$ as in the proof of Lemma 4.

Assume, that condition **a)** does not hold. Applying Lemma 5 four times, we obtain

$$B = \prod_{i=1}^{4}(g_i)(-g_i)\prod_{j=5}^{r} C_j \,.$$

But then $B = CC'\prod_{j=5}^{r} C_j$, a contradiction to $r = \min L(B)$. $\square$

LEMMA 7. *There exists a block $B_3 \in \mathcal{B}(G \setminus K)$ such that for every $B \in \mathcal{B}(G_0)$*

$$L(B_3B) = \{x_0, \ldots, x_{M(G_0)}, u_1, \ldots, u_r\}$$

with $x_0 < \cdots < u_r$ and $|x_{i+1} - x_i| \le 2$ for $0 \le i \le M(G_0) - 1$.

Proof. By Lemma 6 and an inductive argument it follows that every block $B \in \mathcal{B}(G_0)$ with

$$v_g(B) \ge (M(G_0) - 1)2 \cdot D(G_0) + D(G_0) + 7\,,$$

for every $g \in G \setminus K$, has this property. $\square$

Finally, it is easy to see that Proposition 10 holds with $B_0 = B_1B_2B_3$.

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