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Recent results in the theory of constant reductions

by Barry GREEN

The aim of this paper is to give a survey of recent results in the theory of constant reductions and in particular to examine the way in which the approaches via rigid analytic geometry or alternatively function field theory have been used to prove these results. From both these areas there has been an interaction of ideas and approaches to solving problems and in this paper we have attempted to illustrate this by the results we discuss. These results are all of algebraic or geometric nature and in some cases special forms were already known from function field theory, rigid analytic geometry, algebraic geometry and valuation theory. We also illustrate how the model theory of valued function fields is used to provide the framework in which unknown cases or more general forms of these results can be proved. Examples of this phenomenon which are discussed in this paper are:

— a theorem on the existence of regular functions for valued function fields;

— a stable reduction theorem for curves over an arbitrary valuation ring with algebraically closed quotient field and a characterisation of such stable curves by a finite set of constant reductions of its function field;

— a Galois characterisation theorem for function fields of one variable over finitely generated fields.

1. Situation and notations

Let $F|K$ be a function field in 1 variable with K' the exact field of constants. Suppose K is equipped with a valuation v_K and that v is any

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prolongation of v_K to F. We let Fv and Kv (suppressing the subscript K when this does not lead to confusion) denote the residue fields and assume throughout that Fv is a function field in 1 variable over Kv. In the literature such valuations are also called constant reductions of $F|K$. We shall often refer to such function fields equipped with a constant reduction as valued function fields and write $(F|K, v)$. When we refer to the genus of a function field we shall mean the genus over the exact constant field as in Chevalley [C]. The expression $r(F|K)(1 - g_F)$, where $r(F|K) = [K' : K]$ and g_F is the genus, will be denoted by $\chi(F|K)$. We shall assume throughout the paper that K is the exact constant field and so it is only for the reduction that $r(Fv|Kv)$ may be greater than one.

Suppose V is a finite set of constant reductions of F, coinciding on K with a given valuation v_K. Let $f \in F$ be residually transcendental for each $v \in V$. Recall that this is equivalent with the assertion that for each $v \in V$, $v|_{K(f)}$ is the functional valuation $v_{K,f}$; that is for a polynomial of degree n

$$t = a_0 + a_1 f + \ldots + a_n f^n \qquad (a_i \in K,\ a_n \neq 0)$$

its valuation is given by

$$v_{K,f}(t) = \min(v_K(a_0), v_K(a_1), \ldots, v_K(a_n)).$$

Let V_f denote the set of all prolongations of $v_{K,f}$ to F, noting that $V \subseteq V_f$. Then:

(i) $f \in F$ is defined to be an *element with the uniqueness property for* V if $V = V_f$;

(ii) $f \in F$ is defined to be V-*regular* for $F|K$ if

$$\deg f := [F : K(f)] = \sum_{v \in V} [Fv : Kv(fv)] =: \sum_{v \in V} \deg fv.$$

Remark 1.1. In general

$$[F : K(f)] = \sum_{v \in V_f} [F_v^h : K(f)_v^h] = \sum_{v \in V_f} \delta_v \mathrm{e}_v [Fv : Kv(fv)],$$

where F_v^h (respectively $K(f)_v^h$) denotes taking a henselisation of F (respectively the corresponding henselisation of $K(f)$), δ_v the henselian defect and e_v is the ramification index with respect to v. We shall discuss the defect of a valued function field $(F|K,v)$ in some detail in §5 but remark now that δ_v is independent of the choice of the residually transcendental element f (this is a non-trivial theorem – see for example [K] or [G–M–P 1], proposition 2.12). Note that if $f \in F$ is V-regular for $F|K$ then $V = V_f$ and for each $v \in V$, $\delta_v = e_v = 1$.

When the constant field K is algebraically closed, for each valuation v the value group is divisible and so there is no ramification, that is $e_v = 1$. An important theorem of Grauert—Remmert asserts that in this situation the defect $\delta_v = 1$ aswell. Precisely:

The Grauert–Remmert stability theorem [Gr–Re]. *Let $F|K$ and V_f be as above and suppose that K is algebraically closed. Then for each $v \in V_f$, $\delta_v = 1$.*

In the paper of Grauert–Remmert this theorem is only proved for rank 1 valued function fields and the methods used are from rigid analytic geometry. Proofs of this result for arbitrarily valued function fields have been given independently by Ohm in [O3] and Kuhlmann in [K]. The methods of proof found in [O3] and [K] are similar in that both authors deduce the general case by realizing it as a particular constant extension of the rank 1 case. For the rank 1 case Ohm relies on the proof of Grauert–Remmert, while Kuhlmann has given a new proof which replaces the analytic methods by valuation theoretical arguments. In [G–M–P 1] and [G–M–P 2] different proofs of this theorem can be found where it appears as a corollary in a study of the vector space defect of a valued function field and as a consequence of a theorem on the existence of regular functions. However here also the proofs can be traced back to a rank 1 form of the result; in the first case by means of a valuation decomposition and in the second case by means of a model theoretic argument (using the model completeness of the theory of valued fields [Rob]). The proof using model theory is perhaps the most interesting since here in the rank 1 case one only needs to know the result for K the algebraic closure of a local field and this can be deduced directly from the work of Epp [E].

This theorem is important because it assures us that if K is algebraically closed then each function $f \in F \setminus K$ is a V_f-regular function. It also means that for the given set of constant reductions V if we are able to find an element with the uniqueness property for V then it will be a V-regular function. In the next paragraph we shall discuss the existence of elements with the uniqueness property for V.

2. Elements with the uniqueness property

The problem:

> *Given a function field $F|K$ and a set of constant reductions V, coinciding on the constant field K, under what conditions on K do there exist elements with the uniqueness property for V,*

marked a starting point for the recent study of valued function fields. A positive answer to this problem was first obtained by Matignon, [M2], theorem 3, p. 197, when the constant reductions are rank 1 valuations and K is complete with respect to their common restriction. Matignon proved this result in a paper in which he gave a new genus inequality relating the genus of the function field to the genera of the residual function fields. This genus inequality improved the previously known genus inequalites of Mathieu [Ma] and Lamprecht [L] in several respects:

– firstly the inequality was proved for any rank 1 valuation on the constant field, whereas previously a weaker inequality was only known for a discrete valuation on the constant field;

– for each constant reduction $v \in V$ it included a factor corresponding to a vector space defect associated to $(F|K, v)$;

– it contained a term corresponding to the number of constant reductions in V. Because of this term it followed there is at most one $v \in V$ such that with respect to this $F|K$ has good reduction.

In §5 we shall describe the genus inequality precisely, but now we return to the problem on the existence of elements with the uniqueness property. In Matignon's paper this result follows from a structure theorem for affinoid domains, Fresnel–Matignon [F–M], theorem 1, p. 160. It can also be deduced directly from a general contraction lemma of Bosch and Lütkebohmert [B–L].

In valuation theory there is a general philosophy that if an algebraic property holds for complete rank 1 valuations then it probably holds more generally for henselian valuations of arbitrary rank. This is the case with the above problem. The first result in this direction was proved by Polzin in [Po], when the constant reductions are rank 1 valuations and K is henselian with respect to their common restriction. This result of Polzin depended on that of Matignon and appeared as a corollary of his study of the Local Skolem problem and its relationship to the existence of elements with the uniqueness property for valued function fields over a rank

1 henselian constant field. He used methods from rigid analytic geometry. The case of a valued rational function field [1]) having henselian field of constants (arbitrary rank) has been treated in [M–O 2], where a direct proof of the existence of elements with the uniqueness property for this case is given. In the general case the following theorem gives sufficient conditions for the solution of the problem:

Theorem 2.1 [G–M–P 2]. *Let (K, v_K) be a henselian valued field, $F|K$ a function field and V a finite set of constant reductions of F with $v|_K = v_K$ for each $v \in V$. Then there exist elements $t \in F$ with the uniqueness property for V. In particular if K is algebraically closed then t is a V-regular function.*

We shall briefly discuss the methods used to prove this theorem and point out that it is proved without appealing to rigid analytic geometry, but did involve model theory. As a first step we proved a general theorem giving a criterion for the existence of V-regular functions of degree bounded by something depending only on g_F and $s = \text{card}(V)$ provided there exists a V-regular function to begin with.

Theorem 2.2. *Let (K, v_K) be an algebraically closed valued field and $(F|K, v)_{v\in V}$ valued function fields with $v|_K = v_K$ for each $v \in V$ (V is assumed finite). Suppose there exists a V-regular function. Then there exist V-regular functions of degree bounded by $4g_F - 4 + 5s$.*

This theorem is a tool to be used in the proof of 2.1. However one sees from the statement that it only gives information once one knows the existence of a V-regular function and so it is not evident how it can be used to prove 2.1. The way we proceed is by first giving a direct proof of the following very special case of 2.1, namely:

Theorem 2.3. *Let $(F|K, v)$ be a valued function field and suppose that (K, v_K) is the algebraic closure of a local field (K_0, v_{K_0}). Then there is a v-regular function t for $F|K$.*

As a corollary we conclude from the previous theorem that if $(F|K, v)$ is as above with (K, v_K) the algebraic closure of a local field (K_0, v_{K_0})

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[1]) Problems concerned with the constant reductions of a rational function field have been studied recently by Ohm and Matignon in [O1], [O2] and [M–O 1] and by Alexandru, Popescu and Zaharescu in [A–P–Z].

then there is a v-regular function t for $F|K$ with degree bounded by $4g_F + 1$.

The next step is to show that a certain statement concerning the existence of regular functions in a valued function field is elementary. More precisely, let φ be a positive real-valued function of $\mathbb{N}^2$ and (K, v_K) a valued field. We say that (K, v_K) belongs to the class $\mathcal{C}_\varphi$ if: for each unramified valued function field $(F|K, v)$, with v prolonging a functional valuation $v_{K,f}$ for some $f \in F$, there exist v-regular functions for $F|K$ of degree bounded by $\varphi(g_F, \deg f)$. The result is:

> *The class $\mathcal{C}_\varphi$ is either empty or an elementary class. In particular, if (K, v_K) and (L, v_L) are elementary equivalent and (K, v_K) belongs to the class $\mathcal{C}_\varphi$ then so does (L, v_L).*

For the benefit of the reader not familiar with model theory or its applications to algebra we digress briefly in order to explain how this result can be used to conclude that each algebraically closed valued field belongs to $\mathcal{C}_\varphi$ for $\varphi = 4g_F + 1$. In this connection we also draw the reader's attention to the survey article of Roquette [R3], *Some Tendencies in Contemporary Algebra,* which appeared in the 1984 anniversary of Oberwolfach edition of *Perspectives in Mathematics.* In this article Roquette discusses the intrusion of model theoretic notions (in the sense of mathematical logic) and arguments into algebra.

Let $\mathcal{L}$ be an elementary language (i.e. first order which means that the variables in $\mathcal{L}$ denote individuals only; there are no set variables or function variables) and $\mathcal{T}$ any theory of $\mathcal{L}$ (set of sentences of $\mathcal{L}$). A model of the theory is an $\mathcal{L}$-structure K such that all the sentences of $\mathcal{T}$ hold in K. If ϕ is an arbitrary sentence in $\mathcal{L}$ then ϕ may hold in some models but perhaps not in all models of the theory. If ϕ holds in K we say that ϕ defines an algebraic property of K. Two models K and L are said to be elementary equivalent if every algebraic property of K is shared by L and vice versa. If this is so we write $K \equiv L$. A class $\mathcal{C}$ of $\mathcal{L}$-structures is said to be an elementary class if there exists a theory $\mathcal{T}$ such that $\mathcal{C}$ is the class of all models of $\mathcal{T}$. Clearly if $K \in \mathcal{C}$ and L is any $\mathcal{L}$-structure with $L \equiv K$ then also $L \in \mathcal{C}$.

In our situation the language is the language of valued fields. There is a classical theorem of A. Robinson [Rob], saying that the theory of algebraically closed valued fields is model complete. We will not discuss the general notion of model completeness for a theory here; for our purpose it suffices to record that here this means that all algebraically closed valued fields having the same characteristic and residue characteristic (characteristic of the residue field) are elementary equivalent.

By the preceeding discussion, if (K, v_K) is the algebraic closure of a local field then it belongs to the class $\mathcal{C}_\varphi$ for $\varphi = 4g_F + 1$. Since $\mathcal{C}_\varphi$ is an elementary class we conclude that each algebraically closed valued field belongs to $\mathcal{C}_\varphi$. In summary we have:

Theorem 2.4. *Let $(F|K, v)$ be a valued function field with K algebraically closed. Then there exist v-regular functions $t \in F$ for $F|K$. Moreover, t can be chosen with* $\deg t \leq 4g_F + 1$.

Using this result we are able to deduce the existence of an element with the uniqueness property for $(F|K, v)$ when the constant field K is henselian. This is done by extending the field of constants to an algebraic closure $\tilde{K}$ and then showing that a regular function t for $(F\tilde{K}|\tilde{K}, \tilde{v})$ can be chosen in F ($\tilde{v}$ is a prolongation of v from F to $F\tilde{K}|\tilde{K}$). As K is henselian $v_{K,t}$ has a unique prolongation from $K(t)$ to $\tilde{K}(t)$ and so this function is an element with the uniqueness property for v.

The final step in the proof of 2.1 is quite general and consists of showing that if $F|K$ is equipped with a finite set of constant reductions V having common restriction to K and for each $v \in V$ there is an element with the uniqueness property then there is an element with the uniqueness property for V. Indeed, once one has an element with the uniqueness property for a given $v \in V$, it is an easy exercise to construct an element t_v with the uniqueness property for v so that for all other $v' \in V$, $v'(t_v) > 0$. Let $t = \sum_{v \in V} t_v$, then by skillfully using the fundamental inequality of valuation theory one shows that t is an element with the uniqueness property for V and that

$$\deg t = \sum_{v \in V} \deg t_v .$$

Remark 2.5. If each of the t_v are v-regular functions then by the identity above and remark 1.1, t is a V-regular function.

3. The Local Skolem property

We have already mentioned that in the rank 1 situation theorem 2.1 appeared as a corollary of Polzin's study of the Local Skolem property and its relationship to the existence of elements with the uniqueness property for valued function fields over a rank 1 constant field. In the general case the problem of finding such a relationship is still an open question. In this section we shall describe more precisely what this relationship is in the rank 1 situation and formulate what one could expect in general.

Let (K,v) be a rank 1 valued field, $\widehat{K}$ the completion and $\widetilde{\widehat{K}}$ its algebraic closure. Suppose $\mathcal{O}_v$ is the valuation ring of v, $\widehat{\mathcal{O}_v}$ the corresponding ring in the complete field and $\widetilde{\widehat{\mathcal{O}_v}}$ its integral closure in $\widetilde{\widehat{K}}$. Likewise $\widetilde{\mathcal{O}_v}$ denotes the integral closure of $\mathcal{O}_v$ in $\widetilde{K}$. Suppose W is any geometrically irreducible variety defined over K, let $F = F(W)$ denote the field of rational functions of W and to each closed point $P \in W$ set $K(P) = \mathcal{O}_{W,P}/\mathcal{M}_P$ where $\mathcal{O}_{W,P}$ is the local ring associated to P and $\mathcal{M}_P$ its maximal ideal. Let $\mathbf{f} = \{f_1, \ldots, f_n\} \subset F$ and set

$$\mathcal{U}_{\mathbf{f}}(\widehat{K}, v) = \{P \in W(\widehat{K}) : f_i \in \mathcal{O}_{W(\widehat{K}),P} \text{ and } v(f_i(P)) \geq 0,\ 1 \leq i \leq n\},$$

where $W(\widehat{K})$ denotes the set of $\widehat{K}$-rational points of W. Similarly for each prolongation $\tilde{v}$ of v to $\widetilde{K}$ set

$$\mathcal{U}_{\mathbf{f}}(\widetilde{K}, \tilde{v}) = \{P \in W(\widetilde{K}) : f_i \in \mathcal{O}_{W(\widetilde{K}),P} \text{ and } \tilde{v}(f_i(P)) \geq 0,\ 1 \leq i \leq n\}.$$

3.1. *The Local Skolem Property for (K,v) asserts that for each geometrically irreducible variety W defined over K and set $\mathbf{f}$ as above, if $\mathcal{U}_{\mathbf{f}}(\widehat{K}, v) \neq \emptyset$ then $\bigcap_{\tilde{v}|v} \mathcal{U}_{\mathbf{f}}(\widetilde{K}, \tilde{v}) \neq \emptyset$.*

Suppose W is a geometrically irreducible affine variety defined over K , embedded into affine r-space for some natural number r and denote by $W(\widetilde{\mathcal{O}_v})$ the set of v-integral points of W in $\widetilde{K}^r$. Let $\mathbf{f}$ be the set of coordinate functions. Then the Local Skolem Property implies that if $W(\widetilde{\widehat{\mathcal{O}_v}}) \neq \emptyset$ then $W(\widetilde{\mathcal{O}_v}) \neq \emptyset$.

The result of Polzin giving the relationship between the Local Skolem property and the existence of elements with the uniqueness is:

Theorem 3.2. *Let (K, v_K) be a rank 1 valued field. Then the following are equivalent:*

(i) The valued field (K, v_K) satisfies the Local Skolem property;

(ii) *If $F|K$ is a function field with constant field K_1 purely inseparable over K and V a finite set of constant reductions of F with $v|_K = v_K$ for each $v \in V$, then there exist elements $t \in F \setminus K_1$ with the uniqueness property for V.*

Notice that by the previous section it follows that if (K, v_K) is a rank 1 henselian valued field it satisfies the Local Skolem property.

It is possible to formulate the Local Skolem property more generally and to ask whether a relationship similar to that above holds. Let (K, v) be any valued field and $\widetilde{K}$ its algebraic closure. Suppose $\mathcal{O}_v$ is the valuation ring of v, ${\mathcal{O}_v}^h$ a henselisation and $\widetilde{{\mathcal{O}_v}^h}$ its integral closure in $\widetilde{K}$. $\widetilde{\mathcal{O}_v}$ denotes the integral closure of $\mathcal{O}_v$ in $\widetilde{K}$. *The Local Skolem Property for (K, v) asserts that for each geometrically irreducible variety W defined over K and set $\mathbf{f}$ as above, if $\mathcal{U}_{\mathbf{f}}(K^h, v) \neq \emptyset$ then $\bigcap_{\tilde{v}|v} \mathcal{U}_{\mathbf{f}}(\widetilde{K}, \tilde{v}) \neq \emptyset$.*

Once again if W is a geometrically irreducible affine variety over K, embedded into affine r-space for some natural number r and $\mathbf{f}$ is the set of coordinate functions, then the Local Skolem Property implies that if $W(\widetilde{{\mathcal{O}_v}^h}) \neq \emptyset$ then $W(\widetilde{\mathcal{O}_v}) \neq \emptyset$.

Question: *Given a valued field (K, v_K) are the following equivalent?*

(i) *The valued field (K, v_K) satisfies the Local Skolem property;*

(ii) *If $F|K$ is a function field with constant field K_1 purely inseparable over K and V a finite set of constant reductions of F with $v|_K = v_K$ for each $v \in V$, then there exist elements $t \in F \setminus K_1$ with the uniqueness property for V.*

4. The Local-Global-Principle of Rumely

The Skolem problem is closely related to the Local-Global-Principle of Rumely for integer points on varieties. In a recent paper Roquette [R6] has given a proof of this result by constant reduction theory. The situation is the following:

$\mathcal{O}_0$ a Dedekind domain with quotient field K_0, and residue fields at the non zero prime ideals of positive characteristic and absolutely algebraic, ie each algebraic over a finite field.

$K = \tilde{K}_0$ an algebraic closure of K_0.

$\mathcal{O} = \tilde{\mathcal{O}}_0$ the integral closure of $\mathcal{O}_0$ in K, which is assumed to be a Bezout domain. By [vdD–McT], proposition 5.2, this implies that the ideal class group of any Dedekind domain lying between $\mathcal{O}_0$ and $\mathcal{O}$ is torsion. Examples of this situation are given by $\mathcal{O}_0 = \mathbb{Z}$ or $\mathbb{F}_p[x]$.

$\mathcal{V} = \mathcal{V}(K)$ the space of all non-archimedean valuations (primes) v of K; each valuation v is written additively.

W an irreducible affine variety defined over K, embedded into affine r-space for some natural number r.

$W(K)$ the set of K-rational points of W.

$W(\mathcal{O})$ the set of integral points in W.

Let $v \in \mathcal{V}$ with valuation ring $\mathcal{O}_v$. Suppose $\boldsymbol{z} = (z_1, \ldots, z_r)$, a typical point of $W(K)$, and set $v(\boldsymbol{z}) = \min(v(z_1), \ldots v(z_r))$ the v-norm of $\boldsymbol{z}$. If $v(\boldsymbol{z}) \geq 0$ then the point $\boldsymbol{z}$ is an element of $W(\mathcal{O}_v)$ and is called *v-integral*, or *locally integral at v*. If this is so for *all* $v \in \mathcal{V}$ then $\boldsymbol{z}$ is said to be an *integral* point, or *globally integral*; this means that $\boldsymbol{z} \in W(\mathcal{O})$, i.e. all coordinates z_i are contained in $\mathcal{O}$ $(1 \leq i \leq r)$.

Theorem 4.1 (Local-Global-Principle for integer points). *Suppose that locally everywhere, the variety W admits a v-integral point $\boldsymbol{z}_v \in W(K)$. Then W has a globally integral point $\boldsymbol{z} \in W(K)$.*

This theorem is proved by first considering the case when W is a curve, i.e. dim $W = 1$. After that, an induction procedure with respect to the dimension of W leads to the general case. In a previous paper, [C–R], the Local-Global-Principle had been proved for *unirational varieties* W defined over $\tilde{\mathbb{Q}}$, which admit a parameterization by means of rational functions. R. Rumely then discovered the validity of the Local-Global-Principle in full generality for arbitrary irreducible varieties defined over $\tilde{\mathbb{Q}}$. His proof, which has appeared in [Ru1], is based on his deep and elaborate *capacity theory* on algebraic curves of higher genus over number fields. In view of the importance of theorem 4.1 there arises the question whether it is more directly accessible. The purpose of Roquette's article is to exhibit such a direct approach. The proofs are very much along the same lines as in the paper [C–R] for rational varieties. As in [C–R] the situation is reduced to the case where the variety W is a curve; but now one has to deal not only with the rational curves but also with the curves of higher genus. It is not clear *a priori* whether curves of higher genus can be treated in the same way, with respect to the above theorem, as was done with the curves of genus zero in [C–R]. The generalization to curves of higher genus rests essentially on two results:

(i) The Reciprocity Lemma for function fields of arbitrary genus which has been presented in [R4]. Here use is made of a divisor reduction map for a valued function field which has good reduction. We discuss this map in §6.

(ii) The Jacobi Existence Theorem for functions on curves whose zeros are situated near prescribed points on the curve. This theorem, in the non-archimedean case, is entirely due to Rumely [Ru2]. By extending the proof of this result in [R2], F. Pop has established a more general theorem taking rationality conditions into consideration in [P4]. In [R2] both the *Jacobi Existence Theorem* aswell as the *Unit Density Lemma* are presented as applications of a more general density theorem. Both these results are needed in the proof of the Local-Global-Principle.

The Local-Global-Principle derives its importance from the fact that the *solvability of local diophantine equations is decidable.* This has been proved by A. Robinson [Rob]. More precisely, for a given prime $v \in \mathcal{V}$ there exists an effective algorithm which permits to decide whether $W(\mathcal{O}_v)$ is non-empty. It is however not necessary to check this for all primes $v \in \mathcal{V}$. There are finitely many primes $v_1, \dots, v_s$, computable from the defining equations of W, such that the $v_1, \dots, v_s$ are "critical" for W in the following sense:

If $W(\mathcal{O}_{v_i}) \neq \emptyset$ for each of those critical primes v_i then $W(\mathcal{O}_v) \neq \emptyset$ for all primes $v \in \mathcal{V}$ and hence, by the Local-Global-Principle, $W(\mathcal{O}) \neq \emptyset$.

The testing of these finitely many critical primes then leads to an *effective algorithm* to decide whether $W(\mathcal{O}) \neq \emptyset$. If one wishes to include arbitrary varieties, not necessarily irreducible, then one has to observe that there is an effective algorithm for decomposing an arbitrary variety into its irreducible components over K. This yields:

The solvability of arbitrary diophantine equations over $\mathcal{O}$ is decidable. So the 10^{th} problem of Hilbert over $\mathcal{O}$ has a positive answer.

For $K = \tilde{\mathbb{Q}}$ and $\mathcal{O} = \tilde{\mathbb{Z}}$ this is in contrast to the situation over $\mathbb{Z}$ where it is known that the 10^{th} problem has a negative answer. A detailed exposition of the above line of arguments, together with historical remarks and precise references, can be found in Rumely's paper [Ru1].

We remark that similar results for global fields are contained in [M-B 1] and [M-B 2], where they are presented from a scheme theoretic point of view. The idea of proof for the existence theorem is similar to that in [R2].

5. Good reduction and the genus inequality

Let us recall the notion of good reduction in valued function fields: A valued function field $(F|K, v)$ is said to have good reduction at v if,

(i) F has the same genus as Fv, $g_F = g_{Fv}$;

(ii) there exists a non-constant v-regular function $f \in F$;

(iii) Kv is the constant field of Fv.

(Recall that throughout the paper K is assumed to be the exact constant field of F.)

When studying valued function fields with the aim of giving minimum criteria for good reduction and some measure of the deviation from good reduction in general, it is suggestive to study whether the natural invariants, ramification index, residue class degree and a defect associated with the constant reduction will serve this purpose. The ramification index and residue class degree have their classical meanings, but the "defect", although defined algebraically, is a geometric invariant of the reduction, [G–M–P 1]. Together with equality of the genera these three invariants are exactly what is needed to characterise good reduction in all cases. This defect is called the *vector space defect* to distinguish it from the natural notion of defect (see 5.3) for the constant reduction which is obtained as the common henselian defect of a certain family of finite algebraic extensions. More generally for extensions of transcendence degree greater than one this defect is defined as the supremum of the henselian defects of a certain family of finite algebraic extensions. This more general situation is treated by Kuhlman in his doctoral thesis [K], chapter 5; see also [O3]. Both these treatments are algebraic and no connections to the geometric point of view are made.

Let (K, v_K) be a non-archimedian valued field and (N, v) a (K, v_K) valued vector space. A subset $\{x_i\}_{i\in I}$, I an indexing set, of elements of $N \setminus \{0\}$ will be called *valuation independent* if $v(\sum_i a_i x_i) = \min_i v(a_i x_i)$, for any K-linear combination $\sum_i a_i x_i$. One shows that any valuation independent subset $\{x_i\}_{i\in I}$, is contained in a maximal one and further any two maximal valuation independent subsets have the same cardinality. Such a maximal valuation independent subset will be called a valuation basis for N over K. (Note: Although linearly independent over K in general this will not be a basis for N over K.)

Let $\mathcal{V}$ be a set of representatives for vN over vK (vN is a vK-set). For any K-subspace $M \subseteq N$ and $\nu \in \mathcal{V}$ let $M^\nu = \{x \in M : v(x) \geq \nu\}/\{x \in M : v(x) > \nu\}$, a Kv-vector space. If $\mathcal{B}_\nu \subset M$ is a set of representatives for a basis of M^ν then $\mathcal{B}_M = \bigcup_{\nu\in\mathcal{V}} \mathcal{B}_\nu$ is a valuation basis of M over K. Therefore if M is finite dimensional over K then the cardinality of any valuation basis $\mathcal{B}_M$ is exactly $\sum_{\nu\in\mathcal{V}} \dim_{Kv} M^\nu$.

Definition 5.1. Let $(N|K, v)$ be a non-archimedian (K, v_K) valued vector space. The vector space defect, $\delta^{\mathrm{vs}}(N|K, v)$, is defined to be:

$$\sup_{M\subseteq N} \frac{\dim_K M}{\sum_{\nu\in\mathcal{V}} \dim_{Kv} M^\nu},$$

where the supremum is taken over all non-zero finite K-subspaces M of N.

Let $(L|K, v)$ be a finite algebraic extension of valued fields. Then one proves that the vector space defect is related to the classical invariants of the valued field extension by

$$\delta^{\mathrm{vs}}(L|K, v) = \frac{[L : K]}{ef},$$

where e is the ramification index and f the residue class degree of $(L|K, v)$. Using this result one deduces that if $(F|K, v)$ is a valued function field and $f \in F$ a residually transcendental element, then

$$\delta^{\mathrm{vs}} \leq \frac{\deg f}{e \deg fv},$$

where we have simplified notation writing e (resp. δ^{vs}) in place of $e(F|K, v)$ (resp. $\delta^{\mathrm{vs}}(F|K, v)$).

We now describe how the vector space defect of a valued function field can be interpreted geometrically. In order to do this we shall first need to recall briefly certain general definitions and results from the work of Lamprecht [L] and Mathieu [Ma].

Let $F|K$ be a function field in 1 variable, denote by $\mathrm{Div}(F|K)$ the divisor group and for each $f \in F^\times$ let (f), $(f)_0$ and $(f)_\infty$ be the principal divisor , zero divisor and pole divisor associated with f respectively.

For a finite-dimensional K-subspace M of F denote by $\pi(M)$ the minimal divisor D such that $(f) + D \geq 0$ for every $f \in M$. It follows directly from the definitions that for any $t \neq 0$ we have: $\pi(tM) = \pi(M) - (t)$. Moreover, if $1 \in M$ then $\pi(M) \geq 0$ and if in addition K is infinite then there exists $f \in M$ such that $(f)_\infty = \pi(M)$, [Ma], Hilfsatz, S. 599.

If F has a valuation v defined on it so that $(F|K, v)$ is a valued function field this method of associating a divisor with a finite dimensional vector space has been used in the literature to obtain a map $\pi_v : \mathrm{Div}(F|K) \to \mathrm{Div}(Fv|Kv)$. We recall this briefly here.

Let $D \in \mathrm{Div}(F|K)$ and denote by $\mathcal{L}(D)$ its associated linear space, that is $\mathcal{L}(D) = \{f \in F : D + (f) \geq 0\}$. We also set $\dim_K D = \dim_K \mathcal{L}(D)$, supressing the subscripts when there is no confusion. For each $D \in \mathrm{Div}(F|K)$ one defines $\pi_v(D) = \pi(\mathcal{L}(D)v)$. When $F|K$ has good reduction at v then this map is the classical divisor reduction homomorphism of Roquette [R1].

More generally, given a set of representatives $\mathcal{V}$ for vF/vK and $\nu \in \mathcal{V}$, if $\mathcal{L}(D)^\nu \neq 0$ then there exists $x_\nu \neq 0$ in $\mathcal{L}(D)$ with $v(x_\nu) = \nu$. Set $\pi_v(D)^\nu = \pi((\mathcal{L}(D)x_\nu^{-1})v)$. We remark that $\pi_v(D)^\nu$ depends on the special x_ν. Nevertheless, its divisor class does not depend on x_ν, hence $\deg \pi_v(D)^\nu$ and $\dim \pi_v(D)^\nu$ are well defined and do not depend on x_ν. Indeed, let $y_\nu \in \mathcal{L}(D)$ with $v(y_\nu) = \nu$. Then $t_\nu = x_\nu y_\nu^{-1}$ has value 0 and $(\mathcal{L}(D)y_\nu^{-1})v = (t_\nu(\mathcal{L}(D)x_\nu^{-1}))v = (t_\nu v)(\mathcal{L}(D)x_\nu^{-1})v$. We can now give the geometrical interpretation of the vector space defect:

Theorem 5.2. *Let $(F|K, v)$ be a valued function field. Then for each real number $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ depending only on ε, δ^{vs}, $\chi(F|K)$ and $r(Fv|Kv)$ such that for each divisor $D \in \mathrm{Div}(F|K)$, $D \geq 0$, if $\deg D \geq n_0$ then:*

$$\text{(i)} \quad \delta^{\mathrm{vs}} - \varepsilon < \frac{\dim D}{\mathrm{e} \max\limits_{\nu \in \mathcal{V}} \dim \pi_v(D)^\nu} \leq \frac{\dim_K \mathcal{L}(D)}{\sum\limits_{\nu \in \mathcal{V}} \dim_{Kv} \mathcal{L}(D)^\nu} \leq \delta^{\mathrm{vs}} ;$$

(ii) $\delta^{\mathrm{vs}} \le \dfrac{\deg D}{\deg \pi_v(eD)} \le \dfrac{\deg D}{e \max\limits_{\nu\in\mathcal{V}} \deg \pi_v(D)^\nu} < \delta^{\mathrm{vs}} + \varepsilon$,

where $\mathcal{V}$ is a set of representatives for vF/vK. In both cases the value of max *is attained for the same ν.*

As corollaries of this theorem we deduce:

(i) $\chi(F|K) \le e\,\delta^{\mathrm{vs}}\,\chi(Fv|Kv)$;

(ii) $F|K$ has good reduction at v if either,

 (a) $g_F = g_{Fv} > 1$, or

 (b) $g_F = g_{Fv}$ and $e(F|K,v) = \delta^{\mathrm{vs}}(F|K,v) = r(Fv|Kv) = 1$.

We remark that the result above is obtained directly without appealing to rank 1 results derived using methods from rigid analytic geometry. Note also, at this point we are yet unable to deduce that if $(F|K,v)$ has good reduction then v is the unique good prolongation of $v|_K$ to F. For this we shall need a genus inequality which is better than (i) above and contains a term corresponding to the number of reductions, but first we mention how the vector space defect relates to the classical henselian defect. The results in this direction are the following:

5.3. For a valued function field $(F|K,v)$, the henselian defect $\delta(F|K(f),v)$ is independent of the residually transcendental element $f\in F$. This number is called the henselian defect of the valued function field and denoted by $\delta(F|K,v)$ ([K] or [G–M–P 1]). Always we have $\delta(F|K,v)\le\delta^{\mathrm{vs}}$ and in particular when K is henselian we have equality.

5.4. The genus inequality. *Let $F|K$ be a function field in 1 variable equipped with constant reductions v_i, $1\le i\le s$, which have a common restriction to the exact constant field K, say v_K. Then:*

$$\chi(F|K) \le 1 - s + \sum_{1\le i\le s} e_i\,\delta_i\,\chi(Fv_i|Kv_i),$$

where e_i, δ_i are the ramification index and henselian defect of $(F|K,v_i)$ respectively.

Remark. We have mentioned at the beginning of section 2 that in the rank 1 situation this result was proved by Matignon, using methods from rigid analytic geometry and that in his proof the theorem on the existence of elements with the uniqueness property over a complete base field was needed. Our proof of the general case, [G–M–P 1], theorem 3.1, depends on the rank 1 form of this result and a valuation decomposition lemma. If the valuations v_i, $1 \leq i \leq s$, defined on F above are independent then in the theorem above a better inequality holds, namely with the henselian defect δ_i replaced by the vector space defect δ_i^{vs} for each i. Actually when the valuations are independent, an algebraic proof of this inequality, but without the $s-1$ term, can be given without appealing to rank 1 results from rigid analytic geometry, [M3], theorem II.9. The corollary below follows immediately from 5.4 and shows that if $g_F \geq 1$ then there is at most one prolongation of a given valuation on K so that on F it is a good reduction.

Corollary 5.5. *Let (K, v_K) be a valued field and $F|K$ a function field in 1 variable with K the exact constant field. Suppose v is a prolongation of v_K to F so that $(F|K, v)$ is a valued function field with $g_F = g_{Fv} \geq 1$. Then v is unique with this property. If $g_F = g_{Fv} > 1$ then $e_v = \delta_v^{\mathrm{vs}} = r_v = 1$.*

The relative genus inequality. Suppose $F|K$ and $E|K$ are function fields in 1 variable with $E \subseteq F$. Further let F be equipped with a constant reduction v and denote also by v its restrictions to E and K. Now we can ask whether there is any relation between the reductions of F and E. In particular if F has good reduction with respect to v does E also? Questions of this kind have been studied by Taoufik Youssefi, a student of Matignon, in his doctoral thesis [Y], and by Hagen Knaf, a student of Roquette, in his Diplomarbeit [Kn]. Youssefi studied the more general problem, namely that of giving an inequality relating the genera obtained when considering a number of reductions of F and E. Using this inequality the question involving good reduction could be answered as a corollary. His results are proved by rigid geometry using the stable reduction theorem in the rank 1 situation. By means of a valuation decomposition and combinatorial argument they are then extended to the general case. The main result is the relative genus inequality.

Theorem 5.6 [Y]. *Suppose K is an algebraically closed field and $F|K$ and $E|K$ are function fields in 1 variable with $E \subseteq F$. Suppose v_i, $1 \leq i \leq s$, are distinct constant reductions of E with common restriction v_K to K. Suppose v'_j, $1 \leq j \leq t$ is a set of prolongations of the v_i*

to F. Then the following inequality holds:

$$-\chi(F|K)+\sum_{j=1}^{t}\chi(Fv'_j|Kv)-t+1 \geq -\chi(E|K)+\sum_{i=1}^{s}\chi(Ev_i|Kv)-s+1 \geq 0.$$

In particular in terms of the genera this inequality asserts that

$$g_F-\sum_{j=1}^{t}g_{Fv'_j} \geq g_E-\sum_{i=1}^{s}g_{Ev_i} \geq 0.$$

As a corollary one obtains:

Corollary 5.7 [Y]. *Suppose (K,v_K) is a valued field and $F|K$ and $E|K$ are conservative function fields in 1 variable with $E\subseteq F$ and $g_F,g_E\geq 1$. Suppose v is a good reduction of F with $v|_K=v_K$ and so that $Fv|Kv$ and $Ev|Kv$ are conservative. Then E has good reduction at v.*

In the case of K algebraically closed this follows directly from 5.6 and the criteria for good reduction. The general case is deduced from this case. We next mention an interesting question concerning the lifting of morphisms over $\mathbf{P}^1_K$ asked by Youssefi in his thesis:

Question: *Let (K,v_K) is an algebraically closed valued field. Suppose $\overline{F}|\overline{K}$ is a function field in 1 variable over $\overline{K}=Kv_K$ and $\overline{h}\in\overline{F}$ is such that $\overline{F}|\overline{K}(\overline{h})$ is separable and $(\overline{h})_0$ is concentrated in one place of $\overline{F}|\overline{K}$. Then does there exist a valued function field $(F|K,v)$ with v prolonging v_K to F and a v-regular function h, such that*

(i) $g_F=g_{\overline{F}}$;

(ii) $Fv=\overline{F}$;

(iii) $hv=\overline{h}$?

In support of this question Youssefi showed that if (K,v_K) is an algebraically closed complete rank 1 valued field and $\overline{F}|\overline{K}(\overline{h})$ is galois with soluble galois group then the answer is positive. He also gave an example in the non galois case.

In his work Knaf approached the problem more directly by studying the reduction of function fields and their automorphism groups. His results were obtained using general theorems from the theory of function fields, their reductions and general valuation theory. Special properties of rank 1 valuations were not used, however he was able to strengthen one of his main results in an appendix by appealing to the work of Youssefi. The main result he obtained relating to the question of good reduction was:

Theorem 5.8 [Kn]. *Let $(F|K,v)$ be a valued function field with $g_F \geq 1$, having good reduction at v and suppose $Fv|Kv$ is conservative. Let $E|K$ be a normal sub function field of F. Then if the extension $Fv|Fv^{Gv}$ is tamely ramified, where Gv is the automorphism group of $Fv|Ev$, then E has good reduction at v.*

The strengthened form of this theorem asserts that E has good reduction at v for an arbitrary sub function field E of F.

6. The divisor reduction map

When studying constant reductions of function fields one of the first questions one is led to ask is what kind of relationship, if any, exists between the Riemann space of the function field and that of its reduction. This question was first studied by Deuring in [D1] and [D2] in connection with his work on the Riemann hypothesis for the congruence zeta function of function fields over finite fields (For a survey of the work of Deuring we mention the survey paper of Roquette [R5] in the 1989 DMV Jahresbericht). Deuring considered the case of good reduction and made the assumption that the valuation on the constant field was discrete rank 1. For this situation he did more than giving a map between the Riemann spaces of the function fields. He showed that there is a natural degree preserving homomorhism between the divisor group of the function field and that of its reduction. The case of a general valuation on the base has been treated by Roquette in [R1]. As this divisor map is the prototype of a more general reduction map which we shall discuss, we review its properties briefly here.

6.1. The divisor reduction map in the case of good reduction. Let $(F|K,v)$ be a valued function field having good reduction at v, $\mathrm{Div}(F|K)$ denotes the divisor group of $F|K$ and similarly $\mathrm{Div}(Fv|Kv)$. Then by Deuring–Roquette, [D1]-[R1], there is a natural homomorphism called the divisor reduction map

$$\mathbf{r}\colon \mathrm{Div}(F|K) \longrightarrow \mathrm{Div}(Fv|Kv), \quad A \longmapsto Av$$

such that

(i) $\deg A = \deg Av$,

(ii) $A \geq 0 \;\Rightarrow\; Av \geq 0$,

(iii) $A = (f),\; v(f) = 0 \;\Rightarrow\; Av = (fv)$.

Observe that if K is algebraically closed and $P \in \mathrm{S}(F|K)$, the set of prime divisors of $F|K$, then by (i), $\deg Pv = \deg P = 1$ and therefore $Pv \in \mathrm{S}(Fv|Kv)$, i.e. prime divisors reduce to prime divisors.

We can express the condition that $f \in F$ is regular at v in terms of the zero or pole divisors associated to f and fv. Namely writing $(f) = (f)_0 - (f)_\infty$, and taking the reduction gives $(f)v = (f)_0 v - (f)_\infty v = (fv) = (fv)_0 - (fv)_\infty$ by (iii) above. As the zero and pole divisors are positive we conclude from (ii) above that $(f)_0 v \leq (fv)_0$ and $(f)_\infty v \leq (fv)_\infty$. Now the condition that f be regular means $\deg (f)_0 = \deg (fv)_0$ and $\deg (f)_\infty = \deg (fv)_\infty$. Therefore from (i) it follows that f is regular at v if and only if $(f)_0 v = (fv)_0$ or equivalently $(f)_\infty v = (fv)_\infty$.

Notice that by the results of the previous section in the case of good reduction the vector space defect is 1. Therefore for every K-module $M \subset F$ of finite dimension

$$\dim_K M = \dim_{Kv} Mv,$$

where as always Mv denotes the set of all reductions of elements of M of non-negative valuation. Taking this into account we observe that for the divisor reduction:

(i) For each $Z \in \mathrm{Div}(F|K)$, $\mathcal{L}(Z)v \subseteq \mathcal{L}(Zv)$ and $\dim Z \leq \dim Zv$. Here $\dim Z = \dim \mathcal{L}(Z)$.

(ii) If $Z \in \mathrm{Div}(F|K)$ has $\deg Z \geq 2g_F - 1$, then by the Riemann-Roch theorem $\dim Z = \dim Zv$ and $\mathcal{L}(Z)v = \mathcal{L}(Zv)$.

The next two results give information about the degree and support of regular functions:

Proposition. *Let $F|K$ be a function field in 1 variable over K and suppose $Z = \sum_{i=1}^{n} m_i P_i \in \mathrm{Div}(F|K)$, where $m_i > 0$ and P_i are prime* *divisors for* $1 \leq i \leq n$, *and* $\deg(Z - \sum_{i=1}^{n} P_i) \geq 2g_F - 1$. *Then there exists* $f \in F$ *such that* $(f)_\infty = Z$, *where* $(f)_\infty$ *is the pole divisor of* f.

Proof. Let $B = Z - \sum_{i=1}^{n} P_i$ and $Z_i = B + P_i$, $1 \leq i \leq n$. Then since $\deg B \geq 2g_F - 1$, $\deg Z_i \geq 2g_F - 1$ and $\deg Z \geq 2g_F - 1$. By the Riemann-Roch theorem it follows that

$$\dim Z = \dim B + \deg \sum_{i=1}^{n} P_i \quad \text{and} \quad \dim Z_i = \dim B + \deg P_i.$$

From this it follows that $\mathcal{L}(B) \subset \mathcal{L}(Z_i)$. Let $f_i \in \mathcal{L}(Z_i) \backslash \mathcal{L}(B)$. Then:

$$\operatorname{ord}_P(f_i) \begin{cases} = -m_i, & \text{if } P = P_i; \\ \geq -m_j + 1, & \text{if } P = P_j, \quad 1 \leq j \leq n, \ j \neq i; \\ \geq 0, & \text{if } P \neq P_j, \quad 1 \leq j \leq n. \end{cases}$$

Let $f = \sum_{i=1}^{n} f_i$. Then

$$\operatorname{ord}_P(f) \begin{cases} = -m_i, & \text{if } P = P_i, \quad 1 \leq i \leq n; \\ \geq 0, & \text{otherwise.} \end{cases}$$

Hence $f \in \mathcal{L}(Z)$ and $(f)_\infty = Z$.

Remark. Note that by construction $(f) = A - Z$ with $A \geq 0$ and that A and Z have no common divisor. If A would contain P_i for some i then $A \geq P_i$ so that $(f) \geq P_i - Z$ and $f \in \mathcal{L}(Z - P_i)$. This contradicts $\operatorname{ord}_{P_i}(f) = -m_i$. The divisor A, is the divisor of zeros of f; we write $(f)_0 = A$.

Theorem. *Suppose* $(F|K, v)$ *is a valued function field having good reduction at* v. *Let* $Z \in \operatorname{Div}(F|K)$ *be chosen so that* $Z = \sum_{i=1}^{n} m_i P_i$ *with* $m_i > 0$, $1 \leq i \leq n$, *and* $\deg B \geq 2g_F - 1$, *where* $B = Z - \sum_{i=1}^{n} P_i$. *Then there exists a regular function* $f \in F$ *with* $(f)_\infty = Z$.

Proof. Note first that as $B \geq 0$ with $\deg B \geq 2g_F - 1$ it follows that $\mathcal{L}(Z)v = \mathcal{L}(Zv)$ and that if $Zv = \sum_{i=1}^{k} l_i Q_i$ with the $l_i > 0$ and

Q_i prime divisors of $\mathrm{Div}(Fv|Kv)$, then $\deg(Zv - \sum_{i=1}^{k} Q_i) \geq 2g_F - 1$ and $Zv - \sum_{i=1}^{k} Q_i \geq 0$. We apply the proposition above to Zv and obtain $fv \in \mathcal{L}(Zv)$ with $(fv)_\infty = Zv$. Let $f \in \mathcal{L}(Z)$ be a foreimage of fv, so that $(f) = A - Z$ with $A \geq 0$. Observe that A and Z have no common divisor or otherwise so would Av and Zv ($Av = (fv)_0$, the divisor of zeros of fv and $Zv = (fv)_\infty$). We conclude that $A = (f)_0$ and $Z = (f)_\infty$. Hence $(f)_\infty v = Zv = (fv)_\infty$ and f is regular at v.

Before turning to the more general context where we define a divisor map for several constant reductions which have a common restriction to the constant field there is an important point to note following the theorem above. Firstly, it is possible to find v-regular functions with pole divisors having arbitrarily prescribed support and secondly, such functions can be chosen having degree bounded by something depending only on the genus of the function field.

We shall next define a divisor reduction map in a more general situation keeping the prototype above in mind.

6.2. Let $F|K$ be a function field in 1 variable and V a finite set of constant reductions of F having a common restriction to K denoted by v_K. Suppose throughout that $f \in F$ is a V-*regular* function for $F|K$. We define the *inf norm* w as $w(x) = \inf_{v \in V} v(x)$ for $x \in F$ and let $\mathcal{O}_w = \{x \,:\, w(x) \geq 0\}$ and $\mathcal{M}_w = \{x \,:\, w(x) > 0\}$. Then $\mathcal{O}_w = \bigcap_{v \in V} \mathcal{O}_v$, $\mathcal{M}_w = \bigcap_{v \in V} \mathcal{M}_v$, and $Fw := \mathcal{O}_w / \mathcal{M}_w = \prod_{v \in V} \mathcal{O}_v / \mathcal{M}_v = \prod_{v \in V} Fv$ where $\mathcal{O}_v$, respectively $\mathcal{M}_v$, denotes the valuation ring, respectively maximal ideal, of v in F and the second equality means the identification $x + \mathcal{M}_w \mapsto (x + \mathcal{M}_v)_v$.

The integral closure of $K[f]$, respectively $\mathcal{O}_K[f]$, in F will be denoted by R_f, respectively $\mathcal{R}_f$. By [G–M–P 2] proposition 2.1, $\mathcal{R}_f = R_f \cap \mathcal{O}_w$. The reduction of $\mathcal{R}_f$ to Fw is denoted by $R_f w$ and throughout we use $'$ to denote integral closure. In terms of divisors R_f can be written as

$$R_f = \{x \in F \,:\, xP \neq \infty \;\forall P \in \mathrm{S}_f\} = \bigcup_{n \in \mathbb{N}} \mathcal{L}(nD),$$

where $\mathrm{S}_f = \{P \in \mathrm{S}(F|K) \,:\, fP \neq \infty\}$, $xP \in FP$ is the image of x by the place determined by the prime divisor P and $D = (f)_\infty \in \mathrm{Div}(F|K)$

the pole divisor of f.

6.3. The divisor reduction map relative to a regular function.

Let $f \in F$ be a V-regular function. Then the divisor reduction map relative to f is defined as follows: First for $A \in \mathrm{Div}(F|K)$ we set

$$\mathcal{A}_f = \{x \in F \,:\, v_P(x) \geq v_P(A)\ \forall P \in \mathrm{S}_f\},$$

where v_P is the ordinal valuation of $F|K$ associated with P. Let $(\mathcal{A}_f w)'$ denote the fractional $(R_f w)'$-ideal, $\mathcal{A}_f w\,(R_f w)'$, obtained by extending to the integral closure. The conductor of $(R_f w)'$ in $R_f w$ is denoted by $\mathcal{F}_{fw}$.

Next we set

$$\mathrm{S}_f^0 = \{P \in \mathrm{S}(F|K) \,:\, fP \in \mathcal{O}_{FP}^{\times} \text{ and } \mathrm{supp}((\mathrm{N}_{K(f)}^{F}(\mathcal{P}_f))w) \cap \mathrm{supp}(\mathcal{F}_{fw}) = \emptyset\}$$

and denote by $\mathrm{Div}_f^0(F|K)$ the subgroup of $\mathrm{Div}(F|K)$ of divisors with support in S_f^0.

The divisor reduction map relative to f on $\mathrm{Div}_f^0(F|K)$,

$$\mathbf{r}\colon \mathrm{Div}_f^0(F|K) \longrightarrow \mathrm{Div}(Fw|Kw),$$

is defined by $\mathbf{r}(A) = Aw = \sum_{v \in V} Av$, with $Av \in \mathrm{Div}(Fv|Kv)$ for each $v \in V$ and

$$v_Q(Av) = \begin{cases} v_Q(\mathrm{pr}_v((\mathcal{A}_f w)')) & \text{for } Q \in \mathrm{S}(Fv|Kv) \text{ such that } fv(Q) \neq \infty \\ v_Q(\mathrm{pr}_v((\mathcal{A}_{f^{-1}} w)')) & \text{for } Q \in \mathrm{S}(Fv|Kv) \text{ such that } fv(Q) \neq 0 \end{cases}$$

where pr_v denotes the projection from Fw onto Fv for each $v \in V$.

The main properties of this map are:

(i) The map $\mathbf{r}$ is a degree preserving homomorphism ([G–M–P 2] theorem 2.2);

(ii) For $A \in \mathrm{Div}_f^0(F|K)$ it holds $\mathcal{L}(A)w \subseteq \mathcal{L}(Aw) := \prod_{v \in V} \mathcal{L}(Av)$ ([G–M–P 2] theorem 2.2);

(iii) Let $f, g \in F$ be V-regular functions. Then the divisor reduction maps relative to f and g coincide on $\mathrm{S}_f^0 \cap \mathrm{S}_g^0$ and moreover supp$((\mathrm{S}_f^0 \cap \mathrm{S}_g^0)w)$ is an open dense subset of $\mathrm{S}(Fw|Kw)$, the disjoint union of the $\mathrm{S}(Fv|Kv)$ and endowed with the induced topology.

The results above were central in a study of the reductions of algebraic curves (see §7) and were also used to prove the following theorem on the existence of regular functions with prescribed support. In order to discuss this result we need to introduce the following notations. First, for any divisor $A \in \mathrm{Div}_f^0(F|K)$ with $Av > 0$ for each $v \in V$ we define

$$R_A = \bigcup_{n\in\mathbb{N}} \mathcal{L}(nA) \qquad \text{and} \qquad R_{Aw} = \bigcup_{n\in\mathbb{N}} \mathcal{L}(nAw).$$

R_A is the Dedekind ring consisting of all functions in F having poles only in supp(A) and R_{Aw} the ring consisting of the functions in Fw having poles only in supp(Aw). Because of the hypothesis on A, we have $R_{Aw} = \prod_v R_{Av}$ with R_{Av} a non-constant Dedekind ring of Fv for each $v \in V$. The conductor of R_{Aw} in $R_A w$ will be denoted by $\mathcal{F}_{A,w} = \prod_v \mathcal{F}_{A,v}$. Then:

(iv) the integral closure $(R_A w)' = R_{Aw}$ and

$$\begin{aligned}\chi(R_A w) &:= \chi(Fw|Kw) - \dim (R_A w)'/R_A w \\ &= \sum_v \chi(Fv|Kv) - \dim (R_A w)'/R_A w = \chi(F|K) =: \chi(R_A).\end{aligned}$$

saying that the arithmetic genus does not change after reduction, [G–M–P 2].

Now we state the theorem on the existence of V-regular functions:

Theorem [G–M–P 2]. *Let $A \in \mathrm{Div}_f^0(F|K)$ be a positive divisor and suppose that for each $v \in V_f$ there exists a positive divisor $B_v \in \mathrm{Div}(Fv|Kv)$ satisfying:*

(i) $2B_v < Av$,

(ii) $\deg B_v > \max(-2\chi(Fv|Kv), \dim R_{Av}/\mathcal{F}_{A,v} - \chi(Fv|Kv))$.

Then there exist V_f-regular functions h for $F|K$, such that $(h)_\infty = A$ and hence $(hw)_\infty = Aw$.

This theorem contains 2.2 as a special case. Indeed, suppose K is algebraically closed and let $A = \sum_v n_v P^v \in \mathrm{Div}_f^0(F|K)$, where:

$$P^v w \in \mathrm{Div}(Fv|Kv) \quad \text{and} \quad n_v = -4\chi(Fv|Kv) + 2\dim R_{Av}/\mathcal{F}_{A,v} + 5.$$

Then

$$\begin{aligned}\deg A = \sum_v n_v &= \sum_v \big(-4\chi(Fv|Kv) + 2\dim R_{Av}/\mathcal{F}_{A,v} + 5\big)\\ &\le -4\chi(F|K) + 5s, \qquad s = \operatorname{card}(V)\\ &= 4g_F - 4 + 5s,\end{aligned}$$

where the inequality follows from the non-trivial identity

$$\begin{aligned}\sum_v \dim R_{Av}/\mathcal{F}_{A,v} &= \dim(R_A w)'/\mathcal{F}_{A,w}\\ &\le 2\dim(R_A w)'/R_A w = 2(\sum_v \chi(Fv|Kv) - \chi(F|K)).\end{aligned}$$

Here the inequality follows from a classical result of Rosenlicht [Ros] and the final equality from (iv) above.

6.4. Recently in his Diplomarbeit Thomas Kässer, a student of Roquette, has defined and studied a reduction map $\Omega_{F|K} \longrightarrow \Omega_{Fv|Kv}$, for the module of differentials of a valued function field $(F|K,v)$ possessing a regular function under the assumption that $Fv|Kv$ is separable. In particular he has studied the behaviour of the holomorphic, residue free and exact differentials under this map. In this work he has made use of the results above on the divisor reduction map (6.3), in order to find regular-separating functions for the valued function field.

7. Reductions of algebraic curves

Let (K, v_K) be a valued field with valuation ring $\mathcal{O}_K$. The assertion concerning the existence of regular functions, for a given finite set of constant reductions V prolonging v_K to a function field $F|K$, can also be formulated geometrically as an assertion concerning the existence of a finite morphism of $\mathcal{O}_K$-schemes from a certain $\mathcal{O}_K$-curve having function field F to $\mathbf{P}^1_{\mathcal{O}_K}$. In this section we review recent results on these $\mathcal{O}_K$-curves. The proofs can be found in [G–M–P 3].

7.1. The $\mathcal{O}_K$-curve associated to a finite family of constant reductions.
Let V be a finite family of constant reductions of F having common restriction v_K to K and possessing a V-regular function $f \in F$. The $\mathcal{O}_K$-curve $\mathcal{C}_V$ associated to V is defined to be the $\mathcal{O}_K$-scheme

$$\mathcal{C}_V = \operatorname{Spec}\mathcal{R}_f \cup \operatorname{Spec}\mathcal{R}_{f^{-1}}$$

obtained by glueing the affine $\mathcal{O}_K$-schemes $\operatorname{Spec}\mathcal{R}_f$ and $\operatorname{Spec}\mathcal{R}_{f^{-1}}$ along $\operatorname{Spec}\mathcal{O}_K[f,f^{-1}]'$. Observe that $\mathcal{R}_f[f] = \mathcal{O}_K[f,f^{-1}]' = \mathcal{R}_{f^{-1}}[f^{-1}]$.

Equivalently, $\mathcal{C}_V$ is the normal closure of $\mathbf{P}^1_{\mathcal{O}_K}$ relative to the field extension $F|K(f)$.

From the definition above it appears that the $\mathcal{O}_K$-curve $\mathcal{C}_V$ depends on the choice of V-regular function f used for its definition. In fact it only depends on the set of constant reductions V and has the following properties:

(i) $\mathcal{C}_V$ is a projective integral normal flat $\mathcal{O}_K$-scheme of pure relative dimension 1.

More precisely we have:

Let $D = (f)_\infty$ be the pole divisor of f and set $S = \bigoplus_{n\geq 0} \mathcal{L}_w(nD)$, where $\mathcal{L}_w(nD) = \mathcal{L}(nD)\cap\mathcal{O}_w$. Then S is a finitely generated graded $\mathcal{O}_K$-algebra and $\mathcal{C}_V \cong \operatorname{Proj} S$.

(ii) Special fibres: Let $\mathcal{P}$ be any point of $\operatorname{Spec}\mathcal{O}_K$ and $v_{\mathcal{P}}$, respectively $V_{\mathcal{P}}$ with semi-norm $w_{\mathcal{P}}$, be the corresponding coarsening of the valuation v_K, respectively the constant reductions V. Set $\mathcal{O}_{\mathcal{P}} = (\mathcal{O}_K)_{\mathcal{P}}$ and let $Kv_{\mathcal{P}}$ denote the residue field at $\mathcal{P}$. Then $f \in F$ is a $V_{\mathcal{P}}$-regular function and if $\mathcal{C}_{V_{\mathcal{P}}}$ is the $\mathcal{O}_{\mathcal{P}}$-curve associated to $V_{\mathcal{P}}$ as defined above for V then

$$\mathcal{C}_{V_{\mathcal{P}}} \approx \mathcal{C}_V \times_{\mathcal{O}_K} \mathcal{O}_{\mathcal{P}}.$$

In particular $\mathcal{C}_V$ and $\mathcal{C}_{V_{\mathcal{P}}}$ have $\mathcal{O}_K$-isomorphic generic fibres which are K-isomorphic to the non-singular irreducible projective curve $\mathbf{C}$ associated to F. Further the special fibre $\mathcal{C}_{\mathcal{P}}$ of $\mathcal{C}_V$ at $\mathcal{P}$ is $\mathcal{O}_K$-isomorphic to the closed fibre of $\mathcal{C}_{V_{\mathcal{P}}}$.

(iii) $\mathcal{C}_V$ is independent of the V-regular function f used for its definition and hence, depends only on V.

(iv) For each $\mathcal{P} \in \operatorname{Spec}\mathcal{O}_K$, $\mathcal{C}_{\mathcal{P}}$ is reduced, and $\chi(\mathcal{C}_{\mathcal{P}}) = \chi(F|K)$.

(v) $\mathcal{C}_V$ is locally of finite presentation over $\mathcal{O}_K$ and the morphism $\mathcal{C}_V \to \mathbf{P}^1_{\mathcal{O}_K}$ is finite.

This follows from (iii) above and the following algebraic result concerning the rings $\mathcal{R}$ associated to regular functions: *Let V be a finite* *set of constant reductions of F having common restriction to K and suppose $f \in F$ is a V-regular function. Then there exists a V-regular function t such that $\mathcal{R}_t$ is a finite free $\mathcal{O}_K[t]$-module and $\mathcal{R}_{t^{-1}}$ is a finite free $\mathcal{O}_K[t^{-1}]$-module.*

The following question remains: If f is a V-regular function, is $\mathcal{R}_f$ a finite free $\mathcal{O}_K[f]$-module? When v_K is rank 1 and K is complete, this result holds, see [G–vdP], p. 100. In the general case we are only able to show that there exists $s \in \mathcal{O}_K[f]$ with $sw = 1$, such that the localisation $(\mathcal{R}_f)_s$ is a finite free $\mathcal{O}_K[f]_s$-module. This gives an alternative way of proving (v) without appealing to (iii).

A natural question to ask is whether given two sets of constant reductions, $V_1 \subset V$, there exists a contraction morphism between $\mathcal{C}_V$ and $\mathcal{C}_{V_1}$.

Contraction Lemma. *Let $V_1 \subset V$ be finite sets of constant reductions of F prolonging v_K and suppose that V_1 and V have regular functions. Let $\mathcal{C}_{V_1}$, respectively $\mathcal{C}_V$ be the corresponding $\mathcal{O}_K$-curves. Then there exists a canonical surjective $\mathcal{O}_K$-morphism from $\mathcal{C}_V$ to $\mathcal{C}_{V_1}$.*

Finally when (K, v_K) is an algebraically closed valued field the $\mathcal{O}_K$-curves $\mathcal{C}_V$ can be characterised by there scheme theoretic properties as follows:

I. Let $\mathcal{X}$ be any proper, integral, normal $\mathcal{O}_K$-scheme of pure relative dimension 1. Let F be the function field of $\mathcal{X}$. Then $\mathcal{X}$ is isomorphic over $\operatorname{Spec}\mathcal{O}_K$ to an $\mathcal{O}_K$-curve $\mathcal{C}_V$ for some finite set of constant reductions V of F prolonging v_K.

   More precisely, if $\eta_i \in \mathcal{X}$ correspond to the generic points of the irreducible components of the closed fibre of $\mathcal{X}$, then the local rings $\mathcal{O}_{\mathcal{X},\eta_i}$ are valuation rings dominating $\mathcal{O}_K$ and defining the set of constant reductions V of $F|K$ which determine $\mathcal{C}_V$.

   The idea of proof of this result is first to show there exists an birational $\mathcal{O}_K$-morphism $\phi : \mathcal{X} \longrightarrow \mathcal{C}_V$ by using the contraction lemma. Next one shows that ϕ satisfies the conditions (in particular properness and quasi finiteness) needed to use Zariski's Main Theorem EGA IV, 8.12.10, to conclude it is an open immersion. From this it then follows ϕ is an isomorphism as it is proper.

II. Let $\mathcal{X}$ be an integral projective $\mathcal{O}_K$-scheme with normal generic fibre of dimension 1 and reduced closed fibre. Let F be the function field of $\mathcal{X}$. Then $\mathcal{X}$ is isomorphic over $\operatorname{Spec}\mathcal{O}_K$ to an $\mathcal{O}_K$-curve $\mathcal{C}_V$ for some finite set of constant reductions V of F prolonging v_K.

The main ingredient for the proof II is a lemma which asserts that if $\mathcal{X}$ is an integral projective $\mathcal{O}_K$-scheme with normal generic fibre and reduced closed fibre, then $\mathcal{X}$ is normal. This lemma is deduced using methods developed by Roquette in [R1]. Using this result II follows directly from EGA IV, lemma 14.3.10 (Artin's lemma) together with I above.

7.2. Constant reduction of fields and reduction of curves.

In this paragraph we shall relate the theory of reduction of curves as presented above with the other theories of reduction, as presented in the EGA style terminology of arithmetic surfaces, or as presented in the terminology of formal analytic spaces as in [B–L] for example.

7.2.1. Constant reduction and arithmetic surfaces. Let (K_0, v_{K_0}) be a rank 1 discrete henselian valued field with valuation ring $\mathcal{O}_{K_0}$ and $(\tilde{K}, v_{\tilde{K}})$ a valued algebraic closure of (K_0, v_{K_0}).

Suppose $\tilde{F}|\tilde{K}$ is a function field in one variable equipped with a finite family $\tilde{V}$ of constant reductions $\tilde{v}$ coinciding on $\tilde{K}$ with $v_{\tilde{K}}$. Let us denote $\tilde{w} := \inf \tilde{v}$. Then by the results above the following situation is uniquely determined by $\tilde{F}|\tilde{K}$ and $\tilde{V}$:

For any $\tilde{V}$-regular function $f \in \tilde{F}$ and $D_{\tilde{K}} = (f)_\infty$, the $\mathcal{O}_{\tilde{K}}$-curve associated to $\tilde{V}$ with function field $\tilde{F}$ is $\mathcal{C}_{\tilde{V}} \approx \operatorname{Proj}(\bigoplus_n \mathcal{L}(nD_{\tilde{K}}) \cap \mathcal{O}_{\tilde{w}})$.

Now there is a finite extension $K|K_0$, and a function field $F|K$, $F \subset \tilde{F}$ with $F\tilde{K} = \tilde{F}$, such that if $V = \tilde{V}|_F$ the following holds:

(a) $F|K$ is conservative with exact constant field K;

(b) For each $v \in V$, $Fv|Kv$ is conservative with exact constant field Kv;

(c) $f \in F$ is a V- regular function.

Let $D \in \operatorname{Div}(F|K)$ be the pole divisor of f in F. Then the $\mathcal{O}_K$-curve associated to V with function field F is $\mathcal{C}_V \approx \operatorname{Proj}(\bigoplus_n \mathcal{L}(nD) \cap \mathcal{O}_w)$

and $\mathcal{C}_{\tilde{V}} \approx \mathcal{C}_V \times_{\mathcal{O}_K} \mathcal{O}_{\tilde{K}}$. Note that in addition to the other properties of $\mathcal{C}$ listed in **7.1** it is also geometrically connected. We remark that the construction above holds for any function field $F|K$ equipped with a family of constant reductions satisfying (a) , (b) and (c) above.

Suppose $K|K_0$ is finite and that $\mathcal{X}$ is a projective geometrically connected and integral normal $\mathcal{O}_K$-scheme of pure relative dimension 1 having reduced special fibres. We show the existence of a finite $\mathcal{O}_K$-morphism $\mathcal{X} \longrightarrow \mathbf{P}^1_{\mathcal{O}_K}$ by using effective Cartier divisors and the finiteness of cohomology for coherent sheaves on $\mathcal{X}$ as in [B–L–R]. This gives an alternative proof of **7.1** (vii) in this special situation. We have included the proof to make the comparison between the methods.

Let $(\eta_i)_{1\leq i\leq s} \in \mathcal{X}$ be the generic points of the irreducible components of the special fibre of $\mathcal{X}$, then $\mathcal{O}_{\mathcal{X},\eta_i}$ is a discrete valuation ring dominating $\mathcal{O}_K$. Let $V = \{v_i : 1 \leq i \leq s\}$ be the corresponding valuations of the function field $F|K$ of the generic fibre $\mathcal{X}_{(K)}$. Note that Fv_i is the function field of the irreducible component $\mathbf{V}(\eta_i) \subset \mathcal{X}$ of the special fibre, so $(F|K, v_i)$ is a constant reduction.

Now we want to show that $\mathcal{X}$ is isomorphic to the curve $\mathcal{C}_V$ of **7.1** associated to V . For this we have to show the existence of a V-regular function $f \in F$, with pole divisor D such that $\mathcal{X} \approx \operatorname{Proj}(\bigoplus_n \mathcal{L}(nD) \cap \mathcal{O}_w)$. First, one can find D , an effective Cartier divisor on $\mathcal{X}$, i.e. a global section of $\mathcal{M}^*_{\mathcal{X}}/\mathcal{O}^*_{\mathcal{X}}$, $\mathcal{M}_{\mathcal{X}}$ the sheaf of quotient rings of the structural sheaf $\mathcal{O}_{\mathcal{X}}$ and * denoting invertibility, such that the special fibre $\mathrm{D}_{(Kv)}$ is concentrated on regular points and meets each irreducible component of $\mathcal{X}_{(Kv)}$, see [B–L–R] p. 169 proposition 4. We denote by $\mathcal{D}$ the invertible sheaf associated to D , which can also be considered as an invertible subsheaf of the sheaf of quotient rings $\mathcal{M}_{\mathcal{X}}$ of $\mathcal{O}_{\mathcal{X}}$. Then ([B–L–R] p. 167 in the more general context of contractions) $A := \bigoplus_n \mathrm{H}^0(\mathcal{X}, \mathcal{D}^{\otimes n})$ is a graded $\mathcal{O}_K$-algebra of finite type and $\mathcal{X} = \operatorname{Proj} A$. Let $\mathcal{B}_i$ be the minimal homogenous primes, then $A_{(\mathcal{B}_i)} \subset F$ is the valuation ring of v_i and $\mathcal{O}_w = \bigcap_i A_{(\mathcal{B}_i)}$.

Now we show the equality

$$(*) \qquad \left(\mathrm{H}^0(\mathcal{X}, \mathcal{D}^{\otimes n}) \otimes_{\mathcal{O}_K} K\right) \cap \mathcal{O}_w = \mathrm{H}^0(\mathcal{X}, \mathcal{D}^{\otimes n}) =: A_n.$$

Let $1 \in A_1$ be the constant unit function and π a uniformizing element in K. For $z \in A_n \otimes_{\mathcal{O}_K} K$, we write $z = a/\pi^l$, where $a \in A_n$ and l is chosen minimal positive. If furthermore $z \in \mathcal{O}_w^{\downarrow}$ one deduces from $(\pi) = \bigcap_i \mathcal{B}_i$ that $l = 0$, and $(*)$ follows.

The equality $(*)$ shows that the algebraic reduction of global sections of $\mathcal{D}^{\otimes n}$ is the same as the reduction relative to w and consequently to conclude this discussion we need to construct a V-regular function $f \in F$ such that $(f)_\infty = n_1 \mathrm{D}_{(K)}$ and $(fw)_\infty = n_1 \mathrm{D}_{(Kv)}$ for an $n_1 \gg 0$. For such a function one has $\mathcal{X}' \approx \mathcal{C}_V$. Now a V-regular function f is obtained directly by applying the method of [G–M–P 2], proposition 2.5. Precisely due to the choice of $\mathrm{D}_{(Kv)}$ with $\mathrm{supp}(\mathrm{D}_{(Kv)})$ concentrated in regular points one deduces using the theorem of Riemann-Roch that

$$\dim \frac{\bigcup_n \mathcal{L}(nn_0 \mathrm{D}_{(Kv)})}{\bigcup_n \mathcal{L}(nn_0 \mathrm{D}_{(K)})w} = \dim \mathcal{L}(nn_0 \mathrm{D}_{(Kv)}) - \dim \mathcal{L}(nn_0 \mathrm{D}_{(K)})w$$

for $n \gg 0$. Hence there exists a fixed Kv-space M so that for all $n \gg 0$

$$\mathcal{L}(nn_0 \mathrm{D}_{(Kv)}) = \mathcal{L}(nn_0 \mathrm{D}_{(K)})w \oplus M$$

and now it is easy to give a V-regular function f.

7.2.2. Constant reduction and formal analytic spaces. Let (K, v_K) be a rank 1 complete algebraically closed valued field. Suppose $F|K$ is a function field in one variable endowed with constant reductions $V = \{v_i \ : \ 1 \leq i \leq s\}$ prolonging $= v_K$ to F and $w = \inf_i v_i$. Let $f \in F$ be a V-regular function, then the special fibre of $\mathcal{C}_V$ at the closed point $\mathcal{M}_K \in \operatorname{Spec} \mathcal{O}_K$, $\mathcal{C}_{\mathcal{M}_K}$, can be viewed as a glueing of affine shemes affine Kv-schemes $\operatorname{Spec} R_f w$ and $\operatorname{Spec} R_{f^{-1}} w$ along $\operatorname{Spec}(K[f, f^{-1}]')w$ and it is clear from this description that $\mathcal{C}_{\mathcal{M}_K}$ is the analytic reduction of the curve $\mathbf{C}$ (the unique non-singular irreducible projective curve associated with F), with respect to the formal covering $U(f) = \{P \in \mathbf{C} \ : \ w(fP) \geq 0\}$, $U(f^{-1}) = \{P \in \mathbf{C} \ : \ w(fP) \leq 0\}$, see for example [M 1].

Reciprocally we assume $\mathbf{C}$ is a non-singular projective curve over K which is equipped with a formal covering $\mathcal{U}$. One can consider the analytic reduction of $\mathbf{C}$ with respect to $\mathcal{U}$, $r \colon \mathbf{C} \to \overline{\mathbf{C}}^{\mathcal{U}}$. Choose one (regular) point Q_i from each irreducible component of $\overline{\mathbf{C}}^{\mathcal{U}}$, then $U := r^{-1}(\overline{\mathbf{C}}^{\mathcal{U}} - \bigcup_i \{Q_i\})$ is an affinoid space and its canonical reduction is $\overline{U}^{\mathbf{C}} = \overline{\mathbf{C}}^{\mathcal{U}} - \bigcup_i \{Q_i\}$, [B] theorem 3.1. Finally using [F–M] we obtain the existence of $f \in F$ (the function field of $\mathbf{C}$) such that $U = U(f) = \{P \in \mathbf{C} : w(fP) \geq 0\}$, where w is the spectral norm defined on U . Further $(\mathbf{C},\mathcal{U})$ is equivalent to $(\mathbf{C}, \mathcal{V} = \{U(f), U(f^{-1})\})$, because $\overline{\mathbf{C}}^{\mathcal{U}}$ (and $\overline{\mathbf{C}}^{\mathcal{V}}$) is the completion of the affine curve $\overline{U}^{\mathbf{C}} = \overline{U(f)}^{\mathbf{C}}$. This completes the proof, for as in the preceding paragraph if $\mathcal{C}_{V_f}$ is the $\mathcal{O}_K$-curve associated to V_f then $\mathcal{C}_{\mathcal{M}_K} \approx \overline{\mathbf{C}}^{\mathcal{V}}$.

7.3. On the elementary nature of stable reductions. The stable reduction theorem for curves over $\mathcal{O}$, an arbitrary valuation ring with algebraically closed quotient field can be stated as follows:

> *Let $\mathcal{O}$ be a valuation ring with algebraically closed quotient field K, and let $\mathbf{C}$ be a projective non-singular irreducible curve over K of genus $g \geq 2$. Then there exists a unique stable curve $\mathcal{C}$ over $\mathcal{O}$ (in the sense of Deligne – Mumford* [D–M]*) with generic fibre $\mathcal{C}_\eta \cong \mathbf{C}$.*

Recall that if $\mathcal{S}$ is any scheme and $g \geq 2$, then a stable curve of genus g over $\mathcal{S}$ is a proper flat morphism $\pi: \mathcal{X} \to \mathcal{S}$ whose geometric fibres are reduced, connected, 1-dimensional schemes $\mathcal{X}_s$ such that

(i) $\mathcal{X}_s$ has only ordinary double points as singularites;

(ii) each non-singular rational component of $\mathcal{X}_s$ meets the other components of $\mathcal{X}_s$ in more than 2 points;

(iii) $\dim \mathrm{H}^1(\mathcal{O}_{\mathcal{X}_s}) = g$.

It is well known that the results of Deligne – Mumford on the moduli space $\overline{\mathrm{M}}_g$ of stable curves can be used to prove the stable reduction theorem above. Roughly speaking, this theorem is equivalent to the assertion that the morphism

$$\rho : \overline{\mathrm{M}}_g \to \mathrm{Spec}\mathbf{Z}$$

is proper. Indeed Deligne and Mumford prove the properness of ρ by appealing to the stable reduction theorm for curves over *discrete* valuation rings and applying the valuative criterion for properness (which here only needs to be checked for discrete valuation rings). Once properness has been established one can then go in the other direction to conclude (again by the valuation criterion) that the stable reduction theorem holds over an arbitrary valuation ring. A precise proof by this argument can be found in [Ka].

Our interest in the stable reduction theorem has been to show that the assertion that there exists a unique stable $\mathcal{O}$-curve $\mathcal{C}$ with generic fibre

$\mathbf{C}$ of genus $g > 1$, is elementary (that is, it is expressible by an assertion in the first order language of valued fields). Further, to characterise these $\mathcal{O}$-curves as those associated with a finite set of constant reductions of the function field and to show that they can be described effectively by an elementary assertion in the language of valued fields having as parameters the coeffients of certain equations defined over $\mathcal{O}$.

We briefly explain the method used to prove this result. First a curve $\mathcal{X}$ over a scheme $\mathcal{S}$ is defined to be *weak stable* of genus g if $\mathcal{X} \to \mathcal{S}$ is a proper flat morphism whose generic and closed geometric fibres are stable curves of genus g. If $\mathcal{S} = \operatorname{Spec} \mathcal{O}$ with quotient field K algebraically closed and $\mathcal{X}$ is projective with normal generic fibre, then by 7.1 II, $\mathcal{X}$ is normal and so is isomorphic to an $\mathcal{O}$-curve C_V associated to some set of constant reductions V of the function field which prolong the valuation determined by $\mathcal{O}$. Next one shows that for $\mathcal{S} = \operatorname{Spec} \mathcal{O}$ as above and $\mathbf{C}$ a projective non-singular irreducible curve of genus $g \geq 2$ over K, the assertion that there exists a unique projective weak stable model of $\mathbf{C}$ over $\mathcal{O}$ is elementary. Further the projective weak stable model can be described effectively by an elementary assertion in the language of valued fields having as parameters the coeffients of certain equations defined over $\mathcal{O}$. Hence provided one knows the result over a representative class of algebraically closed valued fields, using the model completness of the theory of such fields [Rob], one can deduce the weak stable reduction theorem generally. For the representative class one could appeal either to Deligne – Mumford, [D–M], or Artin – Winters, [A–W], where the theorem is proved for K the algebraic closure of a discretely valued field, or alternatively to van der Put, [vdP], or Bosch – Lütkebohmert, [B–L], where the result is proved when the valuation on the constant field K is rank 1 and K is algebraically closed complete.

The reason for introducing the notion of weak stable curves is that here one only needs to make the assertions for the closed and generic fibres, *not simultaneously for all fibres.* Making the assertion simultaneously for all fibres is not elementary a priori. However we finally conclude that one only needs to make the assertions for the closed and generic fibres, for together with the result on the existence and uniqueness of stable curves over $\mathcal{O}$ it follows aposteriori that the unique projective weak stable model of $\mathbf{C}$ over $\mathcal{O}$ is actually the stable $\mathcal{O}$-curve. Hence one obtains the result stated above and a characterisation of such stable curves as $\mathcal{O}$-curves associated to a finite set of constant reductions.

8. On the Galois Theory of function fields

In the final section of this survey we mention the results of Florian Pop [P2] and [P3] on the Galois Theory of function fields over number fields. The results of Pop complete a program of Neukirch, Iwasawa and Urchida and also provide the first step towards a fundamental *conjecture of the birational "anabelian" geometry* of Grothendieck which, roughly speaking, asserts that any isomorphism of absolute Galois groups of finitely generated infinite fields is actually uniquely defined by an isomorphism of the fields in discussion.

Notation: For an arbitrary field K let $\tilde{K}$ denote an algebraic closure and $G_K = \mathrm{Aut}(\tilde{K}|K)$ the absolute Galois group of K. The first results conjectured by Neukirch and proved by Neukirch, Ikeda, Iwasawa and Uchida assert that:

8.1. If K and L are number fields and $G_K \cong G_L$, then $K \cong L$.

The next step was made by Iwasawa (unpublished) and Uchida [U], who showed that the corresponding assertion for global function fields in the following form is true:

8.2. *Let K and L be global function fields of one variable and $\Phi: G_K \longrightarrow G_L$ an isomorphism of their absolute Galois groups. Then there exists a unique isomorphism $\phi: L^s \longrightarrow K^s$, the separable closures, such that $\Phi(g) = \phi^{-1} g \phi$ for all $g \in G_K$. In particular ϕ maps L isomorphically onto K.*

The result of Pop in [P2] extends this to function fields of one variable over number fields.

Theorem 8.3 [P2]. *Let $F|\mathbb{Q}$, $E|\mathbb{Q}$ be two function fields of one variable ($\mathbb{Q}$ not necessarily the exact constant field of F or E). If G_F and G_E are isomorphic, then F and E are isomorphic.*

More precisely, for each group isomorphism $\Phi: G_F \longrightarrow G_E$ there exists a unique field isomorphism $\phi: \tilde{E} \longrightarrow \tilde{F}$ with the property that $\Phi(g) = \phi^{-1} g \phi$ for all $g \in G_K$. As a consequence ϕ maps E isomorphically onto F.

The main new result used in the proof of this theorem is a Galois characterisation of the constant reductions of the function fields of one variable over number fields. In this study, methods involving the model theory of constant reductions were used, particularly the idea of drawing conclusions about Galois extensions of function fields of one variable from the corresponding properties of their constant reductions. The way this is done is by "interpolating" local information using ultraproducts of function fields of one variable.

In [P3], theorem 8.3 above has been generalised to the case $F|K$, $E|K$ with K a finitely generated field and $\Phi: G_F \longrightarrow G_E$ compatible with the augmentation $G_F \xrightarrow{\text{res}} G_K \xleftarrow{\text{res}} G_E$. This result essentially gives a Galois characterisation of the finitely generated fields, if the Galois group is endowed with some "stratification".

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