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Higher order relations for a numerical semigroup

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Introduction. Numerical semigroups are useful in the study of curve singularities and Weierstrass points of smooth projective curves. The aritmetical classification of numerical semigroups is not an easy question because of the complexity of their inner structure (see [6]). Our aim is to measure this complexity by looking at the (higher order) relations and showing how the complexity increases when one considers more general families of semigroups.

Let S be a numerical semigroup, i.e. an additive subsemigroup of $\mathbb{N}$ with $\text{Card}(\mathbb{N} - S) < \infty$. Let $\{b_0, ..., b_g\}$ be the minimal set of generators of S, i.e. b_{i+1} is the least element in $S - \sum_{j \leq i} b_j \mathbb{N}$ and g being determined by the condition $S = \sum_{j \leq g} b_j \mathbb{N}$. One wants to study the relations for an element $m \in S$, namely the set of expressions $m = i_0 b_0 + ... + i_g b_g$ with $i_0, ..., i_g \in \mathbb{N}$. Assume that for such an expression one has $i_j \neq 0$ for j in a certain subset J of $\Lambda = \{0, 1, ..., g\}$, then $m - b_J \in S$ where $b_J = \sum_{j \in J} b_j$. Conversely, if $m - b_J \in S$ then a relation with $i_j \neq 0$ for $j \in J$ does exist.

Let us denote by Δ_m the set of subsets J of Λ such that $m - b_J \in S$. Since $J \in \Delta_m$ and $J' \subset J$ implies $J' \in \Delta_m$, Δ_m is an abstract simplicial complex on the vertex set Λ. Thus, this arithmetical question can be studied by means of combinatorial tools such as these simplicial complexes. The natural invariants to be considered are the Betti numbers $\tilde{h}_i(\Delta_m)$, i.e. the ranks of the augmented homology for the simplicial complex Δ_m. Thus, asociated to a semigroup one has a square of integer $\tilde{h}_i(\Delta_m)$.

It is obvious that $\tilde{h}_i(\Delta_m) = 0$ for $i \geq g$ and every m and it is also clear that $\tilde{h}_i(\Delta_m) = 0$ for m large enough and every i. In fact, if c is the conductor of S, i.e. the least element such that $n \in S$ for any $n \geq c$, then Δ_m is the full simplex on the $g + 1$ vertices for $m > d = c + b_\Lambda - 1$, so $\tilde{h}_i(\Delta_m) = 0$ for such values of m. Thus, the square can be assumed to be bounded. We will also include in the square the values $\tilde{h}_{-1}(\Delta_m)$ which take

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a sense as one deals with the augmented simplicial chain complex. Thus one has $\tilde{h}_{-1}(\Delta_m) = 0$ if $m \in S$, $m > 0$ and $\tilde{h}_{-1}(\Delta_0) = 1$ as $\Delta_0 = \{\Phi\}$.

In this paper we will describe the properties of the square corresponding to symmetric, complete intersection and plane curve semigroups. Moreover, we prove that the integer $\tilde{h}_i(\Delta_m)$ is exactly the number of degree m lineary independent syzygies of order i for the ideal of the monomial curve in $\mathbb{A}_{g+1}$ given by $x_0 = t^{b_0}, ..., x_g = t^{b_g}$.

1. Minimal resolution for the monomial curve

Consider a field K and let $R = K[S]$ denote the semigroup algebra of S over K, i.e. $R = K[t^m \mid m \in S] \subset K[t]$, t an indeterminate. If A is the polynomial ring $K[X_0, ..., X_g]$ the embedding in $\mathbb{A}_{g+1}$ of the monomial curve is given by the K-algebra homomorphism $\Phi : A \to R$, $X_i \to t^{b_i}$, $i = 0, ..., g$. In fact R and A are graded rings with significative degrees only on $S \subset \mathbb{N}$ in the following way

$$R = \underset{m \in S}{\oplus} R_m \quad \text{with } R_m = Kt^m$$

$$A = \underset{m \in S}{\oplus} A_m \quad \text{with } A_m \text{ the vector space spanned by the monomials}$$

$$X_0^{i_0} ... X_g^{i_g} \quad \text{with } i_0 b_0 + ... + i_g b_g = m,$$

the mapping Φ being a degree zero homomorphism. This graded situation corresponds to the K^*-action on the curve which extends to the affine space.

Now, let us consider the minimal resolution for R as a graded A-module. By the Auslander-Buchbaum theorem it is a finite (free) graded resolution of length dim A - depth $R = g$

$$0 \to A^{l_g} \overset{\Phi_g}{\to} A^{l_g - 1} \to .... \to A^{l_1} \overset{\Phi_1}{\to} A \overset{\Phi_0 = \Phi}{\to} R \to 0$$

the integers l_i being the maximum number of lineary independent syzygies of order i.

Thus, l_1 is the cardinality of a minimal set of homogeneous generators of $I = $ Ker Φ and, according to the graded Nakayama lemma, one has

$$l_1 = \text{dim}_K(I/M_A I)$$

M_A being the irrelevant maximal ideal of A, i.e. $M_A = \oplus_{m>0} A_m$.

For a minimal set of homogeneous generators of I one has the mapping $\Phi_1 : A^{l_1} \to A$, with Im $\Phi_1 = I$, sending the standard set of generators of

A^{l_1} to that given one of I. The mapping Φ_1 is also graded if on A^{l_1} one considers the grading given by

$$(A^{l_1})_m = A_{m-m_1} \times ... \times A_{m-m_{l_1}}$$

$m_1, ..., m_{l_1}$ being the degrees of the elements in the set of generators of I. Now, setting $N_0 = I$, by recurrence one constructs successive homogeneous submodules $N_i \subset A^{l_i}$, minimal sets of homogeneous generators of N_i (with $l_{i+1} = \dim_K(N_i/M_A N_i)$ elements) and degree zero graded mappings Φ_{i+1} : $A^{l_{i+1}} \to A^{l_i}$ with $\mathrm{Im}\Phi_{i+1} = N_i$, $\mathrm{Ker}\Phi_{i+1} = N_{i+1}$.

Now if $N_i = \oplus_{m \in S}(N_i)_m$ is the graded structure of the submodule N_i of A^{l_i}, the dimensionality of the vector space

$$V_i(m) = (N_i)_m/(M_A N_i)_m$$

equals to the number of degree m homogeneous elements in a minimal set of homogeneous generators for N_i. In other words, according to the minimal resolution, $\dim_K V_i(m)$ is the number of lineary independant order i homogeneous syzygies of degree m.

In order to compute the above dimensionalities, let us consider the Koszul complex for the elements $t^{b_0}, ..., t^{b_g}$ in the ring R.

$$(K.) \quad 0 \to \overset{g+1}{\bigwedge} R^{g+1} \overset{d_{g+1}}{\to} \overset{g}{\bigwedge} R^{g+1} \to ...... \to \overset{2}{\bigwedge} R^{g+1} \overset{d_2}{\to} R^{g+1} \overset{d_1}{\to} R \to 0.$$

If $e_0, ..., e_g$ is the standard basis of the R-module R^{g+1}, and for $J \subset \Lambda$, $J = \{j_1 < ... < j_q\}$, one writes $e_J = e_{j_1} \wedge ... \wedge e_{j_q}$, then d_p is given by

$$d_p(e_J) = \sum_{r=1}^{p} (-1)^{j_r} t^{b_{j_r}} e_{J-\{j_r\}}$$

for those J such that $\mathrm{card}(J) = p$.

If on $\overset{p}{\bigwedge} R^{g+1}$ one considers the grading (as R-module) such that $\deg(e_J) = b_J$, then it is clear that each d_p is graded of degree zero, so the Koszul complex is a graded one and therefore it give rise to the family of complexes of vector spaces

$$(K._m) \quad 0 \to K_{g+1,m} \to K_{g,m} \to .... \to K_{1,m} \to R_m \to 0$$

where $K_{p,m} = \oplus_{\mathrm{Car}(J)=p} R_{m-b_J}$ and the differential mappings are given as above. The following result relates the Koszul complexes $(K._m)$ to the combinatorial objects Δ_m.

1.1 LEMMA. *With notations as above one has*

$$H_i(K._m) \cong \tilde{H}_{i-1}(\Delta_m), \quad i = 0, ..., g,$$

$\tilde{H}_i(\Delta_m)$ *being the augmented simplicial homology vector space of* Δ_m *with values in* K.

Proof. According to the above description one has

$$K_{p,m} = \bigoplus_{\substack{J \in \Delta_m \\ \mathrm{Car}(J)=p}} R_{m-b_J} e_J$$

since $R_{m-b_J} = 0$ when $J \notin \Delta_m$. Taking into account that for $J \in \Delta_m$ one has $R_{m-b_J} \cong K$ as K-vector space, it follows that $K_{p,m}$ is isomorphic to the order p chain vector space for the simplicial complex Δ_m. Moreover, it is obvious than the differential maps for $K._m$ correspond by the above isomorphism to the differential maps for the simplicial complex, so the results follows from these facts.

1.2 THEOREM. *With notations as above one has*

$$\tilde{H}_i(\Delta_m) \cong V_i(m) \quad i = 0, 1, ..., g-1 \ ; \ m \in S.$$

Proof. Both R and K are A-modules with structure homomorphisms $A \to R$, $A \to K$, and respective kernels I and M_A. Consider the minimal resolution for R and take tensor product by K. Because of the minimality of the resolution, the obtained complex

$$0 \to K^{l_g} \to .... \to K^{l_1} \to K \to K \to 0$$

has differential mappings $\Phi_i \otimes_A K$ equals to zero for $i \geq 1$. This shows that

$$l_i = \dim_K \mathrm{Tor}_A^i(R, K),$$

and moreover $\dim V_{i-1}(m) = \dim_K \mathrm{Tor}_A^i(R, K)_m.$

On the other hand, the Koszul complex for the regular sequence $x_0, ..., x_g$ of A gives a resolution for $K = A/M_A$ as follows

$$0 \to \overset{g+1}{\bigwedge} A^{g+1} \to .... \to \overset{2}{\bigwedge} A^{g+1} \to A^{g+1} \to A \to K \to 0.$$

Taking tensor product by R one gets exactly the complex $K.$, so the Koszul homology is nothing but $\text{Tor}^i{}_A(K, R)_m$. By the lemma one has

$$\tilde{H}_{i-1}(\Delta_m) \cong \text{Tor}^i{}_A(K, R)_m.$$

Thus the theorem follows from the fact that $\text{Tor}^i{}_A(R, K)_m \cong \text{Tor}^i{}_A(K, R)_m$.

2. The square for symmetric semigroups

A numerical semigroup S is said to be symmetric if for any couple of integers m, n with $m + n = c - 1$ one has either $m \in S$ or $n \in S$. Symmetric semigroups are exactly those for which the K-algebra R is Gorenstein (K any field). To see it, we will compute the Cohen Macaulay type $r(R) = \dim_K \text{Ext}^1(K, R)$ and express the Gorenstein condition as $r(R) = 1$. If $\mathbf{q} = t^{b_0} R$ and $\mathbf{q}' = (m_R : \mathbf{q})_R$, m_R being the irrelevant ideal for R, then one has $r(R) = \dim_K \mathbf{q}'/\mathbf{q}$ as $\text{Ext}^1(K, R) = \text{Hom}_R(R/m_R, R/\mathbf{q})$ (takes into account the exactness of the sequence

$$0 \to R \xrightarrow{t^{b_0}} R \to R/\mathbf{q} \to 0$$

and the fact that $t^{b_0}.Ext^1(K, R) = 0$. Now, both $\mathbf{q}$ and $\mathbf{q}'$ are homogeneous ideals so $\mathbf{q}' = \oplus_{m \in A} Kt^m$, $\mathbf{q} = \oplus_{m \in B} Kt^m$ where $A = \{m \in S | m + s - b_0 \in S, \forall s \in S, s > 0\}$ and $B = \{m \in S \mid m - b_0 \in S\}$. Thus $r(R) = \text{Card}(T)$ with $T = A - B$. Note that the element $c - 1 + b_0$ is always in T, so R is Gorenstein iff $T = \{c - 1 + b_0\}$.

If R is Gorenstein and $m + n = c - 1$ with $n \notin S$ then $n + lb_0 \in S$ and $n + (l-1)b_0 \notin S$ for some l. One has two possibilities, either $n + (l-1)b_0 = c - 1$ or $n + (l-1)b_0 < c - 1$. In the first case $m = (l-1)b_0$ belongs to S ; in the second one $n + lb_0 \notin T$ so for some $s \in S$, $s > 0$ one has $n_1 = n + (l - 1)b_0 + s \notin S$. Now apply to n_1 the same argument and continue until the first possibility occurs ; then m is the sum of differences $n_i - n_{i-1} (n_0 = n)$ which belongs to S by contruction, so $m \in S$, which completes the proof that S is symmetric. Conversely, assume S is symmetric and take $m \in T$; if $m \neq c - 1 + b_0$ then $m - b_0 \notin S$, and therefore $s = c - 1 - m + b_0 \in S$ and $s > 0$, hence $m + s - b_0 = c - 1 \notin S$ which is contradictory with $m \in T$, so T has only one element.

The Cohen Macaulay type is expressed in the square in terms of the last column, i.e. the $(g - 1)$-th column, as follows

2.1 LEMMA. *With notations as above one has* $m \in T$ *if and only if*

$$\tilde{h}_{g-1}(\Delta_{m+b_1+....+b_g}) = 1$$

Proof. For $m' \in S$ one has $\tilde{h}_{g-1}(\Delta_{m'}) = 1$ iff $\Delta_{m'}$ is homeomorphic to the sphere S_{g-1}, i.e. if $\Delta_{m'}$ is the complete simplex except the face Λ. Thus $\tilde{h}_{g-1}(\Lambda_{m'}) = 1$ is equivalent to the conditions

$$m' - (b_0 + ... + b_g) \notin S$$

$$m' - (b_1 + ... + b_g) \in S$$

$$m' - (b_1 + ... + b_g) + b_i - b_0 \in S \quad i = 1, 2, ..., g.$$

Setting $m = m' - (b_1 + ... + b_g)$, above conditions are written as

$$m - b_0 \notin S, \quad m \in S, \quad m + b_i - b_0 \in S \quad i = 1, 2, ..., g$$

which are equivalent to $m \in S$, $m - b_0 \notin S$ and $m + s - b_0 \in S$ for any $s \in S, s > 0$.

Thus the Cohen Macaulay type is the sum of ones in the $(g-1)$th column. Gorenstein means that the only one in that column is that corresponding to $(c - 1 + b_0) + b_1 + ... + b_g = d$. In fact, the symmetry of the semigroup implies a more strong property.

2.2 THEOREM. S is a symmetric semigroup if and only if the square is symmetric relative to its center, i.e. for $-1 \leq i \leq g - 1$ and $m', m'' \in S$ such that $m' + m'' = d$ one has

$$\tilde{h}_{g-2-i}(\Delta_{m'}) = \tilde{h}_i(\Delta_{m''})$$

We will indicate two proofs ; one algebraic and other combinatorial.

Algebraic proof : Take a minimal graded resolution for the A-module R

$$0 \to A^{l_g} \xrightarrow{\Phi_g} A^{l_{g-1}} \to ... \to A^{l_1} \xrightarrow{\Phi_1} A \xrightarrow{\Phi_0 = \Phi} R \to 0,$$

and dualize by taking $\mathrm{Hom}_A(-, A)$; then one has a resolution of the canonical module $KR = \mathrm{Coker}\ \Phi'_g$ for R

$$0 \leftarrow KR \leftarrow A^{l_g} \xleftarrow{\Phi^t_{g-1}} A^{l_{g-1}} \leftarrow .... \xleftarrow{\Phi^t_2} A^{l_1} \xleftarrow{\Phi^t_1} A \leftarrow 0 = \mathrm{Hom}_A(K, A).$$

This resolution is also graded and minimal (hence note that is $M = \oplus M_m$, $N = \oplus N_m$ are finitely generated graded modules then $\mathrm{Hom}_A(M, N)$ is also a graded module with $Hom_A(M, N)_m$ being the set of A-linear homomorphism from M to N which are graded of degree m). Now, in the Gorenstein case KR is isomorphic to R as graded module, so one has two minimal resolutions for R and hence, by comparing the degrees, one concludes

$$\tilde{l}_{g-2-i}(\Delta_{m'}) = \tilde{l}_i(\Delta_{m''})$$

if $m' + m'' = d$ and $-1 \leq i \leq g - 1$.

Combinatorial proof : If $m' + m'' = d$ one has the following relationship between $\Delta_{m'}$ and $\Delta_{m''}$

$$J \in \Delta_{m'} \Rightarrow \Lambda - J \notin \Delta_{m''}$$

as in fact, $m' - b_J \in S$ implies $c - 1 - m' + b_J \notin S$ which can be written as $d - m' - b_{\Lambda - J} \notin S$. In the case that the semigroup is symmetric then the converse is also true, i.e. one has

$$J \in \Delta_{m'} \Leftrightarrow \Lambda - J \notin \Delta_{m''}.$$

In general, for a simplicial subcomplex Δ of the complete simplex $P(\Lambda) = \sum$, the dual simplicial subcomplex Δ^* of $\sum$ is defined to be the set of subsets H such that $\Lambda - H \notin \Delta$. Thus the theorem will follows from the following result

2.3 LEMMA. *With notations as above one has*

$$\tilde{h}_{g-2-i}(\Delta^*) = \tilde{h}_i(\Delta) \quad -1 \leq i \leq g - 1.$$

Proof of the lemma. One has an exact sequence of complexes of augmented simplicial homology as follows

$$0 \to C_*(\Delta) \to C_*(\Sigma) \to C_*(\Sigma)/C_*(\Delta) \to 0.$$

Taking into account that $C_*(\Sigma)$ is acyclic (as Σ is contractible) it follows that

$$\tilde{H}_i(\Delta) \cong \tilde{H}_{i+1}(\Sigma, \Delta)$$

($H_*(\Sigma, \Delta)$ denotes the relative homology, i.e. the homology of the simplex $C_*(\Sigma)/C_*(\Delta)$). Now, one has

$$C_{i+1}(\Sigma)/C_{i+1}(\Delta) = \bigoplus_{\substack{J \notin \Delta \\ \mathrm{Card}(J)=i+2}} K\sigma_J,$$

σ_J meaning the face defined by J, and by considering the cohomology of Δ^*, with dual basis notation one writes

$$C^{g-2-i}(\Delta^*) = \bigoplus_{\substack{H \in \Delta^* \\ \mathrm{Card}(H)=g-1+i}} K\overset{\vee}{\sigma}{}^*_H \; ;$$

so it is clear that the bijective correspondence $J \notin \Delta \leftrightarrow H = \Lambda - J \in \Delta^*$ establishes an isomorphism between the K-vector spaces $C_{i+1}(\Sigma)/C_{i+1}(\Delta)$ and $C^{g-2-i}(\Delta^*)$. Moreover, it is elementary to realize that, up to a sign, the boundary operators for the relative chain complex and for the cohomology of Δ^* are the same, so one concludes

$$\tilde{H}_i(\Delta) \cong \tilde{H}_{i+1}(\Sigma, \Delta) \cong \tilde{H}^{g-2-i}(\Delta^*).$$

Hence, from the fact that the homology and cohomology in the same order have the same rank, one has

$$\tilde{h}_i(\Delta) = \tilde{h}_{g-2-i}(\Delta^*)$$

as required.

Remark. The converse of the theorem is obvious true, as if for $i = -1$ one has
$$\tilde{h}_{g-1}(\Delta_{m'}) = \tilde{h}_{-1}(\Delta_{m''}) \quad m' + m'' = d$$

then $\tilde{h}_{g-1}(\Delta_{m'}) = 1$ iff $m' = d$, so R will be Gorenstein.

3. The square for complete intersection semigroups

A numerical semigroup S is said to be complete intersection when the graded ring $R = K[S]$ be a complete intersection ring, i.e. when the ideal $I = \text{Ker } \Phi, \Phi : A \to R$ as in 1, can be generated by g homogeneous elements.

Assume S is complete intersection and let $f_1, ..., f_g$ a homogeneous set of generators for I. Let us denote by $m_1, ..., m_g$ the degrees of the respective elements $f_1, ..., f_g$. We will also assume $m_1 \leq m_2 \leq ... \leq m_g$. Then one has the following result

3.1 THEOREM. Let S be a complete intersection semigroup and keep the notations as above. Then for $-1 \leq i \leq g - 1$ one has

$$\tilde{h}_i(\Delta_m) = N_m$$

N_m being the number of ways in which m can be written as a sum

$$m = m_{j_0} + ... + m_{j_i} \quad \text{with } j_0 < j_1 ... < j_i.$$

We only give an algebraic proof.

The elements $f_1, ..., f_g$ are a regular sequence for I, so the Koszul complex of $f_1, ..., f_g$ augmented with the mapping $\Phi : A \to R$ gives a graded resolution for R.

$$0 \to \bigwedge^g A^g \to \bigwedge^{g-1} A^g \to ... \to \bigwedge^2 A^g \to A^g \to A \to R \to 0.$$

In terms of the standard basis $E_1, ... E_g$ of A^g the differentials are given by

$$d(E_{j_0} \wedge ... \wedge E_{j_i}) = \sum_{r=0}^{i} (-1)^{j_r} f_{j_r} E_{j_0} \wedge ... \wedge E_{j_{r-1}} \wedge E_{j_{r+1}} \wedge ... \wedge E_{j_i},$$

$$j_0 < ... < j_i$$

so it is clear that the differential operators are zero modulo M_A and therefore this augmented Koszul complex is a minimal resolution for R.

Now it is clear that $E_{j_0} \wedge ... \wedge E_{j_i}$ must be homogeneous of degree $(m_{j_0} + ... + m_{j_i})$ in order to have graded differential operators of degree zero. Thus $V_i(m)$ is generated by the $E_{j_0} \wedge ... \wedge E_{j_i}$ such that $m = m_{j_0} + ... + m_{j_i}$ and theorem 3 follows from theorem 1.

Remarks. 1. Note that in particular for $i = 0$ one has $\tilde{h}_0(\Delta_m) = 0$ iff $m \neq m_i$ and $\tilde{h}_0(\Delta_{m_i})$ is the number of m_j equals to m_i. On the other hand, for $i = g - 1$ one obtains

$$m_1 + m_2 + ... + m_g = d$$

since a complete intersection is Gorenstein, and so is the only possibility for having $\tilde{h}_{g-1}(\Delta_d) = 1$.

2. There is several characterizations of the complete intersection property in the literature. In [3] Delorme gives one in terms of the minimal set of generators and in [5] Herzog and Kunz another one in terms of the sequence $m_1 \leq ... \leq m_{l_1}$ with the above meaning (for general semigroups we have l_1 integers) as follows: One has $m_1 + ... + m_g \geq d$ and equality is true iff S is complete intersection. This criterion gives an idea why the complete intersection semigroups are the only with a table having the structure indicated in theorem 3.

3. A nice class of semigroups which are complete intersection are those for which one has $n_i b_i \in < b_0, ..., b_{i-1} >$ for $i = 1, ..., g$ and $n_i = e_{i-1}/e_i$, with $e_i = g.c.d.(b_0, ..., b_i)$. By a result of the Herzog [4] one has in fact $m_i = n_i b_i$, $1 \leq i \leq g$ and therefore

$$n_1 b_1 + ... + n_g b_g = d.$$

Moreover, Bertin and Carbonne [2] show that for general semigroups one has $n_1 b_1 + \dots + n_g b_g \geq d$ and equality holds if and only if the semigroup satisfies $n_i b_i \in < b_0, \dots, b_{i-1} >$. Thus, in this particular case the square is determined by the generator set.

4. A particular case of 3 are the so called "plane curve semigroups" arising as semigroup of values of analitically irreducible plane curve singularities. For them one has in addition $n_i b_i < b_{i+1}$, $i = 1, 2, \dots, g - 1$, so it is not difficult to see that in this case the $N'_m s$ in the theorem are all equal to 1 or 0. Thus, for plane curve semigroups the square has only zeroes and ones.

5. All the results in the paper are true if one considers general set of generators for the semigroups ; we have taken the minimal one for the sake of simplicity in the exposition.

6. In [8] some combinatorial properties of numerical semigroups are given in terms of the graphs G_m obtained by considering the vertex set Δ_m and taking the one dimensional faces as arcs. The relationship between the complex Δ_m can be also graphically studied showing again the complexity of the semigroup structure (see [7]).

Examples. In the following examples we show the non zero entries in the square.

(i) Non symmetric semigroup $S = < 3, 4, 5 >$

$d = 14$.

$m_1 = 8, m_2 = 9, m_3 = 10$.

$\tilde{h}_{-1}(\Delta_0) = 1;$

$\tilde{h}_0(\Delta_8) = 1, \tilde{h}_0(\Delta_9) = 1, \tilde{h}_0(\Delta_{10}) = 1;$

$\tilde{h}_1(\Delta_{13}) = 1, \tilde{h}_1(\Delta_{14}) = 1.$

(ii) Symmetric non complete intersection $S = < 5, 6, 7, 8 >$

$d = 35$.

$m_1 = 12, m_2 = 13, m_3 = 14, m_4 = 15, m_5 = 16 ; m_1 + m_2 + m_3 = 39 > 35.$

$\tilde{h}_{-1}(\Delta_0) = 1 ;$

$\tilde{h}_0(\Delta_{12}) = 1, \tilde{h}_0(\Delta_{13}) = 1, \tilde{h}_0(\Delta_{14}) = 1, \tilde{h}_0(\Delta_{15}) = 1, \tilde{h}_0(\Delta_{16}) = 1 ;$

$\tilde{h}_1(\Delta_{19}) = 1, \tilde{h}_1(\Delta_{20}) = 1, \tilde{h}_1(\Delta_{21}) = 1, \tilde{h}_1(\Delta_{22}) = 1, \tilde{h}_1(\Delta_{23}) = 1 ;$

$\tilde{h}_2(\Delta_{35}) = 1$.

(iii) Complete intersection without property 3 $S =< 6, 10, 15 >$

$d = 60$.

$m_1 = m_2 = 30$.

$\tilde{h}_{-1}(\Delta_0) = 1$;

$\tilde{h}_0(\Delta_{30}) = 2$;

$\tilde{h}_1(\Delta_{60}) = 1$.

(iv) Property in 3 but non plane curve semigroup $S =< 4, 6, 9 >$

$d = 30$.

$m_1 = 12,\ m_2 = 18$.

$\tilde{h}_{-1}(\Delta_0) = 1$;

$\tilde{h}_0(\Delta_{12} = 1,\ \tilde{h}_0(\Delta_{18}) = 1$;

$\tilde{h}_1(\Delta_{30}) = 1$.

(v) Plane curve semigroup $S =< 4, 6, 13 >$

$d = 38$.

$m_1 = 12, m_2 = 26$.

$\tilde{h}_{-1}(\Delta_0) = 1$;

$\tilde{h}_0(\Delta_{12}) = 1,\ \tilde{h}_0(\Delta_{26}) = 1$;

$\tilde{h}_1(\Delta_{38}) = 1$.

(vi) Two generators semigroup $S =< 3, 4 >$

$d = 12$.

$\tilde{h}_{-1}(\Delta_0) = 1$;

$\tilde{h}_0(\Delta_{12}) = 1$.

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