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## **On the order of vanishing of modular $L$ -functions at the critical point**

par HENRYK IWANIEC

### **1. Introduction**

The nonvanishing of  $L$ -functions at special points is an attractive area of research in contemporary number theory, see [7]-[11]. One example is the Rankin-Selberg zeta-function  $L(f \otimes g_j, s)$  associated with a holomorphic cusp form  $f$  of weight 2 and Maass cusp forms  $g_j$  of eigenvalue  $\lambda_j = s_j(1 - s_j)$ . In this case the nonvanishing of  $L(f \otimes g_j, s)$  at  $s = s_j$  plays a rôle in the work of R. Phillips and P. Sarnak [6] on deformations of groups and was proved to be true for infinitely many cusp forms  $g_j$  by J.-M. Deshouillers and H. Iwaniec [3]. Another example is the Birch-Swinnerton-Dyer conjecture which asserts that the rank of the group of rational points on an elliptic curve  $E$  defined over  $\mathbb{Q}$  is equal to the order of vanishing of the associated Hasse-Weil  $L$ -function  $L(s, E)$  at  $s = 1$  (the center of the critical strip).

Recently V.A. Kolyvagin [4] has proved that the group of rational points on a modular elliptic curve  $E$  is finite if  $L(1, E) \neq 0$  and that the  $L$ -function  $L(s, E, \chi_d)$  twisted by a suitable real character  $\chi_d$  has simple zero at  $s = 1$ . The latter condition was subsequently proved to hold true for infinitely many discriminants  $d$  by D. Bump, S. Friedberg and J. Hoffstein [2] and independently by K. Murty and R. Murty [5]. In these notes we establish (from scratch) quantitative results on Kolyvagin's condition.

### **2 - Statement of results**

Let  $E$  be a modular elliptic curve defined over  $\mathbb{Q}$  and

$$L(s, E) = \sum_1^{\infty} a_n n^{-s}$$

be the Hasse-Weil  $L$ -function associated with  $E$ . Thus

$$f(z) = \sum_1^{\infty} a_n e(nz)$$

is a cusp form of weight 2 which is a newform of level  $N$ , where  $N$  is the conductor of  $E$ . The  $L$ -function is entire and it satisfies the functional equation

$$\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(s, E) = w \left(\frac{\sqrt{N}}{2\pi}\right)^{2-s} \Gamma(2-s)L(2-s, E),$$

where  $w = \pm 1$ . We are interested in curves  $E$  for which  $L(1, E) \neq 0$ , so the functional equation holds with the sign  $w = 1$ . The twisted  $L$ -function

$$L(s, E, \chi_d) = \sum_1^\infty a_n \chi_d(n) n^{-s},$$

where  $\chi_d$  is a real primitive character to modulus  $d$  prime to  $N$  is also entire and it satisfies the functional equation

$$(1) \quad \left(\frac{d\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(s, E, \chi_d) = w_d \left(\frac{d\sqrt{N}}{2\pi}\right)^{2-s} \Gamma(2-s)L(2-s, E, \chi_d)$$

with the sign  $w_d = w\chi_d(-N)$ . In the sequel we let  $d$  range over the set

$$\mathcal{D} = \{d : 0 < d \equiv -\nu^2 \pmod{4N} \text{ for some } \nu \text{ prime to } 4N\}$$

and we let  $\chi_d(n) = \left(\frac{-d}{n}\right)$  be the Kronecker symbol. Thus if  $d$  is squarefree  $\chi_d$  is the primitive character to the modulus  $d$  which is associated with the imaginary quadratic field  $\mathbb{Q}(\sqrt{-d})$ . Every prime dividing  $N$  splits in  $\mathbb{Q}(\sqrt{-d})$ . Moreover we have  $w_d = -1$ , so by (1) it follows that

$$(2) \quad L(1, E, \chi_d) = 0.$$

Our aim is to prove that  $L(s, E, \chi_d)$  has a simple zero at  $s = 1$ , i.e.  $L'(1, E, \chi_d) \neq 0$  for infinitely many  $d$  in  $\mathcal{D}$ . To this end we shall evaluate two sums of type

$$(3) \quad S_4(Y) = \sum_{d \in \mathcal{D}, d \leq Y}^b |L'(1, E, \chi_d)|^4$$

and

$$(4) \quad S_1(Y) = \sum_{d \in \mathcal{D}}^b L'(1, E, \chi_d) F(d/Y),$$

where  $\sum^b$  means that the summation is restricted to squarefree numbers and  $F$  is a smooth function, compactly supported in  $\mathbb{R}^+$  with positive mean value.

THEOREM. For any  $\epsilon > 0$  and  $Y \geq 1$  we have

$$(5) \quad S_4(Y) \ll Y^{2+\epsilon}$$

and

$$(6) \quad S_1(Y) = \alpha Y \log Y + \beta Y + O(Y^{13/14+\epsilon})$$

with some constants  $\alpha \neq 0$  and  $\beta$  which depend on the curve  $E$  and the test function  $F$ .

COROLLARY. Suppose  $\epsilon > 0$  and  $Y > c(\epsilon)$ . Then  $L'(1, E, \chi_d) \neq 0$  for at least  $Y^{2/3-\epsilon}$  real primitive characters  $\chi_d$  to modulus  $d \in \mathcal{D}$ ,  $d \leq Y$ .

### 3. Estimates for the coefficients of $f$

The Fourier coefficients  $a_n$  of the cusp form  $f$  are multiplicative. More exactly, for  $\text{Re } s > 3/2$  we have the Euler product

$$(7) \quad L(s, E) = \prod_p (1 - \alpha_p p^{-s})^{-1} (1 - \beta_p p^{-s})^{-1}$$

with  $\alpha_p = 0, \pm 1, \beta_p = 0$  if  $p|N$  and  $|\alpha_p| = |\beta_p| = p^{1/2}$  if  $p \nmid N$ . In the latter case the result was proved by M. Eichler and P. Deligne. It yields the following bound for the coefficient  $a_n$  (known as the Ramanujan conjecture)

$$(8) \quad |a_n| \leq n^{1/2} \tau(n),$$

where  $\tau(n)$  denotes the divisor function,  $\tau(n) \ll n^\epsilon$ . This bound can be slightly improved on average. Indeed, arguing as G. Hardy and E. Hecke with Parseval's formula and using the boundedness of  $yf(z)$  we get

$$(9) \quad \sum_{m \leq M} |a_m|^2 \ll M^2.$$

Similarly we get

$$(10) \quad \sum_{m \leq M} a_m e(\alpha m) \ll M \log M$$

for any real  $\alpha$  and  $M \geq 2$ , the implied constant depending on  $f$  only. In this section we derive three variations on (10).

LEMMA I. *Let  $\alpha$  be real and  $\psi$  be a periodic function of period  $r$ . We then have*

$$(11) \quad \sum_{m \leq M} a_m \psi(m) e(\alpha m) \ll \Psi M \log M,$$

where

$$\Psi = \frac{1}{r} \sum_{a \pmod r} \left| \sum_{b \pmod r} \psi(b) e\left(\frac{ab}{r}\right) \right|.$$

Moreover, if  $|\psi| \leq 1$  and  $s$  is a positive integer then we have

$$(12) \quad \sum_{m \leq M, (m,s)=1} a_m \psi(m) e(\alpha m) \ll \tau(s) r^{\frac{1}{2}} M \log M$$

and

$$(13) \quad \sum_{m \leq M, (m,s)=1}^b a_m \psi(m) e(\alpha m) \ll \tau(s) r^{\frac{1}{2}} M (\log M)^7$$

PROOF: The sum on the left-hand side of (11) is equal to

$$\frac{1}{r} \sum_{a \pmod r} \left( \sum_{b \pmod r} \psi(b) e\left(\frac{ab}{r}\right) \right) \sum_{m \leq M} a_m e\left(\left(\alpha - \frac{a}{r}\right)m\right),$$

whence the inequality (11) follows by (10). If  $|\psi| \leq 1$  we obtain  $\Psi \leq r^{1/2}$  by Cauchy's inequality. For the proof of (12) we can assume that  $(r, s) = 1$  by changing  $\psi$  suitably. Then we apply (11) for  $\psi\chi_0$  in place of  $\psi$ , where  $\chi_0$  is the principal character to the modulus  $s$ . We obtain

$$\begin{aligned} \Psi &= \frac{1}{rs} \sum_{a \pmod r} \left| \sum_{b \pmod r} \psi(b) e\left(\frac{ab}{r}\right) \right| \\ &\quad \sum_{c \pmod s} \left| \sum_{d \pmod s} \chi_0(d) e\left(\frac{cd}{s}\right) \right| \ll \frac{r^{\frac{1}{2}}}{s} \sum_{c \pmod s} \sum_{d|(c,s)} d = \tau(s) r^{\frac{1}{2}}, \end{aligned}$$

which gives (12). Finally we derive (13) from (12). The sum on the left-hand side of (13) is equal to

$$\begin{aligned} &\sum_{\nu^2 m \leq M, (\nu m, s)=1} \sum \mu(\nu) a_{\nu^2 m} \psi(\nu^2 m) e(\alpha \nu^2 m) \\ &= \sum_{\substack{(\nu, s)=1 \\ \nu^2 \lambda \leq M}} \sum_{\lambda|\nu^\infty} \mu(\nu) a_{\nu^2 \lambda} \sum_{\substack{m \leq M/\lambda \nu^2 \\ (m, \nu s)=1}} a_m \psi(\nu^2 \lambda m) e(\alpha \nu^2 \lambda m) \\ &\ll \tau(s) r^{\frac{1}{2}} M (\log M) \sum_{\substack{(\nu, s)=1 \\ \nu^2 \lambda \leq M}} \sum_{\lambda|\nu^\infty} |a_{\nu^2 \lambda}| \frac{\tau(\nu)}{\nu^2 \lambda}. \end{aligned}$$

Hence (13) follows by (8).

**4. Approximate formulas for  $L'(1, E, \chi_d)$**

We shall express  $L'(1, E, \chi_d)$  in terms of the rapidly convergent sums

$$\mathcal{A}(X, \chi) = \sum_1^\infty a_n \chi(n) n^{-1} V\left(\frac{2\pi n}{X}\right),$$

where  $V$  is the incomplete gamma function defined by

$$V(X) = \int_X^\infty e^{-t} t^{-1} dt = \frac{1}{2\pi i} \int_{(3/4)} \frac{\Gamma(s)}{s} X^{-s} ds.$$

We have

$$\mathcal{A}(X, \chi_d) = \frac{1}{2\pi i} \int_{(3/4)} L(1 + s, E, \chi_d) \frac{\Gamma(s)}{s} \left(\frac{2\pi}{X}\right)^s ds.$$

Moving the integration to the line  $\operatorname{Re} s = -3/4$  we pass a simple pole at  $s = 0$  with residuum  $L'(1, E, \chi_d)$  by virtue of (2). On the other hand the integral over the line  $\operatorname{Re} s = -3/4$  is equal to  $-\mathcal{A}(d^2 N X^{-1}, \chi_d)$  by the functional equation (1). This gives

$$(14) \quad L'(1, E, \chi_d) = \mathcal{A}(X, \chi_d) + \mathcal{A}(d^2 N X^{-1}, \chi_d)$$

for any  $X > 0$  and  $d$  in  $\mathcal{D}$  which is squarefree. In particular we have

$$(15) \quad L'(1, E, \chi_d) = 2\mathcal{A}(d\sqrt{N}, \chi_d).$$

By (9) we infer trivially that  $\mathcal{A}(X, \chi_d) \ll X^{1/2}$  for any  $X > 0$  and inserting this to (14) we obtain

$$(16) \quad L'(1, E, \chi_d) = \mathcal{A}(X, \chi_d) + O(dX^{-1/2}).$$

**5. Estimation of the fourth moment of  $L'(1, E, \chi_d)$**

By the large sieve inequality (see [1]) together with (8) we get

$$\sum_{d \leq Y} \sum_{\chi \pmod{d}}^* |\mathcal{A}(X, \chi)|^4 \ll (X + Y)^{2+\epsilon}.$$

On the other hand by (14) we have for any  $d \in \mathcal{D}, d \leq Y, d$  squarefree that

$$|L'(1, E, \chi_d)|^4 \ll \int_1^{NY} |\mathcal{A}(X, \chi_d)|^4 X^{-1} dX.$$

Combining both results we infer the upper bound (5) for  $S_4(Y)$ .

**6. An approximate formula for the first moment of  $L'(1, E, \chi_d)$**

By (15) we obtain

$$S_1(Y) = 2 \sum_{d \in \mathcal{D}} {}^b \mathcal{A}(d\sqrt{N}, \chi_d) F\left(\frac{d}{Y}\right).$$

Now we relax the condition that  $d$  is squarefree by introducing the factor  $\sum_{a^2|d} \mu(a)$ , then we split the sum according to whether  $a \leq A$  or  $a > A$  and in the latter case we return to squarefree numbers by extracting square divisors of  $a^{-2}d$ . We obtain  $S_1(Y) = S + R$ , say, where

$$S = 2 \sum_{a \leq A, (a, 4N)=1} \mu(a) \sum_{d \in \mathcal{D}} \mathcal{A}(a^2 d\sqrt{N}, \chi_{a^2 d}) F\left(\frac{a^2 d}{Y}\right)$$

and

$$R = 2 \sum_{(b, 4N)=1} \left( \sum_{a|b, a > A} \mu(a) \right) \sum_{d \in \mathcal{D}} {}^b \mathcal{A}(b^2 d\sqrt{N}, \chi_{b^2 d}) F\left(\frac{b^2 d}{Y}\right).$$

Here  $A$  is a large number to be chosen later. In the term  $\mathcal{A}(X, \chi_{b^2 d})$  with  $X = b^2 d\sqrt{N}$  we return to  $L'(1, E, \chi_d)$  by reversing the arguments as follows

$$\begin{aligned} \mathcal{A}(X, \chi_{b^2 d}) &= \sum_{(n, b)=1} a_n \chi_d(n) n^{-1} V\left(\frac{2\pi n}{X}\right) \\ &= \sum_{k|b} \sum_{\ell|b} \alpha_k \beta_\ell \chi_d(k\ell) \frac{\mu(k)\mu(\ell)}{k\ell} \mathcal{A}\left(\frac{X}{k\ell}, \chi_d\right) \\ &= L'(1, E, \chi_d) \prod_{p|b} \left(1 - \chi_d(p) \frac{\alpha_p}{p}\right) \left(1 - \chi_d(p) \frac{\beta_p}{p}\right) + O(\tau(b) dX^{-\frac{1}{2}}) \end{aligned}$$

the second line being obtained by (7) and the third line by (16). Finally applying (5) and the Hölder inequality we conclude that

$$(17) \quad R \ll \sum_b \left( \sum_{a|b, a > A} 1 \right) (b^{-\frac{5}{2}} Y^{\frac{5}{4}} + b^{-4} Y^{\frac{3}{2}}) Y^\epsilon \ll (A^{-\frac{3}{2}} Y^{\frac{5}{4}} + A^{-3} Y^{\frac{3}{2}}) Y^\epsilon.$$

**7. A transformation of  $S$**

It remains to evaluate  $S$ . For  $(a, 4N) = 1$  and  $d \in \mathcal{D}$  we have

$$A(a^2 d \sqrt{N}, \chi_{a^2 d}) = \sum_{(n, a)=1} a_n n^{-1} \chi_d(n) V(2\pi n / a^2 d \sqrt{N}).$$

Every  $n$  can be written uniquely as the product  $n = k\ell^2 m$ , where  $k$  has prime factors in  $4N$ ,  $\ell m$  is prime to  $4N$  and  $m$  is squarefree. For  $n$  written this way and  $d$  in  $\mathcal{D}$  we have  $\chi_d(n) = \chi_d(m)$  subject to  $(d, \ell) = 1$ . The last condition is detected by the familiar formula of Möbius giving

$$S = 2 \sum_{\substack{a \leq A \\ (a, 4N)=1}} \mu(a) \sum_{\substack{n=k\ell^2 m \\ (n, a)=1}} a_n n^{-1} \sum_{q|\ell} \mu(q) \sum_{d q \in \mathcal{D}} \chi_{dq}(m) F\left(\frac{a^2 dq}{Y}\right) V\left(\frac{2\pi n}{a^2 dq \sqrt{N}}\right).$$

Next, by means of Gauss sums we write

$$\chi_d(m) = \bar{\epsilon}_m m^{-\frac{1}{2}} \sum_{2|r| < m} \chi_{Nr}(m) e\left(\frac{\overline{4N}rd}{m}\right),$$

where  $\epsilon_m = 1$  if  $m \equiv 1 \pmod{4}$ ,  $\epsilon_m = i$  if  $m \equiv -1 \pmod{4}$  and  $4N\overline{4N} \equiv 1 \pmod{m}$ . This gives

$$S = 2 \sum_{\substack{a \leq A \\ (a, 4Nn)=1}} \sum_{n=k\ell^2 m} \mu(a) a_n n^{-1} \bar{\epsilon}_m m^{-\frac{1}{2}} \sum_{q|\ell} \mu(q) \sum_{2|r| < m} \chi_{Nr q}(m) \sum_d,$$

where

$$\sum_d = \sum_{dq \in \mathcal{D}} F\left(\frac{a^2 dq}{Y}\right) V\left(\frac{2\pi n}{a^2 dq \sqrt{N}}\right) e\left(\frac{\overline{4N}rd}{m}\right).$$

We put  $\Delta = \min(1/2, a^2 q Y^{\epsilon-1})$  and split  $S = S_0 + S_1 + S_2$ , where  $S_0, S_1, S_2$  denote the partial sums restricted by the conditions  $r = 0, 0 < |r| < \Delta m, \Delta m \leq |r| < m/2$  respectively.



**8. Estimates for  $S_2$  and  $S_1$**

LEMMA 2. Suppose  $g(x)$  is a smooth and integrable function on  $\mathbb{R}$  with derivatives  $g^{(j)}(x) \ll (|x| + X)^{-j}$  for all  $j \geq 1$  the implied constant depending on  $j$  only. Suppose  $\alpha$  is real and  $q$  is a positive integer such that  $\alpha q$  is not an integer. We then have

$$(18) \quad \sum_{n \equiv v \pmod{q}} g(n)e(\alpha n) \ll \frac{X}{q} \left( \frac{q}{X \|\alpha q\|} \right)^j$$

for any  $j \geq 2$ , the implied constant depending on  $j$  only.

PROOF: By Poisson's formula the sum is equal to

$$\frac{1}{q} \sum_{u=-\infty}^{\infty} e\left(\frac{uv}{q}\right) \hat{g}\left(\alpha - \frac{u}{q}\right),$$

where  $\hat{g}(y)$  denotes the Fourier transform of  $g(x)$ . We have  $\hat{g}(y) \ll X(Xy)^{-j}$  by the partial integration  $j$  times, whence (18) follows by trivial summation over  $u$ .

To estimate  $S_2$  we sum over  $d$  first by an appeal to (18). For any  $j \geq 2$  we get  $\sum_d \ll (n + Y)^{-j}$ , whence  $S_2 \ll 1$ .

To estimate  $S_1$  we sum over  $m$  first using (13) and partial summation together with the relation

$$e\left(\frac{4\overline{N}rd}{m}\right) = e\left(\frac{rd}{4Nm} - \frac{\overline{m}rd}{4N}\right)$$

and then we sum over  $r$  trivially getting

$$\begin{aligned} \sum_{0 < |r| < \Delta m} \sum a_n n^{-1} \overline{\epsilon}_m m^{-\frac{1}{2}} \chi_{Nr q}(m) V\left(\frac{2\pi n}{a^2 dq \sqrt{N}}\right) e\left(\frac{4\overline{N}rd}{m}\right) \\ \ll k^{-\frac{3}{2}} \ell^{-3} a^3 q^2 Y^{\epsilon - \frac{1}{2}}. \end{aligned}$$

Hence we conclude that

$$S_1 \ll \sum_{a \leq A} \sum_{k \ell^2} \sum_{q | \ell} \sum_d F\left(\frac{a^2 dq}{Y}\right) k^{-\frac{3}{2}} \ell^{-3} a^3 q^2 Y^{\epsilon - \frac{1}{2}} \ll A^2 Y^{\epsilon + \frac{1}{2}}.$$

**9. Evaluation of  $S_0$**

Since  $r = 0$  we have  $\chi_{Nr}q(m) = 0$  for all  $m > 1$  and the terms with  $m = 1$  yield

$$S_0 = 2 \sum_{\substack{a \leq A \\ (a, 4N)=1}} \mu(a) \sum_{\substack{n=k\ell^2 \\ (n,a)=1}} a_n n^{-1} \sum_{q|\ell} \mu(q) \sum_d,$$

where

$$\sum_d = \sum_{dq \in \mathcal{D}} F\left(\frac{a^2 dq}{Y}\right) V\left(\frac{2\pi n}{a^2 dq \sqrt{N}}\right).$$

We split the summation over  $d$  into residue classes modulo  $4N$ . Each class contributes

$$\frac{Y}{4Na^2q} \int F(t)V\left(\frac{2\pi n}{t\sqrt{NY}}\right) dt + O\left(\left(1 + \frac{n}{Y}\right)^{-j}\right)$$

for any  $j \geq 2$ , and the number of relevant classes is

$$\gamma(4N) = \#\{d \pmod{4N} : d \equiv -\nu^2 \pmod{4N}, (\nu, 4N) = 1\}.$$

Hence

$$\begin{aligned} S_0 &= \gamma(4N)Y \sum_{n=k\ell^2} \frac{a_n \varphi(\ell)}{2Nn\ell} \left(\sum_{\substack{a \leq A, \\ (a, 4N\ell)=1}} \mu(a)a^{-2}\right) \int F(t)V\left(\frac{2\pi n}{t\sqrt{NY}}\right) dt + O\left(AY^{\epsilon+\frac{1}{2}}\right) \\ &= c_N Y \int F(t)\mathcal{B}(t\sqrt{NY})dt + O((AY^{\frac{1}{2}} + A^{-1}Y)Y^\epsilon), \end{aligned}$$

where

$$c_N = \frac{3\gamma(4N)}{\pi^2 N} \prod_{p|4N} \left(1 - \frac{1}{p^2}\right)$$

and

$$\mathcal{B}(X) = \sum_{n=k\ell^2} \frac{b_n}{n} V\left(\frac{2\pi n}{X}\right)$$

with

$$b_n = a_n \prod_{p|n, p \nmid 4N} \left(1 + \frac{1}{p}\right).$$

To evaluate the series  $\mathcal{B}(X)$  we appeal to analytic properties of the zeta-function

$$L(s) = \sum_{n=k\ell^2} b_n n^{-s}.$$

The required properties are inherited from the properties of the Rankin-Selberg zeta-function

$$H(s) = \sum_1^\infty a_n^2 n^{-s} .$$

The Rankin-Selberg zeta-function is meromorphic on  $\mathbb{C}$ , holomorphic on  $\text{Re } s \geq 1$  except for a simple pole at  $s = 2$  with residuum

$$H = \text{res}_{s=2} H(s) > 0 ,$$

and it satisfies a functional equation which connects  $H(s)$  with  $H(2 - s)$ . Moreover, as shown by G. Shimura [12] the function

$$L(s, \text{sym}^2) = \frac{\zeta(2s)}{\zeta(s)} H(s + 1)$$

is entire. By the Phragmén-Lindelöf principle, using the functional equation, it follows that

$$L(s, \text{sym}^2) \ll |s| \quad \text{if } \text{Re } s \geq 1/2 .$$

Since  $L(s)$  agrees with  $L(2s - 1, \text{sym}^2)/\zeta(4s - 2) = H(2s)/\zeta(2s - 1)$  up to an Euler product  $P(s)$ , say, which converges absolutely in  $\text{Re } s \geq 3/4$  we conclude that  $L(s)$  is holomorphic in  $\text{Re } s \geq 3/4$ , it satisfies

$$L(s) \ll |s|^2 \quad \text{if } \text{Re } s \geq 3/4$$

and that

$$(19) \quad L(1) = HP(1) \neq 0 .$$

Now by the contour integration we get

$$\begin{aligned} \mathcal{B}(X) &= \frac{1}{2\pi i} \int_{(3/4)} L(s + 1) \frac{\Gamma(s)}{s} \left(\frac{X}{2\pi}\right)^s ds \\ &= \text{res}_{s=0} L(s + 1) \frac{\Gamma(s)}{s} \left(\frac{X}{2\pi}\right)^s + \frac{1}{2\pi i} \int_{(-1/4)} \\ &= L(1) \left(\log \frac{X}{2\pi} - \gamma\right) + L'(1) + O(X^{-1/4}) \end{aligned}$$

by the expansion  $\Gamma(s) = s^{-1} - \gamma + \dots$ , where  $\gamma$  is the Euler constant. Integrating against  $F(t)$  we conclude that

$$S_0 = \alpha Y \log Y + \beta Y + O((AY^{\frac{1}{2}} + A^{-1}Y)Y^\epsilon)$$

with

$$(20) \quad \alpha = c_N L(1) \int F(t) dt \neq 0$$

and

$$(21) \quad \beta = c_N \int F(t) \left[ L(1) \left( \log \frac{t\sqrt{N}}{2\pi} - \gamma \right) + L'(1) \right] dt .$$

**10. Evaluation of the first moment of  $L'(1, E, \chi_d)$ . Conclusion**

Collecting the established evaluations we infer that

$$S_1(Y) = S_0 + S_1 + S_2 + R = \alpha Y \log Y + \beta Y + O((AY^{\frac{1}{2}} + A^{-1}Y + A^2Y^{\frac{1}{2}} + A^{-\frac{3}{2}}Y^{\frac{5}{4}} + A^{-3}Y^{\frac{3}{2}})Y^\epsilon)$$

which gives (6) on taking  $A = Y^{3/14}$ .

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