

## Extreme value copulas and max-stable processes

**Titre:** Copules des valeurs extrêmes et processus max-stables

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**Abstract:** During the last decades, copulas have been increasingly used to model the dependence across several random variables such as the joint modelling of the intensity and the duration of rainfall storms. When the problem consists in modelling extreme values, i.e., only the tails of the distribution, the extreme value theory tells us that one should consider max-stable distributions and put some restrictions on the copulas to be used. Although the theory for multivariate extremes is well established, its foundation is usually introduced outside the copula framework. This paper tries to unify these two frameworks in a single view. Moreover the latest developments on spatial extremes and max-stable processes will be introduced. At first glance the use of copulas for spatial problems sounds a bit odd but since usually stochastic processes are observed at a finite number of locations, the inferential procedure is intrinsically multivariate. An application on the spatial modelling of extreme temperatures in Switzerland is given. Results show that the use of non extreme value based models can largely underestimate the spatial dependence and the assumptions made on the spatial dependence structure should be chosen with care.

**Résumé :** Les dernières décennies ont vu une utilisation des copules de plus en plus fréquente afin de modéliser la dépendance présente au sein d'un groupe de plusieurs variables aléatoires ; par exemple afin de modéliser simultanément l'intensité et la durée d'un événement pluvieux. Lorsque l'intérêt porte sur la modélisation des valeurs extrêmes, i.e., seulement les queues de la distribution, la théorie des valeurs extrêmes nous dicte quelles distributions considérer. Ces dernières doivent être max-stables et imposent donc des contraintes sur les copules adéquates. Bien que la théorie pour les extrêmes multivariées soit bien établie, elle est généralement introduite en dehors du cadre des copules. Ce papier essaye de présenter la théorie des valeurs extrêmes par le monde des copules. Les derniers développements sur les extrêmes spatiaux et les processus max-stables seront également évoqués. Bien qu'il paraisse étrange au premier abord de parler de copules pour les processus stochastiques, leur utilisation peut être adéquate puisque les processus sont souvent observés en un nombre fini de positions et la procédure d'estimation est alors intrinsèquement multivariée. Une application à la modélisation spatiale des températures extrêmes en Suisse est donnée. Les résultats montrent que l'utilisation de modèles non extrêmes peut largement sous-estimer la dépendance spatiale et que le choix fait sur la structure de dépendance spatiale est primordial.

**Keywords:** Max-stable process, Extreme value copula, Rainfall

**Mots-clés :** Processus max-stable, Copule des valeurs extrêmes, Précipitation

**AMS 2000 subject classifications:** 60G70, 60E05, 62P12

### 1. Introduction

During the last decades, copulas have been increasingly used as a convenient tool to model dependence across several random variables. A particular area of interest is finance where the joint modelling of (large) portfolios is crucial [11, 14]. Clearly for financial applications one is

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mainly interested in modelling the largest expected losses and therefore one might use suitable models for the tails of the distribution. In the meantime, many advances have been made towards a statistical modelling of multivariate extremes using an extreme value paradigm. Although these two frameworks share some connections, only a few authors from the extreme value community adopt the copula framework for multivariate extremes [2, 9, 16] and the use of copulas has been criticized [22].

This work is organized as follows. Section 2 introduces the copula framework with a particular emphasis on extreme value copulas and tail dependence. Section 3 gives a spatial extension of the copula framework and makes some connections with max-stable processes. An application on the spatial modelling of extreme temperatures in Switzerland is given in Section 4.

## 2. Multivariate extremes and copulas

### 2.1. Generalities

The starting point for using copulas in multivariate problems is Sklar's theorem [23, pages 17–24] that states that the cumulative distribution function of a  $k$ -variate random vector  $\mathbf{Z} = (Z_1, \dots, Z_k)$  may be written as

$$\Pr(Z_1 \leq z_1, \dots, Z_k \leq z_k) = C(u_1, \dots, u_k), \tag{1}$$

where  $u_j = \Pr(Z_j \leq z_j)$ ,  $j = 1, \dots, k$ . The  $k$ -dimensional distribution  $C$  defined on  $[0, 1]^k$  is known as the copula and is unique when  $Z$  has continuous margins.

One common choice is the Gaussian copula

$$C(u_1, \dots, u_k) = \Phi \{ \Phi^{-1}(u_1), \dots, \Phi^{-1}(u_k); \Sigma \},$$

where  $\Phi$  is the standard cumulative distribution function of a standard normal random variable and  $\Phi(\cdot; \Sigma)$  is the joint distribution function of a  $k$ -variate standard Gaussian random vector with correlation matrix  $\Sigma$ . Similarly one can consider the Student copula

$$C(u_1, \dots, u_k) = T_\nu \{ T_\nu^{-1}(u_1), \dots, T_\nu^{-1}(u_k); \Sigma \},$$

where  $T_\nu$  denotes the cumulative distribution function of a Student random variable with  $\nu$  degrees of freedom and  $T_\nu(\cdot; \Sigma)$  is the joint distribution of a  $k$ -variate standard Student random vector with  $\nu$  degrees of freedom and dispersion matrix  $\Sigma$ .

### 2.2. Extreme value copulas

Although there exist several copula families such as the Archimedean or the harmonic ones [23], in this paper we restrict our attention to extreme value copulas, i.e., copulas  $C_*$  such that there exists a copula  $C$  satisfying [13]

$$C \left( u_1^{1/n}, \dots, u_k^{1/n} \right)^n \longrightarrow C_*(u_1, \dots, u_k), \quad n \rightarrow \infty, \tag{2}$$

for all  $(u_1, \dots, u_k) \in [0, 1]^k$ . Equation (2) is an asymptotic justification for using an extreme value copula to model componentwise maxima. To see this let  $\mathbf{U}_i = (U_{i,1}, \dots, U_{i,k})$ ,  $i \geq 1$ , be

independent copies of a random vector  $\mathbf{U} = (U_1, \dots, U_k)$  whose joint distribution is  $C$ . Hence (2) may be rewritten as

$$\Pr \left( \max_{i=1, \dots, n} U_{i,1}^n \leq u_1, \dots, \max_{i=1, \dots, n} U_{i,k}^n \leq u_k \right) \longrightarrow C_*(u_1, \dots, u_k), \quad n \rightarrow \infty,$$

and justifies the use of  $C_*$  when modelling pointwise maxima over  $n$  appropriately rescaled independent realizations,  $n$  large enough.

It can be shown using standard extreme value arguments that the class of extreme value copulas corresponds to that of max-stable copulas, i.e., copulas such that

$$C_*(u_1, \dots, u_k)^n = C_*(u_1^n, \dots, u_k^n), \quad n > 0. \quad (3)$$

In [7], de Haan and Resnick derive a characterization for the distribution function of any max-stable random vector which writes in terms of extreme value copulas as

$$C_*(u_1, \dots, u_k) = \exp \left\{ -V \left( -\frac{1}{\log u_1}, \dots, -\frac{1}{\log u_k} \right) \right\}, \quad (4)$$

where the function  $V$  is a homogeneous function of order  $-1$ , i.e.,  $V(nu_1, \dots, nu_k) = n^{-1}V(u_1, \dots, u_k)$  for all  $n > 0$ , and is known as the exponent function.

Two examples of well known extreme value copulas are the Gumbel–Hougaard copula, also known as the logistic family [17],

$$C_*(u_1, \dots, u_k) = \exp \left[ - \left\{ \sum_{j=1}^k (-\log u_j)^{1/\alpha} \right\}^\alpha \right], \quad 0 < \alpha \leq 1,$$

which is the only extreme value copula that belongs to the archimedean family [15] and the Galambos copula, also known as the negative logistic family [12],

$$C_*(u_1, \dots, u_k) = \exp \left[ - \sum_{\substack{J \subset \{1, \dots, k\} \\ |J| \geq 2}} (-1)^{|J|} \left\{ \sum_{j \in J} (-\log u_j)^{-\alpha} \right\}^{-1/\alpha} \right] \prod_{j=1}^k u_j, \quad \alpha > 0,$$

where the outer sum is over all subsets  $J$  of  $\{1, \dots, k\}$  whose cardinality  $|J|$  is greater than 2.

The two models above are likely to be too limited for medium to large dimensional problems since the dependence is driven by a single parameter  $\alpha$ . Although some authors derive asymmetric versions of these copulas [19, 31], these asymmetric versions are still too restrictive or induce a too large number of parameters.

Two other parametric extreme value copulas that do not suffer from this drawback are the extremal- $t$  and Hüsler–Reiss copulas [18, 9]. Although closed forms exist for these two latter copulas in the general  $k$ -variate setting [24], we restrict our attention to the bivariate case only to ease the notations.

It is well known that if in (2)  $C$  is the copula related to (appropriately rescaled) bivariate normal random vectors with correlation  $\rho < 1$ , then

$$C \left( u_1^{1/n}, u_2^{1/n} \right)^n \longrightarrow u_1 u_2, \quad n \rightarrow \infty, \quad (5)$$

i.e., the extreme value copula is the independence copula. To obtain a non trivial extreme value copula, it can be shown [18] that if the correlation increases at the right speed as  $n$  gets large, i.e.,  $\{1 - \rho_n\} \log n \rightarrow a^2$  as  $n \rightarrow \infty$  for some  $a \in [0, \infty]$ , then the corresponding extreme value copula, known as the Hüsler–Reiss copula, is

$$C_*(u_1, u_2) = \exp \left[ \Phi \left\{ \frac{a}{2} + \frac{1}{a} \log \left( \frac{\log u_2}{\log u_1} \right) \right\} \log u_1 + \Phi \left\{ \frac{a}{2} + \frac{1}{a} \log \left( \frac{\log u_1}{\log u_2} \right) \right\} \log u_2 \right], \quad (6)$$

where  $\Phi$  denotes the standard normal cumulative distribution function.

More recently [9] consider the case where  $Z$  is a standard bivariate Student random vector with  $\nu$  degrees of freedom and dispersion matrix whose off-diagonal elements are  $\rho \in (-1, 1)$ . It can be shown that the corresponding extreme value copula, known as the extremal- $t$  copula, is

$$C_*(u_1, u_2) = \exp \left[ T_{\nu+1} \left\{ -\frac{\rho}{b} + \frac{1}{b} \left( \frac{\log u_2}{\log u_1} \right)^{1/\nu} \right\} \log u_1 + T_{\nu+1} \left\{ -\frac{\rho}{b} + \frac{1}{b} \left( \frac{\log u_1}{\log u_2} \right)^{1/\nu} \right\} \log u_2 \right], \quad (7)$$

where  $T_\nu$  is the cumulative distribution function of a Student random variable with  $\nu$  degrees of freedom and  $b^2 = (1 - \rho^2)/(\nu + 1)$ .

Although the Hüsler–Reiss copula is not a special case of the extremal- $t$ , the former can be derived from the latter [24, 5] since by letting  $\rho = \exp\{-a^2/(2\nu)\}$  in (7), we have  $b \sim a/\nu$  for  $\nu$  large enough and

$$b^{-1} \left\{ \left( \frac{\log u_2}{\log u_1} \right)^{1/\nu} - \rho \right\} \sim \frac{\nu}{a} \left\{ \left( \frac{\log u_2}{\log u_1} \right)^{1/\nu} - 1 + \frac{a^2}{2\nu} \right\} \rightarrow \frac{a}{2} + \log \frac{\log u_2}{\log u_1}, \quad \nu \rightarrow \infty.$$

### 2.3. Tail dependence and extremal coefficients

When the interest is in modelling extremes, the tail dependence coefficient is a useful statistic that summarizes how extremes events tend to occur simultaneously. To ease the notations we restrict our attention throughout this section to the bivariate case but extension to higher dimensions is straightforward. Provided the limit exists, the upper tail dependence coefficient is

$$\chi_{\text{up}} = \lim_{u \rightarrow 1^-} \Pr(U_2 > u \mid U_1 > u) = \lim_{u \rightarrow 1^-} \frac{1 - 2u + C(u, u)}{1 - u},$$

and indicates dependence in the upper tail when positive and independence otherwise. The upper tail dependence coefficient of a copula and its related extreme value copula, i.e.,  $C_*$  and  $C$  in (2), are the same [20]; for instance the Student copula and the extremal- $t$  both satisfy

$$\chi_{\text{up}} = 2 - 2T_{\nu+1} \left[ \left\{ \frac{(1 - \rho)(\nu + 1)}{1 + \rho} \right\}^{1/2} \right].$$

Due to (5) and provided  $|\rho| < 1$ , the Gaussian copula has  $\chi_{\text{up}} = 0$  while as expected the Hüsler–Reiss copula allows dependence in the upper tail and has  $\chi_{\text{up}} = 2 - 2\Phi(a/2)$ .

Similarly one can define a lower tail dependence coefficient

$$\chi_{\text{low}} = \lim_{u \rightarrow 0^+} \Pr(U_2 \leq u \mid U_1 \leq u) = \lim_{u \rightarrow 0^+} \frac{C(u, u)}{u},$$

TABLE 1. Parametric families of isotropic correlation functions or semi variograms. Here  $K_\kappa$  denotes the modified Bessel function of order  $\kappa$ ,  $\Gamma(u)$  denotes the gamma function and  $J_\kappa$  denotes the Bessel function of order  $\kappa$ . In each case  $\lambda > 0$ .

Family	Correlation function	Range of validity
Whittle–Matérn	$\rho(h) = \{2^{\kappa-1}\Gamma(\kappa)\}^{-1}(\ h\ /\lambda)^\kappa K_\kappa(\ h\ /\lambda)$	$\kappa > 0$
Cauchy	$\rho(h) = \{1 + (\ h\ /\lambda)^2\}^{-\kappa}$	$\kappa > 0$
Powered exponential	$\rho(h) = \exp\{-(\ h\ /\lambda)^\kappa\}$	$0 < \kappa \leq 2$
Spherical	$\rho(h) = \max\{0, 1 - 1.5\ h\ /\lambda + 0.5(\ h\ /\lambda)^3\}$	—
Family	Semi variogram	Range of validity
Fractional	$\gamma(h) = (\ h\ /\lambda)^\kappa$	$0 < \kappa \leq 2$
Brownian	$\gamma(h) = \ h\ /\lambda$	—

that indicates dependence in the lower tail when positive and independence otherwise. By symmetry of the Gaussian and Student densities, it is clear that the Gaussian and Student copulas have  $\chi_{\text{low}} = \chi_{\text{up}}$ . Further the lower tail dependence coefficient for any extreme value copula is  $\chi_{\text{low}} = 0$  since the homogeneity property of  $V$  in (4) implies

$$\lim_{u \rightarrow 0^+} \frac{C_*(u, u)}{u} = \lim_{u \rightarrow 0^+} u^{V(1,1)-1} = 0,$$

provided  $V(1, 1) \neq 1$ .

When focusing on extreme value copulas, a convenient statistic to summarize the dependence is the extremal coefficient [28, 3]. Let  $C_*$  be an extreme value copula, then due to the homogeneity property of  $V$  in (4) we have

$$C_*(u, u) = u^{V(1,1)}, \tag{8}$$

and the quantity  $\theta = V(1, 1)$  is the (pairwise) extremal coefficient. It takes values in the interval  $[1, 2]$ ; the lower bound indicates complete dependence, and the upper one independence. The extremal coefficient  $\theta$  is strongly connected to  $\chi_{\text{up}}$  since by using (8) and l’Hôpital’s rule we have

$$\chi_{\text{up}} = \lim_{u \rightarrow 1^-} \frac{1 - 2u + C_*(u, u)}{1 - u} = \lim_{u \rightarrow 1^-} \frac{1 - 2u + u^\theta}{1 - u} = \lim_{u \rightarrow 1^-} \frac{2 - \theta u^{\theta-1}}{u} = 2 - \theta.$$

### 3. Spatial extension

At first glance the use of copulas for spatial problems seems odd since spatial problems are often related to stochastic processes while copulas are essentially multivariate models. However most often stochastic processes are observed at a finite number of locations and the inferential procedure is therefore intrinsically multivariate. Further having resort to the Kolmogorov’s extension theorem, one can extend any suitable copula to stochastic processes.

Throughout this section we will consider a stochastic process  $Z$  defined on a spatial domain  $\mathcal{X} \subset \mathbb{R}^d$  and suppose that  $Z$  has been observed at a finite number of locations  $x_1, \dots, x_k \in \mathcal{X}$ .

#### 3.1. Two simple models

Not every copula would extend naturally to the infinite dimensional setting, e.g., the logistic and negative logistic families, and even if they do, the copulas should be flexible enough to capture

the observed spatial dependence. Clearly the Gaussian and Student copulas are appealing since the generalization to the infinite dimensional setting is just matter of taste by using in place of Gaussian or Student random vectors, Gaussian or Student processes with prescribed correlation functions such as the Whittle-Matèrn, powered exponential or Cauchy families—see Table 1.

For instance extending the Gaussian copula amounts to considering the process

$$Z(x) = F_x^{\leftarrow} [\Phi \{ \varepsilon(x) \}], \quad x \in \mathcal{X}, \tag{9}$$

where  $\varepsilon$  is a standard Gaussian process and  $F_x^{\leftarrow}$  is the generalized inverse function of the marginal distribution of  $Z$  at location  $x$  while the Student copula extension considers the process

$$Z(x) = F_x^{\leftarrow} \left[ T_v \left\{ \varepsilon(x) \sqrt{v/X} \right\} \right], \quad x \in \mathcal{X},$$

where  $X$  is a  $\chi^2$  random variable with  $v$  degrees of freedom.

Typically the data one want to model will drive the choice for  $F^{\leftarrow}$ . For instance if our interest is in pointwise maxima, the univariate extreme value theory tells us that the margins should behave as a generalized extreme value distribution [8], i.e., for all  $x \in \mathcal{X}$  we have

$$F_x(z) = \begin{cases} \exp \left[ - \left\{ 1 + \xi(x) \frac{z - \mu(x)}{\sigma(x)} \right\}_+^{-1/\xi(x)} \right], & \xi(x) \neq 0, \\ \exp \left[ - \exp \left\{ - \frac{z - \mu(x)}{\sigma(x)} \right\} \right], & \xi(x) = 0, \end{cases}$$

where  $u_+$  denotes  $\max(u, 0)$  and  $\mu(x), \sigma(x), \xi(x)$  are respectively a real location parameter, a positive scale parameter and a real shape parameter. Typically to allow for prediction at some unobserved locations and to have parsimonious models, one may define trend surfaces for the marginal parameters  $\{\mu(x), \sigma(x), \xi(x)\}$ . For instance one could consider the following trend surface for the location parameter

$$\mu(x) = \beta_{0,\mu} + \beta_{1,\mu} \text{lon}(x) + \beta_{2,\mu} \text{lat}(x), \quad x \in \mathcal{X},$$

where  $\text{lon}(x)$  and  $\text{lat}(x)$  are the longitude and latitude at location  $x$ .

Given data  $z_1, \dots, z_k$  assumed to be a realization from the Gaussian copula process (9) observed at locations  $x_1, \dots, x_k \in \mathcal{X}$ , the contribution to the likelihood is easily seen to be

$$\varphi [\Phi^{\leftarrow} \{F_{x_1}(z_1)\}, \dots, \Phi^{\leftarrow} \{F_{x_k}(z_k)\}; \Sigma] \prod_{j=1}^k \frac{\varphi [\Phi^{\leftarrow} \{F_{x_j}(z_j)\}]}{f_{x_j}(z_j)}, \tag{10}$$

where  $\varphi(\cdot; \Sigma)$  is the  $k$ -variate density of a standard multivariate normal distribution with correlation matrix  $\Sigma = \{\rho(x_i - x_j)\}_{i,j}$ ,  $\varphi$  and  $\Phi^{\leftarrow}$  are the probability density and quantile functions of a standard normal random variable and  $f_{x_j}$  is the density related to the distribution  $F_{x_j}$ . Similar expressions to (9) and (10) hold for the Student copula. As a consequence of (10), the maximum likelihood estimator for the Gaussian or the Student copula processes is easily obtained.

### 3.2. Extreme value copula based models

As stated in Section 3.1, the logistic and negative logistic families are likely to be too restrictive in practice since the dependence is driven by a single parameter. The Hüsler–Reiss and extremal- $t$  copulas seems more appropriate since they are based on Gaussian or Student random vectors and both generalize easily to stochastic processes. Since these copulas are extreme value copulas, their extension to the infinite dimensional setting corresponds to max-stable processes [8].

Although another characterization exists [6], [27, 29] show that a max-stable process  $Z$  with unit Fréchet margins, i.e.,  $\Pr\{Z(x) \leq z\} = \exp(-1/z)$ ,  $z > 0$ ,  $x \in \mathcal{X}$ , can be represented as

$$Z(x) = \max_{i \geq 1} \zeta_i Y_i(x), \quad x \in \mathcal{X}, \quad (11)$$

where  $\{\zeta_i\}_{i \geq 1}$  are the points of a Poisson process on  $(0, \infty)$  with intensity  $d\Lambda(\zeta) = \zeta^{-2} d\zeta$  and  $Y_i$  are independent copies of a nonnegative stochastic process such that  $\mathbb{E}\{Y(x)\} = 1$  for all  $x \in \mathcal{X}$ . It is not difficult to show [29, 5] that, for all  $z_1, \dots, z_k > 0$ ,  $k \in \mathbb{N}$ , the finite dimensional cumulative distribution functions of (11) are

$$\Pr[Z(x_1) \leq z_1, \dots, Z(x_k) \leq z_k] = \exp \left[ -\mathbb{E} \left\{ \max_{j=1, \dots, k} \frac{Y(x_j)}{z_j} \right\} \right]. \quad (12)$$

with exponent function

$$V(z_1, \dots, z_k) = \mathbb{E} \left\{ \max_{j=1, \dots, k} \frac{Y(x_j)}{z_j} \right\}.$$

The corresponding extreme value copula, derived by letting  $u_j = \exp(-1/z_j)$ ,  $j = 1, \dots, k$ , is

$$C_*(u_1, \dots, u_k) = \exp \left[ \mathbb{E} \left\{ \max_{j=1, \dots, k} Y(x_j) \log u_j \right\} \right], \quad u_1, \dots, u_k > 0,$$

and is as expected an extreme value copula since

$$C_*(u_1^n, \dots, u_k^n) = \exp \left\{ n \mathbb{E} \left( \max_{j=1, \dots, k} Y(x_j) \log u_j \right) \right\} = C_*(u_1, \dots, u_k)^n.$$

Based on (11), many parametric max-stable models have been proposed by making different choices for the process  $Y$  [1, 30, 29, 21]. For instance the Brown–Resnick model [21, 1] takes

$$Y(x) = \exp\{\varepsilon(x) - \gamma(x)\},$$

where  $\varepsilon$  is an intrinsically stationary Gaussian process with semi variogram  $\gamma$  and extends the Hüsler–Reiss copula with  $a^2 = 2\gamma(x_i - x_j)$ ,  $x_i, x_j \in \mathcal{X}$ . The extremal- $t$  process extends the extremal- $t$  copula by taking

$$Y(x) = c_\nu \max\{0, \varepsilon(x)\}^\nu, \quad c_\nu = \pi^{1/2} 2^{-(\nu-2)/2} \Gamma\left(\frac{\nu+1}{2}\right)^{-1}, \quad \nu \geq 1,$$

where  $\varepsilon$  is a stationary Gaussian process and  $\Gamma$  is the Gamma function [25].

Another possibility, known as the Schlather model [29], takes

$$Y(x) = \sqrt{2\pi} \max\{0, \varepsilon(x)\}, \quad x \in \mathcal{X},$$

where  $\varepsilon$  is a standard Gaussian process with correlation function  $\rho$ . Its bivariate distribution function is

$$\Pr\{Z(x_1) \leq z_1, Z(x_2) \leq z_2\} = \exp\left[-\frac{1}{2}\left(\frac{1}{z_1} + \frac{1}{z_2}\right)\left(1 + \sqrt{1 - \frac{2\{1 + \rho(x_1 - x_2)\}z_1z_2}{(z_1 + z_2)^2}}\right)\right],$$

where  $x_1, x_2 \in \mathcal{X}$ , and the associated extreme value copula is

$$C_*(u_1, u_2) = \exp\left[\frac{\log u_1 + \log u_2}{2}\left\{1 + \sqrt{1 - \frac{2(1 + \rho)\log u_1 \log u_2}{(\log u_1 + \log u_2)^2}}\right\}\right], \quad -1 \leq \rho \leq 1.$$

As for non extreme value models, trend surfaces can be used to allow prediction at unobserved locations. However as stated by [5, 26], making inferences from max-stable processes is not as simple as for the Gaussian or Student copulas since (12) or equivalently (4) yields a combinatorial explosion for the likelihood. Indeed since any max-stable distribution has a joint cumulative distribution function

$$F(z_1, \dots, z_k) = \exp\{-V(z_1, \dots, z_k)\},$$

and the associated may be written as

$$f(z_1, \dots, z_k) = \{\text{sum of Bell}(k) \text{ terms}\}F(z_1, \dots, z_k),$$

where  $\text{Bell}(k)$  is the  $k$ -th Bell number. Unfortunately, the sequence of Bell numbers increases extremely fast. For example when  $k = 10$  one would need to sum up around 116000 terms to compute the contribution of a single observation to the likelihood.

To bypass this computational burden, a strategy consists in maximizing the pairwise likelihood in place of the full likelihood which gives an estimator that shares the same properties as the maximum likelihood estimator, i.e., consistency and asymptotic normality, but yields to a loss in efficiency [26].

For spatial problems, the (pairwise) extremal coefficient  $\theta$  is extended to the spatial setting as a function  $\theta: \mathbb{R}^d \mapsto [1, 2]$

$$\theta(x_1 - x_2) = -z \log \Pr\{Z(x_1) \leq z, Z(x_2) \leq z\}, \quad x_1, x_2 \in \mathbb{R}^d,$$

and quantifies how the spatial dependence of extremes evolves as the distance between two locations  $x_1, x_2 \in \mathbb{R}^d$  increases.

#### 4. Application

In this section we fit various extreme value and non extreme value models to extreme temperatures. The data considered here were previously analyzed by [4] and consist in annual maximum temperatures recorded at 15 sites in Switzerland during the period 1961–2005, see Figure 1.



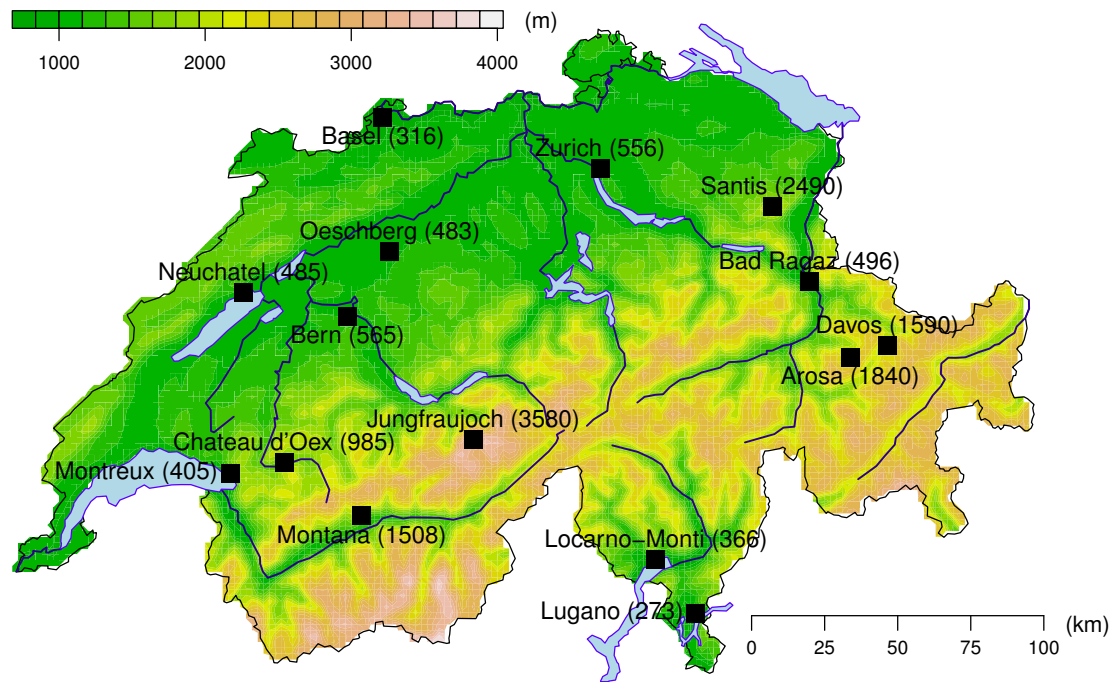


FIGURE 1. Topographical map of Switzerland showing the sites and altitudes in metres above sea level of 15 weather stations for which annual maxima temperature data are available.

For each model and following the work of [4, 10], we consider the following trend surfaces

$$\mu(x, t) = \beta_{0,\mu} + \beta_{1,\mu} \text{alt}(x) + \beta_{2,\mu} \frac{t - 1983}{100}, \quad (13)$$

$$\sigma(x, t) = \beta_{0,\sigma}, \quad (14)$$

$$\xi(x, t) = \beta_{0,\xi}, \quad (15)$$

where  $\text{alt}(x)$  denotes the altitude above mean sea level in kilometres and  $\{\mu(x, t), \sigma(x, t), \xi(x, t)\}$  are the location, scale and shape parameters of the generalized extreme value distribution at location  $x$  and year  $t$ .

To assess the impact on the assumption of max-stability for modelling extremes, we consider the Gaussian copula, the Student copula, the extremal- $t$  and the Brown–Resnick processes. The Brown–Resnick models take as semi variogram  $\gamma(h) = (h/\lambda)^\kappa$ ,  $0 < \kappa \leq 2$ . Each model is fitted by maximizing the likelihood or the pairwise likelihood when the former was not tractable.

As one might expect, the marginal parameter estimates are consistent across all considered models yielding to the estimated trend surfaces

$$\begin{aligned} \hat{\mu}(x, t) &= 34.9_{(0.2)} - 7.35_{(0.06)} \text{alt}(x) + 2.48_{(1.07)} \frac{t - 1983}{100}, \\ \hat{\sigma}(x, t) &= 1.87_{(0.07)}, \\ \hat{\xi}(x, t) &= -0.20_{(0.02)}, \end{aligned}$$

TABLE 2. Summary of the models fitted to the Swiss temperature data. Standard errors are in parentheses and (\*) denotes that the parameter was held fixed.  $h_-$  and  $h_+$  are, respectively, the distances for which  $\theta(h)$  is equal to 1.3 and 1.7. DoF denotes the degrees of freedom when appropriate,  $\ell$  is the maximized log-likelihood and  $\ell_p$  the maximized pairwise log-likelihood. TIC is an analogue of the Akaike information criterion designed for misspecified models.

Gaussian copula							
Correlation		$\lambda$ (km)	$\kappa$	$h_-$ (km)	$h_+$ (km)	$\ell$	AIC
Whittle–Matèrn		57 (17)	0.49 (0.13)	—	—	-1295.6	2605
Stable		61 ( 9)	0.81 (0.14)	—	—	-1294.9	2604
Stable		56 ( 6)	1.00 ( * )	—	—	-1295.7	2603

Student copula							
Correlation	DoF	$\lambda$ (km)	$\kappa$	$h_-$ (km)	$h_+$ (km)	$\ell$	AIC
Whittle–Matèrn	36 (34)	49 (15)	0.54 (0.15)	0.55	3.54	-1294.8	2606
Powered Exponential	54 (72)	58 ( 9)	0.86 (0.17)	0.14	1.36	-1294.5	2605
Powered Exponential	38 (35)	53 ( 6)	1.00 ( * )	0.41	3.02	-1294.8	2604

Extremal- $t$							
Correlation	DoF	$\lambda$ (km)	$\kappa$	$h_-$ (km)	$h_+$ (km)	$\ell_p$	TIC
Whittle–Matèrn	8.5 (3.7)	3793 (7507)	0.18 (0.04)	0.36	70	-19482.7	39338
Powered Exponential	7.7 (2.9)	1296 (2366)	0.44 (0.11)	0.66	71	-19482.1	39337
Powered Exponential	6.9 (1.7)	734 ( 618)	0.50 ( * )	1.21	73	-19482.4	39333

Brown–Resnick							
Variogram		$\lambda$ (km)	$\kappa$	$h_-$ (km)	$h_+$ (km)	$\ell_p$	TIC
Fractional		6.7 (5.5)	0.34 (0.08)	0.19	62	-19486.8	39357
Brownian		33.6 (7.4)	1.00 ( * )	9.9	72	-19539.8	39447

where the standard errors are displayed as subscripts. The effect of elevation is physically plausible since it is known that temperature decreases by an amount of around 7°C for each kilometer of climb. The estimated temporal trend leads to an increase of about 2.5°C per century and is consistent with the values given by the Intergovernmental Panel on Climate Change in their 2007 Fourth Assessment Report (<http://www.ipcc.ch>).

The estimates of the spatial dependence parameters for various models is presented in Table 2. The Gaussian copula model gives a practical range, i.e., the distance at which the correlation function equals 0.05, around 170 km but the extremal practical range  $h_+$ , i.e., the distance at which the extremal coefficient function equals 1.7 [4], does not exist since its corresponding extreme value copula is the independence copula—see (5). Although the Student copula is in the domain of attraction of the extremal- $t$  copula, the Student copula model gives similar results to the Gaussian one since the estimated degrees of freedom is large and gives an extremal practical range  $h_+$  of around 3km which seems to be largely underestimated as typically heat waves impact much larger areas. The max-stable models, i.e., the Hüsler–Reiss and the extremal- $t$  models, give consistent and much more plausible estimates for the extremal practical range  $h_+$ . Possibly due to the strong non orthogonality of its dependence parameters, the extremal- $t$  model has unreasonably large standard errors; the Brown–Resnick model seems to be less impacted since it has a fewer number of parameters.

Following the lines of [5], Table 3 shows estimates of the probabilities that the temperatures observed in year 2003, i.e., during the 2003 European heat wave, would be exceeded in the years 2003, 2010, 2020 and 2050—under the model of linear trend in time. These estimates

TABLE 3. Frequencies (%) of the simulated annual maxima exceeding those observed for the year 2003 at  $k$  stations,  $k = 0, \dots, 14$ , for the Gaussian copula, the Student copula and the extremal- $t$  models, for the years 2003, 2010, 2020 and 2050. Frequencies smaller than 0.01% are omitted for clarity.

		Gaussian copula														
Year	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2003	44.3	31.2	14.6	5.9	2.6	1.0	0.3	0.1								
2010	39.1	32.1	16.2	7.3	3.3	1.3	0.5	0.2								
2020	32.7	31.7	18.3	9.5	4.7	1.9	0.8	0.3	0.1	0.1						
2050	16.2	25.4	22.0	15.7	9.9	5.5	3.2	1.4	0.5	0.3	0.1					

		Student copula														
Year	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2003	56.8	24.0	9.5	3.7	2.0	1.3	0.8	0.5	0.5	0.3	0.3	0.1	0.1	0.1		
2010	52.5	25.8	10.4	4.4	2.2	1.6	1.0	0.6	0.6	0.3	0.3	0.2	0.1	0.1		
2020	45.8	28.0	12.1	5.7	2.5	1.9	1.3	0.8	0.7	0.4	0.5	0.2	0.1	0.1		
2050	26.0	30.8	18.2	9.2	5.1	3.1	2.4	1.6	1.1	0.8	0.7	0.4	0.3	0.2	0.1	

		Extremal- $t$														
Year	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2003	42.3	28.3	13.9	6.7	3.3	2.2	1.1	0.7	0.4	0.5	0.2	0.1	0.2	0.1	0.1	0.1
2010	37.1	28.6	15.6	7.8	4.0	2.7	1.4	0.9	0.6	0.4	0.3	0.2	0.2	0.1	0.1	0.1
2020	30.7	27.9	17.4	9.6	5.2	3.1	2.1	1.5	0.7	0.6	0.4	0.3	0.3	0.2	0.1	0.1
2050	13.1	21.3	20.7	14.7	9.2	6.8	4.3	3.1	2.1	1.5	0.9	0.8	0.7	0.5	0.2	0.2

were obtained from 10000 independent realizations from each model. Results corroborate the ones displayed in Table 2 since the Gaussian copula model shows the weaker spatial dependence structure, followed by the Student copula. The only max-stable model, the extremal- $t$  model, has the strongest spatial dependence structure and is the only one that gives a positive probability that the 2003 temperatures were exceeded for all available weather stations.

Figure 2 shows one realization from each fitted model using the trend surfaces (13)–(15) and extrapolated to the year 2020. These realizations were obtained by taking the 0.99 sample quantile of the temperature average over Switzerland from 10000 independent realizations from each model. Although the estimated trend surfaces are similar for each model, the distribution of the overall average temperature differs appreciably from one model to another—see the top panel of Figure 2. Due to a stronger spatial dependence structure, the extremal- $t$  model shows the largest variability. The 0.99 sample quantiles for the temperature average over Switzerland are respectively 29.1°C, 29.2°C and 30.6°C for the Gaussian copula, Student copula and extremal- $t$  models. Although the differences across the different models appear to be limited, an increase of around 1.5°C might have a considerable impact on the survival of species and the model driving the spatial dependence should be considered with care.

## 5. Discussion

In this paper we tried to make connections between copulas and the extreme value theory. The modelling of multivariate extremes was known to be a difficult task due to the unavailability of flexible yet parsimonious parametric extreme value copulas. The last decade has seen many advances towards a geostatistic of extremes using max-stable processes. Although the connection between stochastic processes and copulas seems odd at first glance, it is straightforward to extend

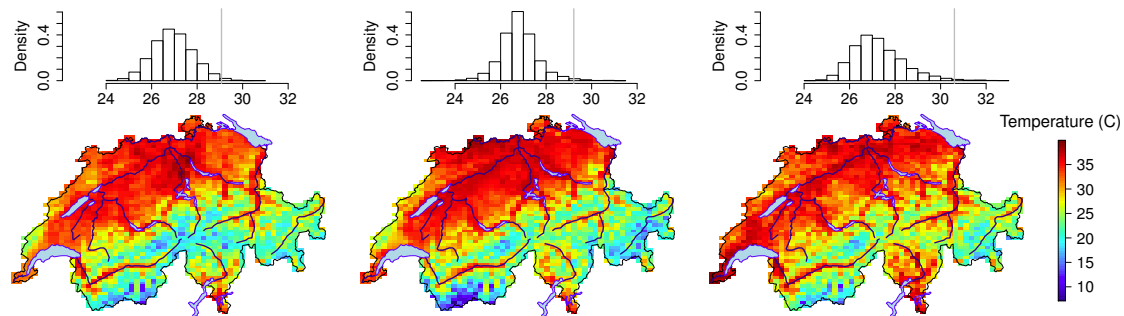


FIGURE 2. One realization from the fitted Gaussian copula, Student copula and extremal- $t$  models. Each simulation corresponds to a situation where the mean temperature over Switzerland for the simulated field is expected to be exceeded once every 100 year—for the year 2020. The top row shows histograms of the mean temperature values obtained from 10000 simulations from each model. The vertical lines denotes the mean temperature corresponding to the simulated field.

suitable copulas to stochastic processes and we make the connection between some well-known extreme value copulas and their spectral characterization. An application to the modelling of extreme temperature was given and we show that the choice of a non extreme value model might underestimate the spatial dependence.

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