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Some expansions in vector space

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Some expansions in vector space:

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1. Introduction. — The general theory of linear vector spaces has been studied by several authors (2). The present paper is mainly concerned with the study of the properties of two expansions B(T) and D(T) in a special linear vector space S, closed under multiplication by numbers of a real or complex system A, and having additional properties abstracted from those of a space of linear transformations. The functions B(T) and D(T) are on S to S and S to A respectively. D(T) is analogous to the Fredholm determinant and B(T) to the first Fredholm minor. A simple derivation of some results on generalized rotations is incidentally given in Section 5.

2. Postulates and Definitions. — Let A denote the system of real or the system of complex numbers.

We consider a space S consisting of a set {T, U, ...} of elements T, U, ... and five opérations

$$(\oplus),\quad (\odot),\quad (\otimes),\quad \|\ldots\|,\quad |\ldots|$$

satisfying the following postulates.

1. The set {T, U, ...} forms a complete linear vector space with

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(2) *See*, for example, S. Banach, *Fundamenta Mathematicae*, vol. 3, 1922, p. 133-181, and M. Fréchet, *Espaces abstraits*, Paris, 1928. *See* also *Fréchet's fundamental papers on abstract spaces in, Rend. Circ. Mat. Palermo*, vol. 22, 1906, p. 1-74, and in *Journal de Mathématiques*, 1929, p. 71-92.

respect to addition, $\oplus$, multiplication, $\odot$, by numbers $a, b, \ldots$ of A, and the norm $\| \ldots \|$.

II. The set $\{T, U, \ldots\}$ forms a ring with respect to $\oplus$ and composition, $\otimes$, having a unique two-sided unit I, that is

$$I \otimes T = T \otimes I = T \qquad (\text{all } T).$$

III. $\|I\| = 1$.

IV. The product $T \otimes U$ is a bilinear operation of modulus unity on the space S^2 to S.

V. $[\ldots]$ is a homogeneous linear continuous operation on S to the number system A, having the property that

$$[T \otimes U] = [U \otimes T] \qquad (\text{all } T, U).$$

That the five postulates are consistent is shown by taking S to be A, the operations

$$(\oplus), \quad (\odot), \quad (\otimes), \quad \| \ldots \|,$$

to be ordinary addition, multiplication, and modulus respectively, and $[a]$ to be a.

With A the real number system and with the following interpretation of the elements and operations, the set of ordered pairs of functions $\left(T(x, y), T(x)\right)$ form a non-trivial instance of a space S. Let

$$T = \left(T(x, y), T(x)\right), \qquad U = \left(U(x, y), U(x)\right),$$
$$I = (0, 1),$$

where $T(x, y)$ and $U(x, y)$ are continuous on the square $a \leqq x, y \leqq b$, and $T(x)$ and $U(x)$ are continuous on the interval (a, b). Take

$$T \oplus U = \left(T(x, y) + U(x, y), T(x) + U(x)\right),$$

$$a \odot T = \left(a\,T(x, y), a\,T(x)\right),$$

$$T \otimes U = \left(\int_a^b T(x, z) U(z, y)\, dz + T(x) U(x, y) + U(y) T(x, y), T(x) U(x)\right),$$

$$\|T\| = (b - a) \max_{x, y} |T(x, y)| + \max_x |T(x)|,$$

$$[T] = \int_a^b T(x, x)\, dx.$$

We may also take

$$|T| = cT(x_1),$$

where c is a constant and x_1 is a fixed number in (a, b).

Unless ambiguity arises we shall write $\oplus$ as simply $+$ and omit to write $\odot$ and $\otimes$. As immediate consequences of postulate (4) we have

$$(2.1) \qquad (a \odot T) \otimes U = T \otimes (a \odot U) = a \odot (T \otimes U)$$

and

$$(2.2) \qquad \| T \otimes U \| \leq \| T \| \cdot \| U \|.$$

For purposes of exposition it is convenient to make several definitions, some of whose further implications have been studied elsewhere [1].

Definition I. — By a polynomial $p(x)$ of degree n on a vector space $V_1(A)$ to a vector space $V_2(A)$ we shall mean a continuous function $p(x)$ such that $p(x + \lambda y)$ is a polynomial of degree n in λ of the number system A, with coefficients in $V_2(A)$. If further

$$p(\lambda x) = \lambda^n p(x),$$

$p(x)$ is said to be homogeneous of degree n. It is convenient to consider a null polynomial as having any degree whatever.

Definition II. — By the modulus of a homogeneous polynomial of degree n we shall mean

$$\max_x \frac{\| p(x) \|}{\| x \|^n}$$

and shall denote it by mp.

Definition III. — We shall say that a function $f(x)$ on $V_1(A)$ to $V_2(A)$ is analytic at a point x_0 if there exists a sequence $\{h_n(x)\}$ of homogeneous polynomials $\big(h_n(x)$ of degree $n\big)$ such that $\Sigma \lambda^n mh_n$ has a positive radius of convergence r, and such that for $\|x - x_0\| < r$, $\Sigma h_n(x - x_0)$ converges (in the norm) to $f(x)$. We shall call r the

(1) R. S. MARTIN, *California Institute thesis*, 1932.

radius of analyticity of $f(x)$ at x_0 and shall often refer to such an $f(x)$ as analytic (r) at $x = x_0$.

3. The Polynomials $B_n(T)$ and $a_n(T)$. — Let T be in S. Let $a_0 = 1$ and $B_0 = 0$ where 0 is the zero element of S. Define a sequence of numbers $\{a_n(T)\}$ and a sequence of elements $\{B_n(T)\}$ of S by means of the recurrence relations

$$B_n(T) = a_{n-1}(T)T - B_{n-1}(T)T, \tag{3.1}$$

$$a_n(T) = \frac{1}{n}[B_n(T)]. \tag{3.2}$$

We have

$$B_n(T) = \sum_{r=1}^{n} (-1)^{r-1} a_{n-r}(T) T^r. \tag{3.3}$$

$$n\, a_n(T) = \sum_{r=1}^{n} (-1)^{r-1} a_{n-r}(T) [T^r]. \tag{3.4}$$

Solving (3.4) (whose determinant is $n\,!$) we have

$$a_n(T) = \frac{1}{n\,!} \begin{vmatrix} [T] & [T^2] & \dots & . & [T^n] \\ n-1 & [T] & \dots & . & [T^{n-1}] \\ 0 & n-2 & \dots & . & [T^{n-2}] \\ . & \dots\dots & \dots & . & \dots\dots \\ 0 & 0 & \dots & 1 & [T] \end{vmatrix} \qquad (n > 0). \tag{3.5}$$

Furthermore

$$B_n(T) = \frac{1}{(n-1)\,!} \begin{vmatrix} T & T^2 & \dots & T^{n-1} & T^n \\ n-1 & [T] & \dots & [T^{n-2}] & [T^{n-1}] \\ 0 & n-2 & \dots & [T^{n-3}] & [T^{n-2}] \\ . & \dots\dots & \dots & \dots\dots & \dots\dots \\ 0 & 0 & \dots & 1 & [T] \end{vmatrix} \qquad (n > 0), \tag{3.6}$$

as can be seen by expanding (3.6) in its first row and using (3.3) and (3.5).

Theorem 3.I. — $B_n(T)$ *is a homogeneous polynomial of degree n on the space* S *to the space* S. *The modulus of* $B_n(T)$ *satisfies the inequality*

$$m\, B_n \leq \frac{(\gamma + 1)(\gamma + 2)\dots(\gamma + n - 1)}{(n-1)\,!} \qquad (n > 0). \tag{3.7}$$

where γ is the modulus of the operation $[\ldots]$. Similarly $a_n(T)$ is a homogeneous polynomial of degree n on S to A and

$$(3.8)\qquad m\, a_n \leqq \frac{\gamma(\gamma+1)(\gamma+2)\ldots(\gamma+n-1)}{n!}\qquad (n>0).$$

Proof : The polynomial property of $a_n(T)$ and $B_n(T)$ is evident for $n=1$, for then $a_1(T)=[T]$ and $B_1(T)=T$. Assuming the theorem true for $n-1$, the continuity of $B_n(T)$ follows from the inequality

$$\begin{aligned}\|B_n(T)-B_n(U)\| \leqq & \|a_{n-1}(T)T-a_{n-1}(U)T\|+\|a_{n-1}(U)T-a_{n-1}(U)U\| \\ & +\|B_{n-1}(U)T-B_{n-1}(T)T\|+\|B_{n-1}(U)U-B_{n-1}(U)T\| \\ \leqq & \big(|a_{n-1}(T)-a_{n-1}(U)|+\|B_{n-1}(T)-B_{n-1}(U)\|\big)\|T\| \\ & +\big(|a_{n-1}(U)|+\|B_{n-1}(U)\|\big)\|T-U\|\end{aligned}$$

and that of $a_n(T)$ from the fact that $[\ldots]$ is a linear continuous operation.

Observing that $a_{n-1}(T+\lambda U)$ and $B_{n-1}(T+\lambda U)$ are polynomials in λ on A to A and A to S respectively we see at once from (3.1) that $B_n(T+\lambda U)$ is a polynomial in λ on A to S, say $\Sigma\lambda^r K_r$. Then

$$a_n(T+\lambda U)=\frac{1}{n}\Big[\sum \lambda^r K_r\Big]=\sum \lambda^r\Big[\frac{K_r}{n}\Big]$$

is a polynomial in λ on A to A. The homogeneity is obvious from (3.1) and (3.2).

The proof of the inequalities (3.7) and (3.8) is by induction and the use of the postulates.

4. The Expansions B(T) and D(T). — Theorem 4.1 :

$$B(T)\equiv\sum_{1}^{\infty} B_n(T)\qquad \textit{and}\qquad D(T)\equiv\sum_{0}^{\infty} a_n(T)$$

are functions on S to S and S to A respectively analytic at $T=0$. Their radii of analyticity are not less than unity.

Proof :

$$\Sigma\lambda^n m B_n \quad\text{and}\quad \Sigma\lambda^n m a_n$$

are dominated respectively by

$$\sum_{1}^{\infty} \lambda^n \frac{(\gamma+1)\ldots(\gamma+n-1)}{(n-1)!} = \lambda(1-\lambda)^{-(\gamma+1)} \qquad (|\lambda|<1)$$

and

$$\sum_{0}^{\infty} \lambda^n \frac{\gamma(\gamma+1)\ldots(\gamma+n-1)}{n!} = (1-\lambda)^{-\gamma} \qquad (|\lambda|<1).$$

THEOREM 4.II. — *If* $|\lambda|<1$ *then*

$$\text{(4.1)} \qquad D(\lambda I) = (1+\lambda)^t$$

and

$$\text{(4.2)} \qquad B(\lambda I) = \lambda(1+\lambda)^{t-1} I$$

where

$$t = [I].$$

Proof : From the homogeneity of $a_n(T)$ we have

$$a_n(\lambda I) = \lambda^n a_n(I)$$

and from (3.4)

$$\text{(4.3)} \qquad n\, a_n(I) = t \sum_{r=1}^{n} (-1)^{r-1} a_{n-r}(I).$$

The recursion formula (4.3) together with the condition $a_0 = 1$ determine the successive coefficients b_n in the expansion of $(1+\lambda)^t$, for differentiating

$$\text{(4.4)} \qquad (1+\lambda)^t = \Sigma b_n \lambda^n$$

we obtain

$$\sum n b_n \lambda^{n-1} = t(1+\lambda)^{t-1} = t\left(\sum b_n \lambda^n\right)\frac{1}{1+\lambda} = t\left(\sum b_n \lambda^n\right)\sum (-1)^r \lambda^r.$$

Equating coefficients of λ^n we see that $b_0 = 1$ and that b_n satisfies the recursion formulæ (4.3) for $a_n(I)$. Hence

$$b_n = a_n(I).$$

Furthermore from (3.3) we see that

$$B_n(I) = \beta_n I$$

where

$$\beta_n = \sum_{r=1}^{n} (-1)^{r-1} a_{n-r}(I) \qquad (\beta_0 = 0).$$

Since $\beta_0 = 0$ and since

$$\lambda(1+\lambda)^{l-1} = \left(\sum_0^\infty a_n(I)\lambda^n\right)\left(\sum_0^\infty (-1)^r \lambda^{r+1}\right)$$

it follows that β_n is precisely the coefficient of λ^n in the expansion of

$$\lambda(1+\lambda)^{l-1}.$$

COROLLARY 1. — *There exist spaces* S *in which the radii of analyticity about* T = 0 *of* B(T) *and* D(T) *are unity.*

By a proper choice of the constant c in the instance of a space S given in section (2) we may secure that l is not a positive integer or zero.

Since $\|I\| = 1$ we have that

$$m\, a_n \geq |a_n(I)|$$

and

$$m\, B_n \geq |\beta_n|.$$

Hence the radii of convergence of the power series $\Sigma m a_n \lambda^n$ and $\Sigma m B_n \lambda^n$ are respectively at most equal to the radii of convergence of $\Sigma |a_n(I)| \lambda^n$ and $\Sigma |\beta_n| \lambda^n$. The latter two series, however, have the same radius of convergence as $\Sigma a_n(I)\lambda^n$ and $\Sigma \beta_n \lambda^n$, namely unity. Combining this result with Theorem 4.I. we have the corollary.

COROLLARY 2. — *If the real part of* [I] *is positive then* D(—I) = 0.

We have

$$a_n(-I) = (-1)^n b_n$$

where b_n is given by (4.4).

If the real part of l is positive then the series $\Sigma(-1)^n b_n$ is known to converge. Hence using Abel's theorem on convergence up to the unit circle we have

$$\Sigma a_n(-I) = \lim_{\lambda \to -1-0} \Sigma b_n \lambda^n = \lim_{\lambda \to -1-0} (1+\lambda)^l = 0.$$

COROLLARY 3. — *If the real part of* $[I]$ *is negative then the series* $\Sigma a_n(-I)$ *representing* $D(-I)$ *is divergent.*

COROLLARY 4. — *If* $[I] \equiv 0$, *then* $D(-I) = 1$.

We observe that for fixed T, $B(\lambda T)$ and $D(\lambda T)$ are analytic functions of λ near $\lambda = 0$ on A to S and A to A respectively and that their radii of analyticity are not less than $\frac{1}{\|T\|}$.

THEOREM 4.III. — *If* $\{E_n\}$ *is a sequence of elements of* S *such that* $E \equiv \Sigma E_n$ *converges in the norm, then* ΣTE_n *and* $\Sigma E_n T$ *converge in the norm and are equal to* TE *and* ET *respectively.*

Proof: Let

$$C_m = \sum^m E_n,$$

then

$$\left\| \sum^m TE_n - TE \right\| = \| T(C_m - E) \| \leq \| T \| . \| C_m - E \|.$$

Since by hypothesis $\| C_m - E \|$ approaches zero with $\frac{1}{m}$, the first part of the theorem is proved. The proof of the second part is similar.

THEOREM 4.IV. — *If* $E(T) = \Sigma E_n(T)$, *where* $E_n(T)$ *is a homogeneous polynomial of degree n, is a function on* S *to* S *analytic at* $T = 0$, *then* $TE(T)$ *and* $E(T)T$ *are analytic on* S *to* S *and their radii of analyticity are at least as large as that of* $E(T)$.

Proof: That $TE_n(T)$ is a homogeneous polynomial of degree $n+1$ is shown by a similar argument to that employed in the proof of Theorem 3.I From the inequality

$$\frac{\| TE_n(T) \|}{\| T \|^{n-1}} \leq \frac{\| T \|}{\| T \|} \frac{\| E_n(T) \|}{\| T \|^n} \leq m E_n$$

and Theorem 4.III the rest of Theorem 4.IV follows The proof is similar for $E(T)T$.

5. INVERSES AND ROTATIONS. — In this section we shall not make use

of postulate 5 for the space S. Hence throughout this section S will denote a space that satisfies the first four postulates.

Definition. — By a left-hand (right hand) inverse to an element T of S we shall understand an element T_1 such that

$$T_1 \otimes T = I \qquad (T \otimes T_1 = I).$$

A two-sided inverse T^{-1} we shall call an inverse of T.

THEOREM 5.I. — *If* T *of* S *has at least one right hand inverse and at least one left hand inverse then it has unique right and left hand inverses and they are identical.*

Proof : Let T_1 be any left hand inverse and T_2 any right hand inverse of T. From

$$T \otimes T_2 = I,$$

we have

$$T_2 = I \otimes T_2 = T_1 \otimes T \otimes T_2 = T_1 \otimes I = T_1.$$

COROLLARY 1. — *If* T *has an inverse, then it is unique.*

COROLLARY 2. — *If* $T_1, T_2, \ldots, T_n$ *of* S *have inverses, then*

$$T_1 \otimes T_2 \otimes \ldots \otimes T_n.$$

has an inverse, namely

$$T_n^{-1} \otimes T_{n-1}^{-1} \otimes \ldots \otimes T_1^{-1}.$$

COROLLARY 3. — *If* T *has an inverse* T^{-1} *then* T^{-1} *has an inverse, namely* T.

THEOREM 5.II. — *If* $\|T\| < 1$, *then* $I + T$ *has an (unique) inverse, namely*

$$I + \sum_{n=1}^{\infty} (-1)^n T^n.$$

Proof : By successive applications of (2.2) we have

$$\|(-1)^n T^n\| \leqq \|T\|^n.$$

It follows from the completeness of S that $\sum_{n=1}^{\infty}(-1)^n T^n$ is in S. By an evident calculation and the use of Theorem 4.III, it is seen that

$$1+\sum_{n=1}^{\infty}(-1)^n T^n$$

is an inverse of $1+T$, which by Corollary 1 of the preceding theorem is unique.

Suppose now there exist an operation $'$, wich we shall call transposition, with the following three properties:

(a) $'T$ is a linear operation on S to S.

(b) $$'T'U = '(UT).$$

(c) $$'('T) = T.$$

It follows immediately from the properties (b) and (c) that $'1 = 1$. From (b) and this result it follows that if T has an inverse then the transposed of T has an inverse given by the transposed of the inverse of T.

Definition. — If $1-\lambda T$ has an inverse $1+\lambda\Gamma(\lambda)$, then $\Gamma(\lambda)$ will be called the resolvent of λT.

THEOREM 5.III. — *Suppose $1-\lambda T$ and $1-(\lambda+\mu)T$ have inverses, then if $\Gamma(\lambda)$ is the resolvent of λT it follows that $\Gamma(\lambda+\mu)$ is the resolvent of $\mu\Gamma(\lambda)$.*

Proof: By hypothesis

$$(1-\lambda T)\big(1+\lambda\Gamma(\lambda)\big)=1.$$

From this relation we obtain

$$(1-\lambda T)\Gamma(\lambda)=T. \tag{5.1}$$

Consider an element J defined by

$$J=\big(1-\mu\Gamma(\lambda)\big)\big(1+\mu\Gamma(\lambda+\mu)\big). \tag{5.2}$$

On applying (5.1) twice successively to $(1-\lambda T)J$ we obtain

$$(1-\lambda T)J=\big(1-(\lambda+\mu)T\big)\big(1+\mu\Gamma(\lambda+\mu)\big)=1-\lambda T.$$

Since $I - \lambda T$ has an inverse it follows that $J = I$ and hence the theorem is proved.

Definition of a rotation. — If $I + T$ has an inverse given by $I + {}'T$ then T will be called a rotation.

THEOREM 5.IV. — *If* T *is a rotation and if* $(\lambda - 1)T$ *has a resolvent* $\Gamma(\lambda - 1)$ *then* $-\lambda T$ *has a resolvent and*

$$\Gamma(\lambda - 1) = - {}'\Gamma(-\lambda). \tag{5.3}$$

Proof : Since T is a rotation

$${}'T = - \Gamma(-1). \tag{5.4}$$

By hypothesis and the previous theorem we have

$$\left(I - \lambda \Gamma(\lambda - 1)\right)(I + \lambda {}'T) = I. \tag{5.5}$$

Applying the properties of transposition, the transposed of (5.5) becomes

$$(I + \lambda T)\left(I + \lambda {}'\Gamma(\lambda - 1)\right) = I. \tag{5.6}$$

Hence the inverse of $I - (-\lambda)T$ exists and we have

$$\left(I - (-\lambda)T\right)\left(I - \lambda \Gamma(-\lambda)\right) = I. \tag{5.7}$$

By Corollary I to Theorem 5.I

$${}'\Gamma(\lambda - 1) = - \Gamma(-\lambda).$$

From this result (5.3) follows immediately.

Corollary. — *If* T *is a rotation and* $I + \frac{1}{2} T$ *has an inverse then*

$$H = \Gamma\left(-\frac{1}{2}\right)$$

is defined and

$$H = - {}'H.$$

THEOREM 5.V. — *If* T *is a rotation such that* $I + \frac{1}{2} T$ *has an inverse, then* T *is the résolvent of* $\frac{1}{2} H$ *where* H *satisfies the relation*

$${}'H = - H.$$

Proof : Take $\lambda = \frac{1}{2}$ in (5.7) and apply the previous corollary.

THEOREM 5.VI. — *If* $I - \frac{1}{2}T$ *has an inverse and if*

$$'T = -T \tag{5.8}$$

then the resolvent $\Lambda\left(\frac{1}{2}\right)$ *of* $\frac{1}{2}T$ *is a rotation.*

Proof : By hypothesis we have

$$\left(I + \frac{1}{2}\Lambda\left(\frac{1}{2}\right)\right)\left(I - \frac{1}{2}T\right) = I. \tag{5.9}$$

Transposing and making use of (5.8) we obtain

$$\left(I + \frac{1}{2}T\right)\left(I + \frac{1}{2}\,'\Lambda\left(\frac{1}{2}\right)\right) = I. \tag{5.10}$$

Since by hypothesis $I - \frac{1}{2}T$ has an inverse and since $I + \frac{1}{2}T$ is the transposed of $I - \frac{1}{2}T$ it follows that $I + \frac{1}{2}T$ has an inverse, namely,

$$I - \frac{1}{2}\Lambda\left(-\frac{1}{2}\right).$$

Hence from (5.10) we have

$$'\Lambda\left(\frac{1}{2}\right) = -\Lambda\left(-\frac{1}{2}\right). \tag{5.11}$$

Since $\Lambda\left(\frac{1}{2}\right)$ and $\Lambda\left(-\frac{1}{2}\right)$ exist we can apply Theorem 5.III and obtain

$$\left(I - \Lambda\left(\frac{1}{2}\right)\right)\left(I - \Lambda\left(-\frac{1}{2}\right)\right) = I.$$

With the aid of (5.11) we can write this relation in the form

$$\left(I + \Lambda\left(\frac{1}{2}\right)\right)\left(I - \,'\Lambda\left(\frac{1}{2}\right)\right) = I.$$

This proves the theorem.

6. A Lemma on Sequences of Real Numbers Defined by Recurrent Inequalities. — The results of the following lemma are necessary for the development of the succeeding sections.

Lemma. — *Let* $\{r_n\}$ *be a sequence of non-negative real numbers such that* Σr_n *converges. Let two sequences of non-negative real numbers* $\{e_n\}$ *and* $\{f_n\}$ *satisfy the recurrent inequalities*

$$(6.1) \qquad n e_n \geqq \mu\{\lambda(e_{n-1} + f_{n-1}) + r_{n-1}\},$$

$$(6.2) \qquad f_n \geqq \lambda(e_{n-1} + f_{n-1}) + r_{n-1},$$

$$e_0 = f_0 = 0,$$

where $\mu \geqq 0$. *Then if* $0 \leqq \lambda \leqq 1$, $\Sigma e_n x^n$ *and* $\Sigma f_n x^n$ *have radii of convergence at least unity. If furthermore* $0 \leqq \lambda < 1$, Σe_n *and* Σf_n *converge and their sums tend to zero with* Σr_n.

Proof : Since e_n, f_n, r_n are non-negative it is clearly sufficient to prove the lemma for the case where the inequalities (6.1) and (6.2) are replaced by the corresponding equalities.

Assume first that $\mu > 0$, $0 \leqq \lambda \leqq 1$. From the hypotheses on $\{r_n\}$ it is clear that $\Sigma r_n x^n$ converge uniformly and absolutely for $|x| \leqq 1$ and hence definies an analytic function $R(x)$ of x in the interior of the unit circle.

Consider the differential equation

$$(6.3) \qquad u'(x) = \frac{\mu}{1 - \lambda x}\left(\lambda u(x) + R(x)\right).$$

Let $E(x) \equiv \Sigma E_n x^n$ be the unique analytic solution satisfying the initial condition $E(0) = 0$. Let

$$(6.4) \qquad F(x) = \sum F_n x^n = x \frac{E'(x)}{\mu}.$$

We have at once

$$(6.5) \qquad \frac{E'(x)}{\mu} = \lambda\big(E(x) + F(x)\big) + R(x),$$

$$(6.6) \qquad \frac{F(x)}{x} = \lambda\big(E(x) + F(x)\big) + R(x).$$

Equating the coefficients of x^{n-1} and observing that $F(0) = E(0) = 0$

we have

$$(6.7)\qquad \begin{cases} nE_n = \mu\{\lambda(E_{n-1} + F_{n-1}) + r_{n-1}\}, \\ F_n = \lambda(E_{n-1} + F_{n-1}) + r_{n-1}, \\ E_0 = F_0 = 0. \end{cases}$$

Since the coefficients of differential equation (6.3) are analytic within the unit circle it follows that the solution $E(x)$, and hence $F(x)$, is analytic within the unit circle. For $|x| < 1$ we may write the solution $E(x)$ in the form

$$(6.8)\qquad E(x) = (1-\lambda x)^{-\mu} \int_0^x (1-\lambda t)^{\mu-1} R(t)\, dt.$$

Furthermore for $0 \leq \lambda < 1$ the sums of the coefficients of x^n in the expansions of $(1-\lambda x)^{-\mu}$ and $(1-\lambda x)^{\mu-1}$ are absolutely convergent. Since the result of Cauchy multiplication of two power series the sums of whose coefficients are absolutely convergent is a power series having the same property, and since the property is preserved under term by term integration, it follows from (6.8) that ΣE_n is convergent.

On differentiating (6.8) and applying a similar argument we see that ΣF_n is convergent.

Applying Abel's theorem on continuity on the circle of convergence we have that $R(x)$ and $E(x)$ are continuous on the left at $x = 1$.

Hence

$$\sum E_n = (1-\lambda)^{-\mu} \int_0^1 (1-\lambda t)^{\mu-1} R(t)\, dt.$$

Since

$$R(t) \leq R(1) \qquad (0 \leq t \leq 1),$$
$$\Sigma E_n \leq M(\lambda, \mu) R(1) = M(\lambda, \mu) \Sigma r_n.$$

By a similar argument we prove that

$$\Sigma F_n \leq N(\lambda, \mu) \Sigma r_n.$$

Assume now that $\mu = 0$. The reccurence relations (6.7) become

$$(6.9)\qquad \begin{cases} E_n = 0, \\ F_n = \lambda F_{n-1} + r_{n-1}. \end{cases}$$

We simply observe that the coefficients in the power series expansion of

$$F(x) = \frac{x\,B(x)}{1-\lambda x}$$

are given by (6.9). The rest of the argument is similar to that for $\mu > 0$.

7. Fréchet Differentiability of B(T) and D(T). — If the number system A of a vector space $V_1(A)$ is the *complex number system* it can be shown by a general argument that a function $f(x) = \sum_0^\infty h_n(x)$ analytic (r) at $x = 0$ on $V_1(A)$ to a complete $V_2(A)$ has for $\|x\| < r$ a Fréchet differential (¹) $\delta f(x)$ given by $\sum_0^\infty \delta h_n(x)$ which, as a function of x, is also analytic (r) at $x = 0$. For the special analytic functions B(T) and D(T) it is possible to give a direct proof of their term by term differentiability for an *unrestricted number system* A.

Theorem 7.1. — *The functions* $D(T) \equiv \Sigma a_n(T)$ *and* $B(T) \equiv \Sigma B_n(T)$ *have Fréchet differentials* δD *and* δB *given for* $\|T\| < 1$ respectively by $\Sigma \delta a_n(T)$ *and* $\Sigma \delta B_n(T)$. *These last two series define analytic functions of* T *whose radii of analyticity are at least unity.*

Proof : By Theorem 3.1 and by a theorem on the differentials of polynomials proved elsewhere (²), it follows that $a_n T$ and $B_n(T)$ possess differentials for all T in S.

From the definition of a Fréchet differential we have that $\varepsilon_n(T, \delta T)$ and $\varepsilon'_n(T, \delta T)$ defined by

$$(7.1)\qquad \begin{cases} \|\delta T\|\,\varepsilon_n = \Delta a_n(T) - \delta a_n(T) & (\delta T \neq 0), \\ \varepsilon_n = 0 & (\delta T = 0); \end{cases}$$

$$(7.2)\qquad \begin{cases} \|\delta T\|\,\varepsilon'_n = \Delta B_n(T) - \delta B_n(T) & (\delta T \neq 0), \\ \varepsilon'_n = 0 & (\delta T = 0) \end{cases}$$

where

$$\Delta f(T) = f(T + \delta T) - f(T).$$

(¹) Fréchet, *La notion de différentielle dans l'Analyse générale* (*Ann. Éc. Norm. sup.*, t. XLII, 1925, p. 293-323).

(²) Martin, *loc. cit.*

are continuous functions of δT at $\delta T = 0$. Incorder to show Fréchet differentiability for $\|T\| < 1$ it is clearly sufficient to prove that for $\|T\| < 1$ $\Sigma \delta a_n(T)$ and $\Sigma \delta B_n(T)$ converge and that

$$\varepsilon(T, \delta T) = \Sigma \varepsilon_n(T, \delta T)$$

and

$$\varepsilon'(T, \delta T) = \Sigma \varepsilon'_n(T, \delta T)$$

converge for

$$\|T\| + \|\delta T\| < 1$$

and represent continuous functions of δT at $\delta T = 0$.

From the recurrence formulæ (3.1) and (3.2), and from the evident identity

$$\Delta\big(U(T)V(T)\big) = \big(\Delta U(T)\big)V(T) + U(T + \delta T)\Delta V(T),$$

we have

$$n\,\Delta a_n(T) = \big(\Delta a_{n-1}(T)\big)[T] + a_{n-1}(T + \delta T)[\delta T] + \Big[\big(\Delta B_{n-1}(T)\big)T + B_{n-1}(T + \delta T)\,\delta T\Big]. \tag{7.3}$$

$$\Delta B_n(T) = \big(\Delta a_{n-1}(T)\big)T + a_{n-1}(T + \delta T)\,\delta T + \big(\Delta B_{n-1}(T)\big)T + B_{n-1}(T + \delta T)\,\delta T. \tag{7.4}$$

Taking norms of (7.3) and (7.4) we obtain

$$n|\Delta a_n(T)| \leqq \gamma\Big\{\big(|\Delta a_{n-1}(T)| + \|\Delta B_{n-1}(T)\|\big)\|T\| + \big(|a_{n-1}(T + \delta T)| + \|B_{n-1}(T + \delta T)\|\big)\|\delta T\|\Big\},$$

$$\|\Delta B_n(T)\| \leqq \big(|\Delta a_{n-1}(T)| + \|\Delta B_{n-1}(T)\|\big)\|T\| + \big(|a_{n-1}(T + \delta T)| + \|B_{n-1}(T + \delta T)\|\big)\|\delta T\|,$$

where γ is the modulus of the operation $[\ldots]$. By Theorem 3.1 the quantities

$$r_n = \big(|a_n(T + \delta T)| + \|B_n(T + \delta T)\|\big)\|\delta T\|$$

are for $\|T\| + \|\delta T\| < 1$ the terms of an absolutely convergent series.

Hence from the preceding Lemma taking

$$e_n = |\Delta a_n(T)|, \qquad f_n = \|\Delta B_n(T)\|, \qquad \lambda = \|T\|, \qquad \mu = \gamma$$

and r_n as defined above we have that $\Sigma|\Delta a_n(T)|$ and $\Sigma\|\Delta B_n(T)\|$ converge and tend to zero with Σr_n and hence with $\|\delta T\|$.

Making use of the formulae, valid when a, U, V are differentiable

$$\delta\left(a \, (\widehat{\cdot}) \, U\right) = (\delta a)(\widehat{\cdot})\, U,$$
$$\delta\left(U \, (\widehat{\times}) \, V\right) = (\delta U)(\widehat{\times})\, V + U \,(\widehat{\times})\, \delta V,$$
$$\delta[U] = [\delta U]$$

we have again from (3.1) and (3.2)

$$(7.5) \qquad n\,\delta a_n(T) = \left(\delta a_{n-1}(T)\right)[T] + a_{n-1}(T)[\delta T]$$
$$\qquad + \left[\left(\delta B_{n-1}(T)\right)T + B_{n-1}(T)\,\delta T\right].$$
$$(7.6) \qquad \delta B_n(T) = \left(\delta a_{n-1}(T)\right)T + a_{n-1}(T)\,\delta T$$
$$\qquad + \left(\delta B_{n-1}(T)\right)T + B_{n-1}(T)\,\delta T.$$

Substracting (7.5) from (7.3) and (7.6) from (7.4) we obtain, using (7.1) and (7.2)

$$(7.7) \qquad n\|\delta T\|\varepsilon_n = \|\delta T\|\varepsilon_{n-1}[T] + \left(\Delta a_{n-1}(T)\right)[\delta T]$$
$$\qquad + \left[\|\delta T\|\varepsilon'_{n-1}T + \left(\Delta B_{n-1}(T)\right)\delta T\right]$$

and

$$(7.8) \quad \|\delta T\|\varepsilon'_n = \|\delta T\|\varepsilon_{n-1}T + \left(\Delta a_{n-1}(T)\right)\delta T + \|\delta T\|\varepsilon'_{n-1}T + \left(\Delta B_{n-1}(T)\right)\delta T.$$

Taking norms of (7.7) and (7.8) we obtain the inequalities

$$(7.9) \qquad n|\varepsilon_n| \leqq \gamma\left(|\varepsilon_{n-1}|\,\|T\| + |\Delta a_{n-1}| + \|\varepsilon'_{n-1}\|\,\|T\| + \|\Delta B_{n-1}\|\right),$$
$$(7.10) \qquad \|\varepsilon'_n\| \leqq |\varepsilon_{n-1}|\,\|T\| + |\Delta a_{n-1}| + \|\varepsilon'_{n-1}\|\,\|T\| + \|\Delta B_{n-1}\|.$$

Again Applying the Lemma to (7.9) and (7.10) and using what we have just proved for $\Sigma|\Delta a_n|$ and $\Sigma\|\Delta B_n\|$ we obtain the result that $\Sigma|\varepsilon_n|$ and $\Sigma\|\varepsilon'_n\|$ converge and tend to zero with $\delta\|T\|$.

Taking norms of (7.5) and (7.6), dividing by $\|T\|^{n-1}$ and taking the maximum of both sides quà T we have

$$(7.11) \quad n(m\,\delta a_n) \leqq \gamma\{(m\,\delta a_{n-1}) + (m\,\delta B_{n-1}) + (m a_{n-1} + m B_{n-1})\|\delta T\|\},$$
$$(7.12) \quad (m\,\delta B_n) \leqq (m\,\delta a_{n-1}) + (m\,\delta B_{n-1}) + (m a_{n-1} + m B_{n-1})\|\delta T\|.$$

Let τ_1 be a positive number less than unity. Multiply both sides of (7.11), (7.12) by τ_1^{n-1} and apply the Lemma taking

$$e_n = \tau_1^{n-1}\, m\,\delta a_n, \qquad f_n = \tau_1^{n-1}\, m\,\delta B_n, \qquad \lambda = \tau_1, \qquad \mu = \gamma$$

and

$$r_n = r_1^{n-1}(m a_{n-1} + m B_{n-1}) \| \delta T \|.$$

Since by Theorem 4.I, Σr_n converges it follows at once that

$$\Sigma r_1^{n-1}(m\, \delta a_n)$$

and

$$\Sigma r_1^{n-1}(m\, \delta B_n)$$

have radii of convergence not less than unity. Thus $\Sigma \delta a_n$ and $\Sigma \delta B_n$ represent functions analytic $(r \geq 1)$ at $T = 0$.

Finally from the convergence of $\Sigma | \varepsilon_n |$ and $\Sigma \| \varepsilon'_n \|$, and the completeness of the space S we have that $\Sigma \varepsilon_n$ and $\Sigma \varepsilon'_n$ converge in the norm, and that $| \Sigma \varepsilon_n |$ and $\| \Sigma \varepsilon'_n \|$ tend to zero with $\| \delta T \|$. Summing (7.1) and (7.2) we have for $\| T \| + \| \delta T \| < 1$

$$\Delta D(T) - \Sigma \delta a_n(T) = \| \delta T \| \, \varepsilon(T, \delta T),$$
$$\Delta B(T) - \Sigma \delta B_n(T) = \| \delta T \| \, \varepsilon'(T, \delta T).$$

This completes the proof of the theorem.

8. The inverse of I + T for $\| T \| < 1$. Theorem 8.I. — *If* $\| T \| < 1$ *then*

$$I - \frac{B(T)}{D(T)}$$

is an inverse to I + T.

Proof: We have for $\| T \| < 1$

$$(8.1) \qquad D(T)T - B(T) - B(T)T = 0,$$

for from the definitions of D(T) and B(T), the recurrence formulae (3.1) and (3.2), Theorem 4.III and the vanishing of $B_0(T)$ we have

$$D(T)T - B(T)T = \sum_{0}^{\infty} \left(a_n(T)T - B_n(T)T \right) = \sum_{0}^{\infty} B_{n+1}(T) = B(T).$$

By a simple calculation we see from (8.1) that the theorem is proved for those values of T for which $D(T) \neq 0$. For all T for which $\| T \| < 1$ and $D(T) \neq 0$ we have by Theorem 5.II

$$(8.2) \qquad \frac{B(T)}{D(T)} = \sum_{1}^{\infty} (-1)^{n-1} T^n.$$

Let T_1 be any chosen T such that $\|T_1\| < 1$. From the remark preceding Theorem 4.III we have that

$$D(\lambda T_1) = \sum_0^\infty \lambda^n a_n(T_1)$$

is an analytic function on A to A for

$$|\lambda| < \frac{1}{\|T_1\|}.$$

From the recursion formula (3.2) we have

$$(8.3) \qquad \lambda \frac{d}{d\lambda} D(\lambda T_1) = \sum_0^\infty n\lambda^n a_n(T_1) = \sum_0^\infty \lambda^n [B_n(T_1)]$$
$$= \sum_0^\infty [B_n(\lambda T_1)] = [B(\lambda T_1)].$$

For all λ in A such that $|\lambda| < \frac{1}{\|T_1\|}$ and $D(\lambda T_1) \neq 0$ we have using (8.3) and (8.2) with $T = \lambda T_1$

$$(8.4) \qquad \frac{\sum_0^\infty n\lambda^{n-1} a_n(T_1)}{\sum_0^\infty \lambda^n a_n(T_1)} - \sum_0^\infty (-1)^n \lambda^n [T_1^{n+1}] = 0.$$

Since we clearly have

$$|(-1)^n [T_1^{n+1}]| \leq \gamma \|T_1\|^{n+1}$$

it follows that

$$\sum_0^\infty (-1)^n \lambda_n [T_1^{n+1}]$$

definies an analytic function of λ for $|\lambda| < \frac{1}{\|T_1\|}$.

The left hand side of (8.4) regarded as a function of a general complex variable λ is analytic in the region $|\lambda| < \frac{1}{\|T_1\|}$ except possibly for poles, and vanishes for λ in A and in the above region except at the zeros of $D(\lambda T_1)$, that is, it has zeros which are not isolated. It follows [1] that (8.4) is an identity for complex λ in the region $|\lambda| < \frac{1}{\|T_1\|}$. Let C be a circle of radius $\rho\left(1 < \rho < \frac{1}{\|T_1\|}\right)$ whose center is at the origin and whose circumference passes trough none of the zeros (if any) of $\sum_0^\infty \lambda^n a_n(T_1)$. Integrating (8.4) around C we have

$$\int_C \frac{\frac{d}{d\lambda}\left(\sum_0^\infty \lambda^n a_n(T_1)\right)}{\sum_0^\infty \lambda^n a_n(T_1)} d\lambda = 0.$$

It follows immediately from a well known theorem that $\sum_0^\infty \lambda^n a_n(T_1)$ has no zeros inside C. Placing $\lambda = 1$ we see that $D(T_1) \neq 0$. But T_1 is any T in the region $\|T\| < 1$. The argument of the first part of the proof is therefore valid for all T in $\|T\| < 1$.

Corollary 1. — *For any chosen* T_1, *if* $|\lambda| < \frac{1}{\|T_1\|}$, *then* $D(\lambda T_1)$ *is not zero.*

Corollary 2. — *For* $\|T\| < 1$

$$\log D(T) = \sum_1^\infty (-1)^{n-1} \frac{|T^n|}{n}. \tag{8.5}$$

Proof : Integrate (8.4) from 0 to 1.

9. The Product Theorem for D(T). Theorem 9.I. — *The differential of the polynomial* $a_n(T)$ *is given by*

$$\delta a_n(T) = a_{n-1}(T)[\delta T] - [B_{n-1}(T)\delta T] \qquad (n > 0). \tag{9.1}$$

Proof : We first establish by induction the formula, évident for $n = 1$.

$$\delta a_n(T) = \sum_0^{n-1} (-1)^p a_{n-p-1}(T)[T^p \delta T]. \tag{9.2}$$

(1) E. Goursat, *Cours d'Analyse mathématique*, 4e édition, t. II, p. 98.

Differentiation of (3.4) gives

$$(9.3)\qquad n\,\delta a_n(T) = \sum_{j=0}^{n-1}(-1)^j(\delta\alpha_{n-j-1})[T^{j+1}] + \sum_{j=0}^{n-1}(-1)^j(j+1)a_{n-j-1}[T^j\delta T].$$

Assuming (9.2) for (1, 2, ..., n-1), substituting the result in the first sum on the right hand side of (9.3), reversing the summations and making use of (3.4), we have

$$n\,\delta a_n(T) = \sum_{r=0}^{n-1}(-1)^r(n-r-1)a_{n-r-1}[T^r\delta T] + \sum_{j=0}^{n-1}(-1)^j(j+1)a_{n-j-1}[T^j\delta T]$$

which gives (9.2)

From (3.3) we have

$$(9.4)\qquad [B_{n-1}(T)\delta T] = \sum_{r=1}^{n-1}(-1)^{r+1}a_{n-r-1}[T^r\delta T].$$

The theorem follows by adding (9.2) and (9.4).

With the aid of Theorems 7.I and 9.I we obtain the corollary.

Corollary. — *The differential* $\delta D(T)$ *is given for* $\|T\| < 1$ *by*

$$\delta D(T) = D(T)[\delta T] - [B(T)\delta T].$$

Theorem 9.II. — *If* $\|T\| < 1$, $\|U\| < 1$ *and if for* $0 \leqq \lambda \leqq 1$

$$\|\lambda(T+U)+\lambda^2 TU\| < 1$$

then

$$(9.5)\qquad D(T+U+TU) = D(T)D(U).$$

Proof : By successive use of a theorem (1) (proved elsewhere) on the differentials of a function of a function in vector spaces, by the

(1) M. Fréchet, *Annales sc. de l'École Normale supérieure*, vol. 42, 1925. *See* also R. S. Martin, *loc. cit.*

corollary to Theorem 9.1 and by Corollary 1 to Theorem 8.1 we have for $|\lambda| \leqq 1$

$$\frac{d}{d\lambda} \log D(\lambda T) = [T] - \left[\frac{B(\lambda T)}{D(\lambda T)} T\right]$$

which may be written in the convenient form

$$(9.6) \qquad \frac{d}{d\lambda} \log D(\lambda T) = [\overline{T}(\lambda T) T],$$

where

$$\overline{T}(\lambda T) = I - \frac{B(\lambda T)}{D(\lambda T)},$$

is the inverse of $I + \lambda T$. Similarly

$$(9.7) \qquad \frac{d}{d\lambda} \log D(\lambda U) = [\overline{U}(\lambda U) U].$$

Let $W(\lambda)$ be defined by means of

$$(9.8) \qquad I + W(\lambda) = (I + \lambda T)(I + \lambda U).$$

Then

$$(9.9) \qquad \frac{d}{d\lambda} \log D(W(\lambda)) = \left[\frac{dW}{d\lambda}\right] - \left[\frac{B(W)}{D(W)} \frac{dW}{d\lambda}\right] = \left[\overline{W} \frac{dW}{d\lambda}\right].$$

where $\overline{W}$ is the inverse of $I + W(\lambda)$. By Corollary 2 to Theorem 5.1

$$\overline{W} = \overline{U}\,\overline{T}.$$

Differentiating (9.8) we have

$$\frac{dW}{d\lambda} = (I + \lambda T) U + T(I + \lambda U).$$

Placing the last two results in (9.9) and making special use of postulate 5 for the space S, we have

$$\frac{d}{d\lambda} \log D(W(\lambda)) = [\overline{T} T] + [\overline{U} U].$$

Hence by (9.6) and (9.7)

$$(9.10) \qquad \frac{d}{d\lambda} \log D(W(\lambda)) = \frac{d}{d\lambda} \log D(\lambda T) + \frac{d}{d\lambda} \log D(\lambda U).$$

Since the derivatives in (9.10) are in the ordinary sense we may integrate from O to 1 and take exponentials. This gives (9.5).

Corollary. — *If* $|T| < \sqrt{2} - 1$ *and* $|U| < \sqrt{2} - 1$ *then the conclusions* (9.5) *hold.*

10. *Concluding Remarks.* — If the elements T of the space S are matrices (t_j^i) of a finite order s and if

$$|T| = \sum_{i,j}^{s} t_j^i$$

then a_r is precisely the coefficient of λ^r in the determinant

$$D = |\delta_j^i + \lambda t_j^i| \quad (^1)$$

and B_r is given by $a_r I - A_r$ where A_r is the coefficient of λ^r in the adjoint of D so that for $r > s$, $B_r = 0$. The equation (3.3) for $n = s + 1$ with the condition $B_{s+1} = 0$ is equivalent to the theorem that T satisfies its characteristic equation. In the case of a general space S, if T is such a point that D(T) and B(T) converge, then $|B_r| \to 0$ with $\frac{1}{r}$ and one may, if he so chooses, regard the limiting form of (3.3) as the generalization of the algebraic theorem.

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(1) Where $\delta_j^i = \begin{cases} 1 & \textit{if } i = j, \\ 0 & \textit{if } i \neq j. \end{cases}$