JOURNAL
DE
MATHÉMATIQUES
PURES ET APPLIQUÉES
FONDÉ EN 1836 ET PUBLIÉ JUSQU'EN 1874
PAR JOSEPH LIOUVILLE

CHARLES H. ROWE

On certain systems of curves in riemannian space

Journal de mathématiques pures et appliquées 9e série, tome 12 (1933), p. 283-308.
<http://www.numdam.org/item?id=JMPA_1933_9_12__283_0>

On certain systems of curves in Riemannian space;

By Charles H. ROWE.

Trinity College, Dublin.

1. Introduction. — Let V_N be a Riemannian space of N dimensions, the metric of which is given by

$$ds^2 = g_{ij}\, dx^i\, dx^j$$

in an arbitrary system of coordinates x^i. By a *system of curves* in V_N we shall mean a set of ∞^{2N-2} curves which can be defined by differential equations of the form

$$\frac{d^2x^i}{ds^2} = F^i\left(\frac{dx^r}{ds}, x^r\right) \quad (^1),$$

where s is the length of the arc of the curve, and which satisfies the condition that a unique curve of the set joins any two sufficiently close points.

If we form a curvilinear triangle with three curves of a given system and allow it to shrink to a point, the excess over π of the sum of its angles will in general be an infinitesimal of the same order as its perimeter. Certain special systems, however (for example, the geodesics of V_N), have the property that the excess is always an infinitesimal of

(1) Throughout this paper, we shall assume that any functions which are introduced (explicitly or implicitly) have a sufficient degree of regularity to ensure the validity of our reasoning from the point of view of the theory of functions.

at least the same order as the area [1]. We shall say that such a system of curves has the *triangle property*. The main problem of the present paper is that of determining the systems of curves in V_N that have this property.

In a previous paper [2], I discussed this question in the case where $N = 2$, and showed that a system of ∞^2 curves on a surface has the triangle property if, and only if, it is a velocity system [3]. It will be shown in what follows that every velocity system in V_N has the triangle property, but that it is only when $N = 2$ that a system possessing the triangle property is necessarily a velocity system. The result at which we shall arrive is the following:

In order that a system of curves in V_N should have the triangle property, it is necessary and sufficient that it should be possible to represent it by differential equations of the form

$$\frac{d^2 x^i}{ds^2} - \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0,$$

where s is the length of the arc of the curve, and where the coefficients Γ^i_{jk} are functions of position which are restricted only by the conditions of compatibility that are implied by the fact that the independent variable is the arc.

We shall see in paragraph **2** that such systems are identical with velocity systems for $N = 2$, but that they form a wider class when $N > 2$. Systems of this kind in V_N do not appear to have been studied [4] or named, and we may refer to them as *quadratic systems*. It is clear that a quadratic system is a system of paths of a special type, but we shall postpone until paragraph **10** the consideration of the position of quadratic systems among systems of paths.

---

(1) It is not necessary here to give any precise meaning to the « area » of the triangle, since we may replace it by any infinitesimal of the same order as the square of the perimeter.

(2) *On certain doubly infinite systems of curves on a surface* (*Bull. American Math. Soc.*, t. **36**, 1930, p. 695-704).

(3) For the definition of a velocity system, see paragraph **2** below.

(4) A particular system of this kind is considered by J. L. Synge in connection with non-holonomic geometry (*Math. Annalen*, t. **99**, 1928, p. 738-751).

2. QUADRATIC SYSTEMS AND VELOCITY SYSTEMS. — The equations of the general quadratic system are

$$(2.1)\qquad \frac{dx'^i}{ds} + \Gamma^i_{jk} x'^j x'^k = 0 \qquad (\Gamma^i_{jk} = \Gamma^i_{kj}),$$

where s is the arc of the curve, and where we use the abbreviation

$$x'^i = \frac{dx^i}{ds}.$$

If we write

$$(2.2)\qquad \Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + T_{jk}{}^i,$$

the equations (2.1) may be given the alternative form

$$(2.3)\qquad \frac{\delta x'^i}{\delta s} + T_{jk}{}^i x'^j x'^k = 0 \qquad (T_{jk}{}^i = T_{kj}{}^i),$$

where $\delta/\delta s$ is the symbol of intrinsic differentiation along the curve with respect to the arc, that is,

$$\frac{\delta \lambda^i}{\delta s} = \frac{d\lambda^i}{ds} + \left\{ {i \atop jk} \right\} \lambda^j x'^k,$$

where λ^i is any vector defined at every point of the curve. It is thus clear that $T_{jk}{}^i$ is a tensor.

Since $\delta x'^i/\delta s$ is the (first) curvature vector of the curve, we may state the definition of a quadratic system by saying that the components of the curvature vector of any curve of the system are equal to homogeneous quadratic functions of the components of the unit tangent vector, the coefficients in these quadratics being functions of position.

The restrictions that are implied by the fact that the independent variable is the arc may be simply expressed in terms of the T's. Since the vectors $\delta x'^i/\delta s$ and x'^i are perpendicular, (2.3) implies that

$$T_{jki} x'^i x'^j x'^k = 0,$$

where

$$T_{jki} = g_{ir} T_{jk}{}^r,$$

and hence that

$$(2.4)\qquad T_{jki} + T_{kij} + T_{ijk} = 0.$$

The conditions (2.4) are necessary and sufficient for the compatibility of the equations (2.1) or (2.3).

On account of these conditions and the symmetry of T_{jki} in the first two indices, a quadratic system depends on $\frac{1}{3}N(N^2-1)$ arbitrary functions of position.

A velocity system in V_N is defined as follows [1]. If f^i is a vector defined at every point of V_N, a curve of the corresponding velocity system has the property that its curvature vector at any point is the component normal to the curve [2] of the vector f^i at the point. If we express this property analytically, we obtain the differential equations of the velocity system in the form

$$(2.5)\qquad \frac{\delta x'^i}{\delta s} = f^i - x'^i(x'^j f_j).$$

It is clear therefore that velocity systems are quadratic systems: for (2.5) may be written in the form (2.3) with

$$(2.6)\qquad T^i_{jk} = \frac{1}{2}(\delta^i_j f_k + \delta^i_k f_j) - g_{jk} f^i,$$

where δ^i_j is a Kronecker delta, or equivalently,

$$(2.7)\qquad T_{jki} = \frac{1}{2}(g_{ji} f_k + g_{ki} f_j) - g_{jk} f_i.$$

A quadratic system is not in general a velocity system. — A velocity system depends on N arbitrary functions of position, whereas a quadratic system depends on $\frac{1}{3}N(N^2-1)$; and of these two numbers the latter is the greater, except when $N=2$, in which case they are equal. The class of quadratic systems is thus a wider one than that of velocity systems, except when $N=2$.

When $N=2$, every quadratic system is a velocity system. — In order

---

(1) J. LIPKA, *Note on velocity systems in curved space of N dimensions* (*Bull. American Math. Soc.*, t. 27, 1920, p. 71-77).

(2) We can express f^i uniquely as the sum of a vector normal to the curve and a vector tangential to the curve. We call the former the component of f^i normal to the curve.

to identify the quadratic system (2.3) with a velocity system, we have to find a vector f_i to satisfy (2.7); and, when $N = 2$, these equations reduce in virtue of (2.4) to the two equations

$$2T_{121} = g_{11} f_2 - g_{12} f_1, \qquad 2T_{212} = g_{22} f_1 - g_{12} f_2,$$

which can be solved for f_1 and f_2.

We may obtain conditions that a quadratic system should reduce to a velocity system in the case where $N > 2$. If there exists a vector f_i which satisfies (2.6), it is given by

$$(2.8) \qquad (N-1) f_i = 2T_{ij}{}^{j},$$

and if we substitute these values in (2.7) we obtain the equations

$$(2.9) \qquad (N-1) T_{ijk} = g_{ij} T_{kr}{}^{r} + g_{ik} T_{jr}{}^{r} - 2 g_{jk} T_{ir}{}^{r},$$

which are necessary conditions that the quadratic system should be a velocity system. They are also sufficient conditions, because they express the fact that it is possible to satisfy (2.7) by giving to f_i the values (2.8). When $N = 2$, the equations (2.9) are consequences of (2.4).

3. INDUCED SYSTEMS OF CURVES. — Our discussion of the problem of determining the systems of curves that have the triangle property involves the notion of the *induction* of a system of curves on a surface by a given system in V_N; and we shall now explain what we mean by this.

Let (Γ) be a given system of curves in V_N, and let Σ be any surface (or V_2) contained in V_N. We can determine on Σ a system (Γ_Σ) of ∞^2 curves by the condition that the geodesic curvature vector of any curve Γ_Σ should coincide with the component in Σ [1] of the curvature

[1] If we are given a vector at a point of a V_M contained in V_N, we can express the vector uniquely as the sum of a vector tangential to V_M and a vector normal to V_M. We call the former the component of the vector in V_M, and the latter its component normal to V_M. There will be no risk of confusing « components » in this sense, which are vectors, with covariant or contravariant components, which are numbers.

vector of the curve Γ that touches it at the point under consideration; or by the equivalent condition that, whenever a curve Γ_Σ and a curve Γ touch, the difference of their curvature vectors at the point of contact should be normal to Σ. We shall say that the system (Γ_Σ) is *induced* [1] on Σ by the system (Γ).

We shall need later the result that *a quadratic system* (Q) *induces a quadratic* (*or velocity*) *system on any surface* Σ. Let the system of coordinates x^i be chosen so that a given surface Σ is represented by the equations $x^3 = x^4 = \ldots = x^N = 0$, and let us adopt the convention that Greek indices take the values 1 and 2 only. Let λ_i be the covariant components of any vector at a point of Σ. If we regard the component in Σ of this vector as a vector of the two-dimensional space Σ, its covariant components in the coordinate system (x^1, x^2) are the two numbers λ_δ. If we apply this to the case where λ_i is the curvature vector of a curve Q which touches Σ, so that

$$\lambda_i = -T_{\beta\gamma i}\dot{x}^\beta \dot{x}^\gamma,$$

we see that the covariant components in the coordinate system (x^1, x^2) of the geodesic curvature vector of the curve Q_Σ of the induced system, whose unit tangent vector is $\dot{x}^\delta$, are the two numbers

$$-T_{\beta\gamma\delta}\dot{x}^\beta \dot{x}^\gamma.$$

The induced system (Q_Σ) is thus quadratic. Its equations are

$$\frac{d^2x^\delta}{ds^2} + \left(\left\{ {\delta \atop \beta\gamma} \right\} - g^{\delta\sigma}T_{\beta\gamma\sigma}\right)\dot{x}^\beta \dot{x}^\gamma = 0 \quad (^2).$$

---

(1) The idea of an induced system of curves (which may be extended at once to the case where the surface is replaced by a V_M) is similar to the idea of an induced affine connection (See J. A. SCHOUTEN, *Der Ricci-Kalkül*, 1924, p. 158). If the system (Γ) consists of the paths of an affine connection, the induced system consists of the paths of the induced affine connection. We may remark that, if (Γ) is a velocity system, it induces a velocity system in any V_M, the vector associated with the induced system being the component in V_M of the vector associated with (Γ). In particular, the extremals in V_N of an integral $\int \mu ds$, in which μ is a function of position, induce in any V_M the extremals of the same integral in V_M.

(2) It is clear of course that in general $g^{\delta\sigma}T_{\beta\gamma\sigma} \neq T_{\beta\gamma}^{\delta}$.

4. Proof of the fact that a system which has the triangle property is necessarily quadratic. — For the purpose of establishing this result, we shall first show that *a system* (Γ) *which has the triangle property induces a quadratic (or velocity) system on any surface* Σ. In order to do this, it will suffice to prove that any three curves of the induced system (Γ_Σ) that pass through an arbitrary point O of Σ have their centres of geodesic curvature [1] at O collinear; for, in two dimensions, the definition of a velocity system is equivalent to the statement that the ∞^1 curves of the system that pass through a point have their centres of geodesic curvature there collinear.

Suppose then that we are given three curves of (Γ_Σ) through a point O of Σ; and let $\Gamma_1, \Gamma_2, \Gamma_3$ be the three curves of (Γ) that touch them there. If S is any surface containing $\Gamma_1, \Gamma_2, \Gamma_3$, S and Σ touch at O; and it follows from the definition of induction that the centres of geodesic curvature at O of $\Gamma_1, \Gamma_2, \Gamma_3$ on S coincide with the centres of geodesic curvature at O of the three given curves on Σ. Suppose now that S is the locus of a variable curve of (Γ) which moves so that it constantly meets Γ_1 and Γ_2, and so that, when it comes to pass through O, it coincides with Γ_3. We may regard the three curves Γ_1, Γ_2, Γ_3 as the limiting positions of the sides of a variable triangle (with two fixed sides) traced on S, which shrinks to the point O; and, since (Γ) has the triangle property, the ratio of the excess to the perimeter of this triangle tends to zero. Hence, in virtue of a theorem which is proved in the paper mentioned in paragraph 1, the centres of geodesic curvature at O of the three curves $\Gamma_1, \Gamma_2, \Gamma_3$ on S are collinear. The centres of geodesic curvature of the three given curves of (Γ_Σ), being the same three points, are therefore collinear; and it thus follows that the system (Γ_Σ) is a quadratic system.

We have now to prove that *a system* (Γ) *in* V_N *which induces a quadratic system on every surface is a quadratic system.* We shall suppose that (Γ) is defined by the differential equations

$$\frac{\delta x'^r}{\delta s} = \varphi^r(x^r, x'^r),$$

[1] We are not in reality introducing elements external to V_N when we replace the consideration of geodesic curvature vectors by that of centres of geodesic curvature. We do so merely to avoid verbal complication in our statements.

in which the functions φ^i satisfy identically the condition of compatibility

$$\dot{x}^i \varphi_i = 0, \tag{4.1}$$

where $\varphi_i = g_{ir}\varphi^r$. On account of the identity $g_{ij}\dot{x}^i\dot{x}^j = 1$, there will be no loss of generality in assuming that the functions φ^i, and therefore also the functions φ_i, are homogeneous of degree 2 in the variables $\dot{x}^r$. We have to prove that they are homogeneous quadratic polynomials in these variables.

Let Σ be any surface in V_N defined by the equations $x^i = x^i(u, v)$, and let O be any point on it, whose coordinates are x^i, or u and v. If we write

$$\lambda^i = \frac{\partial x^i}{\partial u}, \qquad \mu^i = \frac{\partial x^i}{\partial v},$$

the curvature vector η_i of a curve of (Γ) which touches Σ at O is given by

$$\eta_i = \varphi_i(\lambda^r \dot{u} + \mu^r \dot{v}, x^r).$$

Now, the covariant components in the coordinate-system (u, v) of the geodesic curvature vector of the curve of (Γ_Σ) that passes through O in the same direction are $\lambda^i\eta_i$ and $\mu^i\eta_i$. Since (Γ_Σ) is by hypothesis a quadratic system, these must be equal to homogeneous quadratics in the contravariant components of the unit tangent vector, that is, in $\dot{u}$ and $\dot{v}$. We shall write $\varphi_i(\dot{x}^r)$ instead of $\varphi_i(\dot{x}^r, x^r)$, which will cause no confusion since we shall not change the point O. We thus see, on account of the homogeneity of φ_i, that each of the functions

$$\lambda^i \varphi_i(\lambda^r + \mu^r\theta), \qquad \mu^i \varphi_i(\lambda^r + \mu^r\theta) \tag{4.2}$$

is a polynomial of at most the second degree in the variable θ, when λ^i and μ^i have arbitrary fixed values. From (4.1) we have

$$\lambda^i \varphi_i(\lambda^r + \mu^r\theta) + \theta\mu^i \varphi_i(\lambda^r + \mu^r\theta) = 0,$$

and this shows that the second of the two functions (4.2) is linear in θ. We may therefore equate to zero its second and third derivatives: and if we replace θ by zero after differentiation and write $\dot{x}^r$ for λ^r we thus have

$$\mu^i\mu^j\mu^k \varphi_{i,jk}(\dot{x}^r) = 0, \qquad \mu^i\mu^j\mu^k\mu^l \varphi_{i,jkl}(\dot{x}^r) = 0.$$

where a partial differentiation with respect to a variable $\dot{x}^j$ is denoted by an index j following a comma. Since the numbers μ^i are arbitrary, these imply

$$\varphi_{i,(jk)}(\dot{x}^r) = 0, \qquad \varphi_{i,(jkl)}(\dot{x}^r) = 0 \quad [1]. \tag{4.3}$$

The first of the identities (4.3) leads to

$$\varphi_{i,(jk)l}(\dot{x}^r) = 0,$$

and this shows, on comparison with the second, that

$$\varphi_{i,jkl}(\dot{x}^r) = 0.$$

It follows that the functions φ_i are homogeneous quadratic polynomials in the variables $\dot{x}^r$, and therefore that the system (Γ) is a quadratic system.

We have thus proved that *a system of curves in* V_N *which has the triangle property is necessarily a quadratic system* [2].

5. Proof of the fact that every quadratic system has the triangle property. — Consider first the case of a quadratic (or velocity) system for N = 2. If we denote by **f** the vector associated with the system in the manner already explained, and by K the intrinsic curvature of the surface, then, as was shown in the paper referred to in paragraph 1, the excess E of any triangle bounded by three curves of the system is given by the formula

$$E = \int\int (K - \operatorname{div} \mathbf{f})\, dS. \tag{5.1}$$

---

[1] We use Schouten's notation (*Der Ricci-Kalkül*, 1924, p. 25). The symbol formed by enclosing in parentheses p of the indices of a term represents the arithmetic mean of the $p!$ terms corresponding to the $p!$ different arrangements of these indices.

[2] The result, which we have obtained incidentally, that quadratic systems are characterized by the property of inducing a velocity system on any surface takes the following form in a Euclidean space E_3 of three dimensions: *Quadratic systems in* E_3 *are characterized by the property that the polar lines* (or axes of the osculating circles) *of the curves of the system that touch a plane at a point meet the plane in collinear points.*

where the surface integral is taken over the interior of the triangle, and where div **f**, the divergence of the vector **f**, is defined as follows. If f^β $(\beta = 1, 2)$ are the contravariant components of **f** in any system of coordinates x^β on the surface, then

$$\operatorname{div}\mathbf{f} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\beta}(f^\beta\sqrt{g}) = \nabla_\beta f^\beta,$$

where g is the determinant of the fundamental quadratic form, and where $\nabla_\alpha f^\beta$ is the covariant derivative of f^β with respect to x^α [1]. The formula (5.1) proves that every quadratic system on a surface has the triangle property, and shows also that, as a triangle belonging to the system reduces to a point, we have

$$\lim\frac{E}{A} = K + \operatorname{div}\mathbf{f},$$

where A is the area of the triangle. Thus, for a given system, lim (E/A) depends only on the position of the point to which the triangle reduces.

Consider now a quadratic system (Q) in V_N. If Σ is any surface, we shall denote by $\mathbf{f}_\Sigma$ the vector that is associated with the velocity system (Q_Σ) that (Q) induces on Σ, and by div $\mathbf{f}_\Sigma$ the divergence of $\mathbf{f}_\Sigma$ regarded as a vector of the two-dimensional space Σ. Also, we shall denote by K_Σ the intrinsic curvature of Σ.

Allow a triangle formed with three curves of the system (Q) to shrink to a point O so that the curves that form its sides tend to

---

(1) The formula (5.1) may be obtained easily in our present notation. We find from Bonnet's formula that the excess of any curvilinear triangle on the surface is $\int \varkappa\, ds + \int\int K\, dS$ where $\varkappa$ is the geodesic curvature of a side (See DARBOUX, *Leçons sur la théorie des surfaces*, t. 3, p. 127). Now if n_α is the unit vector along the positive normal to a curve, $n_1 = -\dot{x}^2\sqrt{g}$, $n_2 = \dot{x}^1\sqrt{g}$. (See A. J. MC CONNELL, *Applications of the absolute differential calculus*, 1931, p. 170), so that, if the curve belongs to the velocity system,

$$\varkappa\, ds = n_\alpha f^\alpha ds = (f^2 dx^1 - f^1 dx^2)\sqrt{g}.$$

Our formula follows at once on substituting this value for $\varkappa$ in the line integral and transforming into a surface integral.

coincide with three curves Q_1, Q_2, Q_3 of (Q) through O. The tangents at O to these three curves are necessarily coplanar, and determine an elementary plane p. Let Σ be a surface passing through the three curves that form the triangle. Let it vary with the triangle and, as the triangle tends to reduce to the point O, let it tend to coincide with a surface Σ' through O which is regular in the neighborhood of O. The surface Σ' necessarily contains the curves Q_1, Q_2, Q_3, and touches p at O. If A is the area of the portion of Σ bounded by the triangle, we have

$$(5.2)\qquad \frac{E}{A} = \frac{1}{A}\int\int (K_\Sigma - \operatorname{div} f_\Sigma)\, dS,$$

where the surface integral is extended over this portion of Σ. If we impose sufficiently restrictive conditions on the way in which the triangle and the surface Σ tend to their limiting forms, we may infer from (5.2) that

$$\lim \frac{E}{A} = K_\Sigma - \operatorname{div} f_\Sigma.$$

This shows that *every quadratic system has the triangle property*.

We may show further that, for a given system, $\lim (E/A)$ depends only on the ultimate position and orientation of the triangle, that is, on the point O and the elementary plane p. Let S' be the surface generated by the curves of (Q) that touch p at O [1]. S' and Σ' both contain the three curves Q_1, Q_2, Q_3 through O, and they therefore have contact of the second order. By this we mean that the curves in which the two surfaces are cut by a general V_{N-1} through O have

(1) In the case of an arbitrary system of curves, the curves that pass through a point O in the ∞^1 directions of a linear pencil generate a surface which in general has a singularity at the point O; for it is not in general possible to find a representation $x^r = x^r(u, v)$ of the surface such that the functions $x^r(u, v)$ have determinate second partial derivatives at O. In order that O should always be an ordinary point it is sufficient (but not necessary) that the system should be quadratic; we shall see in paragraph 6 that, if the system is quadratic, it is possible to choose the coordinates in V_N so that the surface is represented by linear equations.

contact of the second order (1); or, analytically, that it is possible to represent the surfaces in the forms $x^i = \varphi^i(u, v)$, $x^i = \psi^i(u, v)$ so that the first and second partial derivatives of φ^i are respectively equal to those of ψ^i at O (2).

Now, for any surface S represented by the equations $x^i = x^i(u, v)$, the values of $\mathbf{K}_S$ and div $\mathbf{f}_S$ at a point depend only on the values of the functions $x^i(u, v)$ and their first and second partial derivatives at this point. Hence we have at O

$$K_\Sigma - \operatorname{div} \mathbf{f}_\Sigma = K_S - \operatorname{div} \mathbf{f}_S,$$

so that

$$\lim \frac{E}{A} = K_S - \operatorname{div} \mathbf{f}_S.$$

We have thus completed the proof of the following theorem :

In order that a system of curves in V_N should have the property that the ratio of the excess to the area of a triangle formed with three curves of the system tends to a limit as the triangle shrinks to a point, it is necessary and sufficient that the system should be quadratic. If the system is quadratic, this limit depends only on the limiting position of the triangle and on the limiting orientation of its surface.

In this statement, we interpret the area of the triangle to mean the area bounded by the triangle on any surface containing its sides which satisfies, as the triangle contracts, the conditions of regularity that we imposed above on the surface Σ.

---

(1) See A. J. Mc Connell, *The contact of curves in Riemannian space* (*Proc. London Math. Soc.* (2), t. 28, 1928, p. 510-517).

(2) In order to prove that the surfaces have contact of the second order, we may choose the coordinate system x^i so that $x^i = 0$ at O, and so that one of the surfaces is $x^3 = x^4 = \ldots = x^N = 0$. We may represent the other surface in the form

$$x^i = a^i_{\alpha\beta} x^\alpha x^\beta + a^i_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma + \ldots \qquad (i = 3, 4, \ldots, N),$$

where our convention about Greek indices is observed. It is clear that, if the surfaces have in common more than two curves through O in distinct directions, we must have $a^i_{\alpha\beta} = 0$.

6. Calculation of the intrinsic curvature of the surface S'. — We shall now consider the problem of calculating $\lim (E/A)$ in terms of the coefficients T^i_{jk} in the equations (2.3) of the quadratic system (Q), when we are given the point O to which the triangle reduces and the orientation of the elementary plane p at O. For this purpose, it is convenient first to calculate the intrinsic curvature K_S at O of the surface S' generated by the curves of (Q) that touch p at O.

For simplicity we shall assume that the fundamental quadratic form of V_N is positive-definite. The modifications that are necessary when this is not so can easily be made. Also, we may slightly simplify our notation by writing K', f', div f' instead of K_S, $\mathbf{f}_S$, div $\mathbf{f}_S$.

We shall suppose temporarily that our system of coordinates x^i is a special one defined as follows. Let N fixed mutually perpendicular directions be chosen at O, and let the N numbers l^i be the cosines of the angles between these directions and the direction of the tangent at O to any curve of (Q). Then the coordinates of a point P on this curve are defined by $x^i = sl^i$, where s is the length of the arc OP of the curve. It is known that the coordinate system thus defined is an admissible one : it is in fact a system of normal coordinates [1] for the system of paths (Q).

At O we have

$$(6.1) \qquad g_{ij} = \delta_{ij},$$

and therefore the value of a symbol at O is not altered if an index is raised or lowered. Any curve of (Q) through O is represented by equations of the form $x^i = l^i s$, where the numbers l^i are constants, and therefore satisfies the equations $d^2x^i/ds^2 = 0$. Hence the coefficients Γ^i_{jk} in the equations (2.1) all vanish at O. We thus have at O

$$T_{ijk} = T^i_{jk} = -\left\{ {i \atop jk} \right\} = -[jk, i].$$

---

[1] See O. Veblen and T. Y. Thomas, *The geometry of paths* (*Trans. American Math. Soc.*, t. 25, 1923, p. 551-608). It may be remarked that the coordinate system that we have described is an admissible one only if the system of curves is quadratic. This follows from a theorem due to J. Douglas (*Annals of Mathematics* (2), t. 29, 1928, p. 163).

The surface S' is represented in this system of coordinates by linear equations: and we shall further choose our system of coordinates so that S' is the surface

$$x^3 = x^4 = \ldots = x^N = 0.$$

Let R_{hijk} be the curvature tensor of V_N, that is,

$$R_{hijk} = \frac{\partial}{\partial x^j}[ik, h] - \frac{\partial}{\partial x^k}[ij, h] - \begin{Bmatrix} r \\ ij \end{Bmatrix}[hk, r] - \begin{Bmatrix} r \\ ik \end{Bmatrix}[hj, r] \quad (^1).$$

If we denote by K the curvature of V_N at O corresponding to the orientation of S' or of *p*, we have

$$K = R_{1212}.$$

Similarly, if $R'_{\alpha\beta\gamma\delta}$ is the curvature tensor of the two-dimensional space S' in the system of coordinates x^1, x^2, we have

$$K' = R'_{1212}.$$

In addition to our convention about Greek indices, we shall introduce the further convention that a capital Roman index takes the values 3, 4, ..., N. Then, at any point of S',

$$R'_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \begin{Bmatrix} P \\ \beta\gamma \end{Bmatrix}[\alpha\delta, P] - \begin{Bmatrix} P \\ \beta\delta \end{Bmatrix}[\alpha\gamma, P],$$

and hence at O

(6.2) $$K' - K = T_{11P}T_{22P} - T_{12P}T_{12P}.$$

We can interpret this formula geometrically. Holding P fixed for the moment, we see that $-T_{\beta\gamma P}\dot{x}^\beta\dot{x}^\gamma$ is the projection (²) in the x^P-direction of the curvature vector of one of the curves of (Q) through O that generate S'. Hence $T_{11P}T_{22P} - (T_{12P})^2$ (not summed with respect to P) is the product of the two extreme values that this projection takes as we vary the curve. If, for brevity, we speak of the projection of the curvature vector of a curve in a given direction as the curvature of the curve in that direction, we may express (6.2)

(¹) We follow the notation of EISENHART, *Riemannian geometry*, 1926, p. 20.

(²) That is, the scalar product of the curvature vector and the unit vector whose components are δ^i_P.

in the following geometrical form, which has no reference to any particular coordinate-system : To calculate K′ — K, take N — 2 directions at O perpendicular to each other and normal to S′, and, corresponding to each of these directions, find the product of the extreme values of the curvature at O in this direction of a curve of (Q) on S′, and add the results.

We can now express this result analytically in any coordinate-system. Supposing that our system of coordinates x^i is unrestricted, we may specify the orientation of p by means of any two perpendicular unit vectors [1] at O, λ_1^i and λ_2^i, which lie in p. Let ξ_P^i ($P = 3, 4, \ldots, N$) be N — 2 unit vectors [2] at O which are perpendicular to each other and normal to p or S′. The curvature in the direction of ξ_P of a curve of (Q) through O on S′ is

$$-T_{ijk}(\lambda_1^i \cos\theta + \lambda_2^i \sin\theta)(\lambda_1^j \cos\theta + \lambda_2^j \sin\theta)\xi_P^k,$$

and if we calculate the product of the extreme values of this as θ varies, and use the result that we have stated, we find that

$$\text{(6.3)} \qquad K' - K = T_{ijk}\lambda_1^i\lambda_1^j\xi_P^k\, T_{lmn}\lambda_2^l\lambda_2^m\xi_P^n - T_{ijk}\lambda_1^i\lambda_2^j\xi_P^k\, T_{lmn}\lambda_1^l\lambda_2^m\xi_P^n,$$

or

$$\text{(6.4)} \qquad 2(K' - K) = T_{ikm}T_{jln}\Delta^{ij}\Delta^{kl}\xi_P^m\xi_P^n,$$

where

$$\Delta^{ij} = \lambda_1^i\lambda_2^j - \lambda_1^j\lambda_2^i.$$

We can derive from this a formula which involves only the two vectors λ_1^i, λ_2^i. Assuming for the moment that (6.1) holds at O, we

---

(1) The formulae that we shall obtain may be modified easily if it is preferred to specify p by means of two arbitrary vectors which lie in p.

(2) For these vectors, and for the two vectors that lie in p, we shall agree that the lower index identifies the vector and the upper index the component. Since we shall not need to lower the index i, it will not be necessary to use a notation such as $\xi_P^{\,i}$.

(3) This is of course nothing but the form taken in our particular case by Ricci's formula for the curvature of a V_n in V_N (*Formole fondamentali nella teoria generale delle varietà e della loro curvatura*, *Rend. Lincei* (5), t. 11, 1902, p. 355-362), but it seemed as simple to derive it directly as to adapt Ricci's formula.

may write

$$\xi_\rho^i \xi_\rho^j = \delta_{ij} - \lambda_1^i \lambda_1^j - \lambda_2^i \lambda_2^j$$

in (6.3), which thus leads to

$$(6.5)\qquad 2(K' - K) = T_{hi}^{\ \ \alpha} T_{jk\alpha} \Delta^{hj} \Delta^{ik} - 2(T_{hi\alpha} \lambda_1^h \lambda_2^i \lambda_1^\alpha)^2 - 2(T_{hi\alpha} \lambda_1^h \lambda_2^i \lambda_2^\alpha)^2.$$

The form of this equation shows that it is true when the coordinate-system is unrestricted.

When the quadratic system (Q) reduces to a velocity system we obtain, on using (2.6), the simpler formula

$$(6.6)\qquad K' - K = f_i f_j \xi_\rho^i \xi_\rho^j,$$

where f_i is the vector associated with the system. This agrees with a result obtained by Lipka ([1]).

If it is observed that the second member of this formula is the square of the magnitude of the component of f_i normal to S', a simple way of obtaining the formula by geometrical reasoning suggests itself. According to the definition of a velocity system, the curvature vector of a curve of the system on S' through O is the sum of $\mathbf{f}_i$ and a vector tangential to the curve and therefore to S'. The curvature in the direction of ξ_ρ of any of these curves is thus equal to the projection of $\mathbf{f}_i$ in this direction, so that the product of the extreme values of this curvature reduces to the square of this projection. Hence K' — K is equal to the sum of the squares of the projections of $\mathbf{f}_i$ in the N — 2 directions ξ_ρ, and therefore to the square of the magnitude of the component of $\mathbf{f}_i$ normal to S'.

We see incidentally that, *for a velocity system, the point O is an umbilic on the surface* S', because all of the curves of the system through O that lie on S' have the same curvature in any direction normal to S'. Similar reasoning shows that *the* V_M *generated by the curves of a velocity system that pass through a point O in the* ∞^{M-1} *directions of a linear vector-space has an umbilic at* O.

7. Calculation of lim(E/A). — In order to complete the calcu-

(1) J. Lipka, *Trajectory surfaces*, etc. (*Proc. American Acad. of Arts and Sciences*, t. 59, 1923, p. 51-77).

lation of $\lim(E/A)$, we have to find the value of div $\mathbf{f}$, but we shall not try to do this directly. In view of the way in which the corresponding limit for a geodesic triangle can be expressed by means of the curvature tensor of V_N, it is natural to consider the result of treating the curvature tensor of the system of paths (Q) in the same way. If we do this, we shall find that it is more convenient to use instead a somewhat different tensor, whose relation to the curvature tensor of the paths we shall examine later.

We write

$$(7.1) \qquad C_{hijk} = \nabla_k T_{ijh} - \nabla_j T_{ikh},$$

where ∇_k indicates the operation of covariant differentiation with respect to x^k; and

$$D_{hijk} = C_{hijk} + C_{ihkj}.$$

D_{hijk} is a tensor, which is skew-symmetric in its first two indices, and also in its last two. Hence, with the notation of paragraph **6**, the quantity I defined by

$$(7.2) \qquad \tfrac{1}{4}\mathrm{I} = \tfrac{1}{4} D_{hijk}\lambda^h_1 \lambda^i_2 \lambda^j_1 \lambda^k_2 = D_{hijk}\Delta^{hi}\Delta^{jk}$$

is an invariant, which depends only on the point O and the orientation of p. In order therefore to calculate I, we may suppose that our coordinate-system is of the special type that we introduced in paragraph **6**, and that the positive directions of the x^1-curve and the x^2-curve at O are respectively the directions of the two given vectors λ^i_1 and λ^i_2, so that $\lambda^i_1 = \delta^i_1$, $\lambda^i_2 = \delta^i_2$. Then we have, with the help of (2.4),

$$(7.3) \qquad \mathrm{I} = D_{1212} = \nabla_1(T_{212} - T_{221}) + \nabla_2(T_{121} - T_{112}) = 3(\nabla_1 T_{212} + \nabla_2 T_{121}).$$

We can now express I in terms of div $\mathbf{f}$, and $K' - K$.

As we have seen in paragraph **5**, the covariant components in the system of coordinates x^1, x^2 of the geodesic curvature vector of a curve of the system $(Q_{S'})$ induced by (Q) on S′ are $-T_{\beta\gamma\delta}\dot{x}^\beta\dot{x}^\gamma$; and therefore the numbers $T_{\beta\gamma\delta}$ are the components of the tensor of the surface S′ that defines the induced system $(Q_{S'})$ in the same way as the tensor T_{jki} of V_N defines the system (Q). If f_β are the covariant components of the vector **f**, we have, as in paragraph **2**,

$$(7.4)\qquad T_{\beta\gamma\delta}=\frac{1}{2}(g_{\beta\delta}f_{\gamma}+g_{\gamma\delta}f_{\beta})-g_{\beta\gamma}f_{\delta}.$$

Let $\nabla'_{\alpha}T_{\beta\gamma\delta}$ denote the covariant derivative, formed with reference to the metric of S′, of this tensor, so that at any point of S′

$$\nabla'_{\alpha}T_{\beta\gamma\delta}=\nabla_{\alpha}T_{\beta\gamma\delta}-T_{\rho\gamma\delta}\left\{\begin{matrix}\rho\\ \alpha\beta\end{matrix}\right\}-T_{\beta\rho\delta}\left\{\begin{matrix}\rho\\ \alpha\gamma\end{matrix}\right\}-T_{\beta\gamma\rho}\left\{\begin{matrix}\rho\\ \alpha\delta\end{matrix}\right\},$$

and at O,

$$\nabla'_{\alpha}T_{\beta\gamma\delta}=\nabla_{\alpha}T_{\beta\gamma\delta}-T_{\rho\gamma\delta}T_{\alpha\beta\rho}-T_{\beta\rho\delta}T_{\alpha\gamma\rho}-T_{\beta\gamma\rho}T_{\alpha\delta\rho}.$$

Substituting these values in (7.3) and simplifying by means of (2.4), we find that

$$\frac{1}{3}I=\nabla'_{1}T_{212}+\nabla'_{2}T_{121}-(T_{11\rho}T_{22\rho}-T_{12\rho}T_{12\rho}),$$

or, by (6.2),

$$(7.5)\qquad \frac{1}{3}I=\nabla'_{1}T_{212}+\nabla'_{2}T_{121}-(K'-K).$$

Now, from (7.4), we have

$$\nabla_{\alpha}T_{\beta\gamma\delta}=\frac{1}{2}(g_{\beta\delta}\nabla_{\alpha}f_{\gamma}+g_{\gamma\delta}\nabla_{\alpha}f_{\beta})-g_{\beta\gamma}\nabla_{\alpha}f_{\delta},$$

and from this we find, using (6.1), that at O

$$\nabla'_{1}T_{212}+\nabla'_{2}T_{121}=\frac{1}{2}(\nabla_{1}\mathbf{f}_{1}+\nabla_{2}\mathbf{f}_{2})=\frac{1}{2}(\nabla_{\beta}f^{\beta})=\frac{1}{2}\operatorname{div}\mathbf{f}.$$

Thus (7.5) becomes

$$\frac{1}{3}I=\frac{1}{2}\operatorname{div}\mathbf{f}-(K'-K),$$

and hence

$$\lim\frac{E}{A}=K'-\operatorname{div}\mathbf{f}=K-\frac{2}{3}I-(K'-K).$$

This enables us to calculate lim(E/A) in any system of coordinates, since I is given by (7.2), K′ − K by (6.4), and K by

$$4K=R_{hijk}\Delta^{hi}\Delta^{jk}.$$

If we use the tensor A_{hijk} defined by

$$A_{hijk} = \frac{1}{2} B_{hijk} - \frac{1}{3} D_{hijk},$$

we thus have the final result

$$(7.6) \qquad 2 \lim \frac{E}{A} = (A_{hijk} - g^{mn} g^{rs} T_{hin} T_{jks}) \Delta^{hi} \Delta^{jk}.$$

It will be found that this may also be written in the form

$$2 \lim \frac{E}{A} = \left(A_{hijk} + T_{ij}^{\cdot\cdot r} T_{hkr} - \frac{2}{9} g_{hj} T_{imn} T_{krs} \Delta^{mn} \Delta^{rs} \right) \Delta^{hi} \Delta^{jk}.$$

We may express the tensor A_{hijk} in terms of the curvature tensor of the system of paths (Q), the curvature tensor of V_N, and certain sums of products of the coefficients T^i_{jk}. If we regard the system (Q) as a particular system of paths, the corresponding curvature tensor (¹) B^h_{ijk} is defined by

$$B^h_{\cdot ijk} = \frac{\partial}{\partial x^j} \Gamma^h_{ik} - \frac{\partial}{\partial x^k} \Gamma^h_{ij} - \Gamma^r_{ik} \Gamma^h_{rj} - \Gamma^r_{ij} \Gamma^h_{rk},$$

and we have the formula (²)

$$B^h_{\cdot ijk} = R^h_{\cdot ijk} + \nabla_j T_{ik}^{\cdot\cdot h} - \nabla_k T_{ij}^{\cdot\cdot h} + T_{ik}^{\cdot\cdot r} T_{rj}^{\cdot\cdot h} - T_{ij}^{\cdot\cdot r} T_{rk}^{\cdot\cdot h}.$$

We thus find that

$$6 A_{hijk} = 2 (B_{hijk} - B_{ihkj}) - R_{hijk} - 2 (T_{ik}^{\cdot\cdot r} T_{rjh} - T_{ij}^{\cdot\cdot r} T_{rkh} + T_{hj}^{\cdot\cdot r} T_{rki} - T_{hk}^{\cdot\cdot r} T_{rji}).$$

8. A CHARACTERISTIC PROPERTY OF VELOCITY SYSTEMS IN V^N. — Consider the form taken by (7.6) for a quadratic system in a space of three dimensions V_3. If we suppose that $g_{ij} = \delta^i_j$ at O, we may specify the orientation of p by means of the direction cosines l^i of the normal to p at O with reference to the directions of the coordinate curves at O. Since we have $l^1 = \Delta^{23}$, etc., the formula (7.6) expresses lim (E/A) as a homogeneous quartic in the direction cosines l^i. We may contrast this with the corresponding formula in the case where the quadratic

---

(¹) VEBLEN and THOMAS, *loc. cit.*, p. 554.

(²) See SCHOUTEN, *Der Ricci-Kalkül*, p. 86, or EISENHART, *Non-Riemannian geometry*, 1927, p. 10.

system reduces to the geodesics of V_3. In this case $\lim(E/A)$ is equal to a homogeneous quadratic in these direction cosines ([1]), the quartic reducing to a quadratic in virtue of the identity

$$(l^1)^2+(l^2)^2+(l^3)^2=1.$$

The problem thus suggests itself of determining the quadratic systems in V_3 for which $\lim(E/A)$ is a homogeneous quadratic in the direction cosines of the normal to p; or, more generally, of determining the quadratic systems in V_N for which the formula (7.6) reduces to one of the type

$$(8.1) \qquad \lim \frac{E}{A} = F_{hijk}\,\Delta^{hi}\,\Delta^{jk}.$$

We shall indicate briefly how this problem may be discussed.

Referring to (6.5), we see that a quadratic system has the property in question only if

$$(T_{hiu}\lambda_1^h\lambda_2^i\lambda_1^u)^2+(T_{hiu}\lambda_1^h\lambda_2^i\lambda_2^u)^2$$

reduces to something of the form

$$G_{hijk}\lambda_1^h\lambda_2^i\lambda_1^j\lambda_2^k,$$

in virtue of the equations

$$g_{ij}\lambda_1^i\lambda_1^j=1, \qquad g_{ij}\lambda_2^i\lambda_2^j=1, \qquad g_{ij}\lambda_1^i\lambda_2^j=0.$$

It is natural to expect, and it may be proved without difficulty, that this can happen only if a relation of the form

$$T_{hiu}\lambda_1^h\lambda_2^i\lambda_1^u=(a_h\lambda_1^h)(g_{iu}\lambda_1^i\lambda_2^u)+(b_i\lambda_2^i)(g_{hu}\lambda_1^h\lambda_1^u)$$

holds identically. If we equate coefficients in this, and use (2.4), we find that $a_i=-b_i$, and that

$$T_{jki}=g_{ji}b_k+g_{ki}b_j-2g_{jk}b_i,$$

which shows that the system must be a velocity system.

To show that a formula of the type (8.1) holds for any velocity system, we may substitute the values (2.7) in our formula for

(1) See LEVI-CIVITA, *The absolute differential calculus*, 1927, p. 203.

lim (E/A). We find that

$$(8.2)\qquad \lim \frac{E}{A} = \frac{1}{4} R_{hijk}\Delta^{hi}\Delta^{jk} - \nabla_r f^r + \frac{\xi^u\xi^v}{\xi^p\xi_p}(\nabla_u f_v - f_u f_v),$$

and this may readily be expressed in the form (8.1) [1].

We thus see that *a formula of the type* (8.1) *is characteristic of velocity systems.*

In the same way it is clear that *velocity systems are the only quadratic systems for which* K′ *is equal to an expression of the same form as the second member of* (8.1). This result forms a complement to the result of Lipka that we mentioned in paragraph **6**.

9. Systems in which the excess of every triangle is zero. — It will be of interest to compare the results that we have obtained with the following theorem which is due to Douglas : If a system of curves in V_N has the property that the excess of every triangle formed by curves of the system is zero, then, if $N > 2$, V_N must be conformal to a Euclidean space E_N in such a way that the curves of the system correspond to the straight lines of E_N [2]. We shall show how our results may be used in order to give an alternative proof of this theorem.

If the system (Γ) has the property in question, it is clearly a quadratic system for which lim (E/A) is always zero. In virtue of what has been said in paragraph **8**, it is thus a velocity system.

We shall use again the coordinate-system of paragraph **6** with an arbitrary point O as origin. The surface generated by the curves of (Γ) that touch an elementary plane at O is then represented by homogeneous linear equations. As Douglas shows, such a surface contains

---

(1) Where

$$4 F_{hijk} = R_{hijk} - 2(f_r f^r)\, g_{hj} g_{ik} + 4 g_{hj}(f_i f_k - \nabla_i f_k).$$

It is perhaps easier to establish the formula (8.2) directly : it can be shown without difficulty that

$$\operatorname{div} \mathbf{f}' = \nabla_r f^r + \frac{\xi^u\xi^v}{\xi^p\xi_p}(2 f_u f_v - \nabla_u f_v),$$

and this, combined with (6.6), leads at once to (8.2).

(2) J. Douglas, *Criterion for the conformal equivalence of a Riemann space to a Euclidean space* (*Trans. American Math. Soc.*, t. **27**, 1925, p. 299-306).

wholly any curve of (Γ) that touches it: and this implies that along any curve of (Γ) we have

$$\left(\frac{d\dot{x}^1}{ds}\Big/\dot{x}^1\right)=\left(\frac{d\dot{x}^2}{ds}\Big/\dot{x}^2\right)=\ldots=\left(\frac{d\dot{x}^N}{ds}\Big/\dot{x}^N\right).$$

Consequently, there must exist numbers φ_i such that

$$(9.1)\qquad \Gamma^i_{jk}=\tfrac{1}{2}(\delta^i_j\varphi_k-\delta^i_k\varphi_j)\ \text{[1]}.$$

If f_i is the vector associated with the velocity system (Γ), we thus have

$$\left\{{i \atop jk}\right\}=g_{jk}f^i-\tfrac{1}{2}\delta^i_j(f_k-\varphi_k)-\tfrac{1}{2}\delta^i_k(f_j-\varphi_j).$$

Hence

$$(N-1)\varphi_i-(N-1)f_i=2\left\{{r \atop ri}\right\}=\frac{\partial\log g}{\partial x^i},$$

and, if $i\neq j$,

$$g^{jj}(f_i-\varphi_i)=-2g^{jr}\left\{{j \atop ri}\right\}=\frac{\partial g^{jj}}{\partial x^i},$$

where there is no summation with respect to j. These equations show that f_i is a gradient provided that $N>2$, so that we may write $f_i=\partial\sigma/\partial x^i$. The system (Γ) is consequently a natural system, and consists of the extremals of the integral $\int e^{\sigma}ds$.

Consider a second Riemannian space V'_N in which the element of length ds' is given by

$$ds'^2=e^{2\sigma}g_{ij}\,dx^i\,dx^j,$$

and consider the correspondence between V_N and V'_N in which corresponding points have the same coordinates. This correspondence is conformal, and therefore the system (Γ') in V'_N that corresponds to (Γ) in V_N has the property of forming triangles with zero excess. Now the curves of (Γ') are the geodesics of V'_N, since they are the extremals of $\int ds'$: and therefore every geodesic triangle in V'_N has

---

(1) We may use instead the fact that it is possible to find a coordinate-system in which the curves (Γ) are represented by linear equations. In such a coordinate system equations of the form (9.1) are satisfied.

zero excess. Hence V'_N is a Euclidean space, and its straight lines correspond conformally to the curves of (Γ) in V_N.

When $N > 3$, it is possible to give a different proof of this theorem, based on the statement at the end of paragraph 6 and on the theorem of Schouten [1] that, for $N > 3$, a V_N must be conformal to a Euclidean space if it is possible to find in it a V_{N-1}, all of whose points are umbilics, passing through an arbitrary point normal to an arbitrary direction.

10. Quadratic systems and systems of paths. — We have still to consider the relation of quadratic systems to systems of paths in V_N.

In our Riemannian space V_N, let us consider an arbitrary affine (symmetric) connection which is independent of the metric of V_N and is distinct from the Levi-Civita parallelism which is associated with the metric. Let this affine connection be defined by the formulae

$$(10.1) \qquad d\lambda^i + \Gamma^i_{jk}\lambda^j dx^k = 0 \qquad (\Gamma^i_{jk} = \Gamma^i_{kj})$$

for the parallel displacement of an arbitary vector λ^i, the coefficients Γ^i_{jk} being functions of position. The equations of the paths (or auto-parallels) of (10.1) are formed by expressing that a vector which is tangent to a path remains so when it is given a displacement along the path in accordance with (10.1). Taking this tangent vector to be the unit tangent vector, we have the differential equations of the paths in the form

$$(10.2) \qquad \frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = \theta\frac{dx^i}{ds},$$

where s is the arc of the curve, and θ is a factor of proportionality. We may write these in the equivalent form

$$(10.3) \qquad \frac{\delta x'^i}{\delta s} + T'^i_{jk}x'^j x'^k = \theta x'^i,$$

where

$$\Gamma^i_{jk} = T'^i_{jk} + \left\{ {i \atop jk} \right\}.$$

[1] J. A. Schouten, *Ueber die konforme Abbildung*, etc. (*Math. Zeitschrift*, t. 11, 1921, p. 86).

The fact that $\delta \dot{x}^i/\delta s$ and $\dot{x}^i$ are perpendicular vectors shows that

(10.4) $$\theta = T_{jki}\dot{x}^j\dot{x}^k\dot{x}^i.$$

In order that (10.2) should have the same form as (2.1) when θ is replaced by the value (10.4), this value must reduce to the form $q_i\dot{x}^i$ in virtue of the fact that $\dot{x}^i$ is a unit vector, the numbers q_i being functions of position. We must therefore have identically

$$T_{jki}\dot{x}^j\dot{x}^k\dot{x}^i = g_{jk}q_i\dot{x}^j\dot{x}^k\dot{x}^i,$$

and hence

(10.5) $$T_{(jki)} = g_{(jk}q_{i)} \quad (^1).$$

If this condition is satisfied, the paths are represented by the equations (2.1) if

$$\Gamma'^i_{jk} = \Gamma^i_{jk} - \frac{1}{2}(\delta^i_j q_k - \delta^i_k q_j).$$

The existence of numbers q_i which satisfy (10.5) is thus necessary and sufficient in order that the paths of (10.1) should form a quadratic system. If such numbers exist, then

(10.6) $$(N-2)q_i = 2T_{ir}^{\cdot\,r} - T^r_{\cdot\, ri},$$

and hence, on substituting these values for q_i in (10.5),

(10.7) $$(N+2)T_{jki} = 2g_{jk}T_{ir}^{\cdot\,r} - T^r_{\cdot\, ri}g_{jk}.$$

The equations (10.7) are thus necessary conditions that the paths should form a quadratic system. They are also sufficient: for they express the fact that it is possible to satisfy (10.5) by giving the values (10.6) to the numbers q_i.

We may look at the relation of quadratic systems to systems of paths in a slightly different way. By choosing a suitable parameter t along each path (the affine parameter), we may write the equations of the paths of (10.1) in the form

(10.8) $$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0.$$

---

(1) See the first foot-note in paragraph 4.

Now the paths are unaltered if we effect a projective change of the affine connection, that is, if we replace Γ^i_{jk} in (10.1) by $\overline{\Gamma}^i_{jk}$, where

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \psi_k + \delta^i_k \psi_j,$$

the numbers ψ_i being arbitrary functions of position. The corresponding equations of the paths are then

$$\frac{d^2x^i}{d\bar{t}^2} + \overline{\Gamma}^i_{jk} \frac{dx^j}{d\bar{t}} \frac{dx^k}{d\bar{t}} = 0, \tag{10.9}$$

where $\bar{t}$ is a different parameter; and these represent the same curves as (10.2) or (10.3). Now, if we can choose the numbers ψ_i so that the equations (10.9) admit the first integral

$$g_{ij} \frac{dx^i}{d\bar{t}} \frac{dx^j}{d\bar{t}} = \text{const.}, \tag{10.10}$$

we may replace $\bar{t}$ by s in these equations, and the system is therefore quadratic. It will be seen conversely that we can do this whenever the paths form a quadratic system. Hence, *in order that the paths of* (10.1) *should form a quadratic system, it is necessary and sufficient that it should be possible to effect a projective change of the affine connection so that the corresponding equations of the paths admit the first integral* (10.10). Eisenhart gives conditions that this should be possible (¹), which will be found to be equivalent to (10.5).

Incidentally we have the following theorem, which gives a geometrical interpretation of the existence of a homogeneous quadratic first integral:

If, in a geometry of paths, it is possible to write the equations of the paths in the form (10.9) *so that these equations admit a homogeneous quadratic first integral, then it is possible to introduce a Riemannian metric with reference to which the system of paths has the triangle property, and conversely,*

It may be of interest to remark that *velocity systems in* V_N *are identical with the systems of paths that arise when the affine connection that*

(¹) *Non-Riemannian geometry*, 1927, p. 118.

we are imposing on V_N reduces to one of Weyl's type. If the affine connection (10.1) is of Weyl's type, there exists a vector Q_i such that

$$\Gamma^i_{jk} = \frac{1}{2}(\delta^i_j Q_k - \delta^i_k Q_j - g_{jk} Q^i) \quad [1],$$

so that the corresponding equations of the paths are

$$\frac{d^2x^i}{ds^2} - Q_j x'^j x'^i - \frac{1}{2} Q^i = \theta x'^i.$$

If we replace θ by its value, found as in (10.4), these become

$$(10.11) \qquad \frac{d^2x^i}{ds^2} - \frac{1}{2} Q_j x'^j x'^i - \frac{1}{2} Q^i = 0.$$

Now, the differential equations of a velocity system are

$$(10.12) \qquad \frac{d^2x^i}{ds^2} = f^i - f_j x'^j x'^i,$$

and we see that (10.11) and (10.12) are equivalent if $Q_i = 2 f_i$.

J. M. Thomas has given conditions that an affine geometry whose paths admit the quadratic first integral (10.10) should have the same paths as a Weyl geometry [2]. If an index following a comma indicates covariant differentiation with reference to the affine connection, so that, for any tensor P_{jk},

$$P_{jk,l} = \frac{\partial}{\partial x^l} P_{jk} - P_{pk}\Gamma^p_{jl} - P_{jp}\Gamma^p_{kl},$$

these conditions are

$$(N-1) [illegible] = 0.$$

It follows from what we have said that these conditions must be equivalent to the conditions (2.9) that a quadratic system should reduce to a velocity system. That this is so may be verified at once on remarking that

$$g_{ij,k} = T_{ijk}.$$

---

(1) SCHOUTEN, *Der Ricci-Kalkül*, p. 75.

(2) J. M. THOMAS, *First integrals in the geometry of paths* (*Proc. Nat. Acad. of Sciences*, t. 12, 1926, p. 123).