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# On the Bethe-Sommerfeld conjecture

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**Abstract**

We consider the operator in $\mathbb{R}^d, d \geq 2$, of the form $H = (-\Delta)^l + V, l > 0$ with a function $V$ periodic with respect to a lattice in $\mathbb{R}^d$. We prove that the number of gaps in the spectrum of $H$ is finite if $8l > d + 3$. Previously the finiteness of the number of gaps was known for $4l > d+1$. Various approaches to this problem are discussed.

## 1. Introduction

Under very broad conditions, spectra of elliptic differential operators with periodic coefficients have a band structure, i.e. they consist of a union of closed intervals called *bands*, possibly separated by spectrum-free intervals called *gaps* (see [19] and [12]). It was conjectured by H. Bethe and A. Sommerfeld in the 30's that the number of gaps in the spectrum of the Schrödinger operator $-\Delta + V$ with a periodic electric potential $V$ in dimension three must be finite. In the present paper we shall address the issue of finiteness for the more general operator in $L^2(\mathbb{R}^d)$:

$$H = H_0 + V,\ H_0 = (-\Delta)^l,\ l > 0, \tag{1.1}$$

and general dimensions $d \geq 2$. Here $V$ is a multiplication by a bounded real-valued function, which is periodic with respect to a lattice $\mathbf{\Gamma}$.

Due to its physical relevance, the case of the Schrödinger operator, i.e. $l = 1$, has been studied better than the general one. It is known that the number of gaps is generically infinite if $d = 1$ (see [19]). For $d \geq 2$ there are at least three different approaches which lead in one way or another to the justification of the conjecture. For the first time it was rigorously proved in [18] by V.N.Popov and M. Skriganov in the case $d = 2$. Then M. Skriganov (see [21] and references therein) obtained a proof for all dimensions $d \geq 3$ for rational lattices $\mathbf{\Gamma}$. For $d = 3$ the result was extended to arbitrary $\mathbf{\Gamma}$ in [22]. A slightly simpler proof in the case $d = 2$ was given in [2]. Skriganov's approach presents a combination of number-theoretic ideas and analytic tools and was designed specifically to address this issue.

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The other two methods are indirect in the sense that the finiteness of the number of gaps in both of them is inferred from other spectral properties of the operator in question, the study of which presents a separate difficult problem. The first of these "indirect" approaches is based on the high energy asymptotics of the Bloch eigenvalues. It was first applied by O. Veliev (see [23]) to prove the validity of the Bethe-Sommerfeld conjecture for $d = 3$. Another proof can be found in the book [9] by Yu. Karpeshina to which we refer for a comprehensive account of this approach and further bibliography. We point out that in [9] the conjecture was proved for a wide class of singular potentials, including Coulomb potentials.

The third method was developed by B. Helffer and A. Mohamed in [6]. Using microlocal machinery, they derived a suitable two-term asymptotic formula for the integrated density of states of the operator $H$ at large energies, which implied the validity of the Bethe-Sommerfeld conjecture for $d = 2, 3, 4$. Notice that this is the paper where the result for $d = 4$ is obtained without any restrictions on the lattice for the first time.

The case of arbitrary $l$ was studies in much less detail. The first result is due to M. Skriganov (see [20], [21]), who showed that the number of gaps is finite if $2l > d, d \geq 3$. Later the polyharmonic operator was studied by Yu. Karpeshina in [8] (see also [9] and references therein) in the framework of the analytic perturbation theory. The high energy asymptotics of the Bloch eigenvalues found in [8] implied the Bethe-Sommerfeld conjecture for $4l > d+1, d \geq 2$. Under the same restriction on $l, d$ the number of gaps was announced to be finite in the note [24] by N.N.Yakovlev (see also [25]) for operators of the form $P_0 + V$ with an elliptic pseudo-differential operator $P_0$ with constant coefficient and a convex homogeneous symbol of order $2l$. However, we have not been able to reproduce Yakovlev's proof in full.

The aim of the present paper is to present a new result in this direction which states that the Bethe-Sommerfeld conjecture holds for the operator (1.1) if $8l > d + 3, d \geq 2$. For the Schrödinger case $l = 1$ the latter condition is equivalent to the requirement that $d = 2, 3$ or 4. These are exactly the dimensions for which the conjecture was justified in the papers cited above. At the same time, our approach is based on the elementary perturbation theoretic argument, and it is simpler. Besides, it treats all admissible dimensions $d$ and orders $l$ in a unified fashion. The complete proof of this result can be found in [16], [17]. Here we intend to describe the main steps of the proof with an emphasis on its pivotal points.

The finiteness of the number of gaps is a legitimate question for other periodic operators as well, but the list of relevant results is very short. For the Schrödinger operator with a periodic magnetic potential the Bethe-Sommerfeld conjecture was justified by A. Mohamed (see [15]) in the case $d = 2$. E.L. Green (see [5]) constructed an example of the Laplace-Beltrami operator with a periodic metric having arbitrarily many gaps in the spectrum. The scarcity of available information demonsrates that the study of subtle spectral characteristics of periodic operators is a technically challenging problem, which calls for development of new methods. In connection with this one should remember that even the "standard" case $-\Delta + V$ has not yet been studied in full. One of the questions which still remain open, is the validity of the conjecture for irrational lattices in dimensions $d \geq 5$.

## 2. Results and discussion

### 2.1. Definitions and Main Theorems

Using a linear non-degenerate change of variables, we can transform the operator (1.1) to the form

$$H = H_0 + V,\ H_0 = H_0^{(l)} = (\mathbf{DGD})^l.$$

Here $\mathbf{D} = -i\nabla$, $\mathbf{G}$ is a positive definite constant $d \times d$-matrix with real-valued entries, and $V$ is a multiplication by a bounded real-valued function, which is periodic with respect to the lattice $\mathbf{\Gamma} = (2\pi\mathbb{Z})^d$. Let

$$\mathcal{O} = [0, 2\pi)^d,\ \mathcal{O}^\dagger = [0, 1)^d$$

be the fundamental domains of the lattice $\mathbf{\Gamma}$ and the dual lattice $\mathbf{\Gamma}^\dagger = \mathbb{Z}^d$ respectively.

To introduce necessary characteristics of the spectrum we need to describe the Floquet decomposition for the operator $H$. Identify the space $L^2(\mathbb{R}^d)$ with the direct integral

$$\mathfrak{G} = \int_{\mathcal{O}^\dagger} \mathfrak{H} d\mathbf{k},\ \mathfrak{H} = L^2(\mathcal{O}),$$

using the unitary Gelfand transform $U : L^2(\mathbb{R}^d) \to \mathfrak{G}$:

$$(Uu)(\mathbf{x}, \mathbf{k}) = e^{-i\mathbf{kx}} \sum_{\mathbf{m}\in\mathbb{Z}^d} e^{-i2\pi\mathbf{km}} u(\mathbf{x} + 2\pi\mathbf{m}),\ \ \mathbf{x} \in \mathcal{O}, \mathbf{k} \in \mathcal{O}^\dagger.$$

Then

$$UHU^* = \int_{\mathcal{O}^\dagger} H(\mathbf{k}) d\mathbf{k},$$

$$H(\mathbf{k}) = H_0(\mathbf{k}) + V,\quad H_0(\mathbf{k}) = (\mathbf{D} + \mathbf{k})\mathbf{G}(\mathbf{D} + \mathbf{k}).$$

The operators $H(\mathbf{k})$ are defined on the common domain $H^{2l}(\mathbb{T}^d)$ and have compact resolvents. Therefore their spectra are purely discrete, and, by a simple perturbation theory argument, their eigenvalues $\lambda_j(\mathbf{k})$ (arranged in increasing order) are continuous functions of $\mathbf{k} \in \mathbb{T}^d$. The spectrum of the initial opetator $H$ can be now represented as the union

$$\sigma(H) = \bigcup_j \ell_j,\ \ell_j = \bigcup_{\mathbf{k}\in\overline{\mathcal{O}^\dagger}} \lambda_j(\mathbf{k}),$$

where the closed intervals $\ell_j$ are called *spectral bands.* Define two quantitative characteristics of overlapping of the bands:

- *multiplicity of overlapping*, which measures the number of bands covering given point $\lambda$:

$$\mathfrak{m}(\lambda) = \#\{j : \lambda \in \ell_j\},$$

and

- *Overlapping function*, which shows how far the bands penetrate into each other:

$$\varsigma(\lambda) = \begin{cases} \max_j \max\{t : [\lambda - t, \lambda + t] \subset \ell_j\}, & \lambda \in \sigma(H), \\ 0, & \lambda \notin \sigma(H). \end{cases}$$

Both these functions were first introduced by M. Skriganov (see [21]). The quantities $\mathfrak{m}(\lambda)$ and $\varsigma(\lambda)$ can be linked with the counting function

$$N(\lambda) = N(\lambda; H(\mathbf{k})) = \#\{j : \lambda_j(\mathbf{k}) \leq \lambda\}$$

of the operator $H(\mathbf{k})$:

$$\begin{cases} \mathfrak{m}(\lambda) \geq \max_{\mathbf{k}} N(\lambda; H(\mathbf{k})) - \min_{\mathbf{k}} N(\lambda; H(\mathbf{k})), \\ \varsigma(\lambda) = \sup\{t : \min_{\mathbf{k}} N(\lambda + t; H(\mathbf{k})) < \max_{\mathbf{k}} N(\lambda - t; H(\mathbf{k}))\}. \end{cases} \tag{2.1}$$

Now we are in a position to state the main results:

**Theorem 2.1 ([16]).** *Let $V$ be a bounded function and $4l > d + 1$. Then there exists a number $\lambda_l = \lambda_l(V, d)$ such that*

$$\mathfrak{m}(\lambda) \geq c_\delta \lambda^{\frac{d-1}{4l} - \delta}, \quad \varsigma(\lambda) \geq c_\delta \lambda^{1 - \frac{d+1}{4l} - \delta} \tag{2.2}$$

*for all $\lambda \geq \lambda_0$, where*

$$\delta = \delta_d = \begin{cases} 0, & d \neq 1 (\operatorname{mod} 4); \\ \textit{arbitrary positive number}, & d = 1 (\operatorname{mod} 4). \end{cases} \tag{2.3}$$

Clearly, either of the estimates (2.2) implies that there are no gaps in the spectrum starting from the point $\lambda_0$, i.e. that the Bethe-Sommerfeld conjecture holds.

Note that the conditions of Theorem 2.1 can be relaxed to allow any periodic bounded self-adjoint perturbation $V$, not necessarily multiplication by a function. In this case we call $V$ periodic if it commutes with shifts by vectors of the lattice. Also, the theorem remains true if $4l = d+1$, $d \neq 1(\operatorname{mod} 4)$, and $\|V\|$ is small enough. We do not provide the full statement for the sake of brevity (see [16] for details). If the condition $4l \geq d + 1$ is not satisfied, one can construct examples of non-local perturbations $V$ for which the number of gaps is infinite (see Subsect. 2.3).

If the perturbation is smooth, then the condition $4l > d + 1$ can be generalized to $8l > d + 3$:

**Theorem 2.2 ([17]).** *Let $V \in C^\infty(\mathbb{T}^d)$ and $8l > d + 3$. Then the assertion of Theorem 2.1 remains true.*

As was indicated in the Introduction, the case of arbitrary dimensions $d \geq 5$ can be dealt with if one assumes that the matrix $\mathbf{G}$ in the definition of the operator $H_0$ is rational, i.e. $\mathbf{G} = \alpha \mathbf{G}'$ where $\mathbf{G}'$ is a matrix with rational entries and $\alpha$ is a positive number. This requirement is equivalent to Skriganov's condition that the lattice should be rational (see [21]). Namely, under this assumption M. Skriganov proves that

$$\mathfrak{m}(\lambda) \geq c\lambda^{\frac{d}{2}-1}, \ d \geq 5, \tag{2.4}$$

for all sufficiently large $\lambda$. Note that the power of $\lambda$ in (2.4) is bigger than in (2.2). The estimate (2.4) is derived in [21] from number-theoretic estimates specific for rational lattices. For general lattices such estimates do not exist. Moreover, one can show that (2.4) is incorrect if $\mathbf{G}$ is not rational and $d$ is large (see Subsect. 2.3).

The proof of Theorems 2.1 and 2.2 also relies on number-theoretic estimates, but of a different nature. We discuss these below.

## 2.2. Integer points in the ellipsoid

Let $\mathcal{C} \subset \mathbb{R}^d$ be a measurable set, and

$$\mathcal{C}^{(\mathbf{k})} = \{\boldsymbol{\xi} \in \mathbb{R}^d : \boldsymbol{\xi} + \mathbf{k} \in \mathcal{C}\}, \ \mathbf{k} \in \mathcal{O}^\dagger.$$

Denote by $\chi(\ \cdot\ ; \mathcal{C})$ the characteristic function of the set $\mathcal{C}$, and by

$$\#(\mathbf{k}; \mathcal{C}) = \sum_{\mathbf{m} \in \mathbb{Z}^d} \chi(\mathbf{m} + \mathbf{k}; \mathcal{C})$$

the number of integer points in $\mathcal{C}^{(\mathbf{k})}$. Using the notation

$$\langle f \rangle = \int_{\mathcal{O}^\dagger} f(\mathbf{k}) d\mathbf{k},$$

for any function $f \in L^1(\mathcal{O}^\dagger)$, one can show that

$$\langle \#(\mathcal{C}) \rangle = \mathrm{vol}(\mathcal{C}). \tag{2.5}$$

We shall be interested in the number of integer points in the ellipsoid determined by the matrix $\mathbf{G}$. Let $\rho > 0, \mathbf{F} = \sqrt{\mathbf{G}}$, and denote by $\mathcal{E}(\rho) = \mathcal{E}(\rho, \mathbf{F})$, the ellipsoid

$$\{\boldsymbol{\xi} \in \mathbb{R}^d : |\mathbf{F}\boldsymbol{\xi}| \leq \rho\}.$$

Notice that the eigenvalues of the unperturbed operator $H_0(\mathbf{k})$ coincide with $|\mathbf{F}(\mathbf{m}+\mathbf{k})|^{2l}$, $\mathbf{m} \in \mathbb{Z}^d$, so that

$$N\big(\rho^{2l}; H_0(\mathbf{k})\big) = \#(\mathbf{k}; \mathcal{E}(\rho)). \tag{2.6}$$

Formula (2.5) shows that the average value of $\#(\mathbf{k}; \mathcal{E}(\rho))$ equals $\mathrm{w}_d\, \rho^d$, where $\mathrm{w}_d = K_d (\det \mathbf{G})^{-1/2}$ and $K_d$ is the volume of the unit ball in $\mathbb{R}^d$. We shall need bounds on the deviation of the number $\#(\mathbf{k}; \mathcal{E}(\rho))$ from its average value. Denote

$$\begin{aligned} \sigma_p(\rho) &= \langle |\#(\mathcal{E}(\rho)) - \mathrm{w}_d\, \rho^d|^p \rangle^{\frac{1}{p}}, \ p \in [1, \infty), \\ \sigma_\infty(\rho) &= \sup_{\mathbf{k}} |\#(\mathbf{k}; \mathcal{E}(\rho)) - \mathrm{w}_d\, \rho^d|. \end{aligned} \tag{2.7}$$

The question of estimating the quantities $\sigma_p$ falls in the same category with the famous circle problem (see e.g. [3]). The circle problem is usually associated with estimating the number $\sigma_\infty$ from above and has been investigated quite well. On the contrary, for our purposes we shall need bounds from below for $\sigma_1(\rho)$:

**Theorem 2.3 ([16]).**

- *Let $\delta = \delta_d$ be as defined in* (2.3). *Then for sufficiently large $\rho$*

$$\sigma_1(\rho) \geq c\rho^{\frac{d-1}{2}-\delta}. \tag{2.8}$$

- *If $d = 1 (\mathrm{mod}\, 4)$ then there exists a sequence $\rho_j \to \infty, j \to \infty$ such that*

$$\sigma_2(\rho_j) \leq C_\varepsilon \rho_j^{\frac{d-1}{2}} (\ln \rho_j)^{(-1+\varepsilon)/d}, \tag{2.9}$$

*where $\varepsilon > 0$ is arbitrary.*

As was shown in [10], $\sigma_2(\rho) \leq C\rho^{(d-1)/2}$ for all $d$, so that (2.8) is sharp if $d \neq 1 (\mathrm{mod}\, 4)$. For these values of $d$ the estimate (2.8) can be easily derived using an argument due to B.E.J. Dahlberg and E. Trubowitz (see [2] and also [6]). The proof for $d = 1 (\mathrm{mod}\, 4)$ is much more involved and was obtained in [16] for the first time. The upper bound (2.9) demonstrates that (2.8) cannot be improved within the power scale if $d = 1 (\mathrm{mod}\, 4)$. The lower bound (2.8) will be crucial for the proof of Theorems 2.1 and 2.2.

Note that (2.8) ensures the estimates (2.2) for the unperturbed operator $H_0(\mathbf{k})$. Indeed, it follows from (2.8) and (2.6) that

$$\begin{cases} \max_{\mathbf{k}} N(\rho^{2l}; H_0(\mathbf{k})) = & \max_{\mathbf{k}} \#(\mathbf{k}; \mathcal{E}(\rho)) \geq \mathrm{w}_d\, \rho^d + c\rho^{\frac{d-1}{2}-\delta}, \\ \min_{\mathbf{k}} N(\rho^{2l}; H_0(\mathbf{k})) = & \min_{\mathbf{k}} \#(\mathbf{k}; \mathcal{E}(\rho)) \leq \mathrm{w}_d\, \rho^d - c\rho^{\frac{d-1}{2}-\delta}. \end{cases} \tag{2.10}$$

Using the formulae (2.1), one immediately obtains (2.2) for the functions $\mathrm{m}(\lambda)$ and $\zeta(\lambda), \lambda = \rho^{2l}$ with $V = 0$.

## 2.3. Circle problem

In [21] M. Skriganov obtained a number of conditional results which relate the behaviour of $\sigma_\infty(\rho)$ as $\rho \to \infty$, with the number of gaps in the spectrum of the Schrödinger operator $H = \mathbf{DGD} + V$ with a periodic perturbation, which is not supposed to be local. In this subsection we comment on some of these results in the light of recent advances in the circle problem.

**Proposition 2.4 ([21], Theorem 15.3).** *Suppose that $d \geq 5$.*

- *Assume that for a rational* $\mathbf{G}$

$$\sigma_\infty(\rho) = O(\rho^{d-2}), \ \ \rho \to \infty. \tag{2.11}$$

*Then there exists a number $t > 0$ and a (non-local) periodic perturbation $V$ with the norm $\|V\| = t$ such that the number of gaps in the spectrum of $H$ is infinite.*

- *Assume that for some* $\mathbf{G}$

$$\sigma_\infty(\rho) = o(\rho^{d-2}),\ \rho \to \infty. \tag{2.12}$$

*Then for any $t > 0$ there exists a (non-local) periodic perturbation $V$ such that $\|V\| < t$ and the number of gaps in the spectrum of $H$ is infinite.*

Note that the bound (2.11) cannot be improved for $\mathbf{G} = \mathbf{I}$. It has been known for a long time (see [13] and [3]) that for $d \geq 2$

$$|\#(\mathbf{k}; \mathcal{E}(\rho)) - \mathrm{w}_d\, \rho^d| = O(\rho^{d-2+b}),\ b = \frac{2}{d+1}. \tag{2.13}$$

If $\mathbf{k} = 0$, then one can take a smaller value of $b > 0$ (see [11] for details). Also, for $d \geq 5$ the estimate (2.11) with $\mathbf{k} = 0$ was shown to hold for rational matrices $\mathbf{G}$ (see [14]) and then for arbitrary diagonal matrices $\mathbf{G}$ (see [7]). In the recent paper [1] (see also [4]) it was shown that (2.11) and (2.12) (for irrational $\mathbf{G}$'s) hold uniformly in $\mathbf{k}$ if $d \geq 9$. Thus the conclusions of Proposition 2.4 are valid in dimensions $d \geq 9$.

In conclusion we remark that (2.13) leads to the following upper bounds for the multiplicity $\mathfrak{m}(\lambda)$ and the overlapping function $\zeta(\lambda)$ for the operator $H = H_0^{(l)} + V$:

$$\mathfrak{m}(\lambda) \leq C\lambda^{\frac{d-2+b}{2l}},\ \zeta(\lambda) \leq C\lambda^{1-\frac{1-b}{l}}.$$

The proof is elementary and can be found in [21]. Moreover, using the same argument, one readily concludes that (2.12) implies the inequality

$$\mathfrak{m}(\lambda) = o\big(\lambda^{\frac{d-2}{2l}}\big),\quad d \geq 9.$$

The last two estimates clearly show that in the case $l = 1$ the bound (2.4) is sharp if $d \geq 9$ and that it cannot hold for irrational matrices $\mathbf{G}$.

## 3. Sketch of the proof of Theorems 2.1, 2.2

Throughout this Section we assume for the sake of simplicity that $V$ satisfies the conditions of Theorem 2.2.

### 3.1. General idea

The proof is based on the following observation. If a point $\lambda$ belongs to a spectral gap of the operator $H$, then $N(\lambda, \mathbf{k}) = \text{const}$, $\forall \mathbf{k} \in \mathcal{O}^\dagger$. Therefore, for any $p > 0$ the $\|N(\lambda;\ \cdot\ ) - \langle N(\lambda)\rangle\|_p = 0$ for all $p \geq 1$. Here by $\|\ \cdot\ \|_p$ we have denoted the usual $L^p(\mathcal{O}^\dagger)$-norm. Thus, in order to prove the Bethe-Sommerfeld conjecture it suffices to check that

$$\|N(\lambda) - \langle N(\lambda)\rangle\|_p > 0,\ \forall \lambda \geq \lambda_0, \tag{3.1}$$

for some $p \geq 1$ and $\lambda_0 \in \mathbb{R}$. Estimate this quantity from below:

$$\begin{aligned} \|N(\lambda) - \langle N(\lambda)\rangle\|_p &\geq \|N_0(\lambda) - \langle N_0(\lambda)\rangle\|_p \\ &\qquad - \|N(\lambda) - N_0(\lambda)\|_p - |\langle N(\lambda)\rangle - \langle N(\lambda)\rangle| \\ &\geq \sigma_p(\rho) - 2T_p(\rho), \quad \lambda = \rho^{2l}, \end{aligned} \tag{3.2}$$

Here $\sigma_p$ is defined as in (2.7) and

$$T_p(\rho) = \|N(\rho^{2l}) - N_0(\rho^{2l})\|_p.$$

Recall that we already have the lower bound (2.8) for $\sigma_1(\rho)$. Now, to verify (3.1), it remains to establish a suitable upper bound for $T_1(\rho)$. It is provided by

**Theorem 3.1.** *Let $V$ satisfy the conditions of Theorem 2.2 and $2l > 1$. Then for any $\beta > d + 1 - 4l$ there exists a number $\rho_0 = \rho_0(\beta)$ such that*

$$T_1(\rho) \leq C_\beta \rho^\beta, \tag{3.3}$$

*for $\rho \geq \rho_0$.*

It follows immediately that

$$|\langle N(\rho^{2l})\rangle - \mathrm{w}_d\, \rho^d| = |\langle N(\rho^{2l})\rangle - \langle N_0(\rho^{2l})\rangle| \leq C_\beta \rho^\beta, \forall \rho \geq \rho_0. \tag{3.4}$$

This Theorem implies Theorem 2.2. Indeed, since $8l > d+3$, we have $d+1-4l < (d-1)/2$. Choose $\beta$ strictly between these two numbers. Then by virtue of (2.8) and Theorem 3.1, the inequality (3.2) ensures that

$$\|N(\lambda) - \langle N(\lambda)\rangle\|_1 \geq c_\delta \rho^{\frac{d-1}{2}-\delta} - C\rho^\beta \geq c' \rho^{\frac{d-1}{2}-\delta}, \tag{3.5}$$

thus yielding (3.1) for $8l > d + 3$. To prove the estimates (2.2) one notes that the above lower bound leads to the following two estimates similar to (2.10):

$$\begin{cases} \max_{\mathbf{k}} N(\rho^{2l}; H(\mathbf{k})) \geq & \langle N(\rho^{2l})\rangle + c\rho^{\frac{d-1}{2}-\delta} \geq \mathrm{w}_d\, \rho^d + c'\rho^{\frac{d-1}{2}-\delta}, \\ \min_{\mathbf{k}} N(\rho^{2l}; H(\mathbf{k})) \leq & \langle N(\rho^{2l})\rangle - c\rho^{\frac{d-1}{2}-\delta} \leq \mathrm{w}_d\, \rho^d - c'\rho^{\frac{d-1}{2}-\delta}. \end{cases}$$

Here we have used (3.4). Now the same argument as the one employed at the end of Subsect. 2.2, provides (2.2).

*Remark 3.2.* One could attempt to use $L^p$-estimates with other values of $p \neq 1$. For instance, one could try to establish a suitable upper bound on $T_\infty(\rho)$, and then apply the bounds (2.10) directly. This approach was adopted in [22] for $d = 3, l = 1$, where the proof of an appropriate estimate for $T_\infty(\rho)$ was based essentially on the upper bound (2.13). This method does not work for higher dimensions as the discrepancy between the orders of $\rho$ in (2.13) and (2.10) becomes too large. On the contrary, the use of $L^1$-estimates for $l = 1$ allows to extend the validity of (2.2) to $d = 4$ as well.

The rest of the paper concentrates on the proof of Theorem 3.1.

## 3.2. Operators with constant coefficients

The proof of Theorem 3.1 will be reduced to the study of $H_0(\mathbf{k})$ with a perturbation having contstant coefficents, instead of $V$. Let $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n$ be a collection of non-intersecting sets in $\mathbb{R}^d$, and let $g_j \in \mathbb{R}$, $j = 1, 2, \dots n$ be a sequence of real numbers such that $|g_j| \le g_0$. Define the operator

$$X(\mathbf{k}) = H_0(\mathbf{k}) + \sum_{j=1}^{n} g_j \chi(\mathbf{D} + \mathbf{k}; \mathcal{C}_j), \mathbf{k} \in \mathcal{O}^\dagger.$$

Using the fact that $X(\mathbf{k})$ has constant coefficients, and relying upon (2.5), we shall estimate the $L^1$-norm of $N(\lambda; X(\mathbf{k})) - N(\lambda; H_0(\mathbf{k}))$ via the volume of the sets

$$\mathcal{D}_j(\rho) = \begin{cases} (\mathcal{E}(\rho) \setminus \mathcal{E}(\tau_j)) \cap \mathcal{C}_j, g_j \ge 0; \\ (\mathcal{E}(\tau_j) \setminus \mathcal{E}(\rho)) \cap \mathcal{C}_j, g_j < 0; \end{cases} \qquad \rho^{2l} = \lambda, \ \tau_j^{2l} = \lambda - g_j.$$

**Lemma 3.3 ([17]).** *Let $X(\mathbf{k})$ be as defined above. Then*

$$\langle |N(\lambda; X) - N(\lambda; H_0)| \rangle \le \sum_{j=1}^{n} \mathrm{vol}(\mathcal{D}_j(\rho))$$

*for any $\lambda = \rho^{2l} \ge g_0$.*

*Sketch of the proof.* Suppose for simplicity that $n = 1$. Denote $\mathcal{P} = \chi(\mathbf{D} + \mathbf{k}; \mathcal{C}_1)$, $\mathcal{Q} = \mathbf{I} - \mathcal{P}$. Since $X = H_0\mathcal{Q} \oplus \mathcal{P}X\mathcal{P}$, it is clear that

$$\begin{aligned} N(\lambda; X(\mathbf{k})) &= N(\lambda; H_0(\mathbf{k})\mathcal{Q}) + N(\lambda; X(\mathbf{k})\mathcal{P}) \\ &= N(\lambda; H_0(\mathbf{k})\mathcal{Q}) + N(\lambda - g_1; H_0(\mathbf{k})\mathcal{P}). \end{aligned}$$

By definition of $\mathcal{P}$ it is straightforward to rewrite this formula as follows:

$$N(\lambda; X(\mathbf{k})) = \#(\mathbf{k}; \mathcal{C}'(\lambda)), \ \mathcal{C}'(\lambda) = (\mathcal{E}(\rho) \setminus \mathcal{C}_1) \bigcup (\mathcal{E}(\tau_1) \cap \mathcal{C}_1).$$

Using the set $\mathcal{D}_1$ one concludes that

$$\mathcal{C}' = \begin{cases} \mathcal{E}(\rho) \cup \mathcal{D}_1(\rho), \ g_1 < 0; \\ \mathcal{E}(\rho) \setminus \mathcal{D}_1(\rho), \ g_1 \ge 0. \end{cases}$$

Consequently,

$$\#(\mathbf{k}; \mathcal{C}') = \#(\mathbf{k}; \mathcal{E}(\rho)) \mp \#(\mathbf{k}; \mathcal{D}_1(\rho)), \ \pm g_1 \ge 0.$$

By (2.5)

$$\langle |N(\lambda; X) - N(\lambda; H_0)| \rangle = \langle \#(\mathcal{D}_1(\rho)) \rangle = \mathrm{vol}(\mathcal{D}_1(\rho)),$$

which completes the proof. $\square$

### 3.3. Reduction to the operator $X$

Before stating the precise result which is used in the proof of Theorem 3.1, we shall give a simple illustration of application of Lemma 3.3, which will lead to Theorem 2.1.

Let us write a rough estimate for $H(\mathbf{k}) = H_0(\mathbf{k}) + V$:

$$H_0(\mathbf{k}) - v \le H_0(\mathbf{k}) + V \le H_0(\mathbf{k}) + v, \ v = \|V\|_{L^\infty}, \tag{3.6}$$

and apply Lemma 3.3 with $n = 1$, $\mathcal{C}_1 = \mathbb{R}^d$ and $g_1 = \pm v$ to both sides of this inequality. Then $\mathrm{vol}(\mathcal{D}_1(\rho)) \le Cv\rho^{d-2l}$ for both $+$ and $-$ case. Consequently

$$\begin{aligned} T_1(\rho) &\le \langle |N(\rho^{2l}; H_0 + v) - N(\rho^{2l}; H_0)| \rangle + \langle |N(\rho^{2l}; H_0 - v) - N(\rho^{2l}; H_0)| \rangle \\ &\le Cv\rho^{d-2l}. \end{aligned}$$

This estimate is certainly worse than (3.3). Nevertheless, similarly to (3.5), we have

$$\|N(\lambda) - \langle N(\lambda) \rangle\|_1 \ge c_\delta \rho^{\frac{d-1}{2}-\delta} - Cv\rho^{d-2l},$$

so that the r.h.s. is bounded from below by $c\rho^{(d-1)/2-\delta}$ if $4l > d+1$ and $\delta > 0$ is sufficiently small. As before, this leads to (2.2), thereby proving Theorem 2.1. Note that the only information on the perturbation $V$ we used, was the boundedness of $V$.

To obtain more precise estimate (3.3) we should make a more subtle choice of the sets $\mathcal{C}_j$. To avoid cumbersome calculations assume that the potential $V$ is the following trigonometric function:

$$V(\mathbf{x}) = 2\,\mathrm{Re}\big(V_{\boldsymbol\theta} e^{i\boldsymbol\theta\mathbf{x}}\big), \ V_{\boldsymbol\theta} \in \mathbb{C},$$

with some $\boldsymbol\theta \in \mathbb{Z}^d$. Let $\gamma_j, j = 0, 1, \dots, n+1$, be a strictly decreasing sequence of real numbers such that $\gamma_{n+1} = 0$, and

$$\begin{aligned} \gamma_j &= 2l - 1 - j\varepsilon, \ \varepsilon = \beta - (d + 1 - 4l), \\ \eta_j &= \ - j\varepsilon, \ j = 0, 1, 2, \dots, n. \end{aligned}$$

The number $n$ is chosen from the restriction $0 < 2l - 1 - n\varepsilon \le \varepsilon$. Let $\Lambda_0 = \mathbb{R}^d$ and

$$\Lambda_j = \Lambda_j^{(+)} \bigcup \Lambda_j^{(-)},$$
$$\Lambda_j^{(\pm)} = \{\boldsymbol\xi \in \mathbb{R}^d : \big||\mathbf{F}\boldsymbol\xi| - \rho\big| \le \sigma\rho^{\eta_j}, \ |\boldsymbol\theta \mathbf{G}(\boldsymbol\xi \pm \boldsymbol\theta/2)| \le 3\sigma\rho^{\eta_j+1}\},$$

where the parameter $\sigma$ will be chosen later. Since $\eta_j$ form a decreasing sequence, we have $\Lambda_n \subset \Lambda_{n-1} \subset \dots \subset \Lambda_0$. Define the non-intersecting sets

$$\mathcal{C}_n = \Lambda_n, \ \mathcal{C}_j = \Lambda_j \setminus \Lambda_{j+1}, \ \ j = 0, 1, \dots, n-1.$$

Now instead of the elementary rough estimate (3.6) we have

**Theorem 3.4 ([17]).** *Let $V$ be as above, and let*

$$X_{\pm}(\rho;\mathbf{k}) = H_0(\mathbf{k}) \pm v \sum_{j=0}^{n} \rho^{-\gamma_{j+1}} \chi(\mathbf{D}+\mathbf{k};\mathcal{C}_j),$$

*with $v = \max|V|$. Then there exists a number $\sigma = \sigma(v)$ such that*

$$X_{-}(\rho;\mathbf{k}) \le H_0(\mathbf{k}) + V \le X_{+}(\rho;\mathbf{k}),$$

*for all sufficiently large $\rho$.*

The definition of the sets $\Lambda_j$ may look artificial, but they are natural for the problem at hand. Similar sets were used by M. Skriganov (see [21], [22]) and also by Yu. Karpeshina (see [9]). In the literature they are sometimes referred to as "resonant" sets.

Now, to complete the proof of Theorem 3.1 it remains to apply Lemma 3.3 to the operators $X_+$ and $X_-$. As in the proof of Theorem 2.1 above, $\mathrm{vol}(\mathcal{D}_0(\rho)) \le C\rho^{-\gamma_1}\rho^{d-2l} = C\rho^{\beta}$. Furthermore, it is a simple geometrical exercise to realize that for $j \ge 1$

$$\begin{aligned}\mathcal{D}_j(\rho) \subset \{\boldsymbol{\xi} \in \mathbb{R}^d : \rho - Cg_j\rho^{1-2l} \le |F\boldsymbol{\xi}| \le \rho + Cg_j\rho^{1-2l},\\ |\boldsymbol{\theta}\mathbf{G}(\boldsymbol{\xi}+\boldsymbol{\theta}/2)| \le 3\sigma\rho^{\eta_j+1} \ \text{ or } \ |\boldsymbol{\theta}\mathbf{G}(\boldsymbol{\xi}-\boldsymbol{\theta}/2)| \le 3\sigma\rho^{\eta_j+1}\},\end{aligned}$$

with $g_j = \rho^{-\gamma_{j+1}}$. It is now easy to conclude that

$$\begin{aligned}\mathrm{vol}(\mathcal{D}_j(\rho)) &\le C\sigma g_j \rho^{d-1-2l}\rho^{\eta_j+1} = C\sigma\rho^{-\gamma_{j+1}+d-1-2l+\eta_j+1}\\ &= C'\rho^{-2l+1+\varepsilon+d-1-2l+1} = C'\rho^{\beta},\end{aligned}$$

for all $j \ge 1$. Now by Lemma 3.3

$$\langle |N(\lambda;X_{\pm}) - N(\lambda;H_0)|\rangle \le \sum_{j=0}^{n} \mathrm{vol}(\mathcal{D}_j(\rho)) \le C\rho^{\beta}.$$

By virtue of Theorem 3.4 this implies (3.3).

In the case of a general potential $V \in C^{\infty}(\mathbb{R}^d)$ one approximates $V$ by a truncated Fourier series, and then constructs the sets $\mathcal{C}_j$ for each Fourier component.

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