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Journées Équations aux dérivées partielles (1999), p. 1-10

<http://www.numdam.org/item?id=JEDP_1999____A8_0>

Journées *Équations aux dérivées partielles*
Saint-Jean-de-Monts, 31 mai–4 juin 1999
GDR 1151 (CNRS)

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Abstract

After introducing the notion of capacity in a general Hilbert space setting we look at the spectral bound of an arbitrary self-adjoint and semi-bounded operator H. If H is subjected to a domain perturbation the spectrum is shifted to the right. We show that the magnitude of this shift can be estimated in terms of the capacity. We improve the upper bound on the shift which was given in *Capacity in abstract Hilbert spaces and applications to higher order differential operators* (*Comm. P. D. E.*, 24:759–775, 1999) and obtain a lower bound which leads to a generalization of Thirring's inequality if the underlying Hilbert space is an L^2-space. Moreover, a similar capacitary upper bound for the second eigenvalue is established. The results are finally applied to higher-order partial differential operators.

1. Introduction.

During the last decades many mathematicians studied the problem of analyzing the connection between domain perturbations of a given self-adjoint operator and changes of its spectrum. Among others Rauch and Taylor [Rau75], [Tay76], [Tay79], [RT75a], [RT75b] worked on that subject and pointed out the relevance of this kind of problem for various areas of Mathematical Physics.

In particular, the bottom eigenvalue of the Laplacian is an important quantity since the Laplacian plays a fundamental role in Quantum mechanics, theory of heat, theory of vibrations and other areas.

The following result due to Taylor [Tay76] shows that the capacity can be used to give quantitative upper and lower bounds for the bottom eigenvalue of the Laplacian.

Theorem 1.1 *If $\Omega \subset \mathbb{R}^d$ is a bounded domain, λ is the lowest eigenvalue for $-\Delta$ on $\Omega \setminus K$ with Dirichlet boundary conditions on ∂K, Neumann boundary conditions on $\partial\Omega$, $K \subset \Omega$ compact, then there are constants $c_1, c_2 > 0$ such that for small* $\mathrm{Cap}_0(K)$

$$c_1 \mathrm{Cap}_0(K) \leq \lambda \leq c_2 \mathrm{Cap}_0(K).$$

I would like to thank Sven Eder and Walter Renger for fruitful discussions and Michael Demuth, Guy Métivier and Didier Robert for giving me the chance to take part in the conference.

where $\mathrm{Cap}_0(\cdot)$ *is the classical zero-order capacity in* $\mathbb{R}^d$*, which is defined by*

$$\mathrm{Cap}_0(K) = \inf\left\{ \int_{\mathbb{R}^d} |(\nabla u)(x)|^2 dx : \begin{array}{l} u \in C_c^\infty(\mathbb{R}^d), u \geq 1 \ \ a.e. \\ on\ some\ neighborhood\ of\ K \end{array} \right\}.$$

Taylor's result has been generalized in many directions. For example Gesztesy and Zhao proved in [GZ94] that the bottom eigenvalue of certain Schrödinger operators remains unaffected by domain perturbations if and only if the perturbed domain differs from the unperturbed one only by a set of zero capacity. The proof is based on Brownian motion and on the Feynman Kac formula. One year later Arendt and Monniaux gave an analytical proof of this result which uses a domination argument for semigroups as its main ingredient [AM95]. Their result allowed the potential to vary as well. We refer to the survey article [DMN97] for precise statements and the proofs as well as for further literature on this subject. Let us also remark that similar estimates for the bottom eigenvalue also hold for the Laplace-Beltrami operator on a Riemannian manifold, see [CF78], [CF88], [Cou95], [Oza82], [Oza83]. Another generalization of Taylor's Theorem was given by McGillivray [McG96] who proved the same estimate in the context of regular Dirichlet forms.

Most of the known estimates for the bottom eigenvalue only apply to second order differential operators and some of them [Szn98], [Tay79], [Oza81], [MR84], can only handle the Laplacian. This is due to the fact that in the case of second order differential operators there is an interplay between analysis and stochastics via the theory of Dirichlet forms, but there is no such interplay for higher order differential operators. On the other hand higher order differential operators are important as well. For instance the dynamics of the clamped plate is described by the bi-potential equation

$$\Delta^2 u = 0,$$

where Δ^2 is the biharmonic operator, subject to Dirichlet boundary conditions.

In order to include these operators one needs a more general method. The results for self-adjoint operators in general Hilbert spaces (i.e. not necessarily L^2-spaces) which are described in Section 4 turn out to be an appropriate tool in estimating eigenvalues for a wide class of self-adjoint operators.

2. Domain perturbations in general Hilbert spaces.

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be an arbitrary real or complex Hilbert space and let H be a self-adjoint operator in $\mathcal{H}$ which is semi-bounded from below with spectral bound $\lambda := \inf \sigma(H)$. Let $(\mathcal{E}, \mathcal{F})$ be the non-negative closed quadratic form which corresponds to $H - \lambda$ in the usual sense. We then have $\mathcal{F} = \mathrm{dom}((H-\lambda)^{1/2})$ and

$$\mathcal{E}(u,v) := \langle (H-\lambda)^{1/2} u, (H-\lambda)^{1/2} v\rangle \quad u, v \in \mathcal{F}.$$

In what follows we will use the abbreviations

$$\mathcal{E}_1(u,v) := \mathcal{E}(u,v) + \langle u, v\rangle, \quad \mathcal{E}[u] := \mathcal{E}(u,u), \quad \mathcal{E}_1[u] := \mathcal{E}_1(u,u).$$

Since $(\mathcal{E}, \mathcal{F})$ is closed, the space $(\mathcal{F}, \mathcal{E}_1(\cdot,\cdot))$ is a Hilbert space.

Suppose that $\mathcal{G}$ is a closed subspace of the Hilbert space $(\mathcal{F}, \mathcal{E}_1(\cdot, \cdot))$. Then the form $(\mathcal{E} + \lambda, \mathcal{G})$ is semi-bounded from below and closed. Moreover it is densely defined in $\mathcal{H}^{\mathcal{G}} := \overline{\mathcal{G}}$, the closure being taken with respect to the topology of $\mathcal{H}$. Hence there is a unique self-adjoint operator $H^{\mathcal{G}}$ in $\mathcal{H}^{\mathcal{G}}$ that corresponds to $(\mathcal{E} + \lambda, \mathcal{G})$.

Definition 2.1 We call the operator $H^{\mathcal{G}}$ the *domain perturbation of* H *with respect to the subspace* $\mathcal{G}$.

Let us illustrate this construction in the case of the Dirichlet Laplacian.

Example 2.2 Let Ω and Λ be open subsets of $\mathbb{R}^d$ with $\Lambda \subset \Omega$ and let $H = -\Delta_\Omega$ be the Dirichlet Laplacian in $\mathcal{H} = L^2(\Omega)$. The form $(\mathcal{E}, \mathcal{F})$ which corresponds to H is given by

$$\mathcal{F} = H_0^1(\Omega), \quad \mathcal{E}[u] = \int_\Omega |\nabla u|^2 \mathrm{d}x \quad \text{for } u \in \mathcal{F}.$$

Clearly the Sobolev space $\mathcal{G} := H_0^1(\Lambda)$ may be viewed as a closed subspace of $\mathcal{F}$. Hence we can apply the above construction. By definition the Hilbert space $\mathcal{H}^{\mathcal{G}}$ equals the L^2-closure of $\mathcal{G}$, i.e. we have $\mathcal{H}^{\mathcal{G}} = L^2(\Lambda)$. Moreover the restriction of $(\mathcal{E}, \mathcal{F})$ to $\mathcal{G}$ is obviously just the form which is associated to the Dirichlet Laplacian in $L^2(\Lambda)$. Hence $H^{\mathcal{G}} = -\Delta_\Lambda$.

3. The capacity of a subspace.

We now want to introduce a notion of capacity in a general Hilbert space setting. With the notation of the previous section the definition of the abstract capacity reads as follows.

Definition 3.1 Let $\mathcal{G}$ be a closed subspace of $(\mathcal{F}, \mathcal{E}_1(\cdot, \cdot))$ and let $u \in \mathcal{F}$. The *u-capacity of* $\mathcal{G}$ is defined by

$$\mathrm{Cap}_u(\mathcal{G}) := \mathcal{E}_1[P_{\mathcal{G}} u],$$

where $P_{\mathcal{G}}$ is the orthogonal projection onto $\mathcal{G}$ in $(\mathcal{F}, \mathcal{E}_1(\cdot, \cdot))$.

Let us examine what $\mathrm{Cap}_u(\mathcal{G})$ looks like if $\mathcal{H}$ is an L^2-space and $\mathcal{G}$ consists of functions that vanish on a prescribed set. As we shall see in Proposition 3.2 (a) below, the capacity of Definition 3.1 admits a description similar to the classical zero-order capacity in $\mathbb{R}^d$.

Let X be a Hausdorff space which is equipped with a strictly positive Borel measure m (i.e. each open set has positive measure) and let A be an arbitrary subset of X. Define $\mathcal{F}_A$ as the closure of the space

$$\mathcal{F}_A^\circ := \{v \in \mathcal{F} : v = 0 \ \ m\text{-a.e. on some neighborhood of } A\}, \tag{1}$$

where the closure is taken with respect to the topology induced by the scalar product $\mathcal{E}_1(\cdot, \cdot)$.

Proposition 3.2 *For any $u \in \mathcal{F}$ the following equations hold.*

(a) $\mathrm{Cap}_u(\mathcal{F}_A^\perp) = \inf\{\mathcal{E}_1[v] : v \in \mathcal{F}, v = u$ *m-a.e. on some nhd. of* $A\}$,

(b) $\mathrm{Cap}_u(\mathcal{F}_A^\perp) = \inf\{\mathrm{Cap}_u(\mathcal{F}_U^\perp) : U \supset A, U$ *open*$\}$.

For a proof see [Nol97]. From this proposition it is easy to obtain the following result:

Proposition 3.3 *If* H *is such that* $(\mathcal{E}, \mathcal{F})$ *is a regular Dirichlet form in* $L^2(X, m)$, $B \subset X$ *is any Borel set and* $u \in \mathcal{F}$ *is such that* $u = 1$ *in some neighborhood of* B, *then*

$$\mathrm{Cap}(B) = \mathrm{Cap}_u(\mathcal{F}_B^\perp),$$

where $\mathrm{Cap}(B)$ *is the capacity of the regular Dirichlet form* $(\mathcal{E}, \mathcal{F})$, *see [FOT94].*

4. Capacitary estimates for domain perturbations.

This section contains the main results of the article. We shall overcome the restrictions mentioned in the motivation by proving abstract eigenvalue estimates which use the capacity of Definition 3.1. To this aim we assume from now that $H^{\mathcal{G}}$ is the domain perturbation of some self-adjoint and semi-bounded operator H with respect to the subspace $\mathcal{G}$. Our first result gives an upper bound for the spectral bound $\lambda^{\mathcal{G}} = \inf \sigma(H^{\mathcal{G}})$ of $H^{\mathcal{G}}$ in terms of the ϕ_n-capacities of $\mathcal{G}^\perp$ where the ϕ_n's are normalized eigenfunctions of H. The proof can be found in [Nol98].

Theorem 4.1 *Let* $\{\lambda_n\}$ *be the (finite or infinite) set of eigenvalues for* H *with corresponding normalized eigenfunctions* ϕ_n. *Then*

$$\lambda^{\mathcal{G}} \leq \inf_n \left(\lambda_n + \frac{\mathrm{Cap}_{\phi_n}(\mathcal{G}^\perp)}{1 - \frac{\mathrm{Cap}_{\phi_n}(\mathcal{G}^\perp)}{1+\lambda_n-\lambda}} \right).$$

As an immediate consequence we note the following corollary.

Corollary 4.2 *(a) Suppose that* λ *is an eigenvalue of* H *with normalized eigenfunction* $\phi \notin \mathcal{G}^\perp$. *Then*

$$\lambda^{\mathcal{G}} - \lambda \leq \frac{\mathrm{Cap}_\phi(\mathcal{G}^\perp)}{1 - \mathrm{Cap}_\phi(\mathcal{G}^\perp)}. \tag{2}$$

(b) Fix $\delta \in [0, 1)$. *For each closed subspace* $\mathcal{G}$ *of* $(\mathcal{F}, \mathcal{E}_1(\cdot, \cdot))$ *with* $\mathrm{Cap}_\phi(\mathcal{G}^\perp) \leq \delta$ *we have*

$$\lambda^{\mathcal{G}} - \lambda \leq \frac{1}{1-\delta} \mathrm{Cap}_\phi(\mathcal{G}^\perp).$$

Corollary 4.2 (a) can be used to prove a capacitary criterion for the perturbed operator to have a bottom eigenvalue.

Corollary 4.3 *If the spectral bound* λ *is an isolated eigenvalue of* H *of finite multiplicity and some normalized* ϕ *in the eigenspace* $\ker(H - \lambda)$ *satisfies*

$$\lambda + \frac{\mathrm{Cap}_\phi(\mathcal{G}^\perp)}{1 - \mathrm{Cap}_\phi(\mathcal{G}^\perp)} < \inf(\sigma(H) \setminus \{\lambda\}),$$

then $\lambda^{\mathcal{G}}$ *is an isolated eigenvalue of* $H^{\mathcal{G}}$ *of finite multiplicity.*

Proof. Just use Corollary 4.2 (a) and the minimax principle. □
It is possible to obtain capacitary lower bounds for the shift $\lambda^{\mathcal{G}} - \lambda$ similar to those of Corollary 4.2 if λ is a simple and isolated eigenvalue of H.

Theorem 4.4 *Suppose that λ is a simple eigenvalue of H with normalized eigenfunction ϕ. Let $\mu \in [0, \infty)$ be the spectral gap, i.e.*

$$\mu := \inf\{\sigma(H) \setminus \{\lambda\}\} - \lambda. \tag{3}$$

Then

$$\lambda^{\mathcal{G}} - \lambda \geq \frac{\mu \mathrm{Cap}_\phi(\mathcal{G}^\perp)}{1 + \mu(1 - \mathrm{Cap}_\phi(\mathcal{G}^\perp))} \geq \frac{\mu}{\mu + 1} \mathrm{Cap}_\phi(\mathcal{G}^\perp).$$

The proof is carried out in [Nol99]. ¿From Corollary 4.2 (a) and Theorem 4.4 we immediately obtain the following characterization for the shift $\lambda^{\mathcal{G}} - \lambda$ to be positive.

Corollary 4.5 *Suppose that λ is a simple and isolated eigenvalue of H with eigenfunction ϕ. Then $\lambda^{\mathcal{G}} > \lambda$ if and only if* $\mathrm{Cap}_\phi(\mathcal{G}^\perp) > 0$.

In practice it seems to be difficult to compute the capacities involved in Theorems 4.1 and 4.4 since both $\mathcal{G}$ and $\mathcal{G}^\perp$ are typically infinite dimensional, e.g. if H is an operator in $L^2(\mathbb{R}^d)$ and $H^{\mathcal{G}}$ arises from H by imposing boundary conditions on a set with non-empty interior. It is possible, however, to prove another lower bound for the shift of the spectral bound which involves the u-capacity of the eigenspace $\mathcal{K} := \ker(H - \lambda)$ for $u \in \mathcal{G}$ rather than the ϕ-capacity of $\mathcal{G}$ for $\phi \in \mathcal{K}$ as in Theorems 4.1 and 4.4. Since $\mathcal{K}$ happens to be one-dimensional in many interesting cases, e.g. if H is the Dirichlet Laplacian on some bounded domain in $\mathbb{R}^d$, it is possible to compute the u-capacity of $\mathcal{K}$ explicitly in these cases. The proofs of the remaining results in this section can be found in [Nol98].

Theorem 4.6 *Suppose that λ is an eigenvalue of H. Denote the corresponding eigenspace by $\mathcal{K}$ and let*

$$\delta := \sup\{\mathrm{Cap}_u(\mathcal{K}) : u \in \mathcal{G}, \|u\| = 1\}.$$

Then $\lambda^{\mathcal{G}} - \lambda \geq \mu(1 - \delta)$, where μ is as in (3) the spectral gap.

As an application of Theorem 4.6 let us state a generalization of Thirring's inequality which reads as follows: Let $\Omega, \Lambda \subset \mathbb{R}^d$ be bounded domains with $\Lambda \subset \Omega$. Denote by $(\lambda_j(\Omega))_{j\geq 1}$ and $(\lambda_j(\Lambda))_{j\geq 1}$ the eigenvalues of the Dirichlet Laplacian in $L^2(\Omega)$ and in $L^2(\Lambda)$ respectively. Then

$$\lambda_1(\Lambda) \geq \lambda_1(\Omega) \int_\Lambda |\phi|^2 \mathrm{d}x + \lambda_2(\Omega) \int_{\Omega\setminus\Lambda} |\phi|^2 \mathrm{d}x.$$

A proof of this result can be found in [Szn98]. For the generalization of Thirring's inequality we use the previous notation and assume additionally that (X, m) is a measure space, $\mathcal{H} = L^2(X, m)$ and $\mathcal{G}$ is a subspace of $\mathcal{F}$ such that $\mathcal{H}^{\mathcal{G}} = L^2(Y, m)$ with some measurable set $Y \subset X$.

Corollary 4.7 *Suppose that λ is a simple eigenvalue of H with normalized eigenfunction ϕ. Then*

$$\lambda^{\mathcal{G}} \geq \lambda \int_Y |\phi|^2 \mathrm{d}m + (\lambda + \mu) \int_{X \setminus Y} |\phi|^2 \mathrm{d}m,$$

where μ is as in (3) the spectral gap.

There are only few results which allow the treatment of higher eigenvalues for general classes of operators. One of those is the previously mentioned work of McGillivray [McG96] who proved in the context of regular Dirichlet forms that under an ultracontractivity condition the shift of the higher eigenvalues may be estimated from above by some constant times the capacity of the set on which the domain perturbation takes place. Next we state a similar estimate for the second eigenvalue in our general Hilbert space setting.

To keep things simple we assume from now on that H has purely discrete spectrum, i.e. there is an orthonormal basis $(\phi_j)_{j\in\mathbb{N}}$ of $\mathcal{H}$ and a non-decreasing sequence $(\lambda_j)_{j\in\mathbb{N}}$ of real numbers such that $\lambda_n \to \infty$,

$$\mathrm{dom}(H) = \{u \in \mathcal{H} : (\lambda_j \langle \phi_j, u\rangle)_{j\in\mathbb{N}} \in \ell^2\}$$

and

$$Hu = \sum_{j\in\mathbb{N}} \lambda_j \langle \phi_j, u\rangle \phi_j$$

for all $u \in \mathrm{dom}(H)$.

Since we are interested in the shift of the eigenvalues we may assume without loss of generality that the bottom eigenvalue λ_1 is equal to zero. It is a consequence of the minimax principle that the spectrum of $H^{\mathcal{G}}$ is again discrete and that each eigenvalue, not only the lowest, gets shifted to the right. The following result gives an upper bound for the shift of the second eigenvalue.

Theorem 4.8 *Let λ_1 be a simple eigenvalue of H. Then, for each $c > 0$ there are constants $c_1, \varepsilon > 0$ such that for all closed subspaces $\mathcal{G}$ of $\mathcal{F}$ with $\mathrm{Cap}_{\phi_2}(\mathcal{G}^\perp) < \varepsilon$ and satisfying*

$$\mathrm{Cap}_{\phi_1}(\mathcal{G}^\perp) \leq c\mathrm{Cap}_{\phi_2}(\mathcal{G}^\perp)$$

we have

$$\lambda_2^{\mathcal{G}} - \lambda_2 \leq c_1 \mathrm{Cap}_{\phi_2}(\mathcal{G}^\perp).$$

The constant c_1 can be given explicitly:

$$c_1 = 2(1+\lambda_2)^2(1 + 2c(1+\lambda_2)^2).$$

5. Applications to differential operators of arbitrary order.

In this section the general results of Section 4 are applied to the special case of differential operators on open subsets of $\mathbb{R}^d$. We shall treat operators of arbitrary even order $2m$. Let $\Omega \subset \mathbb{R}^d$ be an open set. The operators we are interested in are formally given by

$$(Hu)(x) = \sum_{|\alpha|,|\beta|\leq m} (-1)^\alpha D^\alpha(a_{\alpha\beta}(x)D^\beta u(x)) \tag{4}$$

with coefficient functions $a_{\alpha\beta} = \overline{a_{\beta\alpha}} \in L^1_{loc}(\Omega)$. To give a rigorous definition of H we consider the quadratic form

$$a(u,v) := \sum_{|\alpha|,|\beta|\leq m} \int_\Omega a_{\alpha\beta}(x) D^\alpha u D^\beta \overline{v}\, dx, \tag{5}$$

initially defined on $C_c^\infty(\Omega)$, the space of smooth functions with compact support. In order to obtain a self-adjoint operator we need to know that a is closable. The requirement of closability is an implicit assumption on the coefficients $a_{\alpha\beta}$ and there are many criteria known on the coefficients $a_{\alpha\beta}$ implying closability of a, see e.g. Davies [Davb], [Dava] or Agmon [Agm65], Section 7. We will always assume that for some constants $b \in \mathbb{R}$, $c > 0$, and all $u \in C_c^\infty(\Omega)$ the following inequality holds.

$$c^{-1}\|u\|_m^2 \leq a(u,u) + b\|u\|^2 \leq c\|u\|_m^2, \tag{6}$$

where $\|\cdot\|_m$ is the Sobolev norm

$$\|u\|_m := \left(\sum_{|\alpha|\leq m} \int_\Omega |D^\alpha u|^2 dx \right)^{1/2}.$$

Observe that a quadratic form satisfying inequality (6) is always closable and that (6) is still valid for $u \in H_0^m(\Omega)$, the Sobolev space of order m, because $H_0^m(\Omega)$ is by definition the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_m$. This also makes clear that the closure $\overline{a}$ of a has domain $\mathcal{F} = H_0^m(\Omega)$. Let H be the self-adjoint operator corresponding to $(\overline{a}, \mathcal{F})$, put $\lambda = \inf \sigma(H)$ and let $\mathcal{E} := \overline{a} - \lambda$ be the non-negative form associated to $H - \lambda$.

We now want to look at domain perturbations arising from subspaces of $\mathcal{F}$ which consist of functions that vanish on some set $A \subset \Omega$. Therefore we define $\mathcal{F}_A$ as in (1). If $A = \overline{B}_r(x_0) := \{x \in \Omega : |x - x_0| \leq r\}$ we write $\mathcal{F}_{r,x_0}$ for short. In the following theorems H_{r,x_0} denotes the self-adjoint operator in $L^2(\Omega \setminus \overline{B}_r(x_0))$ which is associated to $(\overline{a}, \mathcal{F}_{r,x_0})$ and $\lambda_{r,x_0} := \inf \sigma(H_{r,x_0})$ its spectral bound.

Theorem 5.1 *Let $d > 2m$ and let $a_{\alpha\beta} = \overline{a_{\beta\alpha}} \in L^1_{loc}(\Omega)$ such that inequality (6) holds. Moreover suppose that $\lambda = \inf\sigma(H)$ is a simple and isolated eigenvalue with eigenfunction $\phi \notin \mathcal{F}_{r,x_0}^\perp$. If $x_0 \in \Omega$ is such that ϕ is continuous in x_0 with $\phi(x_0) \neq 0$, then there are constants $c_1, c_2 > 0$ such that for small $r > 0$*

$$c_1 r^{d-2m} \leq \lambda_{r,x_0} - \lambda \leq c_2 r^{d-2m}. \tag{7}$$

The proof can be found in [Nol99].

Remark 5.2 This result is already contained in [Maz85], Chapter 10 in the case of operators being defined on the unit cube in $\mathbb{R}^d$. Due to the ignorance of the speaker this fact was not mentioned during the talk.

Next we treat the second eigenvalue. For simplicity we restrict ourselves to constant coefficient operators. Let Ω be an open and bounded subset of $\mathbb{R}^d$ and let

$$H = \sum_{|\alpha|,|\beta|\leq m} (-1)^\alpha D^\alpha(a_{\alpha_\beta} D^\beta u), \quad a_{\alpha\beta} = \overline{a_{\beta_\alpha}} \in \mathbb{C}$$

be an elliptic constant coefficient operator of order $2m$, defined as the closure of the form

$$(u,v) \mapsto \sum_{|\alpha|,|\beta|\leq m} \int_\Omega a_{\alpha\beta} D^\alpha u D^\beta \overline{v} dx,$$

initially defined on $C_c^\infty(\Omega)$. Then the spectrum of H is discrete by Theorem 14.6 of [Agm65] and all eigenfunctions are analytic (see e.g. [Joh49], [Hör83a], Section 8 and [Hör83b], Section 11). Assume that the bottom eigenvalue λ is simple which happens to be true e.g. in the case of second order differential operators on a bounded domain.

Theorem 5.3 *In the situation just described suppose that $d > 2m$ and that the second eigenfunction ϕ_2 of H does not vanish in x_0. Then there is a constant $c > 0$ such that for small enough r*

$$\lambda_{2,r,x_0} - \lambda_2 \leq cr^{d-2m},$$

where λ_2 and λ_{2,r,x_0} are the second eigenvalues of H and H_{r,x_0} respectively.

The proof is contained in [Nol98].

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