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SERGIÙ KLAINERMAN
DAMIANO FOSCHI

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On Bilinear estimates for wave equations

Sergiu Klainerman Damiano Foschi

Abstract

I will start with a short review of the classical restriction theorem for the sphere and Strichartz estimates for the wave equation. I then plan to give a detailed presentation of their recent generalizations in the form of " Bilinear Estimates" . In addition to the L^2 theory, which is now quite well developed, I plan to discuss a more general point of view concerning the L^p theory. By investigating simple examples I will derive necessary conditions for such estimates to be true. I also plan to discuss the relevance of these estimates to nonlinear wave equations.

1. Introduction.

In this lecture we plan to give an account of our recent work [1] concerning general bilinear estimates for solutions to homogeneous wave equations.

Consider the homogeneous wave equation

$$\Box\phi = 0 \qquad (\Box = -\partial_t^2 + \Delta_x, \quad t \in \mathbb{R}, x \in \mathbb{R}^n),$$

subject to the reduced initial conditions at $t = 0$,

$$\phi = f, \quad \partial_t\phi = 0.$$

The solution ϕ to this initial value problem possesses some important space-time integrability properties known under the name of "Strichartz estimates".

Theorem 1.1 (Strichartz [12], Pecher [11], Ginibre-Velo [2], Keel-Tao [4].). *We have*

$$\|\phi\|_{L^q_t L^r_x} \lesssim \|f\|_{\dot{H}^\alpha}, \tag{1}$$

whenever the exponents q, r, α satisfy the following conditions:

$$\frac{2}{q} \leq \gamma(r) = (n-1)\left(\frac{1}{2} - \frac{1}{r}\right), \qquad \textit{(concentration)},$$

$$\left(\frac{2}{q}, \gamma(r), \frac{n-1}{2}\right) \neq (1,1,1), \qquad \textit{(endpoint)},$$

$$\frac{1}{q} + \frac{n}{r} = -\alpha + \frac{n}{2}, \qquad \textit{(scaling)}.$$

These estimates have turned out to play a decisive role in the study of both regularity and long time behavior for solutions to semilinear wave equations of the type

$$\Box u = F(u).$$

They apply, in particular, to the question of minimal regularity of initial conditions for which one can construct meaningful solutions to these equations ([3], [10], [13]). The Strichartz inequalities are however not well adapted to equations, such as Yang-Mills, Wave Maps etc, which contain derivatives in the nonlinear terms

$$\Box u = F(u, \partial u).$$

In recent years, new estimates, based on a bilinear point of view ([5]- [8]), have allowed us to make some progress on this type of problems. At the heart of the new approach lie certain estimates which can be viewed as a bilinear generalization of (1). Observe that (1) has in fact an equivalent bilinear formulation as

$$\|\phi\psi\|_{L_t^{q/2}L_x^{r/2}} \lesssim \|f\|_{\dot{H}^\alpha}\|g\|_{\dot{H}^\alpha},$$

where ϕ and ψ are both solutions of the homogeneous wave equation with initial data at time $t = 0$,

$$\phi = f, \quad \partial_t\phi = 0, \qquad \psi = g, \quad \partial_t\psi = 0.$$

One way to generalize these estimate is to consider estimates for derivatives of products, or more general bilinear expressions, in particular those which possess some cancellation properties (null condition).

More precisely, we want to investigate what are the possible estimates of the type,

$$\|Q(\phi,\psi)\|_{L_t^{q/2}L_x^{r/2}} \lesssim \|f\|_{\dot{H}^{\alpha_1}}\|g\|_{\dot{H}^{\alpha_2}}, \tag{2}$$

when Q is one of the quadratic null forms Q_0, Q_{ij}, Q_{0j}, defined by

$$\begin{aligned} Q_0(\phi,\psi) &= -\partial_t\phi\partial_t\psi + \nabla_x\phi\cdot\nabla_x\psi, \\ Q_{ij}(\phi,\psi) &= \partial_i\phi\partial_j\psi - \partial_j\phi\partial_i\psi, \quad 1 \le i < j \le n, \\ Q_{0j}(\phi,\psi) &= \partial_t\phi\partial_j\psi - \partial_j\phi\partial_t\psi, \quad 1 \le j \le n. \end{aligned}$$

We will also consider derivatives of products,

$$\left\| D^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}(\phi\psi)\right\|_{L_t^{q/2}L_x^{r/2}} \lesssim \|f\|_{\dot{H}^{\alpha_1}}\|g\|_{\dot{H}^{\alpha_2}}, \tag{3}$$

where

$$\begin{aligned} (D^\alpha f)\hat{}\,(\xi) &= |\xi|^\alpha \hat{f}(\xi), \\ (D_+^\alpha F)\tilde{}\,(\tau,\xi) &= (|\tau| + |\xi|)^\alpha \tilde{F}(\tau,\xi), \\ (D_-^\alpha F)\tilde{}\,(\tau,\xi) &= ||\tau| - |\xi||^\alpha \tilde{F}(\tau,\xi). \end{aligned}$$

At least for the case $q/2 = r/2 = 2$, using Plancherel's theorem we can see that the estimates (2) follow from (3) by making use of the following lemma.

Lemma 1.2. *Let ϕ and ψ be solutions of the homogeneous wave equation. Then*

$$Q_0(\phi,\psi) = \frac{1}{2}\Box(\phi\psi) \approx \frac{1}{2}D_+D_-(\phi\psi).$$

Moreover, let $\underset{\sim}{\phi} = |\tilde{\phi}|$ and $\underset{\sim}{\psi} = |\tilde{\psi}|$. Then $\underset{\sim}{\phi}$ and $\underset{\sim}{\psi}$ are still solution of the homogeneous wave equation whose data have the same $\dot{H}^\alpha$ norms as those for ϕ and ψ, and

$$\left|\tilde{Q}_{ij}(\phi,\psi)(\tau,\xi)\right| \lesssim \left|\left(D_+^{\frac{1}{2}}D_-^{\frac{1}{2}}\left(D^{\frac{1}{2}}\underset{\sim}{\phi}D^{\frac{1}{2}}\underset{\sim}{\psi}\right)\right)^{\tilde{}}(\tau,\xi)\right|.$$

We present here a summary of our main results.

1. Complete necessary and sufficient conditions when $q/2 = r/2 = 2$.
2. Necessary conditions and conjectures for general (q,r).
3. Some special cases:
   (a) $q = \infty$, $r = 2$;
   (b) $2/q = \gamma(r)$, $\beta_+ = \beta_- = 0$, $\beta_0 < 0$.
4. Applications to the bilinear null forms Q_0, Q_{ij}, and also $D^{-1}Q_{ij}(\phi,\psi)$ and $Q_{ij}(D^{-1}\phi,\psi)$, the last two being important for the theory of Yang-Mills equations.

2. Restriction theorems for the sphere.

The proofs of these results are based on the Fourier transform and techniques of harmonic analysis, such as restriction theorems and dyadic decompositions. The main observation is that the Fourier transform of a solution to the homogeneous wave equation is a measure supported on the null cone, which can be seen as a linear combinations of distributions of the form

$$\tilde{\phi}(\tau,\xi) = \delta(\tau \mp |\xi|)\frac{f(\xi)}{|\xi|^\alpha},$$

with $f \in L^2$.

A pedagogically interesting, and simpler, situation is to consider functions whose Fourier transform is supported on the unit sphere. What we want is to generalize the restriction theorem of Stein and Tomas to a bilinear setting. Let us first recall the content of the restriction theorem.

Let $\mathbb{S} = \mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ and define the restriction operator $Rf = \hat{f}\Big|_{\mathbb{S}}$.

Theorem 2.1. *Assume*

$$1 \le p \le \frac{2(n+1)}{n+3} < 2.$$

Then Rf is well defined for any $f \in L^p(\mathbb{R}^n)$ and we have

$$\|Rf\|_{L^2(\mathbb{S})} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

This theorem has an equivalent dual formulation. Consider the Stein operator, defined as the adjoint of the Fourier restriction operator $Rf = \hat{f}\big|_{\mathbb{S}}$,

$$Sf(x) = R^*f(x) = \int_{\mathbb{S}} e^{ix\cdot\xi} f(\xi)\, \mathrm{d}S_\xi \simeq (f\, \mathrm{d}S)^\vee(x).$$

Theorem 2.2. *Let $f \in L^2(\mathbb{S})$ and*

$$\frac{2(n+1)}{n-1} \leq p \leq \infty.$$

Then $Sf \in L^p(\mathbb{R}^n)$ and

$$\|Sf\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{S})}. \tag{1}$$

Since the exponent $\frac{2(n+1)}{n-1}$ is greater than 2, we also have a bilinear formulation, which, in its sharpest case, reads

$$\|Sf \cdot Sg\|_{L^{\frac{n+1}{n-1}}} \lesssim \|f\|_{L^2(\mathbb{S})} \|g\|_{L^2(\mathbb{S})}.$$

The condition $p \geq \frac{2(n+1)}{n-1}$ is sharp in view of the Knapp example.

Example 2.3. *Consider $f = g = \chi_{C_\varepsilon}$, the characteristic function of the set C_ε, where C_ε is the cap defined as the intersection of the sphere $\mathbb{S}$ and the rectangular box*[1]

$$D_\varepsilon = \left\{\xi \in R^n : |\xi_1 - 1| \leq \varepsilon^2,\, |\xi'| \leq \varepsilon\right\},$$

for a small $\varepsilon > 0$. We write

$$Sf(x) = e^{ix_1} \int_{C_\varepsilon} e^{ix_1(\xi_1 - 1)} e^{x'\cdot\xi'}\, \mathrm{d}S_\xi,$$

and observe that it is possible to choose a region R_ε defined by

$$R_\varepsilon = \left\{x \in \mathbb{R}^n : |x_1| \lesssim \varepsilon^{-2},\, |x'| \lesssim \varepsilon^{-1}\right\},$$

such that $|Sf(x)| \gtrsim |C_\varepsilon|$ when $x \in R_\varepsilon$. (See Figure 1.)

Therefore,

$$\frac{\|Sf\|_{L^p}}{\|f\|_{L^2(\mathbb{S})}} \gtrsim \frac{|C_\varepsilon||R_\varepsilon|^{\frac{1}{p}}}{|C_\varepsilon|^{\frac{1}{2}}} \approx \varepsilon^{\frac{n-1}{2} - \frac{n+1}{p}}.$$

In the limit $\varepsilon \to 0$ an inequality like (1) implies $p \geq \frac{2(n+1)}{n-1}$.

[1]We use the following convention for vectors in R^n and their components: $x = (x_1, x') = (x_1, x_2, x'')$, $x \in \mathbb{R}^n$, $x' \in \mathbb{R}^{n-1}$, $x'' \in \mathbb{R}^{n-2}$. If $n = 2$ then $x'' = \varnothing$.

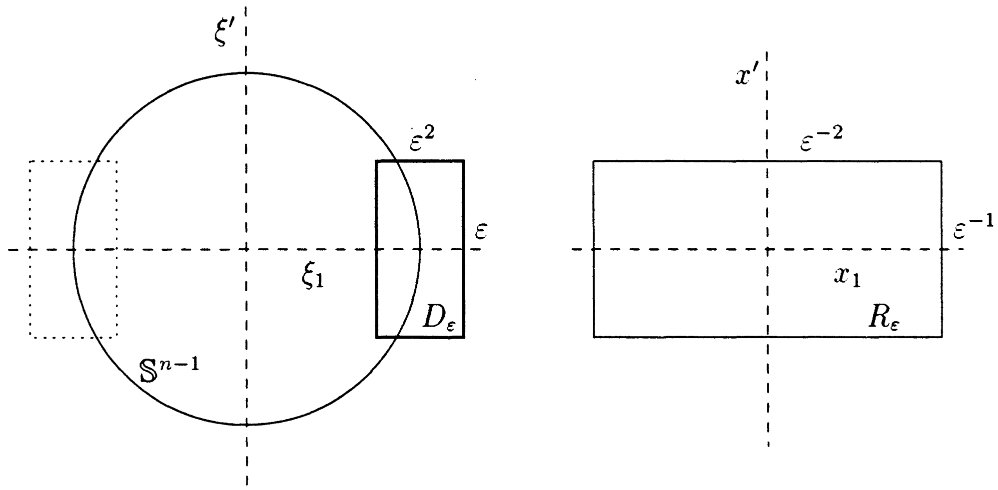

Figure 1: Knapp example.

A similar result holds if, in the bilinear version, we take f and g to be supported in diametrically opposite caps on the sphere.

On the other hand, if the supports of f and g are projectionally disjoint, we may improve the exponent p.

Example 2.4. *Consider $f = \chi_{C_\varepsilon}$ and $g = \chi_{C'_\varepsilon}$, where C_ε and C'_ε are the caps obtained intersecting the sphere with the the boxes*

$$D_\varepsilon = \left\{\xi \in R^n : |\xi_1 - 1| \leq \varepsilon^2,\ |\xi_2| \leq \varepsilon,\ |\xi''| \leq \varepsilon\right\},$$
$$D'_\varepsilon = \left\{\xi \in R^n : |\xi_1| \leq \varepsilon,\ |\xi_2 - 1| \leq \varepsilon^2,\ |\xi''| \leq \varepsilon\right\}.$$

Then, proceeding as before, we have $|Sf(x)| \gtrsim |C_\varepsilon|$ when $x \in R_\varepsilon$ and $|Sg(x)| \gtrsim |C'_\varepsilon|$ when $x \in R'_\varepsilon$, where

$$R_\varepsilon = \left\{x \in \mathbb{R}^n : |x_1| \lesssim \varepsilon^{-2},\ |x_2| \lesssim \varepsilon^{-1},\ |x''| \lesssim \varepsilon^{-1}\right\},$$
$$R'_\varepsilon = \left\{x \in \mathbb{R}^n : |x_1| \lesssim \varepsilon^{-1},\ |x_2| \lesssim \varepsilon^{-2},\ |x''| \lesssim \varepsilon^{-1}\right\}.$$

(See Figure 2.)

We can estimate the product $Sf \cdot Sg$ only on the intersection of R_ε with R'_ε and we have,

$$\frac{\|Sf \cdot Sg\|_{L^{p/2}}}{\|f\|_{L^2(\mathbb{S})}\|g\|_{L^2(\mathbb{S})}} \gtrsim \frac{|C_\varepsilon||C'_\varepsilon||R_\varepsilon \cap R'_\varepsilon|^{\frac{2}{p}}}{|C_\varepsilon|^{\frac{1}{2}}|C'_\varepsilon|^{\frac{1}{2}}} \approx \varepsilon^{n-1-\frac{2n}{p}}.$$

In the limit $\varepsilon \to 0$, we obtain the necessary condition $p \geq \frac{2n}{n-1}$.

This improvement, however, is not optimal. We are clearly losing something, due to the fact that the regions R_ε and R'_ε do not coincide. This suggests the following modification.

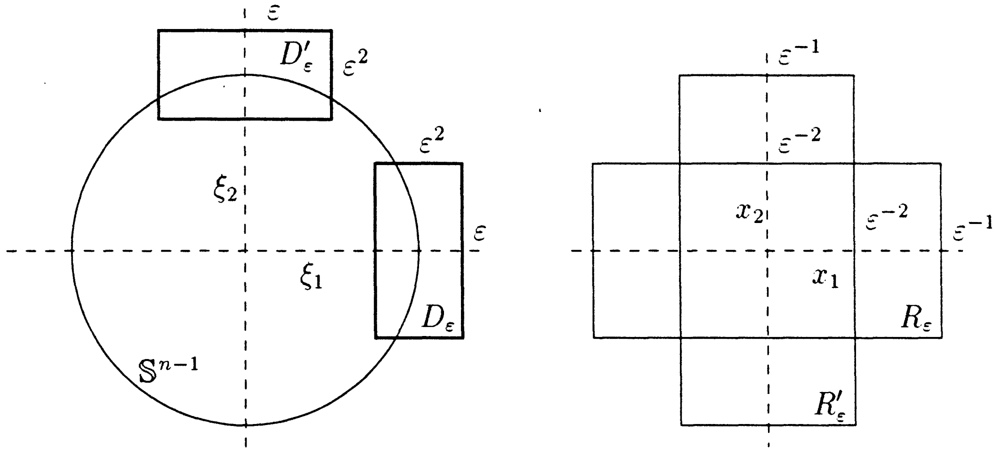

Figure 2: Bilinear Knapp example.

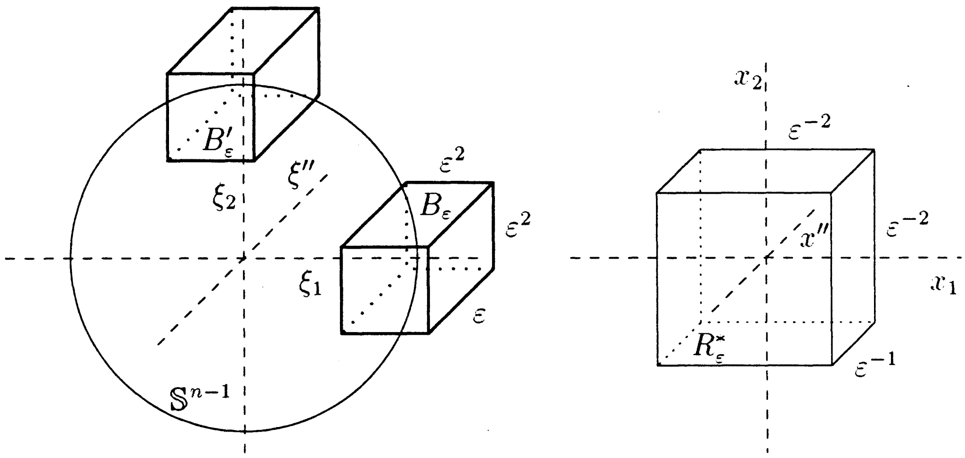

Figure 3: Sharp bilinear example.

Example 2.5. *Let $f = \chi_{A_\varepsilon}$ and $g = \chi_{A'_\varepsilon}$, where A and A' are the sets obtained intersecting the sphere with the boxes*

$$B_\varepsilon = \left\{\xi \in R^n : |\xi_1 - 1| \le \varepsilon^2,\ |\xi_2| \le \varepsilon^2,\ |\xi''| \le \varepsilon\right\},$$
$$B'_\varepsilon = \left\{\xi \in R^n : |\xi_1| \le \varepsilon^2,\ |\xi_2 - 1| \le \varepsilon^2,\ |\xi''| \le \varepsilon\right\}.$$

This time we have $|Sf(x)| \gtrsim |C_\varepsilon|$ and $|Sg(x)| \gtrsim |C'_\varepsilon|$ simultaneously when $x \in R^*_\varepsilon$, where

$$R^*_\varepsilon = \left\{x \in \mathbb{R}^n : |x_1| \lesssim \varepsilon^{-2},\ |x_2| \lesssim \varepsilon^{-2},\ |x''| \lesssim \varepsilon^{-1}\right\}.$$

(See Figure 3.)

We have

$$\frac{\|Sf \cdot Sg\|_{L^{p/2}}}{\|f\|_{L^2(\mathbb{S})}\|g\|_{L^2(\mathbb{S})}} \gtrsim \frac{|A_\varepsilon||A'_\varepsilon||R^*_\varepsilon|^{\frac{2}{p}}}{|A_\varepsilon|^{\frac{1}{2}}|A'_\varepsilon|^{\frac{1}{2}}} \approx \varepsilon^{n - \frac{2(n+2)}{p}}.$$

When $\varepsilon \to 0$ we obtain the (sharp) necessary condition $p \geq \frac{2(n+2)}{n}$.

The previous example motivates the following conjecture.

Conjecture 2.6 (Bilinear restriction conjecture.). *Let Ω_1, Ω_2 two disjoint subsets of $\mathbb{S}^{n-1}$ such that*

$$\operatorname{dist}(\Omega_1, \Omega_2) > 0, \quad \operatorname{dist}(\Omega_1, -\Omega_2) > 0.$$

Then

$$\|Sf \cdot Sg\|_{L^{\frac{n+2}{n}}(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\Omega_1)} \|g\|_{L^2(\Omega_2)},$$

for all f supported in Ω_1 and g supported in Ω_2.

Instead of imposing a condition on the supports of f and g we may consider bilinear forms with special cancellation properties. Let $Q(f,g) = (\partial_i Sf)(\partial_j Sg) - (\partial_j Sf)(\partial_i Sg)$. If we play with the above examples, replacing $Sf \cdot Sg$ with $Q(f,g)$, then we are lead to formulate a similar conjecture.

Conjecture 2.7. *The estimate*

$$\|Q(f,g)\|_{L^{\frac{n+2}{n}}(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{S})} \|g\|_{L^2(\mathbb{S})}.$$

holds for any $f, g \in L^2(\mathbb{S})$, (with no assumptions on the supports).

These conjecture can be easily proved to be true in the case $n = 2$. The two dimensional case is somehow special, due to the fact that $\frac{n+2}{n} \geq 2$. In fact, using Plancherel's theorem and Cauchy-Schwarz inequality, we can derive the following result,which implies the conjectures.

Proposition 2.8. *Let $n = 2$. We have*

$$\|Sf \cdot Sg\|_{L^2(\mathbb{R}^2)} \lesssim \iint_{\mathbb{S}^1 \times \mathbb{S}^1} \frac{|f(\eta)|^2 |g(\zeta)|^2}{(1 - (\eta \cdot \zeta)^2)^{\frac{1}{2}}} \, d\sigma_\eta \, d\sigma_\zeta.$$

3. Bilinear estimates for the wave equation.

We consider now two solutions, ϕ and ψ, of the homogeneous wave equation in $\mathbb{R}^{1+n}$,

$$\Box \phi = 0, \quad \Box \psi = 0, \tag{1}$$

subject to the initial conditions at $t = 0$,

$$\phi(0,\cdot) = f, \ \partial_t \phi(0,\cdot) = 0, \quad \psi(0,\cdot) = g, \ \partial_t \psi(0,\cdot) = 0. \tag{2}$$

We want to investigate the regularity, in terms of differentiability as well as space-time integrability, of the product $\phi\psi$. Let $n \geq 2$, $1 \leq q, r \leq \infty$ and $\alpha_1, \alpha_2, \beta_0, \beta_+, \beta_- \in \mathbb{R}$. Consider the bilinear estimate

$$\left\| D^{\beta_0} D_+^{\beta_+} D_-^{\beta_-} (\phi\psi) \right\|_{L_t^q L_x^r} \lesssim \|D^{\alpha_1} f\|_{L^2} \|D^{\alpha_2} g\|_{L^2}. \tag{3}$$

Proposition 3.1 ([1]). *If the bilinear estimate* (3) *is true then the exponents must satisfy the following conditions:*

$$\beta_0 + \beta_+ + \beta_- = \alpha_1 + \alpha_2 + \frac{1}{q} - n\left(1 - \frac{1}{r}\right), \tag{4}$$

$$\frac{1}{q} \leq \frac{n+2}{2}\left(1 - \frac{1}{r}\right), \tag{5}$$

$$\beta_- \geq \frac{1}{q} - \frac{n-1}{2}\left(1 - \frac{1}{r}\right), \tag{6}$$

$$\beta_0 \geq \frac{1}{q} - n\left(1 - \frac{1}{r}\right), \tag{7}$$

$$\beta_0 \geq \frac{2}{q} - (n+1)\left(1 - \frac{1}{r}\right), \tag{8}$$

$$\alpha_1 + \alpha_2 \geq \frac{1}{q}, \tag{9}$$

$$\alpha_i \leq \beta_- + \frac{n}{2} - \frac{1}{q} + n\left(\frac{1}{2} - \frac{1}{r}\right), \tag{10}$$

$$\alpha_i \leq \beta_- + \frac{n}{2} - \frac{1}{q} + \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{r}\right), \tag{11}$$

$$\alpha_i \leq \beta_- + \frac{n}{2}, \qquad (i = 1, 2). \tag{12}$$

The first one of these condition follows immediately by dimensional analysis, since by homogeneity of the norms involved the estimate has to be scaling invariant.

The Fourier transform of the product $\phi\psi$ is a convolution of two measures supported on the light cone $|\tau| = |\xi|$. The light cone is the union of the $(+)$ component, $\tau = |\xi|$, and the $(-)$ component, $\tau = -|\xi|$. Accordingly, we decompose the solutions of the wave equation as the sum of two pieces, $\phi = \phi^+ + \phi^-$, where

$$\phi^\pm(t,x) \simeq \int_{\mathbb{R}^n} e^{\pm it|\xi|} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi.$$

Notice that the Fourier transform of $\phi^\pm$ is supported on the $(\pm)$ component of the light cone. The product $\phi\psi$ is then the sum of four terms

$$\phi\psi = \phi^+\psi^+ + \phi^+\psi^- + \phi^-\psi^+ + \phi^-\psi^-.$$

By symmetry, it is enough to consider only the $(++)$ and $(+-)$ cases.

The conditions of Proposition 3.1 are obtained considering specific examples which test the regularity properties of the products $\phi^+\psi^\pm$ when the data are localized in frequency space. The method is similar to that used for the examples on the bilinear restriction theory.

Same frequency examples

Example 3.2. *This example is the analogous of the Knapp example, Example 2.3, adapted here for the wave equation. Consider the* $(++)$ *case with data defined by* $\hat{f} = \hat{g} = \chi_{A_\varepsilon}$ *where*

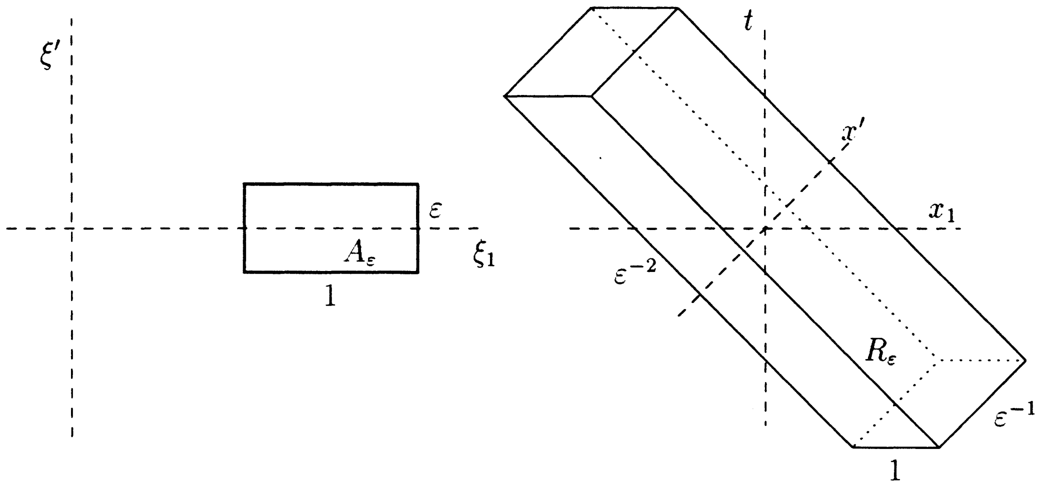

Figure 4: Knapp example for the wave equation.

A_ε is the set

$$A_\varepsilon = \{\xi \in \mathbb{R}^n : 1 \le \xi_1 \le 2,\ |\xi'| \le \varepsilon\}.$$

The (++) interaction then concentrate on a set R_ε given by

$$R_\varepsilon = \left\{(t,x) : |t| \lesssim \varepsilon^{-2},\ |t + x_1| \lesssim 1,\ |x'| \lesssim \varepsilon^{-1}\right\}.$$

(See Figure 4.)

The condition that follows when testing the estimate (3) on this example is

$$2\beta_- - \frac{2}{q} + (n-1)\left(1 - \frac{1}{r}\right) \ge 0,$$

which corresponds to (6).

Example 3.3. *Consider the (++) case with data defined by $\hat{f} = \chi_{F_\varepsilon}$, $\hat{g} = \chi_{G_\varepsilon}$, where $F_\varepsilon, G_\varepsilon$ are the sets*

$$F_\varepsilon = \{\xi \in \mathbb{R}^n : |\xi_1 - 1| \le \varepsilon,\ |\xi'| \le \varepsilon\},$$
$$G_\varepsilon = \{\xi \in \mathbb{R}^n : |\xi_1 + 1| \le \varepsilon,\ |\xi'| \le \varepsilon\}.$$

The (++) interaction then concentrate on a set R_ε given by

$$R_\varepsilon = \left\{(t,x) : |t| \lesssim \varepsilon^{-1},\ |x| \lesssim \varepsilon^{-1}\right\}.$$

(See Figure 5.)

The condition that follows in this case is

$$\beta_0 - \frac{1}{q} + n\left(1 - \frac{1}{r}\right) \ge 0,$$

which corresponds to (7).

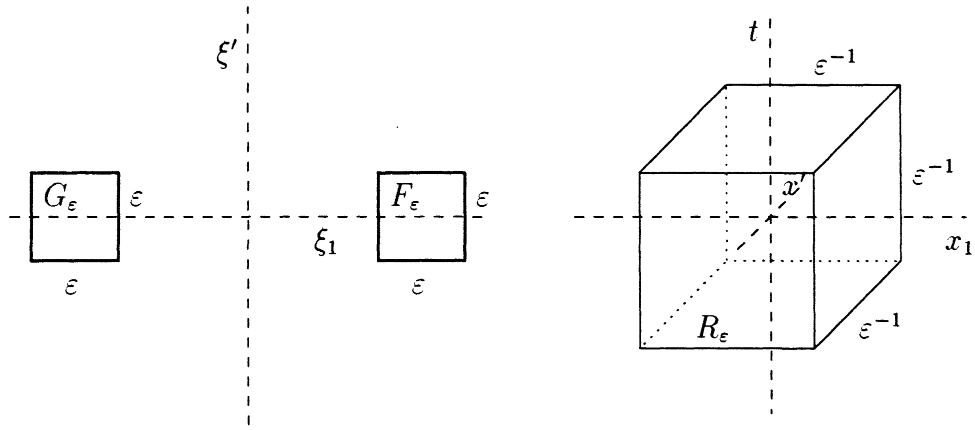

Figure 5:

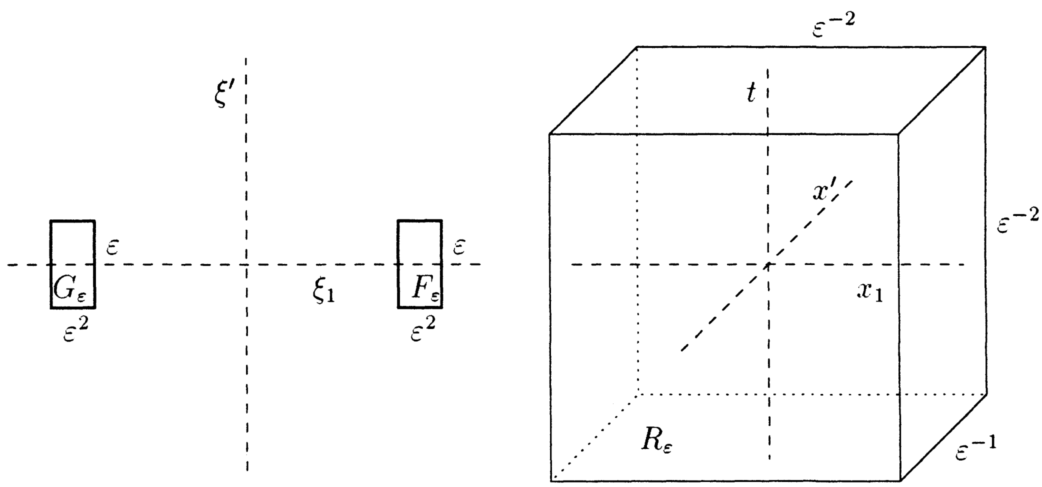

Figure 6:

Example 3.4. *Consider the* $(++)$ *case with data defined by* $\hat{f} = \chi_{F_\varepsilon}$, $\hat{g} = \chi_{G_\varepsilon}$, *where* $F_\varepsilon, G_\varepsilon$ *are the sets*

$$F_\varepsilon = \left\{\xi \in \mathbb{R}^n : |\xi_1 - 1| \le \varepsilon^2,\ |\xi'| \le \varepsilon\right\},$$
$$G_\varepsilon = \left\{\xi \in \mathbb{R}^n : |\xi_1 + 1| \le \varepsilon^2,\ |\xi'| \le \varepsilon\right\}.$$

The $(++)$ *interaction then concentrate on a set* R_ε *given by*

$$R_\varepsilon = \left\{(t,x) : |t| \lesssim \varepsilon^{-2},\ |x_1| \lesssim \varepsilon^{-2},\ |x'| \lesssim \varepsilon^{-1}\right\}.$$

(See Figure 6.)

The condition that follows in this case is

$$\beta_0 - \frac{2}{q} + (n+1)\left(1 - \frac{1}{r}\right) \ge 0,$$

which corresponds to (8).

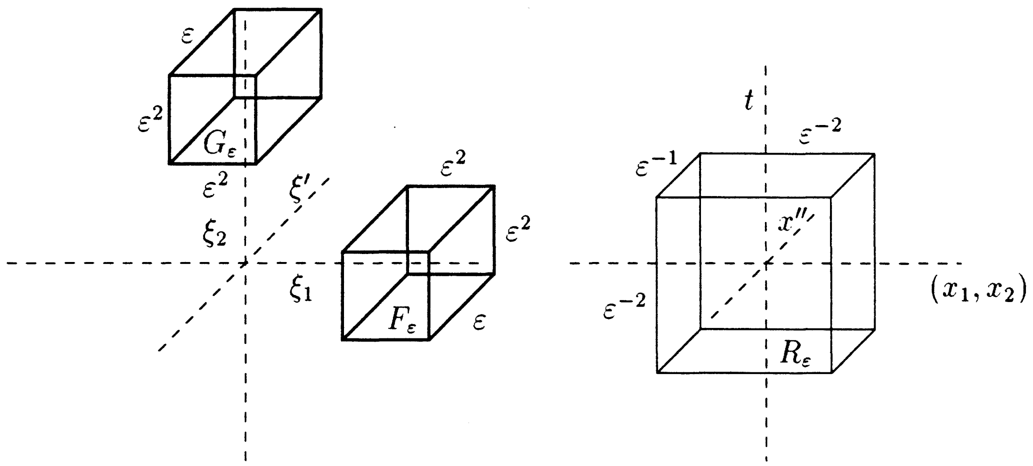

Figure 7:

Example 3.5. *This example is the analogous of Example 2.5. Consider the* $(++)$ *case with data defined by* $\hat{f} = \chi_{F_\varepsilon}$, $\hat{g} = \chi_{G_\varepsilon}$, *where* $F_\varepsilon, G_\varepsilon$ *are the sets*

$$F_\varepsilon = \left\{\xi \in \mathbb{R}^n : |\xi_1 - 1| \leq \varepsilon^2, \, |x_2| \leq \varepsilon^2, \, |\xi''| \leq \varepsilon\right\},$$
$$G_\varepsilon = \left\{\xi \in \mathbb{R}^n : |\xi_1| \leq \varepsilon^2, \, |x_2 - 1| \leq \varepsilon^2, \, |\xi''| \leq \varepsilon\right\}.$$

The $(++)$ *interaction then concentrate on a set* R_ε *given by*

$$R_\varepsilon = \left\{(t,x) : |t| \lesssim \varepsilon^{-2}, \, |x_1| \lesssim \varepsilon^{-2}, \, |x_2| \lesssim \varepsilon^{-2}, \, |x''| \lesssim \varepsilon^{-1}\right\}.$$

(See Figure 7.)

The condition we derive in this case is,

$$-\frac{2}{q} + (n+2)\left(1 - \frac{1}{r}\right) \geq 0,$$

which corresponds to (5).

Example 3.6. *Consider now the* $(+-)$ *case with data defined by* $\hat{f} = \chi_{F_\varepsilon}$, $\hat{g} = \chi_{G_\varepsilon}$, *where* $F_\varepsilon, G_\varepsilon$ *are the sets*

$$F_\varepsilon = \{\xi \in \mathbb{R}^n : |\xi_1 - 1| \leq \varepsilon, \, |\xi'| \leq \varepsilon\},$$
$$G_\varepsilon = \{\xi \in \mathbb{R}^n : |\xi_1 + 1| \leq \varepsilon, \, |\xi'| \leq \varepsilon\}.$$

The $(+-)$ *interaction then concentrate on a set* R_ε *given by*

$$R_\varepsilon = \left\{(t,x) : |t| \lesssim \varepsilon^{-2}, \, |t + x_1| \lesssim \varepsilon^{-1}, \, |x'| \lesssim \varepsilon^{-1}\right\}.$$

(See Figure 8.)

Following the same steps as in the previous examples we derive,

$$\beta_0 + \beta_+ + \beta_- - \frac{2}{q} + n\left(1 - \frac{1}{r}\right) \geq 0,$$

which, according to (4), *corresponds to* (7).

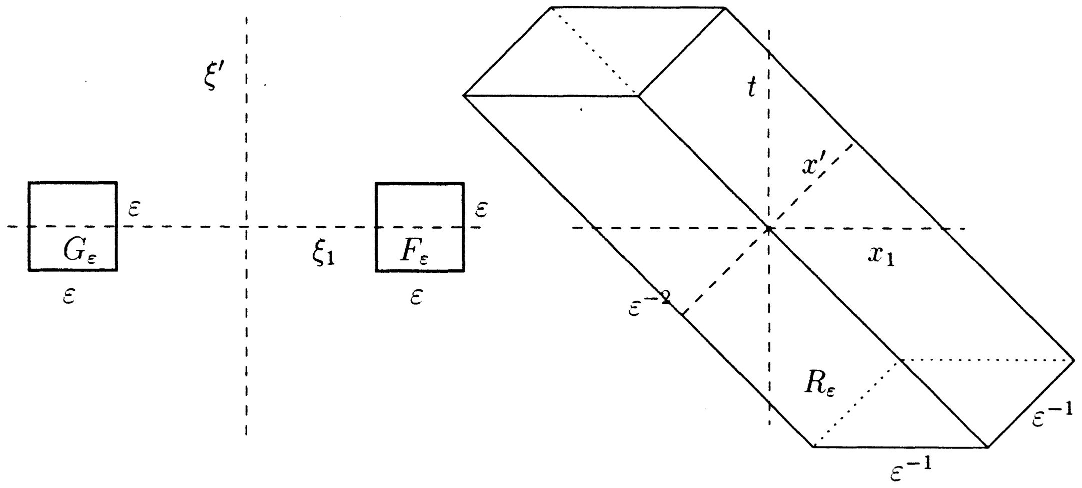

Figure 8: (+−) case.

Mixed frequencies examples

Example 3.7. *Consider the* (++) *case with data defined by* $\hat{f} = \chi_{F_L}$, $\hat{g} = \chi_{G_L}$, *where* F_L, G_L *are the sets*

$$F_L = \{\xi \in \mathbb{R}^n : L \leq \xi_1 \leq L+1,\ |\xi'| \leq 1\},$$
$$G_L = \{\xi \in \mathbb{R}^n : 1 \leq \xi_1 \leq 2,\ |\xi'| \leq 1\}.$$

The (++) *interaction then concentrate on a set* R_L *given by*

$$R_L = \{(t,x) : |t| \lesssim 1,\ |x| \lesssim 1\}.$$

The condition that follows in the limit $L \to \infty$ *is*

$$-\beta_0 - \beta_+ + \alpha_1 \geq 0,$$

which corresponds to (10).

Example 3.8. *Consider the* (++) *case with data defined by* $\hat{f} = \chi_{F_L}$, $\hat{g} = \chi_{G_L}$, *where* F_L, G_L *are the set*

$$F_L = \left\{\xi \in \mathbb{R}^n : L \leq \xi_1 \leq 2L,\ |\xi'| \leq L^{\frac{1}{2}}\right\},$$
$$G_L = \{\xi \in \mathbb{R}^n : 1 \leq \xi_1 \leq 2,\ |\xi'| \leq 1\}.$$

The (++) *interaction then concentrate on a set* R_L *given by*

$$R_L = \left\{(t,x) : |t| \lesssim 1,\ |t+x_1| \lesssim L^{-1},\ |x'| \lesssim L^{-\frac{1}{2}}\right\}.$$

The condition that follows in the limit $L \to \infty$ *is*

$$-\beta_0 - \beta_+ + \alpha_1 - \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{r}\right) \geq 0,$$

which corresponds to (11).

Example 3.9. *Consider the* $(++)$ *case with data defined by* $\hat{f} = \chi_{F_L}$, $\hat{g} = \chi_{G_L}$, *where* F_L, G_L *are the set*

$$F_L = \{\xi \in \mathbb{R}^n : L \leq \xi_1 \leq 2L,\ |\xi'| \leq L\},$$
$$G_L = \{\xi \in \mathbb{R}^n : 1 \leq \xi_1 \leq 2,\ |\xi'| \leq 1\}.$$

The $(++)$ *interaction then concentrate on a set* R_L *given by*

$$R_L = \left\{(t,x) : |t| \lesssim 1,\ |x| \lesssim L^{-1}\right\}.$$

The condition that follows in the limit $L \to \infty$ *is*

$$-\beta_0 - \beta_+ + \alpha_1 + \frac{1}{q} - n\left(\frac{1}{2} - \frac{1}{r}\right) \geq 0,$$

which corresponds to (12).

We conjecture that the set of necessary conditions in Proposition 3.1 is complete, with the exception of some limiting cases.

It is not difficult to verify it for the case when $q = \infty$ and $r = 2$.

Theorem 3.10. *The estimate*

$$\left\| D^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}(\phi\psi) \right\|_{L_t^\infty L_x^2} \lesssim \|D^{\alpha_1} f\|_{L^2} \|D^{\alpha_2} g\|_{L^2},$$

holds whenever the exponents verify the conditions

$$\beta_0 + \beta_+ + \beta_- = \alpha_1 + \alpha_2 + \frac{n}{2}, \tag{13}$$

$$\beta_- > -\frac{n-1}{4}, \tag{14}$$

$$\beta_0 > \frac{n}{2}, \tag{15}$$

$$\alpha_1 + \alpha_2 > 0, \tag{16}$$

$$\alpha_i \leq \beta_- + \frac{n}{2}, \qquad (i = 1, 2). \tag{17}$$

All the strict inequalities are necessary, with the possible exception of (14) for which we still don't know what happens if we have equality.

Theorem 3.10 can be used to analyze the first iterate in the problem of optimal well posedness for nonlinear wave equations. (For example $\Box\phi = (\partial_t\phi)^2$.)

Another situation, where we can completely settle the conjecture, is the case of $q = r = 2$. This is the case which turns out to be very important in applications. We have the following general theorem:

Theorem 3.11 ([1]). *The estimate*

$$\left\| D^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}(\phi\psi) \right\|_{L_t^2 L_x^2} \lesssim \|D^{\alpha_1} f\|_{L^2} \|D^{\alpha_2} g\|_{L^2},$$

holds if and only if the exponents verify the conditions

$$\beta_0 + \beta_+ + \beta_- = \alpha_1 + \alpha_2 - \frac{n-1}{2}, \tag{18}$$

$$\beta_- \geq -\frac{n-3}{4}, \tag{19}$$

$$\beta_0 > -\frac{n-1}{2}, \tag{20}$$

$$\alpha_i \leq \beta_- + \frac{n-1}{2}, \quad i = 1, 2, \tag{21}$$

$$\alpha_1 + \alpha_2 \geq \frac{1}{2}, \tag{22}$$

$$(\alpha_i, \beta_-) \neq \left(\frac{n+1}{4}, -\frac{n-3}{4}\right), \quad i = 1, 2, \tag{23}$$

$$(\alpha_1 + \alpha_2, \beta_-) \neq \left(\frac{1}{2}, -\frac{n-3}{4}\right). \tag{24}$$

The estimates of Theorem 3.11 have been used to analyze the optimal well posedness for nonlinear wave equations. For the equations of Wave Maps the reader is referred to [7], [8], [14]. For Yang-Mills equations see [6], [9].

We single out some special cases, which are important in applications to the Wave Maps problem.

$$\left\|D^{\frac{n}{2}} D_-^{\frac{1}{2}}(\phi\psi)\right\|_{L_t^2 L_x^2} \lesssim \left\|D^{\frac{n}{2}} f\right\|_{L^2} \left\|D^{\frac{n}{2}} g\right\|_{L^2}, \tag{25}$$

$$\left\|D^{1-\frac{n}{2}} D_-^{\frac{1}{2}}(\phi\psi)\right\|_{L_t^2 L_x^2} \lesssim \left\|D^{1-\frac{n}{2}} f\right\|_{L^2} \left\|D^{\frac{n}{2}} g\right\|_{L^2}. \tag{26}$$

To give an idea of how to prove Theorem 3.11 we show below a short proof, using our methods, of the classical Stricartz estimate in dimension $n = 3$.

$$\|\phi\psi\|_{L_t^2 L_x^2} \lesssim \|f\|_{\dot{H}^{\frac{1}{2}}} \|g\|_{\dot{H}^{\frac{1}{2}}}.$$

Proof. Since there are no derivatives in front of the product, it is enough to consider just the $(++)$ case. We have

$$\widetilde{\phi^+\psi^+}(\tau, \xi) = \int_{\mathbb{R}^3} \delta(\tau - |\xi - \eta| - |\eta|) \hat{f}(\xi - \eta) \hat{g}(\eta) \, d\eta.$$

By Cauchy-Schwarz with respect to the measure $\delta(\tau - |\xi - \eta| - |\eta|) \, d\eta$ we obtain

$$\left|\widetilde{\phi^+\psi^+}(\tau, \xi)\right|^2 \leq J(\tau, \xi) \int \delta(\tau - |\xi - \eta| - |\eta|) |\xi - \eta| \left|\hat{f}(\xi - \eta)\right|^2 |\eta| \, |\hat{g}(\eta)|^2 \, d\eta,$$

where the quantity

$$J(\tau, \xi) = \int_{\mathbb{R}^3} \frac{\delta(\tau - |\xi - \eta| - |\eta|)}{|\xi - \eta||\eta|} \, d\eta \leq C$$

is uniformly bounded for $\tau \geq |\xi|$. Integrating with respect to τ and ξ we find

$$\|\phi^+\psi^+\|^2_{L^2_t L^2_x} \lesssim \iiint \delta(\tau - |\xi-\eta| - |\eta|)|\xi-\eta| \left|\hat{f}(\xi-\eta)\right|^2 |\eta| \, |\hat{g}(\eta)|^2 \, \mathrm{d}\tau \, \mathrm{d}\xi \, \mathrm{d}\eta =$$
$$= \left\|D^{\frac{1}{2}} f\right\|^2_{L^2} \left\|D^{\frac{1}{2}} g\right\|^2_{L^2}.$$

□

The proof of Theorem 3.11 in the (++) case is not much more involved. The (+−) case, which one doe not have to treat in the case of the Strichartz inequality, cannot be treated in the same manner. It is in fact considerably more complicated.

Results outside the L^2 theory are more difficult to obtain. Beside Theorem 1.1, we mention the following estimates which appear in [9].

Theorem 3.12. *Suppose q, r are such that*

$$\frac{1}{q} = \frac{n-1}{2}\left(1 - \frac{1}{r}\right), \qquad \left(\frac{1}{q}, \frac{n-1}{2}\right) \neq (1,1).$$

Then the estimate

$$\left\|D^{\beta_0}(\phi\psi)\right\|_{L^q_t L^r_x} \lesssim \|f\|_{\dot{H}^\alpha} \, \|g\|_{\dot{H}^\alpha}$$

holds when

$$-\left(1 - \frac{1}{r}\right) < \beta_0 \leq 0, \quad \beta_0 = 2\alpha - \frac{n+1}{n-1}\frac{1}{q}.$$

4. Conjectures for null forms.

To conclude this brief exposition, we state some conjectures for quadratic null forms. We believe that a proof of these conjectures will provide new light and new tools to attack many unanswered problems in the theory of regularity of nonlinear wave equations.

Conjecture 4.1. *Let $n \geq 2$, $1 \leq q, r \leq \infty$. Then*

$$\|Q_0(\phi, \psi)\|_{L^q_t L^r_x} \lesssim \|f\|_{\dot{H}^\alpha} \, \|g\|_{\dot{H}^\alpha},$$

when

$$\alpha = 1 - \frac{1}{2q} + \frac{n}{2}\left(1 - \frac{1}{r}\right),$$
$$\frac{1}{q} \leq \frac{n+1}{2}\left(1 - \frac{1}{r}\right),$$
$$\frac{1}{q} \leq \frac{n-1}{2} + \frac{1}{r}.$$

A special case of this conjecture is when $q = r = \frac{n+3}{n+1}$.

Conjecture 4.2. *Let $n \geq 2$, $1 \leq q, r \leq \infty$. Then*

$$\|Q_{ij}(\phi, \psi)\|_{L_t^q L_x^r} \lesssim \|f\|_{\dot{H}^\alpha} \|g\|_{\dot{H}^\alpha},$$

when

$$\begin{aligned}
\alpha &= 1 - \frac{1}{2q} + \frac{n}{2}\left(1 - \frac{1}{r}\right), \\
\frac{1}{q} &\leq \frac{n+2}{2}\left(1 - \frac{1}{r}\right), \\
\frac{1}{q} &\leq \frac{1}{2} + \frac{n-1}{2}\left(1 - \frac{1}{r}\right), \\
\frac{1}{q} &\leq \frac{n-1}{2} + \frac{1}{r}.
\end{aligned}$$

A special case of this conjecture is when $q = r = \frac{n+4}{n+2} < \frac{n+3}{n+1}$.

References

[1] Damiano Foschi and Sergiu Klainerman, *Bilinear space-time estimates for homogeneous wave equations*, preprint (1998).

[2] Jean Ginibre and Giorgio Velo, *Generalized Strichartz inequalities for the wave equation*, J. Funct. Anal. **133** (1995), no. 1, 50–68.

[3] Lev Kapitanski, *Weak and yet weaker solutions of semilinear wave equations*, Comm. Partial Differential Equations **19** (1994), no. 9-10, 1629–1676.

[4] Mark Keel and Terence Tao, *Endpoint Strichartz estimates*, American Journal of Mathematics (1998), to appear.

[5] Sergiu Klainerman and Matei Machedon, *Space-time estimates for null forms and the local existence theorem*, Comm. Pure Appl. Math. **46** (1993), no. 9, 1221–1268.

[6] ______, *Finite energy solutions of the Yang-Mills equations in $\mathbb{R}^{3+1}$*, Annals of Mathematics **142** (1995), 39–119.

[7] ______, *Estimates for null forms and the spaces $H_{s,\delta}$*, Int. Math. Res. Not. (1996), no. 17, 853–865.

[8] Sergiu Klainerman and Sigmund Selberg, *Remark on the optimal regularity for equations of wave maps type*, Comm. Partial Differential Equations **22** (1997), no. 5-6, 901–918.

[9] Sergiu Klainerman and Daniel Tataru, *On the optimal regularity for Yang-Mills equations in $\mathbb{R}^{4+1}$*, preprint (1998).

[10] Hans Lindblad and Christopher D. Sogge, *On existence and scattering with minimal regularity for semilinear wave equations*, J. Funct. Anal. **130** (1995), no. 2, 357–426.

[11] Hartmut Pecher, *Nonlinear small data scattering for the wave and Klein-Gordon equation*, Math. Z. **185** (1984), no. 2, 261–270.

[12] Robert S. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. **44** (1977), no. 3, 705–714.

[13] Terence Tao, *Low regularity semi-linear wave equations*, preprint (1998).

[14] Daniel Tataru, *Local and global results for wave maps i*, to appear, Comm. PDE (1998).

PRINCETON UNIVERSITY & I.H.É.S.
dfoschi@math.princeton.edu

PRINCETON UNIVERSITY & I.H.É.S.
seri@math.princeton.edu