JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

ENRIQUE ZUAZUA

Some uniqueness and observability problems arising in the control of vibrations

Journées Équations aux dérivées partielles (1999), p. 1-8

<http://www.numdam.org/item?id=JEDP_1999____A19_0>

Journées *Équations aux dérivées partielles*
Saint-Jean-de-Monts, 31 mai–4 juin 1999
GDR 1151 (CNRS)

Uniqueness problems for the Lamé system arising in the control of vibrations

Enrique Zuazua

Abstract

We discuss a control problem for the Lamé system which naturally leads to the following uniqueness problem: Given a bounded domain of $\mathbf{R}^3$, are there non-trivial solutions of the evolution Lamé system with homogeneous Dirichlet boundary conditions for which the first two components vanish? We show that such solutions do not exist when the domain is Lipschitz. However, in two space dimensions one can build easily polygonal domains in which there are eigenvibrations with the first component being identically zero. These uniqueness problems do not feet in the context of the classical Cauchy problem. They are of global nature and, therefore, the geometry of the domain under consideration plays a key role. We also present a list of related open problems.

1. Problem formulation.

Let Ω be a bounded domain of $I\!R^3$ and ω an open non-empty subset of Ω. Consider the Lamé system:

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u = f\chi_\omega & \text{in} \quad \Omega\times(0,T) \\ u = 0 & \text{on} \quad \partial\Omega\times(0,T) \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{in} \quad \Omega. \end{cases} \tag{1.1}$$

Here $\lambda, \mu > 0$ are the (constant) Lamé coefficients, $u = (u_1, u_2, u_3)$ is the displacement, χ_ω denotes the characteristic function of the set ω and $f = (f_1, f_2, f_3)$ is the control.

Given $(u_0, u_1) \in H = (H_0^1(\Omega))^3 \times (L^2(\Omega))^3$ and $f \in L^1\left(0,T;(L^2(\Omega))^3\right)$, system (1.1) admits a unique finite energy solution

$$(u, u_t) \in C([0,T]; H). \tag{1.2}$$

One of the simplest control problems one may address for (1.1) is the so-called *approximate controllability problem.* It can be stated as follows: Is the set of reachable states

$$R(T) = \left\{(u(T), u_t(T)) \in H : f \in L^1\left(0,T;\left(L^2(\Omega)\right)^3\right)\right\} \tag{1.3}$$

Supported by grant PB96-0663 of the DGES (Spain)

dense in H?

This problem is equivalent to a unique-continuation problem for the adjoint uncontrolled system

$$\begin{cases} \varphi_{tt} - \mu\Delta\varphi - (\lambda+\mu)\nabla \operatorname{div}\varphi = 0 & \text{in} \quad \Omega\times(0,T) \\ \varphi = 0 & \text{on} \quad \partial\Omega\times(0,T). \end{cases} \tag{1.4}$$

Namely, the problem may be reformulated as follows: Does the fact the solution φ of (1.4) satisfies

$$\varphi = 0 \text{ in } \omega\times(0,T) \tag{1.5}$$

guarantee that $\varphi \equiv 0$?

Using Holmgren's Uniqueness Theorem it is easy to see that, if T is sufficiently large, this uniqueness property does indeed hold. Therefore, the approximate controllability property holds as well.

However, in applications, one is interested on controlling the system using the minimal amount of control. In this context it is therefore natural to impose the following constraint on the control f:

$$f_3 = 0 \text{ in } \omega\times(0,T). \tag{1.6}$$

We are then trying to control the $3-d$ system of elasticity by means of *plannar forces.*

One again, the problem of approximate controllability may be reduced to a uniqueness problem. However, this time we have the following partial information on φ:

$$\varphi_1 = \varphi_2 = 0 \text{ in } \omega\times(0,T). \tag{1.7}$$

In other words, the condition (1.5) has to be replaced by the weaker one (1.7).

Obviously, because of the lack of information on φ_3, this uniqueness problem does not feet in the frame of the classical Cauchy problem. However, due to the boundary conditions ($\varphi = 0$ on $\partial\Omega\times(0,T)$) and the coupling of φ_3 with φ_1 and φ_2 through the system of PDE, one may still expect the uniqueness property to hold, at least for some geometries. But in any case, the problem under consideration is not of local nature. It is a global problem in which the geometry of the domain Ω plays a key role.

This type of problems arises naturally in the context of *systems* of PDE. We refer to [LiZ] for a closely related problem on the Stokes system.

The problem under consideration was addressed for the first time in [Z]. There it was proved if $T > T^*$ (with T^* a minimal time that may be computed explicitly in terms of Ω, ω and the Lamé coefficients) the uniqueness problem (under the weaker information (1.7)) may be reduced to the following one, related to the eigensolutions of the Lamé system:

$$\begin{cases} -\mu\Delta\psi - (\lambda+\mu)\nabla \operatorname{div}\psi = \alpha^2\psi & \text{in} \quad \Omega \\ \psi = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.8}$$

The reduced uniqueness problem is as follows: Assume that the eigensolution ψ of (1.8) has the particular structure below

$$\begin{cases} \psi_1 \equiv \psi_2 \equiv 0 \\ \psi_3(x_1,x_2,x_3) = a(x_1,x_2) + b(x_3). \end{cases} \tag{1.9}$$

Does this imply that $\psi \equiv 0$?

In [Z] this uniqueness property was shown to hold generically with respect to the domain Ω. More recently, in [SZ] it was proved that, in fact, this uniqueness property holds for any bounded Lipschitz domain Ω.

It is interesting to compare this result with the $2-d$ analogue. As we shall see in section 2, there are $2-d$ polygonal bounded domains in which the corresponding uniqueness property fails to be true.

In section 3 we briefly describe some of the key ingredients of the proof of [SZ] and, in particular, we explain why exceptional Lipschitz domains may exist in dimension 2.

As we mentioned above, the problem we discuss in these notes is just an example of a large class of uniqueness problems for systems with partial information on the data that arise naturally in the context of the control of vibrations and fluids. In section 4 we present some closely related open problems.

To close this introduction we would like to mention that this type of uniqueness problems arises also in other contexts. For instance, the problem we have discussed above appears when analyzing the decay of magneto-elastic waves. More generally, in a wide class of systems in thermo-elasticity, the thermal component induces some dissipation on the elastic ones. However, very often, this dissipation is partial in the sense that not all the components of the elastic part of the solution are dissipated. The existence of non-trivial equilibrium solutions leads naturally to the type of uniqueness problems we are considering here. The problem of the decay rate of solutions is much more subtle. At this respect we refer to [LZ] and to [BuL] where the analysis of how the polarization of singularities is reflected on the boundary plays a crucial role.

2. Discussion of the $2-d$ problem.

Let Ω be a bounded domain of $\mathbb{R}^2$ and consider the Lamé eigenvalue problem with Dirichlet boundary conditions

$$\begin{cases} -\mu\Delta\varphi - (\lambda+\mu)\nabla \operatorname{div}\varphi = \gamma^2\varphi & \text{in} \quad \Omega \\ \varphi = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{2.1}$$

where $\varphi = (\varphi_1, \varphi_2)$ and $\varphi_j = \varphi_j\,(x_1, x_2)\,,\, j = 1,2.$

We look for solutions of (2.1) such that

$$\begin{cases} \varphi_1 \equiv 0 \\ \varphi_2 = a(x_1) + b(x_2). \end{cases} \tag{2.2}$$

It is then easy to see that a and b satisfy the following second order differential equations:

$$-\mu a'' = \gamma^2 a;\ -(\lambda+2\mu)b'' = \alpha^2 b. \tag{2.3}$$

Therefore, it is easy to build non-trivial solutions of this form. We set for instance

$$a(x_1) = \cos\left(2\pi k x_1 \,/\sqrt{\mu}\right)\,;\; b(x_2) = -\cos\left(2\pi k x_2 \Big/ \sqrt{\lambda+2\mu}\right)\,,$$

with $k \in I\!N$. Then,

$$\varphi_1 \equiv 0,\ \varphi_2(x_1,x_2) = \cos\left(2\pi x_1 / \sqrt{\mu}\right) - \cos\left(2\pi x_2 \Big/ \sqrt{\lambda + 2\mu}\right)$$

is a solution of (2.1) with $\gamma = 4\pi^2 k^2$.

Moreover the boundary condition $\varphi\, |_{\partial\Omega} = 0$ is satisfied if Ω is the polygon:

$$\Omega = \left\{ (x_1, x_2) \in I\!R^2 : \left| \frac{x_1}{\sqrt{\mu}} \right| - 1 < \frac{x_2}{\sqrt{\lambda + 2\mu}} < \left| \frac{x_1}{\sqrt{\mu}} \right| + 1 \right\}.$$

Consequently, we deduce that, in space dimension 2, there are polygonal domains Ω for which there are uniaxial eigenvibrations of the Dirichlet Lamé system of the form (2.2).

Obviously, once such an eigensolution is built, $u = e^{i\gamma t}\varphi(x_1, x_2)$ provides a solution of the evolution Lamé system, with Dirichlet boundary conditions, such that $u_1 \equiv 0$ for all time.

As shown in [Z] these polygons are the only exceptional domains in which this kind of vibrations may exist. In particular, if domain Ω is of class C^1, there are no non-trivial uniaxial vibrations for the $2 - d$ Lamé system.

3. The $3 - d$ problem.

Let now Ω be a bounded domain of $I\!R^3$ and consider the Lamé system

$$\begin{cases} -\mu\Delta\varphi - (\lambda + \mu)\nabla \operatorname{div} \varphi = \gamma^2 \varphi & \text{in} \quad \Omega \\ \varphi = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.1}$$

Now $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ and $\varphi_j = \varphi_j(x)$, $j = 1, 2, 3$, $x = (x_1, x_2, x_3)$.

We look for particular solutions of (3.1) of the form

$$\begin{cases} \varphi_1 \equiv \varphi_2 \equiv 0 & \text{in} \quad \Omega \\ \varphi_3(x) = a(x_1, x_2) + b(x_3) & \text{in} \quad \Omega. \end{cases} \tag{3.2}$$

If we set $\psi = \varphi_3$, ψ has to satisfy

$$\begin{cases} -\mu\Delta\psi - (\lambda + \mu)\partial_3^2\psi = \gamma^2\psi & \text{in} \quad \Omega \\ \psi = 0 & \text{on} \quad \partial\Omega \\ \psi = a(x_1, x_2) + b(x_3) & \text{in} \quad \Omega. \end{cases} \tag{3.3}$$

By a linear change on the variable x_3 the elliptic operator involved in (3.3) may be transformed into the Laplacian. The problem is then reduced to analyze the existence of bounded domains Ω of $I\!R^3$ such that there is a non-trivial eigenfunction of the Dirichlet Laplacian

$$\begin{cases} -\Delta u = \gamma^2 u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{3.4}$$

of the form

$$u(x_1, x_2, x_3) = a(x_1, x_2) + b(x_3). \tag{3.5}$$

The problem may be reduced further. We can assume, without loss of generality, that

$$u = \varphi(x_1, x_2) - \cos(\sqrt{\gamma} x_3) \tag{3.6}$$

but then $\varphi = \varphi(x_1, x_2)$ satisfies

$$\begin{cases} -\Delta' \varphi = \gamma^2 \varphi & \text{in} \quad \mathcal{O} \\ \varphi = 1 & \text{on} \quad \partial\mathcal{O} \\ -1 \leq \varphi \leq 1 & \text{in} \quad \mathcal{O} \end{cases} \tag{3.7}$$

where $\mathcal{O}$ is a bounded $2-d$ domain.

More precisely, if φ is a solution of (3.7) with $\gamma \neq 0$, then (3.6) provides a solution of (3.4) in the $3-d$ domain

$$\Omega = \{(x_1, x_2, x_3) : (x_1, x_2) \in \mathcal{O}, \ -\arccos(\varphi(x_1, x_2)) \leq \sqrt{\gamma} x_3 \leq \arccos(\varphi(x_1, x_2))\}. \tag{3.8}$$

Reciprocally, if u as in (3.6) satisfies (3.4), then φ satisfies (3.7) for a suitable cross section $\mathcal{O}$ of Ω parallel to $x_3 = 0$.

In (3.7) and in the sequel, Δ' denotes the Laplacian in the variables x_1 and x_2.

Before analyzing (3.7) it is interesting to study the corresponding $1-d$ problem arising when Ω is a $2-d$ domain as in Section 2 above.

Obviously, for any $\gamma \neq 0$, $\varphi(x_1) = \cos(\gamma x_1)$ satisfies

$$\begin{cases} -\varphi'' = \gamma^2 \varphi & \text{in} \quad (0, 2\pi) \\ \varphi(0) = \varphi(2\pi) = 1 & \\ -1 \leq \varphi \leq 1 & \text{in} \quad (0, 2\pi). \end{cases} \tag{3.9}$$

It is precisely the existence of the non-trivial solution φ of (3.9) which allows to build (as in (3.8)) the polygonal domains of Section 2 for which there exist uni-axial vibrations in plannar elasticity.

Let us now go back to (3.7) when $\mathcal{O}$ is a bounded $2-d$ domain.

The following result from [SZ] shows that the situation is rather different from what we have seen in the corresponding $1-d$ problem.

Theorem 3.1 *([SZ]) Whatever the bounded Lipschitz domain $\mathcal{O}$ of $\mathbb{R}^2$ is, the unique solution of* (3.7) *is the trivial one, i.e.* $\gamma = 0$ *and* $\varphi \equiv 1$.

The proof of Theorem 3.1 uses the classical method of comparing the solution φ of (3.7) with a reflected and translated function φ^* at the point of contact by means of the maximum principle. The argument goes back at least to [M].

As an immediate consequence of Theorem 3.1 the following holds:

Corollary 3.1 *Let Ω be a bounded, Lipschitz domain of $\mathbb{R}^3$. Then, the unique eigenfunction of the Dirichlet Laplacian*

$$\begin{cases} -\Delta \varphi = \gamma^2 \varphi & \text{in} \quad \Omega \\ \varphi = 0 & \text{on} \quad \partial\Omega \end{cases}$$

of the form

$$\varphi = a(x_1, x_2) + b(x_3),$$

is the trivial one, i.e. $\varphi \equiv 0$.

An then, as a consequence of Corollary 3.1 we get:

Corollary 3.2 *Let Ω be a bounded domain of $\mathbb{R}^3$. Then, the unique uni-axial eigensolution of the $3-d$ Dirichlet system*

$$\begin{cases} -\mu\Delta\varphi - (\lambda+\mu)\nabla \operatorname{div}\varphi = \gamma^2\varphi & \text{in} \quad \Omega \\ \varphi = 0 & \text{on} \quad \partial\Omega \end{cases}$$

such that

$$\varphi_1 \equiv \varphi_2 \equiv 0 \text{in } \Omega$$

is the trivial one, i.e. $\varphi \equiv 0$.

Finally, as a consequence of Corollary 3.2, it follows that the unique solution of the evolution Lamé system (1.4) such that (1.7) holds is the trivial one provided $T > T^*$, T^* being a minimal time that depends on Ω, ω and the Lamé coefficients.

4. Open problems.

4.1 Quantitative results

We have seen that when Ω is a bounded, Lipschitz domain of $\mathbb{R}^3$, and $T > 0$ is large enough, the following uniqueness problem holds for the solutions of the Lamé system (1.4): If $\varphi_1 = \varphi_2 = 0$ in $\omega \times (0,T)$, then $\varphi \equiv 0$.

This shows that the quantity

$$\left\{ \int_0^T \int_\omega \left[|\varphi_1|^2 + |\varphi_2|^2 \right] dxdt \right\}^{1/2} \tag{4.1}$$

defines a norm for the solutions of (1.4).

Assuming that Ω is smooth enough, is it true that

$$\int_0^T \int_\Omega |\varphi|^2 \, dxdt \leq C \int_0^T \int_\omega \left[|\varphi_1|^2 + |\varphi_2|^2 \right] dxdt \tag{4.2}$$

with a constant C that is independent of the solution φ of (1.4) under consideration?

If not, does a weaker version of (4.2) hold true if the left hand side is replaced by the norm in some negative Sobolev space H^{-s}?

Obviously, for (4.2) (or a weaker version of it) to be true a "geometric control condition" in the spirit of [BLR] needs to be imposed. Namely, if there is a ray of geometric optics that does not intersect ω in a time interval of length T, then (4.2) may not hold (not even in the weaker form mentioned above). This is due to the existence of gaussian beam solutions concentrated along rays as in [R]. (We refer to [LZ] for the analysis of gaussian beam solutions for the Lamé system).

Accordingly, the problem (4.2) (or its weaker form) has to be analyzed for domains ω satisfying this kind of geometric control condition. At this respect, the results of [BuL] on the polarization of singularities may play a crucial role.

Obviously, the same problems arise in space-dimension 2. Assuming that Ω is not one of the exceptional polygonal domains described in Section 2 and that ω satisfies a geometric control condition, can we guarantee that a inequality of the following form holds

$$\| \varphi \|_{H^{-s}}^2 \leq C \int_0^T \int_\omega |\varphi_1|^2 \, dxdt?$$

4.2 Uniqueness under information on one component

When analyzing the controllability of the Lamé system in $3-d$ by uniaxial forces the following uniqueness problem arises naturally: If φ is a solution of (1.4) such that

$$\varphi_1 = 0 \text{ in } \omega \times (0, T),$$

can we guarantee that $\varphi \equiv 0$?

As far as we know, this problem has not been treated in the literature.

For those domains Ω and ω in which this uniqueness property holds and under a natural geometric control condition, the next problem to be studied would be quantitavie version like those we have raised in section 4.1 but now replacing the right hand side of (4.2) by the weaker quantity $\int_0^T \int_\omega \varphi_1^2 dxdt$.

4.3 Uniqueness under information on the boundary traces

One of the most classical problems in Control Theory is the one in which the control acts on the system through the boundary of the domain. In this context, this gives rise to the following uniqueness problem. If φ solves (1.4) and satisfies

$$\frac{\partial \varphi_1}{\partial \nu} = \frac{\partial \varphi_2}{\partial \nu} = 0 \text{ on } \partial\Omega \times (0, T),$$

can we guarantee that $\varphi \equiv 0$?

Once again, as far as we know, this problem is completely open.

Quantitative versions of this problem may also be considered.

The $2-d$ analogue would be: If the solution $\varphi = (\varphi_1, \varphi_2)$ of the $2-d$ Lamé system (1.4) is such that

$$\partial \varphi_1 / \partial \nu = 0 \text{ on } \partial\Omega \times (0, T),$$

can we guarantee that $\varphi \equiv 0$? Of course, this property fails for the exceptional polygonal domains described in section 2. But we do not know whether the property is true for other domains.

The same problem may also be formulated in $3-d$. Is $\partial \varphi_1 / \partial \nu = 0$ on $\partial\Omega \times (0, T)$ sufficient to guarantee that $\varphi \equiv 0$?

References

[BLR] C. Bardos, G. Lebeau and J. Rauch, Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary, SIAM J. Cont. Optim., **30** (1992), 1024-1065.

[BuL] N. Burq and G. Lebeau, work in preparation.

[LZ] G. Lebeau and E. Zuazua, Decay rates for the linear system of three-dimensional thermoelasticity, Archives Rat. Mech. Anal., to appear (C. R. Acad. Sci. Paris, **324** (1997), 409-415).

[LiZ] J.L. Lions and E. Zuazua, A generic uniqueness result for the Stokes system and its control theoretical consequences, in "Partial Differential Equations and Applications", P. Marcellini, G. Talenti and E. Visentini eds., Marcel-Dekker Inc., LNPAS **177**, 1996, p. 221-235.

[M] A. McNabb, Strong comparison theorems for elliptic equations of second order, J. Math. Mech., 10 (1961), 431-440.

[PZ] G. Perla-Menzala and E. Zuazua, Energy decay of magnetoelastic waves in a bounded conductive medium, Asymptotic Anal., **18** (1998), 349-362.

[R] J. Ralston, Solution of the wave equation with localized energy, Comm. Pure Appl. Math., **22** (1969), 807-823.

[SZ] G. Sweers and E. Zuazua, On the nonexistence of some special eigenfunctions for the Dirichlet Laplacian and the Lamé system, J. Elasticity, **52** (1999), 111-120.

[Z] E. Zuazua, A uniqueness result for the linear system of elasticity and its control theoretical consequences, SIAM J. Cont. Optim., **34** (5) (1996), 1473-1495 & **37** (1) (1998), 330-331

Departamento de Matemática Aplicada, Universidad Complutense, Facultad de Matematicas,28040 Madrid. Spain
`zuazua@eucmax.sim.ucm.es`