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# On the $L^2$-instability and $L^2$-controllability of steady flows of an idealincompressible fluid

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**Abstract**

In the existing stability theory of steady flows of an ideal incompressible fluid, formulated by V. Arnold, the stability is understood as a stability with respect to perturbations with small in $L^2$ vorticity. Nothing has been known about the stability under perturbation with small energy, without any restrictions on vorticity; it was clear that existing methods do not work for this (the most physically reasonable) class of perturbations. We prove that in fact, every nontrivial steady flow is unstable in $L^2$; moreover, every flow may be transformed into any other one, with the same energy and momentum, with the help of an appropriately chosen perturbation with arbitrary small energy. This phenomenon reminds the Arnold's diffusion. This result is proven by the direct construction of a growing perturbation, which is done by a variational method.

**1.** In this work we are studying the flows of an ideal incompressible fluid in a bounded 2-d domain $M \subset \mathbf{R}^2$, described by the Euler equations

$$\frac{\partial u}{\partial t} + (u, \nabla)u + \nabla p = 0; \tag{1}$$

$$\nabla \cdot u = 0. \tag{2}$$

Here $u = u(x,t),\ x \in M,\ t \in [0,T]$, and $u|_{\partial M}$ is tangent to $\partial M$.

It has been known for a long time, that if the initial velocity field $u(x,0)$ is smooth, then there exists unique smooth solution $u(x,t)$ of the Euler equations, which is defined for all $t \in \mathbf{R}$; see [M-P]. The next natural question is, what may be the behavior of this solution, as $t \to \infty$. This is a problem of indefinite complexity. A restricted problem is the following: suppose that the initial flow field $u_0(x)$ is in close to a steady solution $u_0(x)$. What may happen with this flow for big $t$? Does

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it stay always close to $u_0$, or it can escape far away? Which flows are available, if we start from different initial velocities, close to $u_0$?

These are problems of a global, nonlinear perturbation theory of steady solutions of the Euler equations. The first idea is to develop a linear stability theory. The spectrum of a linearized operator is always symmetric w.r.t. both the real and the imaginary axes, for the system is Hamiltonian. Therefore we can never prove an asymptotic stability by the linear method; at best we can prove the absence of a linear instability, which, in its turn, may be a tricky business.

The true, nonlinear stability of some classes of steady flows was first proven by V. Arnold (see [A1, A2, A-K]). He considered a very strong restriction on perturbation: the perturbation of the vorticity, $\omega(0)-\omega_0 = \nabla\times u(\cdot,0)-\nabla\times u_0(\cdot)$, should be small in $L^2(M)$. There are three classes of steady flows which are stable under perturbations small in this sense. The first class contains only one flow with constant vorticity; its stability is obvious. The second and the third classes consist of steady flows, corresponding to a strong local maximum, resp. minimum, of the kinetic energy on the leaf of equivortical fields in the space of all smooth velocity fields in $M$ (see [A-K]).

But this theory breaks down, if we drop the condition on the vorticity perturbation, and consider all (smooth) velocity fields $u(x,0)$, which are close to $u_0(x)$ in $L^2$, without any conditions on the derivatives. Note that this class of perturbations is no less physically significant than the previous one, because it describes perturbations with small energy. In this case, there is apparently no obstacle preventing the flow from going far away from $u_0$. So, it is likely that the flow is unstable. (but this does not prove instability, for there may be some other reasons for stability, like, for example in the KAM theory. This is analogous to the situation in the 3-d Euler equations. In the 2-d case, the vorticity is transported by the flow; this means that there exist infinity of integrals of motion, namely the moments of vorticity. These integrals prevent the solution from forming singularities. In the 3-d case, the vorticity field is transformed by the flow as a frozen-in vector field, and we can't extract additional integrals of motion from the vorticity field. This means that we don't know any obstacle to the formation of a finite-time singularity from a smooth initial flow. But we have no examples yet of such singularities. May be, this work may give some hints.)

**2.** In this work we consider a weaker stability problem. Consider the Euler equations with a nonzero right hand side (i.e. external force):

$$\frac{\partial u}{\partial t} + (u,\nabla)u + \nabla p = f; \tag{3}$$

$$\nabla\cdot u = 0. \tag{4}$$

Here $f = f(x,t)$ is a smooth in $x$ vector field, such that $\nabla\cdot f = 0$, and $f(x,t)|_{\partial M}$ is parallel to $\partial M$. Consider the behavior of $u(x,t)$, if $f$ is small in the following sense: $\int_0^T \| f(\cdot,t) \|_{L^2}\, dt$ is small, where $[0,T]$ is the time interval (assumed to be long), where the flow is considered. For example, if $f$ has the form $f(x,t) = F(x)\delta(t)$, we return to the initial stability problem.

**Definition 1** *Suppose that $u(x_1, x_2)$ and $v(x_1, x_2)$ are two steady flows. We say that the force $f$ transfers the flow $u$ into the flow $v$ during the time interval $[0, T]$, if the following is true: if $w(x, t)$ is the solution of the nonhomogeneous Euler equations (3), (4) with the initial condition $w(x, 0) = u(x)$, then $w(x, T) = v(x)$.*

Consider the simplest basic steady flow, namely a parallel flow. Let $M$ be a strip $0 \leq x_2 \leq 1$ in the $(x_1, x_2)$-plane. We restrict ourselves to the flows having period $L$ along the $x_2$-axis; this period is the same for all flows that are considered below. Suppose that the velocity field $u_0(x)$ has the form $(U(x_2), 0)$, where $U$ is a given smooth function (the velocity profile). The original problem was, for which profiles $U$ the flow $u_0$ is stable. Our main result is the following

**Theorem 1** *For every nontrivial (i.e. different from constant) velocity profile $U$ the flow $u_0$ is $L^2$-unstable. This means that for every function $U(x_2) \neq$ const there exists $C > 0$, such that for every $\varepsilon > 0$ the following is true. There exist $T > 0$ and a smooth force $f(x, t)$, defined in $M \times [0, T]$, such that $\int_0^T \| f(\cdot, t) \|_{L^2} \, dt < \varepsilon$, and $f$ transfers the flow $u_0$ during the time interval $[0, T]$ into a steady flow $u_1$, such that $\| u_0 - u_1 \|_{L^2} > C$.*

So, the flow may be considerably changed by arbitrarily small force, provided the time interval is sufficiently long.

Note that if the force $f$ satisfies a stronger condition $\int_0^T \| \omega(\cdot, t) \|_{L^2} \, dt < \varepsilon$, where $\omega = \nabla \times f$ is the vorticity, then, for every Arnold stable flow $u_0$, the resulting perturbation at time $t$ will be small, too.

This theorem is implied by a much stronger assertion.

**Theorem 2** *Suppose that $U(x_2)$ and $V(x_2)$ are two velocity profiles, such that $\int_0^1 U(x_2) dx_2 = \int_0^1 V(x_2) dx_2$, and $\int_0^1 \frac{1}{2} |U(x_2)|^2 dx_2 = \int_0^1 \frac{1}{2} |V(x_2)|^2 dx_2$; let $u_0(x_1, x_2) = (U(x_2), 0)$, $v_0(x_1, x_2) = (V(x_2), 0)$ be corresponding steady parallel flows (having equal momenta and energies). Then for every $\varepsilon > 0$ there exist $T > 0$ and a smooth force $f(x, t)$, such that $\int_0^T \| f(\cdot, t) \|_{L^2} < \varepsilon$, and $f$ transfers $u$ into $v$ during the time interval $[0, T]$.*

This means that the flow of an ideal incompressible fluid is perfectly controllable by arbitrarily small force.

**3.** Theorems 1, 2 are proven by an explicit construction of the flow.

Note first, that if $U_1, U_2, \cdots, U_N$ are velocity profiles, and Theorem 2 is true for every pair $(U_i, U_{i+1})$ of velocity profiles, then we can pass from $U_1$ to $U_N$, simply concatenating the flows connecting $U_i$ and $U_{i+1}$; thus Theorem 2 is true for the pair $(U_1, U_N)$. Therefore it is enough to construct the sequence of steady flows with profiles $U_1, \cdots, U_N$, and the intermediate nonsteady flows connecting every two successive steady ones.

Note also, that it is enough to construct a sequence of piecewise-smooth flows, for it is not difficult to smoothen them, so that the necessary force will have arbitrarily small norm in $L^1(0, T; L^2(M))$.

As a first step, we change the flow with the profile $U = U_1$ by a piecewise-constant profile $U_2$ with sufficiently small steps; this may be done by a force with arbitrarily small norm.

Thus, $U_2(x_2)$ is a step function, $U_2(x_2) = U_2^{(k)}$ for $x_2^{(k-1)} < x_2 < x_2^{(k)}$, $k = 1, \cdots, K$. Every next profile $U_i$ is also a step-wise function. We are free to subdivide the steps and change a little the values of velocity, if these changes are small enough.

Every flow $u_k$ is obtained by the previous one $u_{k-1}$ by one of two operations, described in the following theorems.

**Theorem 3** *Let $U(x_2)$ be a step function, $U(x_2) = U^{(k)}$ for $x_2^{(k-1)} < x_2 < x_2^{(k)}$; let $V(x_2)$ be another step function, obtained by transposition of two adjacent segments $[x_2^{(k-1)}, x_2^{(k)}]$ and $[x_2^{(k)}, x_2^{(k+1)}]$. Let $u(x_1, x_2)$, $v(x_1, x_2)$ be parallel flows with velocity profile $U(x_2), V(x_2)$. Then for every $\varepsilon > 0$ there exist $T > 0$ and a piecewise-smooth force $f(x,t)$, such that $\int_0^T \| f(\cdot, t) \|_{L^2} < \varepsilon$, and the force $f$ transfers the flow $u$ into the flow $v$ during the time interval $[0, T]$.*

To formulate the next theorem, remind the law of an elastic collision of two bodies. Suppose that two point masses $m_1$ and $m_2$, having velocities $u_1$ and $u_2$, collide elastically. Then their velocities after collision will be $v_1 = 2u_0 - u_1$, $v_2 = 2u_0 - u_2$, where $u_0 = (m_1 u_1 + m_2 u_2)/(m_1 + m_2)$ is the velocity of the center of masses. The transformation $(u_1, u_2) \to (v_1, v_2)$ is called a *transformation of elastic collision.*

**Theorem 4** *Suppose that the profile $U(x_2)$ is like in Theorem 3, and the profile $V(x_2)$ is equal to $U(x_2)$ outside the segment $x_2^{(k-1)} < x_2 < x_2^{(k+1)}$; on the last segment, $V(x_2) = v^{(k)}$, if $x_2^{(k-1)} < x_2 < x_2^{(k)}$, and $V(x_2) = v^{(k+1)}$, if $x_2^{(k)}) < x_2 < x_2^{(k+1)}$, where $(v^{(k)}, v^{(k+1)})$ is obtained from $(u^{(k)}, u^{(k+1)})$ by the transformation of elastic collision, the lengths $x_2^{(k)} - x_2^{(k-1)}$, $x_2^{(k+1)} - x_2^{(k)}$ playing the role of masses $m_1, m_1$. Let $u(x_1, x_2), v(x_1, x_2)$ be parallel flows with profiles $U(x_2), V(x_2)$. Then for every $\varepsilon > 0$ there exist $T > 0$ and a force $f(x,t)$, such that $\int_0^T \| f(\cdot, t) \|_{L^2} < \varepsilon$, and the force $f$ transfers the flow $u$ into flow $v$.*

Suppose now, that $U(x_2)$ and $V(x_2)$ are two velocity profiles, having equal momenta and energies. Then it is not difficult to construct a sequence of step functions $U_2(x_2), U_3(x_2), \cdots, U_N(x_2)$, so that $U_2$ is $L^2$-close to $U_1 = U$, $U_N$ is $L^2$-close to $V$, and every profile $U_k$ is obtained from $U_{k-1}$ by one of two operations, described in Theorems 3 and 4. Using these theorems and the notes above, we construct a piecewise-smooth force $f(x,t)$, such that $\int_0^T \| f(\cdot, t) \|_{L^2} \, dt < \varepsilon$, and $f$ transfers $U$ into $V$ during the time interval $[0, T]$.

**4.** Theorems 3 and 4 are proven by the variational method.

Let $\mathcal{D}(\mathcal{M}) = \mathcal{D}$ be the group of the volume-preserving diffeomorphisms of the flow domain $M$. These diffeomorphisms may be identified with fluid configurations: every configuration is obtained from some fixed one by a permutation of fluid particles, which is assumed to be a smooth, volume preserving diffeomorphism. The flow is a family $g_t$ of elements of $\mathcal{D}$, depending on time $t$, $0 \le t \le T$. The Lagrangian velocity of the flow is a vector-function $V(x,t) = \frac{\partial}{\partial t} g_t(x) = \dot{g}_t(x)$, while the Eulerian velocity is the vector field $v(x,t) = \dot{g}_t(g_t^{-1}(x))$. The action of the flow is defined as $J\{g_t\}_0^T = \int_0^T \frac{1}{2} \parallel \dot{g}_t \parallel_{L^2}^2 dt$, and the length $L\{g_t\}_0^T = \int_0^T \parallel \dot{g}_t \parallel_{L^2} dt$.

The solution $u(x,t)$ of the homogeneous Euler equations (1), (2) is an Eulerian velocity field of a geodesic trajectory $g_t$ on the group $\mathcal{D}$: $u(x,t) = \dot{g}_t(g_t^{-1}(x))$, such that $\delta J\{g_t\}_0^T = 0$, provided $g_0$, $g_T$ are fixed. This implies that also $\delta L\{g_t\}_0^T = 0$. This is the classical Hamiltonian principle (see [A3]). The evident idea is to try to construct solutions of the Euler equations by fixing $g_0$, $g_T \in \mathcal{D}$, and looking for the shortest trajectory, connecting these fluid configurations. If the minimum is attainable, then we have constructed some nontrivial solution of the Euler equations.

But this idea does not work well. If $g_0$ and $g_T$ are $C^2$-close, the minimum is assumed at some smooth trajectory. But if $g_0$ and $g_T$ are far away from each other, which is the only interesting case, then it is possible that the minimum is no more attainable (see [S, A-K]). In the 3-d case there are examples of $g_0, g_T$, such that for every smooth flow $g_t$, connecting $g_0$ and $g_T$, there exists another smooth flow $g'_t$, connecting the same fluid configurations, such that $J\{g'_t\}_0^T < J\{g_t\}_0^T$; so, the minimum is unattainable.

The existence of a minimal geodesic connecting two configurations of a 2-d fluid is neither proven nor disproven, while some physical considerations show that sometimes the minimal smooth flow does not exist .

If there is no smooth solution of the variational problem, we may look for a generalized solution, which is no longer a smooth flow, but belongs to a wider class of object. The appropriate notion of a generalized flow was introduced by Y. Brenier [B]. Generalized flow is a probability measure $\mu$ in the space $X = C(0,T;M)$ of all continuous trajectories in the flow domain $M$, satisfying the following two conditions:

1. For every $t \in [0,T]$ and every Borel set $A \subset M$,

$$\mu\{x(\cdot)|x(t) \in A\} = \text{mes}A;$$

2.

$$J\{\mu\} = \int_X \int_0^T |\dot{x}(t)|^2 dt \mu\{dx\} < \infty$$

The meaning of the first condition is that the generalized flow is incompressible; the second condition expresses the finiteness of the mean action (and that $\mu$-almost all trajectories belong to $H^1$).

Every smooth flow may be regarded as a generalized one; but there is a lot of truly generalized flows.

The generalized variational problem may be posed as follows: given a diffeomorphism $g \in \mathcal{D}$; consider all generalized flows which, in addition to the above conditions, satisfy the following:

3. For $\mu$-almost all trajectories $x(t)$, $x(T) = g(x(0))$.

We are looking for a generalized flow, which satisfies all three conditions and minimizes the functional $J\{\mu\}_0^T = \int_X \int_0^T \frac{1}{2} |\dot{x}(t)|^2 dt \mu\{dx\}$.

Y. Brenier has proved, using the simple ideas of weak compactness of a family of measures and semicontinuity of the action functional in $X$, that this problem has a solution for every $g \in \mathcal{D}$. Simple examples show that this solution may be very far from any smooth, or even measurable, flow.

But in the 2-d case the situation is much better, because there is an additional structure. To see it, consider a smooth incompressible flow $g_t$, $0 \leq t \leq T$, $g_0 = \mathrm{Id}$, $g_T = g$. Let $Q = M \times [0, T]$ be a cylinder in the $(x, t)$-space. Every trajectory $\lambda_x = \{(g_t(x), t)\}$, $x \in M$, is a smooth curve in $Q$, connecting the points $(x, 0)$ and $(g(x), T)$. For different points $x, x'$, the curves $\lambda_x, \lambda_{x'}$ do not intersect. So, the lines $\lambda_x$ form a *braid*, containing continuum of threads. Such a braid is called a smooth braid.

Now let us define a generalized braid. Let us fix a volume-preserving diffeomorphism $g \in \mathcal{D}$ and a piecewise-smooth incompressible flow $G_t$, such that $G_0 = \mathrm{Id}$, $G_T = g$. The bundle of lines $(G_t(x), t)$, $x \in M$, is called a reference braid and denoted by $\mathbf{B}_0$.

**Definition 2** *A generalized flow $\mu$ is called a generalized braid, and denoted by $\mathbf{B}$, if it satisfies conditions 1, 2, 3 above, and the following condition*

*4. For any $N$, let us pick trajectories $x^1(t), \cdots, x^N(t)$ by random and independently, i.e. with a probability distribution $\mu \otimes \cdots \otimes \mu$ ($N$ times). Then for almost all such $N$-tuples of trajectories, the lines $(x^i(t), t)$ for different $i$ do not intersect, and the finite braid, formed by the curves $(x^1(t), t), \cdots, (x^N(t), t)$, is isotopic to the braid, formed by the curves $(G_t(x^1(0)), t), \cdots, (G_t(x^N(0)), t)$ (these braids have the same endpoints, so it is possible to define their isotopy).*

The braid $\mathbf{B}$ is called a braid weakly isotopic to a piecewise-smooth braid $\mathbf{B}_0$.

We are discussing the following variational problem: given a map $g$ and a reference braid $\mathbf{B}_0$; find a generalized braid $\mathbf{B}$, isotopic to $\mathbf{B}_0$, which minimizes the functional $J\{\mathbf{B}\}$.

**Theorem 5** *The variational problem has a solution for every data $g, \mathbf{B}_0$.*

To prove this theorem, we consider a sequence $\mathbf{B}_i$ of braids, such that $J\{\mathbf{B}_i\}_0^T \searrow J_0$, where $J_0 = \inf J\{\mathbf{B}\}$ for all braids $\mathbf{B}$, isotopic to a given braid $\mathbf{B}_0$. This sequence is weakly compact; its subsequence converges to a generalized flow $\mu$, such that $J\{\mu\} = J_0$, exactly as for the generalized flows. But the generslized flow $\mu$ is, in fact, a braid isotopic to $\mathbf{B}_0$, which we shall denote by $\bar{\mathbf{B}}$. This is implied by the fact that the isotopy class of a finite subbraid with fixed endpoints is "weakly continuous", and therefore we can pass to the limit and conclude that the weak limit of the braids $\mathbf{B}_i$, regarded as generalized flows, is a braid, isotopic to $\mathbf{B}_0$. Let us call $\bar{\mathbf{B}}$ a minimal braid.

Generally, braids are as nonregular locally as generalized flows. In particular, they have, in general, no definite velocity field: for almost every $(x, t)$ there are different trajectories passing through this point with different velocities. But the *minimal* braid is much more regular. Recall that a measurable flow is defined as a family $h_{s,t}$ of measurable maps of $M$ into itself, preserving the Lebesgue measure, and such that $h_{s,t} \circ h_{r,s} = h_{r,t}$ for all $r, s, t \in [0, T]$.

**Theorem 6** *Let* $\bar{\mathbf{B}}$ *be a minimal braid, isotopic to the reference braid* $\mathbf{B}_0$. *Then there exists a measurable flow* $h_{s,t}$ *in* $M$, *such that for* $\mu$-*almost all trajectories* $x(t)$, $x(t) = h_{s,t}x(s)$. *Moreover, there exists a vector field* $u(x,t) \in L^2$, *divergence free and tangent to* $\partial M$, *such that for almost all trajectories* $x(t)$, $\dot{x}(t) = u(x(t),t)$ *for almost all* $t$.

The next fact about the minimal braids is the following

**Theorem 7** *The velocity field* $u(x,t)$, *corresponding to the minimal braid* $\bar{\mathbf{B}}$, *is a weak solution of the Euler equations. This means that for every vector field* $v(x,t) \in C_0^\infty$, *such that* $\nabla \cdot v = 0$, *and for every scalar function* $\varphi(x,t)$,

$$\int\limits_Q [(u, \frac{\partial v}{\partial t}) + (u \otimes u, \nabla v)]dxdt = 0, \tag{5}$$

$$\int\limits_Q (u, \nabla\varphi)dxdt = 0. \tag{6}$$

The last fact which we need is the following approximation theorem.

**Theorem 8** *Suppose that* $\bar{\mathbf{B}}$ *is a minimal braid, and* $u(x,t)$ *is its velocity field. Then for every* $\varepsilon > 0$ *there exists a smooth incompressible flow with velocity field* $w(x,t)$, *which is a solution of the nonhomogeneous Euler equations with the force* $f(x,t)$, *such that* $\| w(x,t) - u(x,t) \|_{L^2} < \varepsilon$ *for all* $t$, *and* $\int_0^T \| f(\cdot,t) \|_{l^2} dt < \varepsilon$.

**5.** The brais are used to construct the flows, described in Theorems 3 and 4. Consider a flow with a piecewise-constant profile $U(x_2)$; then the cylinder $Q = M \times [0,T]$ may be divided into slices $Q_k$, so that $U|_{Q_k} = U^{(k)}$. In every such slice the trajectories are parallel lines with the same slope. Thus, they form a simple, piecewise-smooth braid.

Now let us describe the braids corresponding to the flows described in Theorem 3. Let us divide the domains $M_k$, $M_{k+1}$, the bases of the cylinders $Q_k$, $Q_{k+1}$, into small subdomains $M_{k,j}$, $M_{k+1,l}$. Let us pick one point $x_{k,j}$ in every domain $M_{k,j}$, and one point $x_{k+1,l}$ in every domain $M_{k+1,l}$. Let $\lambda_{k,j}$, $\lambda_{k+1,l}$ be the trajectories, passing through the points $(x_{k,j},0)$ and $(x_{k+1,l},0)$. Their endpoints in $M \times \{T\}$ are denoted by $(y_{k,j},T)$ and $(y_{k+1,l},T)$. The trajectories, passing through $M_{k,j} \times \{0\}$, and through $M_{k+1,l} \times \{0\}$, form subbraids $\mathbf{B}_{k,j}$ and $\mathbf{B}_{k+1,l}$.

Now let us define a new braid $\mathbf{B}'_0$. First let us define its threads $\lambda'_{k,j}$ and $\lambda'_{k+1,l}$, passing through the points $(x_{k,j},0)$ and $(x_{k+1,l},0)$. They are straight lines, passing through the points $(y'_{k,j},T)$ and $(y'_{k+1,l},T)$, obtained from the points $(y_{k,j},T)$ and $(y_{k+1,l},T)$ by the shift in the $x_2$-direction by, resp., $(x_2^{(k+1)} - x_2^{(k)})$ and $(x_2^{(k-1)} - x_2^{(k)})$. These lines do not, generally, intersect.

Now let us define a piecewise-smooth braid $\mathbf{B}'_0$. It coincides with $\mathbf{B}$ outside $Q_k \bigcup Q_{k+1}$. In the last domain the braid $b'_0$ consists of smooth incompressible subbraids $\mathbf{B}'_{k,j}$ and $\mathbf{B}'_{k+1,l}$ with the bases $M_{k,j} \times \{0\}$ and $M_{k+1,l} \times \{0\}$. Each subbraid contains one line $\lambda'_{k,j}$ and $\lambda'_{k+1,l}$. The interfaces between these subbraids are piecewise-smooth. It is easy to construct such subbraids, while they are not unique.

Now let us use $\mathbf{B}'_0$ as a reference braid, and construct a minimal braid $\mathbf{B}'$, isotopic to $\mathbf{B}'_0$. Using Theorems 5–8, we construct a smooth flow $w(x,t)$, supported by a smooth force $f(x,t)$, such that $\int_0^T \| f(\cdot,t) \|_{L^2} \, dt$ is arbitrarily small, provided $T$ is big enough. This flow is $L^2$-close to $U$ at $t = 0$ and to $V$ at $t = T$; after a small modification of $w(x,t)$, requiring an $L^2$-small correcting force, we obtain a flow described in Theorem 3.

The proof of Theorem 4 is similar; but in this case the curves $\lambda'_{k,j}$ and $\lambda'_{k+1,l}$ are going from points $(x_{k,j}, 0)$ $((x_{k+1,l}, 0))$ to the points $(y_{k,j}, 0)$ $((y_{k+1,l}, 0))$ , and some of the curves $\lambda'_{k,j}$ and $\lambda'_{k+1,l}$ are linked.

**6.** Theorems 3 and 4 are true also for circular flows in a disk, with the angular momentum staying in place of momentum in Theorem 4. But for generic 2-d domains the situation is not so clear. We don't know, whether there is an integral of motion, similar to the angular momentum, in any domain different from the disk. If such integral does not exist, which is the most likely, then the natural conjecture is that for any two flows with equal energies the conclusion of Theorem 4 is true. But this behavior is paradoxical: just imagine a nearly circular flow in a nearly circular domain (e.g. ellipse), which after some long time changes the sign of the angular velocity. This question requires more thinking.

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