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Kinetic and Hydrodynamical equations for one-dimensional granular media

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Abstract

In this lecture I present some open mathematical problems concerning some PDE arising in the study of one-dimensional models for granular media.

1. The microscopic model.

Roughly speaking a granular media is a collection of particles (grains, seeds,...) interacting via inelastic collisions. Although each individual can have a complex internal structure, we shall consider it as the basic microscopic object. The main purpose in the study of such systems is to outline its collective behavior, attempting more fruitful descriptions that those arising by the mere consideration of the equation of the motion for the microscopic system. In other words we try to introduce and investigate reduced descriptions similar to that used in kinetic theory to derive macroscopic (thermo-dynamical or hydro-dynamical) behaviors of the usual non-dissipative mechanical systems.

It is surprising how a small inelasticity on the system can change so drastically its macroscopic behavior. As a consequence we face aspects of the theory which are absolutely new and challenging.

In the present lecture we review some of the very few rigorous results we are aware and discuss some open problems.

The simplest model of granular media (see e.g. [?, ?, DLK]...) is constituted by a system of point particles on the line, interacting inelastically.

More precisely, we consider a system of N particles in $\mathbb{R}$. Let x_i, $v_i \in \mathbb{R}$ be the position and the velocity of the i-th particle and $Z^N = (X^N, V^N) = (x_1, v_1, \dots x_N, v_N)$ be a state of the system. The dynamics is the following. The particles move freely up to the first instant in which two of them are in the same point. Then they collide according to the following rule:

$$\begin{aligned} v' &= v - \varepsilon(v - v_1) \\ v_1' &= v_1 + \varepsilon(v - v_1), \end{aligned} \tag{1}$$

where v', v_1' and v, v_1 are the outgoing and incoming velocity respectively, and $\varepsilon \in [0, 1/2]$ is a parameter measuring the degree of inelasticity of the collision.

Note that the collision preserves the total momentum and dissipates the kinetic energy. Moreover for $\varepsilon = 0$ we have the free particle system, while for $\varepsilon = 1/2$ we have the so called sticky particle model in which the particle pair remains attached after the collision.

The particles are assumed to be identical so that it is completely arbitrary how to call the particles after a collision. The collision law is useful for many purpose being a small perturbation of the free motion, however a more intuitive representation could be:

$$\begin{aligned} v' &= v_1 + \varepsilon(v - v_1) \\ v_1' &= v - \varepsilon(v - v_1), \end{aligned} \tag{2}$$

which is a slight modification of a perfectly elastic collision.

The most relevant qualitative feature of the systems is the possibility of delivering collapses in a finite time for a suitable values of the main parameters ε and N, namely the particles may suffer infinitely many collisions in a finite time in which some of them reach the same position with the same velocity. After a collapse the dynamics is not prolongable anymore.

The first example of collapse has been shown for three particles (see Ref. [CGM]). They can collapse if and only if $\varepsilon > 2\sqrt{3} - 3$. Then (see Ref. [BC]) it was proved the existence of collapses for suitable values of the main parameters.

Since we are mostly interested in the collective behavior of the system rather than to specific solutions of the dynamical equations, it seems natural to apply the methods of the kinetic theory to understand the general behavior of the system under suitable scaling limits.

According to such a prescription we rescale suitably the degree of inelasticity, as well as the total number of particles (which is assumed to diverge), to obtain a kinetic equation for the one-particle probability density. Such a (purely formal) drivation can be done (See [BCP] and [BCPe]). The result is the following kinetic equation:

$$(\partial_t + v\partial_x) f(x, v) = -\lambda \partial_v (F f), \tag{3}$$

where:

$$F(v, t) = \int d\bar{v} \phi(\bar{v} - v) f(\bar{v}, t). \tag{4}$$

2. The mean-field equation.

Eq.3-4 of the previous section is sometimes called mean-field equation because it is obtained in the so called quasielastic limit: $N \to \infty$ $\varepsilon \to 0$ $N\varepsilon \to \lambda$ which is a sort of a mean field limit.

Such an equation possesses a system of characteristics which allows a rigorous study of the behavior of the solutions. The existence problem is not an easy task. It is not hard to show that homogeneous solutions exist globally in time for a fairly general class of initial conditions. For a general (non-homogeneous) initial datum f_0 however, we cannot exclude the appearance of singularities in a finite time. These (if any) should correspond to the presence of collapses for the particle system in the regime in which $\lambda = N\varepsilon$ is sufficiently large. As matter of fact we are not able to answer (even at heuristic level) to the following questions:

i) Do weak solutions exist globally in time for the non-homogeneous equation?

ii) Can occur singularities in a finite time and if it is so, can the solution concentrate (that is produce a component of δ type in the (x, v)-space)?

iii) If there are many weak solutions, do exists a criterion which select the physical relevant solution ?

Notice that the only extra a-priori information we have on the solution, is a reverse H–theorem, which describes the obvious tendency of the system to concentrate. Namely, defining $H(f) = \int dxdvh(f(x, v, t))$, with h convex function, then:

$$\dot{H}(f) \geq 0. \tag{5}$$

Unfortunately inequality 5 does not seem to exclude or guarantee the concentration feature in a finite time.

The only regularity result we have, (beyond the local existence thorem for classical solutions which is, of course, rather easy to obtain) concerns the case in which λ is sufficiently small compared with the size of the initial condition f_0 (in terms of the L_∞ norm of f_0 and its gradient), because the dispersivity of the free flow allow us to prevent singularities (see Ref. [BCP]).

The features of the dynamics of inelastic particles has been widely investigated in numerical simulations. Let us briefly describe what happen for a large number N of particles in a box, or with periodic spatial boundary conditions, initially distributed homogeneously in x and v. For $\lambda = \varepsilon N$ sufficiently large the system exhibits an inelastic collapse in a finite time. If λ is small, as the time goes on, the particle distribution remains spatially homogeneous, while the velocity distribution concentrates toward two peaks, at distance of order $\frac{1}{t}$. After this, for a larger number of collisions, the space homogeneity breaks down and correlation between particles becomes important.

The two-peaks feature can indeed be explained in terms of the behavior of the homogeneous solution via a careful rigorous asymptotic analysis when $t \to \infty$ (see [BCP]).

This universal behavior suggests that it can be interesting to consider the evolution of initial data for the kinetic equation concentrated into two hydrodynamic profiles. This can be easily done at a formal level. Let $\rho_{1,2}(x, t)$ and $u_{1,2}(x, t)$ with $u_1 < u_2$ be solutions of

$$\begin{aligned}
&\partial_t \rho_1 + \partial_x(\rho_1 u_1) = 0, \\
&\partial_t \rho_2 + \partial_x(\rho_2 u_2) = 0, \\
&\partial_t(\rho_1 u_1) + \partial_x(\rho_1 u_1^2) = \lambda \rho_1 \rho_2 (u_2 - u_1)^2, \\
&\partial_t(\rho_2 u_2) + \partial_x(\rho_2 u_2^2) = -\lambda \rho_1 \rho_2 (u_2 - u_1)^2.
\end{aligned} \tag{6}$$

Then the following measure in the phase space

$$f(x, v, t) dx\, dv = (\rho_1(x, t)\delta(v - u_1(x, t)) + \rho_2(x, t)\delta(v - u_2(x, t)))\, dx\, dv, \tag{7}$$

is a weak solution of the mean field equation, as it follows by a direct inspection.

Eq.s 6 describe the evolution of two profiles of density and velocity, interacting via a friction term. Obviously Eq.s 6 can be generalized by considering k profiles of density and velocity ρ_i, u_i, interacting via the friction terms $\lambda \rho_i \rho_j (u_i - u_j)^2$.

The conservative part of system 6 is not symmetrizable and the reaction term $\lambda\rho_1\rho_2(u_2-u_1)^2$ can favorite the occurrence of shocks in presence of concentrations of the densities. Nevertheless an existence theorem for regular solutions, locally in time, can be obtained by considering a particular change of variables which transforms eq. 6 in a linear wave equation. (Work in progress).

3. One-dimensional Boltzmann equation.

Standard arguments of kinetic theory will lead us to consider the following equation for the unknown $f = f(x, v, t)$ that is the probability density of a single particle:

$$\partial_t f(x,v,t) + v\partial_x f(x,v,t) = \\ l\int dv_1\, |v-v_1| \left(\frac{f(x,v^*,t)f(x,v_1^*,t)}{(1-2\varepsilon)^2} - f(x,v,t)f(x,v_1,t) \right), \tag{8}$$

where $v^* = v + \frac{\varepsilon}{1-2\varepsilon}(v-v_1)$, $v_1^* = v_1 - \frac{\varepsilon}{1-2\varepsilon}(v-v_1)$, are the pre-collisional velocity and $l > 0$ is the mean free time inverse of the theory.

How to justify the introduction of this equation on the basis of logically well founded arguments? One can just say that Eq. 8 is a simplified model of the more difficult two and three dimensional Boltzmann equation for rarefied gas of inelastic balls in the so called Boltzmann-Grad limit .

Let us now pass to the mathematical analysis of Eq. 8. The main difference between Eq. 8 and analogous one-dimensional kinetic equations relative to systems without energy dissipation is that, here, we do not have an H-theorem as a consequence of the tendency of the system to clusterize. As a technical consequence we do not know how to construct global solutions for large data. However we can construct global solutions under suitable smallness assumptions (λ small compared with the L_1 norm of the initial datum f_0. Note the difference with the smallness assumption for the mean-field equation !) exploiting the one-dimensionality of the system.

The proof (see Ref. [BCP2]) is similar to that presented in Ref. [A] for conservative systems. In this last case there is a standard way, based on the H-theorem, which allows us to extend theorems for data with small mass to general situations. The lack of an H-Theorem however, prevent us to apply these arguments to the present situation.

It is not worthless to mention that also the Bony's approach to one-dimensional kinetic models (see Ref. [B]) which do not make use the entropy functional, does not prevent, in our case, a blow up in a finite time.

Therefore we cannot exclude, for a general initial datum, the possibility of a blow-up of the solutions in a finite time. We believe that this cannot happen because we can exclude at least a total concentration of the solution in a finite time, namely that $f(t) \to \delta(x, v)$ weakly when $t \to t_0$ for some t_0. See [BCP2].

There is an interesting connection between the one-dimensional Boltzmann equation and the mean-field equation introduced in Sect. 4.

If we consider the limit $l \to \infty$, $\varepsilon \to 0$, $l\varepsilon = \lambda$, we find that he collision operator of the Boltzmann equation we are considering in this section, converge to

$$-\lambda\partial_v \left(f(x,v,t)F(x,v,t)\right),$$

where:

$$F(x,v,t) = \int d\bar{v}\,(\bar{v}-v)|\bar{v}-v|f(x,\bar{v},t).$$

It is not difficult to show that this limit holds for solutions of the homogeneous Boltzmann equation.

4. Heating the system.

The usual hydrodynamical description for rarefied gases does not apply to granular media. The major point is that, due to the energy dissipation, the system has no thermal equilibria described by Gibbs states. Moreover it is also well known that it is not sufficient to pump energy to the system locally, for instance by means of an hot wall, to thermalize the system (see Ref.[DLK]).

As a consequence we cannot consider local equilibria and a hydrodynamical picture.

A way to overcome this difficulty is to put the system in a thermal bath at a constant temperature. As usual the action of the reservoir on the system will be described by a Fokker-Planck term. The system itself will be considered in the kinetic picture arising by the quasielastic limit formalized in the mean-field equation. Therefore the equation which we are interested on reads:

$$(\partial_t + v\partial_x)f = -\lambda\partial_v(Ff) + \beta\partial_v vf + \sigma\partial_{vv}f. \tag{9}$$

The physical motivation of such a study is to detect the response of the system to a strong thermalization action. Moreover we want also to investigate the equilibria of the system in view of a possible definition of a local equilibrium concept which will be fundamental for establishing hydrodynamical equations. To this purpose we first consider the homogeneous version of Eq. 9 because locally, in the fast thermalization scale, the homogeneous regime is dominant.

We shall see that any solution of the homogeneous version of Eq. 9 converges, as $t \to \infty$, to an equilibrium which is not Maxwellian. Such stationary solution is described implicitly by an equation of mean-field type, and, for large v, behaves as $\exp -C|v|^3$. In other words, the inelastic interaction makes the equilibrium distribution more picked around zero with respect to the usual Maxwellian distribution.

We do not know whether this solution is a stable stationary solution for the general non homogeneous case. Also we do not know whether the thermal bath is sufficient to prevent the solution to have a collapse in a finite time.

A trivial calculation shows that the (formal) equilibria of Eq. 9 , satisfy the following equation:

$$\bar{f}(v) = \frac{1}{Z} e^{-(\frac{\beta}{2\sigma}v^2 + \frac{\lambda}{3\sigma}\int |v-\bar{v}|^3 \bar{f}(\bar{v})d\bar{v})}, \tag{10}$$

where

$$Z = \int dv e^{-(\frac{\beta}{2\sigma}v^2 + \frac{\lambda}{3\sigma}\int |v-\bar{v}|^3 \bar{f}(\bar{v})d\bar{v})}.$$

It is possible to show (see Ref. [BCCP]) that there exists a solution to Eq. 10 which minimize the free-energy functional :

$$\eta(f) = \int f(v)\log f(v)dv + \frac{\lambda}{6\sigma}\int |v-\bar{v}|^3 f(v)f(\bar{v})dvd\bar{v} + \frac{\beta}{2\sigma}\int dvv^2 f(v).$$

A remarkable property of the free energy functional is that it decreases along the solutions as follows by a direct computation and an integration by parts. As a consequence one can prove that any solution to Eq. 9 converges in $L_1(x, v)$ to the unique minimum $\bar{f}$ of the free-energy functional (which satisfies Eq. 10). See Ref. [BCCP] for details.

5. A hydrodynamical picture.

A natural problem arising in the study of a granular medium is that of the possibility of describing the system in terms of a hydrodynamical picture.

It is well known in kinetic theory that a large dense system is often conveniently described by hydrodynamic (macroscopic) variables rather than by kinetic (microscopic or mezoscopic) variables. The transition is nothing else than a change of variables motivated by the scaling properties of the system. If we apply the same procedure to the granular systems, we face at once a difficulty when considering the mean-field or the Boltzmann description. The mass and the momentum are preserved by the collision mechanism so that we hope to write down the continuity equation, expressing the mass conservation and an equation for the momentum field. However we do not have a good notion of a local equilibrium: the only steady states are the delta functions around the mean velocity, so that we get the pressureless gas. Nevertheless if we pump energy to the system by a thermal bath, we can hope to extract a reacher hydrodynamical behavior. Thus let us consider the mean-field equation with the additional Fokker-Planck term:

$$\partial_t f + v\partial_x f + \partial_v[(\lambda F - \beta v)f] = \sigma\partial_v^2 f \tag{11}$$

Let us remark, preliminary, that if $\beta > 0$, the momentum is not conserved anylonger. To have a non-trivial hydrodynamics we assume $\beta = 0$ (the infinite temperature case) and study:

$$\partial_t f + v\partial_x f + [(\lambda\partial_v(Ff) - \sigma\partial_v^2 f] = 0. \tag{12}$$

The usual space-time scaling and the (formal) Hilbert expansion techniques, lead us to the following hydrodynamics:

$$\begin{aligned} &\partial_t\rho(x,t) + \partial_x(\rho u)(x,t) = 0, \\ &\partial_t(\rho u)(x,t) + \partial_x \textstyle\int dv v^2 f(x,v,t) = 0, \end{aligned}$$

that is

$$\partial_t(\rho u)(x,t) + \partial_x(\rho u^2)(x,t) = -\partial_x \int dv[v - u(x,t)]^2 f(x,v,t).$$

The system is not in a closed form because we have not yet used the local equilibrium hypothesis. In doing this we can compute the pressure term and find the following apparently familiar system:

$$\begin{aligned} &\partial_t\rho(x,t) + \partial_x(\rho u)(x,t) = 0 \\ &\partial_t(\rho u)(x,t) + \partial_x(\rho u^2 + p)(x,t) = 0 \end{aligned} \tag{13}$$

with $p = \rho^\gamma, \gamma = 1/3$.

Eq.s 13 are exactly the equations of the gas dynamics of one-dimensional isentropic gases. The only important difference with the gas dynamic case is that, in general, $\gamma \geq 1$ while in the granular case $\gamma = 1/3 < 1$.

A mathematical study of Eq.s 13 is not standard. In fact the qualitative properties of the isentropic gas dynamics change radically if $\gamma < 1$ (see Ref. [BCGP]). Now the sound speed diverges if the density vanishes, and it is not straightforward to establish even a local existence theorem for the case of finite mass in the vacuum. Moreover it is possible to find solutions which collapse in finite time, i.e. the density concentrates in a delta function.

Finally let us notice that the derivation of the system 13 is formal, and it is not easy to prove the validity of the Hilbert expansion even for short times.

A parabolic scaling is also possible to get a nonlinear diffusion equation for the density ($\beta \neq 0$):

$$\partial_t \rho(x,t) - \frac{1}{\beta}\partial^2_{xx} \int dv v^2 f_0(x,v,t), \tag{14}$$

where f_0 is the local equilibrium. Thanks to the structure of the local equilibrium solution, it is clear that the above integral is of the form:

$$\int dv v^2 f_0(x,v,t) = \psi(\rho(x,t))$$

for some function ψ.

To obtain Eq. 14 we use an Hilbert expansion which makes a rigorous sense at least term by term (work in progress).

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